Texts in Applied Mathematics 17

Editors

J.E. Marsden
L. Sirovich
M. Golubitsky
W. Jäger

Advisors

D. Barkley
M. Dellnitz
P. Holmes
G. Iooss
P. Newton

Texts in Applied Mathematics

1. Sirovich: Introduction to Applied Mathematics.
2. Wiggins: Introduction to Applied Nonlinear Dynamical Systems and Chaos.
3. Hale/Kocak: Dynamics and Bifurcations.
4. Chorin/Marsden: A Mathematical Introduction to Fluid Mechanics, 3rd ed.
5. Hubbard/West: Differential Equations: A Dynamical Systems Approach: Ordinary Differential Equations.
6. Sontag: Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd ed.
7. Perko: Differential Equations and Dynamical Systems, 2nd ed.
8. Seaborn: Hypergeometric Functions and Their Applications.
9. Pipkin: A Course of Integral Equations.
10. Hoppensteadt/Peskin: Mathematics in Medicine and the Life Sciences.
11. Braun: Differential Equations and Their Applications, 4th ed.
12. Stoer/Bulirsch: Introduction to Numerical Analysis, 2nd ed.
13. Renardy/Rogers: A First Graduate Course in Partial Differential Equations.
14. Banks: Growth and Diffusion Phenomena: Mathematical Frameworks and Applications.
15. Brenner/Scott: The Mathematical Theory of Finite Element Methods.
16. Van de Velde: Concurrent Scientific Computing.
17. Marsden/Ratiu: Introduction to Mechanics and Symmetry, 2nd ed.
18. Hubbard/West: Differential Equations: A Dynamical Systems Approach: Higher-Dimensional Systems.
19. Kaplan/Glass: Understanding Nonlinear Dynamics.
20. Holmes: Introduction to Perturbation Methods.
21. Curtain/Zwart: An Introduction to Infinite-Dimensional Linear Systems Theory.
22. Thomas: Numerical Partial Differential Equations: Finite Difference Methods.
23. Taylor: Partial Differential Equations: Basic Theory.
24. Merkin: Introduction to the Theory of Stability of Motion.
25. Naber: Topology, Geometry, and Gauge Fields: Foundations.
26. Polderman/Willems: Introduction to Mathematical Systems Theory: A Behavioral Approach.
27. Reddy: Introductory Functional Analysis: with Applications to Boundary Value Problems and Finite Element Methods.
28. Gustafson/Wilcox: Analytical and Computational Methods of Advanced Engineering Mathematics.
29. Tveito/Winther: Introduction to Partial Differential Equations: A Computational Approach.
30. Gasquet/Witomski: Fourier Analysis and Applications: Filtering, Numerical Computation, Wavelets.
31. Brémaud: Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues.
32. Durrant: Numerical Methods for Wave Equations in Geophysical Fluid Dynamics.
33. Thomas: Numerical Partial Differential Equations: Conservation Laws and Elliptic Equations.

Jerrold E. Marsden

Tudor S. Ratiu

Introduction to Mechanics and Symmetry

A Basic Exposition of
Classical Mechanical Systems

Second Edition

With 54 Illustrations

Springer

Jerrold E. Marsden
Control and Dynamical Systems, 107-81
California Institute of Technology
Pasadena, CA 91125
USA

Tudor S. Ratiu
Ecole polytechnique fédérale
de Lausanne
Département de mathématiques
CH-1015 Lausanne
Switzerland

Series Editors

J.E. Marsden
Control and Dynamical Systems, 107-81
California Institute of Technology
Pasadena, CA 91125
USA

L. Sirovich
Division of Applied Mathematics
Brown University
Providence, RI 02912
USA

M. Golubitsky
Department of Mathematics
University of Houston
Houston, TX 77204-3476
USA

W. Jäger
Department of Applied Mathematics
Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg
Germany

Mathematics Subject Classification (1991): 58-01, 70-01, 70Hxx

Library of Congress Cataloging-in-Publication Data
Marsden, Jerrold E.

Introduction to mechanics and symmetry / Jerrold E. Marsden,
Tudor S. Ratiu. — 2nd ed.

p. cm. — (Texts in applied mathematics ; 17)
Includes bibliographical references and index.

ISBN 978-1-4419-3143-6 ISBN 978-0-387-21792-5 (eBook)
DOI 10.1007/978-0-387-21792-5

1. Mechanics, Analytic. 2. Symmetry (Physics) I. Ratiu, Tudor S.

II. Title. III. Series.

QA808.M33 1999
531—dc21

98-44773

Printed on acid-free paper.

©1999 Springer Science+Business Media New York

Originally published by Springer-Verlag New York, Inc. in 1999

Softcover reprint of the hardcover 2nd edition 1999

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher Springer Science+Business Media, LLC, except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by MaryAnn Cottone; manufacturing supervised by Joe Quatela.
Camera-ready copy prepared from the authors' $\text{\LaTeX}$ files.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4419-3143-6

SPIN 10695792

To Barbara and Lilian for their love and support

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

Symmetry and mechanics have been close partners since the time of the founding masters: Newton, Euler, Lagrange, Laplace, Poisson, Jacobi, Hamilton, Kelvin, Routh, Riemann, Noether, Poincaré, Einstein, Schrödinger, Cartan, Dirac, and to this day, symmetry has continued to play a strong role, especially with the modern work of Kolmogorov, Arnold, Moser, Kirillov, Kostant, Smale, Souriau, Guillemin, Sternberg, and many others. This book is about these developments, with an emphasis on concrete applications that we hope will make it accessible to a wide variety of readers, especially senior undergraduate and graduate students in science and engineering.

The geometric point of view in mechanics combined with solid analysis has been a phenomenal success in linking various diverse areas, both within and across standard disciplinary lines. It has provided both insight into fundamental issues in mechanics (such as variational and Hamiltonian structures in continuum mechanics, fluid mechanics, and plasma physics) and provided useful tools in specific models such as new stability and bifurcation criteria using the energy–Casimir and energy–momentum methods, new numerical codes based on geometrically exact update procedures and variational integrators, and new reorientation techniques in control theory and robotics.

Symmetry was already widely used in mechanics by the founders of the subject, and has been developed considerably in recent times in such diverse phenomena as reduction, stability, bifurcation and solution symmetry breaking relative to a given system symmetry group, methods of finding explicit solutions for integrable systems, and a deeper understanding of spe-

cial systems, such as the Kowalewski top. We hope this book will provide a reasonable avenue to, and foundation for, these exciting developments.

Because of the extensive and complex set of possible directions in which one can develop the theory, we have provided a fairly lengthy introduction. *It is intended to be read lightly at the beginning and then consulted from time to time as the text itself is read.*

This volume contains much of the basic theory of mechanics and should prove to be a useful foundation for further, as well as more specialized, topics. Due to space limitations we warn the reader that many important topics in mechanics are not treated in this volume. We are preparing a second volume on general reduction theory and its applications. With luck, a little support, and yet more hard work, it will be available in the near future.

Solutions Manual. A solution manual is available for instructors. It contains complete solutions to many of the exercises, as well as other supplementary comments. For further information, see

<http://www.cds.caltech.edu/~marsden/books/>.

Internet Supplements. To keep the size of the book within reason, we are making some material available (free) on the Internet. These are a collection of sections whose omission does not interfere with the main flow of the text. See <http://www.cds.caltech.edu/~marsden/books/>. Updates and information about the book can also be found at this website.

What Is New in the Second Edition? In this second edition, the main structural changes are the creation of a solutions manual (along with many more exercises in the text) and the Internet supplements. The Internet supplements contain, for example, the material on the Maslov index that was not needed for the main flow of the book. As for the substance of the text, much of the book was rewritten throughout to improve the flow of material and to correct inaccuracies. Some examples: The material on the Hamilton–Jacobi theory was completely rewritten, a new section on Routh reduction (§8.9) was added, Chapter 9 on Lie groups was substantially improved and expanded. The presentation of examples of coadjoint orbits (Chapter 14) was improved by stressing matrix methods throughout.

Acknowledgments. We thank Rudolf Schmid, Rich Spencer, and Alan Weinstein for helping with an early set of notes that helped us on our way. Our many colleagues, students, and readers, especially Henry Abarbanel, Vladimir Arnold, Larry Bates, Michael Berry, Tony Bloch, Dong-Eui Chang, Hans Duistermaat, Marty Golubitsky, Mark Gotay, George Haller, Aaron Hershman, Darryl Holm, Phil Holmes, Sameer Jalnapurkar, Edgar Knobloch, P.S. Krishnaprasad, Naomi Leonard, Debra Lewis, Robert Littlejohn, Richard Montgomery, Phil Morrison, Richard Murray, Peter Olver, Oliver O’Reilly, Juan-Pablo Ortega, George Patrick, Octavian Popp, Mason Porter, Matthias Reinsch, Shankar Sastry, Tanya Schmäh, Juan Simo,

Hans Troger, Loc Vu-Quoc, and Steve Wiggins, have our deepest gratitude for their encouragement and suggestions. We also collectively thank all our students and colleagues who have used these notes and have provided valuable advice.

We are also indebted to Carol Cook, Anne Kao, Nawoyuki Gregory Kubota, Sue Knapp, Barbara Marsden, Marnie McElhiney, June Meyermann, Teresa Wild, and Ester Zack for their dedicated and patient work on the typesetting and artwork for this book. We want to single out with special thanks Hendra Adiwidjaja, Nawoyuki Gregory Kubota, and Wendy McKay for their special effort with the typesetting, the scripts for automatic conversion of references, the macros for indexing, and the figures (including the cover illustration). We also thank the staff at Springer-Verlag, especially Achi Dosanjh, Laura Carlson, MaryAnn Cottone, David Kramer, Ken Dreyhaupt, and Rüdiger Gebauer for their skillful editorial work and production of the book.

JERRY MARSDEN
Pasadena, California

TUDOR RATIU
Santa Cruz, California

December, 1998

About the Authors

Jerrold E. Marsden is professor of control and dynamical systems at Caltech. He received his B.Sc. at Toronto in 1965 and his Ph.D. from Princeton University in 1968, both in applied mathematics. He has done extensive research in mechanics, with applications to rigid-body systems, fluid mechanics, elasticity theory, plasma physics, as well as to general field theory. His primary current interests are in the area of dynamical systems and control theory, especially how it relates to mechanical systems with symmetry. He is one of the founders in the early 1970s of reduction theory for mechanical systems with symmetry, which remains an active and much studied area of research today. He was the recipient of the prestigious Norbert Wiener prize of the American Mathematical Society and the Society for Industrial and Applied Mathematics in 1990, and was elected a fellow of the American Academy of Arts and Sciences in 1997. He has been a Carnegie Fellow at Heriot-Watt University (1977), a Killam Fellow at the University of Calgary (1979), recipient of the Jeffrey-Williams prize of the Canadian Mathematical Society in 1981, a Miller Professor at the University of California, Berkeley (1981–1982), a recipient of the Humboldt Prize in Germany (1991), and a Fairchild Fellow at Caltech (1992). He has served in several administrative capacities, such as director of the Research Group in Nonlinear Systems and Dynamics at Berkeley, 1984–86, the Advisory Panel for Mathematics at NSF, the Advisory committee of the Mathematical Sciences Institute at Cornell, and as director of the Fields Institute, 1990–1994. He has served as an editor for Springer-Verlag's *Applied Mathematical Sciences* series since 1982 and serves on the editorial boards of several journals in mechanics, dynamics, and control.

Tudor S. Ratiu is professor of mathematics at the University of California, Santa Cruz and the Swiss Federal Institute of Technology, in Lausanne. He received his B.Sc. in mathematics and M.Sc. in applied mathematics, both at the University of Timișoara, Romania, and his Ph.D. in mathematics at Berkeley in 1980. He has previously taught at the University of Michigan, Ann Arbor, as a T. H. Hildebrandt Research Assistant Professor (1980–1983) and at the University of Arizona, Tucson (1983–1987). His research interests center on geometric mechanics, symplectic geometry, global analysis, and infinite-dimensional Lie theory, with applications to integrable systems, nonlinear dynamics, continuum mechanics, plasma physics, and bifurcation theory. He was a National Science Foundation Postdoctoral Fellow (1983–86), a Sloan Foundation Fellow (1984–87), a Miller Research Professor at Berkeley (1994), and a recipient of the Humboldt Prize in Germany (1997). Since his arrival at UC Santa Cruz in 1987, he has been on the executive committee of the Nonlinear Sciences Organized Research Unit. He is managing editor of the AMS *Surveys and Monographs* series and on the editorial board of the *Annals of Global Analysis* and the *Annals of the University of Timișoara*. He was also a member of various research institutes such as MSRI in Berkeley, the Center for Nonlinear Studies at Los Alamos, the Max Planck Institute in Bonn, MSI at Cornell, IHES in Bures-sur-Yvette, The Fields Institute in Toronto (Waterloo), the Erwin Schrödinger Institute for Mathematical Physics in Vienna, the Isaac Newton Institute in Cambridge, and RIMS in Kyoto.

Contents

Preface	ix
About the Authors	xiii
1 Introduction and Overview	1
1.1 Lagrangian and Hamiltonian Formalisms	1
1.2 The Rigid Body	6
1.3 Lie–Poisson Brackets, Poisson Manifolds, Momentum Maps	9
1.4 The Heavy Top	16
1.5 Incompressible Fluids	18
1.6 The Maxwell–Vlasov System	22
1.7 Nonlinear Stability	29
1.8 Bifurcation	43
1.9 The Poincaré–Melnikov Method	47
1.10 Resonances, Geometric Phases, and Control	49
2 Hamiltonian Systems on Linear Symplectic Spaces	61
2.1 Introduction	61
2.2 Symplectic Forms on Vector Spaces	66
2.3 Canonical Transformations, or Symplectic Maps	69
2.4 The General Hamilton Equations	74
2.5 When Are Equations Hamiltonian?	77
2.6 Hamiltonian Flows	80

2.7	Poisson Brackets	82
2.8	A Particle in a Rotating Hoop	87
2.9	The Poincaré–Melnikov Method	94
3	An Introduction to Infinite-Dimensional Systems	105
3.1	Lagrange’s and Hamilton’s Equations for Field Theory . .	105
3.2	Examples: Hamilton’s Equations	107
3.3	Examples: Poisson Brackets and Conserved Quantities . .	115
4	Manifolds, Vector Fields, and Differential Forms	121
4.1	Manifolds	121
4.2	Differential Forms	129
4.3	The Lie Derivative	137
4.4	Stokes’ Theorem	141
5	Hamiltonian Systems on Symplectic Manifolds	147
5.1	Symplectic Manifolds	147
5.2	Symplectic Transformations	150
5.3	Complex Structures and Kähler Manifolds	152
5.4	Hamiltonian Systems	157
5.5	Poisson Brackets on Symplectic Manifolds	160
6	Cotangent Bundles	165
6.1	The Linear Case	165
6.2	The Nonlinear Case	167
6.3	Cotangent Lifts	170
6.4	Lifts of Actions	173
6.5	Generating Functions	174
6.6	Fiber Translations and Magnetic Terms	176
6.7	A Particle in a Magnetic Field	178
7	Lagrangian Mechanics	181
7.1	Hamilton’s Principle of Critical Action	181
7.2	The Legendre Transform	183
7.3	Euler–Lagrange Equations	185
7.4	Hyperregular Lagrangians and Hamiltonians	188
7.5	Geodesics	195
7.6	The Kaluza–Klein Approach to Charged Particles	200
7.7	Motion in a Potential Field	202
7.8	The Lagrange–d’Alembert Principle	205
7.9	The Hamilton–Jacobi Equation	210
8	Variational Principles, Constraints, & Rotating Systems	219
8.1	A Return to Variational Principles	219
8.2	The Geometry of Variational Principles	226

8.3	Constrained Systems	234
8.4	Constrained Motion in a Potential Field	238
8.5	Dirac Constraints	242
8.6	Centrifugal and Coriolis Forces	248
8.7	The Geometric Phase for a Particle in a Hoop	253
8.8	Moving Systems	257
8.9	Routh Reduction	260
9	An Introduction to Lie Groups	265
9.1	Basic Definitions and Properties	267
9.2	Some Classical Lie Groups	283
9.3	Actions of Lie Groups	309
10	Poisson Manifolds	327
10.1	The Definition of Poisson Manifolds	327
10.2	Hamiltonian Vector Fields and Casimir Functions	333
10.3	Properties of Hamiltonian Flows	338
10.4	The Poisson Tensor	340
10.5	Quotients of Poisson Manifolds	349
10.6	The Schouten Bracket	353
10.7	Generalities on Lie–Poisson Structures	360
11	Momentum Maps	365
11.1	Canonical Actions and Their Infinitesimal Generators	365
11.2	Momentum Maps	367
11.3	An Algebraic Definition of the Momentum Map	370
11.4	Conservation of Momentum Maps	372
11.5	Equivariance of Momentum Maps	378
12	Computation and Properties of Momentum Maps	383
12.1	Momentum Maps on Cotangent Bundles	383
12.2	Examples of Momentum Maps	389
12.3	Equivariance and Infinitesimal Equivariance	396
12.4	Equivariant Momentum Maps Are Poisson	403
12.5	Poisson Automorphisms	412
12.6	Momentum Maps and Casimir Functions	413
13	Lie–Poisson and Euler–Poincaré Reduction	415
13.1	The Lie–Poisson Reduction Theorem	415
13.2	Proof of the Lie–Poisson Reduction Theorem for $GL(n)$	418
13.3	Lie–Poisson Reduction Using Momentum Functions	419
13.4	Reduction and Reconstruction of Dynamics	421
13.5	The Euler–Poincaré Equations	430
13.6	The Lagrange–Poincaré Equations	440

14	Coadjoint Orbits	443
14.1	Examples of Coadjoint Orbits	444
14.2	Tangent Vectors to Coadjoint Orbits	451
14.3	The Symplectic Structure on Coadjoint Orbits	453
14.4	The Orbit Bracket via Restriction of the Lie–Poisson Bracket	459
14.5	The Special Linear Group of the Plane	465
14.6	The Euclidean Group of the Plane	467
14.7	The Euclidean Group of Three-Space	472
15	The Free Rigid Body	481
15.1	Material, Spatial, and Body Coordinates	481
15.2	The Lagrangian of the Free Rigid Body	483
15.3	The Lagrangian and Hamiltonian in Body Representation	485
15.4	Kinematics on Lie Groups	489
15.5	Poinsot’s Theorem	490
15.6	Euler Angles	493
15.7	The Hamiltonian of the Free Rigid Body	495
15.8	The Analytical Solution of the Free Rigid-Body Problem .	498
15.9	Rigid-Body Stability	503
15.10	Heavy Top Stability	507
15.11	The Rigid Body and the Pendulum	512
	References	519
	Index	553