Undergraduate Texts in Mathematics

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(continued after index)

Stephanie Frank Singer

Linearity, Symmetry, and Prediction in the Hydrogen Atom

Springer

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To my mother, Maxine Frank Singer,
who always encouraged me to follow my own instincts:
I think I may be ready to learn some chemistry now.

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Preface

> It just means so much more to so much more people when you're rappin' and you know what for.
>
> — Eminem, "Business" [Mat]

This is a textbook for a senior-level undergraduate course for math, physics and chemistry majors. This one course can play two different but complementary roles: it can serve as a capstone course for students finishing their education, and it can serve as motivating story for future study of mathematics.

Some textbooks are like a vigorous regular physical training program, preparing people for a wide range of challenges by honing their basic skills thoroughly. Some are like a series of day hikes. This book is more like an extended trek to a particularly beautiful goal. We'll take the easiest route to the top, and we'll stop to appreciate local flora as well as distant peaks worthy of the vigorous training one would need to scale them.

Advice to the Student

This book was written with many different readers in mind. Some will be mathematics students interested to see a beautiful and powerful application of a "pure" mathematical subject. Some will be students of physics and chemistry curious about the mathematics behind some tools they use, such as spherical harmonics. Because the readership is so varied, no single reader should be put off by occasional digressions aimed at certain other readers. For instance, in Chapter 2, we include some examples from quantum mechanics; students unfamiliar with quantum mechanics should feel free to skip these paragraphs. Similarly, readers who do not intend to continue their mathematical studies should feel free to skip the brief discussions of more advanced mathematical concepts. We have tried to label these digressions and their intended audiences clearly. In particular, readers should feel free to skip the footnotes. Some exercises require knowledge of another subject (such as topology). These exercises are clearly marked. See, e.g., Exercise 4.28. *Italicized* terms are defined close by; terms "in quotation marks" are not.

The prerequisite for this course is solid understanding of calculus and familiarity with either linear algebra or advanced quantum mechanics. We discuss prerequisites in more detail in Section 1.5.

Finally, the author wishes to offer some broader advice to students: snap out of the one course, one book mode. Talk to people in other fields. Read related material in other sources. The more you can synthesize different points of view, the more powerfully creative you will be.

Advice to the Instructor

Although this book can be used for a homogeneous audience, the author hopes that it will encourage mixed classrooms: mathematics students working with students in the physical sciences. The author has found that students in such classrooms respond well to assignments that allow them to share their particular expertise with the class. One model that has worked well in the author's experience is to replace timed tests with a final project (paper and class presentation) on a related topic of the student's choice. We have listed some paper topic suggestions in Appendix C.

The minimum plan for a semester course should be to teach Chapters 1 through 7. Chapters 8, 9, 10 and 11 (each of which depends on Chapters 1 through 7) are independent from one another and can be used to fill out the semester. Note, however, that Section 11.4 depends on the idea that the state space for the spin of the electron is $\mathbb{C}^2$. This idea (and much more) can be found in Chapter 10.

The representation theory of finite groups is not presented anywhere in this text, setting this book apart from most undergraduate books on representation theory. The author urges instructors to resist the temptation to present the theory of finite group representations before starting the text. While some students find the finite group material helpful, others find it distracting or even downright off-putting. Students interested in the finite group theory can be encouraged to study it and its beautiful physical applications (to the spectroscopy of molecules, for example) as a related topic or final project.

This is a rigorous text, except for certain parts of Chapter 3 and Chapter 4. We state Fubini's theorem and the Stone–Weierstrass theorem without proof. We do not define the Lebesgue integral or manifolds rigorously, choosing instead to write in such a way that readers familiar with the theory will find only true statements while readers unfamiliar will find intuitive, suggestive, accessible language. Finally, in the proof of Proposition 10.6, we appeal to techniques of topology that are beyond the scope of the text.

Group Theory vs. Representation Theory

The phrase "group theory" says different things to different people. To a physicist, "group theory" means what a mathematician would call "representation theory." For example, the physicists' "group theory" includes what mathematicians would call the "representation theory of algebras"; never mind that algebras are not "groups" in the technical mathematical sense. On the other hand, mathematicians use the phrase "group theory" to refer to the study of groups and groups alone. The mathematicians' "group theory" encompasses the properties and classifications of groups and subgroups, and does not often include the study of representations of Lie algebras or classifications of representations of groups. In mathematics departments, representations of groups and other objects are the subject of books, courses and lectures in "representation theory."

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They say that behind every successful man is a woman; I say that behind every successful woman is a housekeeper. Many thanks to Emily Lam for keeping my home clean for many years. Thanks also to Dr. Andrew D'Amico and Dr. Julia Uffner, for keeping me alive and healthy.

The deepest and most heartfelt thanks go to my readers. Keep reading, and keep in touch!

Stephanie Frank Singer
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