Controlled Markov Processes and Viscosity Solutions

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Second Edition

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We dedicate this edition to
Florence Fleming
Serpil Soner

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Preface to Second Edition

This edition differs from the previous one in several respects. The use of stochastic calculus and control methods to analyze financial market models has expanded at a remarkable rate. A new Chapter X gives an introduction to the role of stochastic optimal control in portfolio optimization and in pricing derivatives in incomplete markets. Risk-sensitive stochastic control has been another active research area since the First Edition of this book appeared. Chapter VI of the First Edition has been completely rewritten, to emphasize the relationships between logarithmic transformations and risk sensitivity. Risk-sensitive control theory provides a link between stochastic control and H-infinity control theory. In the H-infinity approach, disturbances in a control system are modelled deterministically, instead of in terms of stochastic processes. A new Chapter XI gives a concise introduction to two-controller, zero-sum differential games. Included are differential games which arise in nonlinear H-infinity control and as totally risk-averse limits in risk-sensitive stochastic control. Other changes from the First Edition include an updated treatment in Chapter V of viscosity solutions for second-order PDEs. Material has also been added in Section I.11 on existence of optimal controls in deterministic problems. This simplifies the presentation in later sections, and also is of independent interest.

We wish to thank D. Hernandez-Hernandez, W.M. McEneaney and S.-J. Sheu who read various new chapters of this edition and made helpful comments. We are also indebted to Madeline Brewster and Winnie Isom for their able, patient help in typing and revising the text for this edition.

W.H. Fleming
May 1, 2005 *H.M. Soner*

Preface

This book is intended as an introduction to optimal stochastic control for continuous time Markov processes and to the theory of viscosity solutions. We approach stochastic control problems by the method of dynamic programming. The fundamental equation of dynamic programming is a nonlinear evolution equation for the value function. For controlled Markov diffusion processes on n - dimensional euclidean space, the dynamic programming equation becomes a nonlinear partial differential equation of second order, called a Hamilton – Jacobi – Bellman (HJB) partial differential equation. The theory of viscosity solutions, first introduced by M. G. Crandall and P.-L. Lions, provides a convenient framework in which to study HJB equations. Typically, the value function is not smooth enough to satisfy the HJB equation in a classical sense. However, under quite general assumptions the value function is the unique viscosity solution of the HJB equation with appropriate boundary conditions. In addition, the viscosity solution framework is well suited to proving continuous dependence of solutions on problem data.

The book begins with an introduction to dynamic programming for deterministic optimal control problems in Chapter I, and to the corresponding theory of viscosity solutions in Chapter II. A rather elementary introduction to dynamic programming for controlled Markov processes is provided in Chapter III. This is followed by the more technical Chapters IV and V, which are concerned with controlled Markov diffusions and viscosity solutions of HJB equations. We have tried, through illustrative examples in early chapters and the selection of material in Chapters VI – VII, to connect stochastic control theory with other mathematical areas (e.g. large deviations theory) and with applications to engineering, physics, management, and finance. Chapter VIII is an introduction to singular stochastic control. Dynamic programming leads in that case not to a single partial differential equation, but rather to a system of partial differential inequalities. This is also a feature of other important classes of stochastic control problems not treated in this book, such as impulsive control and problems with costs for switching controls.

Value functions can be found explicitly by solving the HJB equation only in a few cases, including the linear–quadratic regulator problem, and some special problems in finance theory. Otherwise, numerical methods for solving the HJB equation approximately are needed. This is the topic of Chapter IX.

Chapters III, IV and VI rely on probabilistic methods. The only results about partial differential equations used in these chapters concern classical solutions (not viscosity solutions.) These chapters can be read independently of Chapters II and V. On the other hand, readers wishing an introduction to viscosity solutions with little interest in control may wish to focus on Chapter II, Secs. 4–6, 8 and on Chapter V, Secs. 4–8.

We wish to thank M. Day, G. Kossioris, M. Katsoulakis, W. McEneaney, S. Shreve, P. E. Souganidis, Q. Zhang and H. Zhu who read various chapters and made helpful comments. Thanks are also due to Janice D'Amico who typed drafts of several chapters. We are especially indebted to Christy Newton. She not only typed several chapters, but patiently helped us through many revisions to prepare the final version.

W.H. Fleming

June 1, 1992 *H.M. Soner*

Notation

In this book the following system of numbering definitions, theorems, formulas etc. is used. Roman numerals are used to refer to chapters. For example, Theorem II.5.1 refers to Theorem 5.1 in Chapter II. Similarly, IV(3.7) refers to formula (3.7) of Chapter IV; and within Chapter IV we write simply (3.7) for such a reference.

$I\!R^n$ denotes n-dimensional euclidean space, with elements $x = (x_1, \cdots, x_n)$. We write

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and $|x| = (x \cdot x)^{\frac{1}{2}}$ for the euclidean norm. If A is a $m \times n$ matrix, we denote by $|A|$ the operator norm of the corresponding linear transformation from $I\!R^n$ into $I\!R^d$:

$$|A| = \max_{|x| \leq 1} |Ax|.$$

The transpose of A is denoted by A'. If a and A are $n \times n$ matrices,

$$\operatorname{tr} aA = \sum_{i,j=1}^{n} a_{ij} A_{ij}.$$

$\mathcal{S}^n$ denotes the set of symmetric $n \times n$ matrices and $\mathcal{S}^n_+$ the set of nonnegative definite $A \in \mathcal{S}^n$. The interior, closure, and boundary of a set B are denoted by $intB, \bar{B}$ and ∂B respectively. If Σ is a metric space,

$\mathcal{B}(\Sigma) = \sigma -$ algebra of Borel sets of Σ

$\mathcal{M}(\Sigma) = \{$all real − valued functions on Σ which are bounded below$\}$

$C(\Sigma) = \{$all real − valued continuous functions on $\Sigma\}$

$C_b(\Sigma) =$ bounded functions in $C(\Sigma)\}$.

If Σ is a Banach space

$C_p(\Sigma) = \{$polynomial growing functions in $C(\Sigma)\}$.

A function ϕ is called polynomially growing if there exist constants $K, m \geq 0$ such that

$$|\phi(x)| \leq K(1 + |x|^m), \ \forall x \in \Sigma.$$

For an open set $O \subset I\!R^n$, and a positive integer k,

$C^k(O) = \{$all $k-$ times continuously differentiable functions on $O\}$

$C_b^k(O) = \{\phi \in C^k(O) :$ ϕ and all partial derivatives of ϕ or orders $\leq k$ are bounded$\}$

$C_p^k(O) = \{\phi \in C^k(O) :$ all partial derivatives of ϕ of orders $\leq k$ are polynomially growing$\}$.

For a measurable set $E \subset I\!R^n$, we say that $\phi \in C^k(E)$ if there exist $\tilde{E}$ open with $E \subset \tilde{E}$ and $\tilde{\phi} \in C^k(\tilde{E})$ such that $\phi(x) = \tilde{\phi}(x)$ for all $x \in E$. Spaces $C_b^k(E), C_p^k(E)$ are defined similarly. $C^\infty(E), C_b^\infty(E), C_p^\infty(E)$ denote the intersections over $k = 1, 2, \cdots$ of $C^k(E), C_b^k(E), C_p^k(E)$.

We denote the gradient vector and matrix of second order partial derivatives of ϕ by

$$D\phi = (\phi_{x_1}, \cdots, \phi_{x_n})$$

$$D^2\phi = (\phi_{x_i x_j}), i, j = 1, \cdots, n.$$

Sometimes these are denoted instead by ϕ_x, ϕ_{xx} respectively.

If ϕ is a vector-valued function, with values in $I\!R^m$, then we write $\phi \in C^k(E), \phi \in C_b^k(E)$ etc if each component of ϕ belongs to $C^k(E), C_b^k(E)$ etc. For vector-valued functions, $D\phi$ and $D^2\phi$ are identified with the differentials of ϕ of first and second orders. For vector-valued ϕ, $|D\phi|, |D^2\phi|$ are the operator norms. We denote intervals of $I\!R^1$, respectively closed and half-open to the right, by

$$[a, b], \quad [a, b).$$

Given $t_0 < t_1$

$$Q_0 = [t_0, t_1) \times I\!R^n, \qquad \overline{Q}_0 = [t_0, t_1) \times I\!R^n.$$

Given $O \subset I\!R^n$ open

$$Q = [t_0, t_1) \times O, \qquad \overline{Q} = [t_0, t_1] \times \overline{O}$$

$$\partial^* Q = ([t_0, t_1] \times \partial O) \cup (\{t_1\} \times O).$$

We call $\partial^* Q$ the *parabolic boundary* of the cylindrical region Q. If $\Phi = \phi(t,x), G \subset I\!R^{n+1}$, we say that $\Phi \in C^{\ell,k}(G)$ if there exist $\tilde{G}$ open with $G \subset \tilde{G}$ and $\tilde{\Phi}$ such that $\tilde{\Phi}(t,x) = \Phi(t,x)$ for all $(t,x) \in G$ and all partial derivatives of $\tilde{\Phi}$ or orders $\leq \ell$ in t and of orders $\leq k$ in x are continuous on $\tilde{G}$. For example, we often consider $\Phi \in C^{1,2}(G)$, where either $G = Q$ or $G = \overline{Q}$. The spaces $C_b^{\ell,k}(G), C_p^{\ell,k}(G)$ are defined similarly as above.

The gradient vector and matrix of second-order partial derivatives of $\Phi(t,\cdot)$ are denoted by $D_x\Phi, D_x^2\Phi$, or sometimes by Φ_x, Φ_{xx}.

If F is a real-valued function on a set U which has a minimum on U, then

$$\arg\min_{v \in U} F(v) = \{v^* \in U : F(v^*) \leq F(v)\ \forall v \in U\}.$$

The supnorm of a bounded function is denoted by $\|\ \|$, and L_p-norms are denoted by $\|\ \|_p$.