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Numerical Methods for Elliptic and Parabolic Partial Differential Equations

With 67 Figures

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series Texts in Applied Mathematics (TAM).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the Applied Mathematical Sciences (AMS) series, which will focus on advanced textbooks and research-level monographs.

Pasadena, California — J.E. Marsden
Providence, Rhode Island — L. Sirovich
College Park, Maryland — S.S. Antman

Preface to the English Edition

Shortly after the appearance of the German edition we were asked by Springer to create an English version of our book, and we gratefully accepted. We took this opportunity not only to correct some misprints and mistakes that have come to our knowledge[1] but also to extend the text at various places. This mainly concerns the role of the finite difference and the finite volume methods, which have gained more attention by a slight extension of Chapters 1 and 6 and by a considerable extension of Chapter 7. Time-dependent problems are now treated with all three approaches (finite differences, finite elements, and finite volumes), doing this in a uniform way as far as possible. This also made a reordering of Chapters 6–8 necessary. Also, the index has been enlarged. To improve the direct usability in courses, exercises now follow each section and should provide enough material for homework.

This new version of the book would not have come into existence without our already mentioned team of helpers, who also carried out first versions of translations of parts of the book. Beyond those already mentioned, the team was enforced by Cecilia David, Basca Jadamba, Dr. Serge Kräutle, Dr. Wilhelm Merz, and Peter Mirsch. Alexander Prechtel now took charge of the difficult modification process. Prof. Paul DuChateau suggested improvements. We want to extend our gratitude to all of them. Finally, we

[1] Users of the German edition may consult
http://www.math.tu-clausthal.de/~mala/publications/errata.pdf

thank senior editor Achi Dosanjh, from Springer-Verlag New York, Inc., for her constant encouragement.

Remarks for the Reader and the Use in Lectures

The size of the text corresponds roughly to four hours of lectures per week over two terms. If the course lasts only one term, then a selection is necessary, which should be orientated to the audience. We recommend the following "cuts":

Chapter 0 may be skipped if the partial differential equations treated therein are familiar. Section 0.5 should be consulted because of the notation collected there. The same is true for Chapter 1; possibly Section 1.4 may be integrated into Chapter 3 if one wants to deal with Section 3.9 or with Section 7.5.

Chapters 2 and 3 are the core of the book. The inductive presentation that we preferred for some theoretical aspects may be shortened for students of mathematics. To the lecturer's taste and depending on the knowledge of the audience in numerical mathematics Section 2.5 may be skipped. This might impede the treatment of the ILU preconditioning in Section 5.3. Observe that in Sections 2.1–2.3 the treatment of the model problem is merged with basic abstract statements. Skipping the treatment of the model problem, in turn, requires an integration of these statements into Chapter 3. In doing so Section 2.4 may be easily combined with Section 3.5. In Chapter 3 the theoretical kernel consists of Sections 3.1, 3.2.1, 3.3–3.4.

Chapter 4 presents an overview of its subject, not a detailed development, and is an extension of the classical subjects, as are Chapters 6 and 9 and the related parts of Chapter 7.

In the extensive Chapter 5 one might focus on special subjects or just consider Sections 5.2, 5.3 (and 5.4) in order to present at least one practically relevant and modern iterative method.

Section 8.1 and the first part of Section 8.2 contain basic knowledge of numerical mathematics and, depending on the audience, may be omitted.

The appendices are meant only for consultation and may complete the basic lectures, such as in analysis, linear algebra, and advanced mathematics for engineers.

Concerning related textbooks for supplementary use, to the best of our knowledge there is none covering approximately the same topics. Quite a few deal with finite element methods, and the closest one in spirit probably is [21], but also [6] or [7] have a certain overlap, and also offer additional material not covered here. From the books specialised in finite difference methods, we mention [32] as an example. The (node-oriented) finite volume method is popular in engineering, in particular in fluid dynamics, but to the best of our knowledge there is no presentation similar to ours in a mathematical textbook. References to textbooks specialised in the topics of Chapters 4, 5 and 8 are given there.

Remarks on the Notation

Printing in *italics* emphasizes definitions of notation, even if this is not carried out as a numbered definition.

Vectors appear in different forms: Besides the "short" space vectors $x \in \mathbb{R}^d$ there are "long" representation vectors $\boldsymbol{u} \in \mathbb{R}^m$, which describe in general the degrees of freedom of a finite element (or volume) approximation or represent the values on grid points of a finite difference method. Here we choose **bold type**, also in order to have a distinctive feature from the generated functions, which frequently have the same notation, or from the grid functions.

Deviations can be found in Chapter 0, where vectorial quantities belonging to $\mathbb{R}^d$ are boldly typed, and in Chapters 5 and 8, where the unknowns of linear and nonlinear systems of equations, which are treated in a general manner there, are denoted by $x \in \mathbb{R}^m$.

Components of vectors will be designated by a subindex, creating a double index for indexed quantities. Sequences of vectors will be supplied with a superindex (in parentheses); only in an abstract setting do we use subindices.

Erlangen, Germany — Peter Knabner
Clausthal-Zellerfeld, Germany — Lutz Angermann
January 2002

Preface to the German Edition

This book resulted from lectures given at the University of Erlangen–Nuremberg and at the University of Magdeburg. On these occasions we often had to deal with the problem of a heterogeneous audience composed of students of mathematics and of different natural or engineering sciences. Thus the expectations of the students concerning the mathematical accuracy and the applicability of the results were widely spread. On the other hand, neither relevant models of partial differential equations nor some knowledge of the (modern) theory of partial differential equations could be assumed among the whole audience. Consequently, in order to overcome the given situation, we have chosen a selection of models and methods relevant for applications (which might be extended) and attempted to illuminate the whole spectrum, extending from the theory to the implementation, without assuming advanced mathematical background. Most of the theoretical obstacles, difficult for nonmathematicians, will be treated in an "inductive" manner. In general, we use an explanatory style without (hopefully) compromising the mathematical accuracy.

We hope to supply especially students of mathematics with the information necessary for the comprehension and implementation of finite element/finite volume methods. For students of the various natural or engineering sciences the text offers, beyond the possibly already existing knowledge concerning the application of the methods in special fields, an introduction into the mathematical foundations, which should facilitate the transformation of specific knowledge to other fields of applications.

We want to express our gratitude for the valuable help that we received during the writing of this book: Dr. Markus Bause, Sandro Bitterlich, Dr. Christof Eck, Alexander Prechtel, Joachim Rang, and Dr. Eckhard Schneid did the proofreading and suggested important improvements. From the anonymous referees we received useful comments. Very special thanks go to Mrs. Magdalena Ihle and Dr. Gerhard Summ. Mrs. Ihle transposed the text quickly and precisely into TEX. Dr. Summ not only worked on the original script and on the TEX-form, he also organized the complex and distributed rewriting and extension procedure. The elimination of many inconsistencies is due to him. Additionally he influenced parts of Sections 3.4 and 3.8 by his outstanding diploma thesis. We also want to thank Dr. Chistoph Tapp for the preparation of the graphic of the title and for providing other graphics from his doctoral thesis [70].

Of course, hints concerning (typing) mistakes and general improvements are always welcome.

We thank Springer-Verlag for their constructive collaboration.

Last, but not least, we want to express our gratitude to our families for their understanding and forbearance, which were necessary for us especially during the last months of writing.

Erlangen, Germany — Peter Knabner
Magdeburg, Germany — Lutz Angermann
February 2000

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