Appendix 1

Functional Analysis

Hilbert space functional analysis plays for quantum probability the same role measure theory plays for classical probability. Many papers in quantum probability are unreadable by a non–specialist, because of their heavy load of references to advanced functional analysis, and in particular to von Neumann algebras. However, one can do a lot (not everything, but still a great deal) with a few simple tools. Such tools will be summarily presented in these Appendices, essentially in three parts : here, elementary results of functional analysis in Hilbert space : later, the basic theory of C^*–algebras, and finally, the essentials of von Neumann algebras. For most of the results quoted, a proof will be presented, sometimes in a sketchy way.

These sections on functional analysis have nothing original : my main contribution has consisted in choosing the *omitted* material — this is almost as important as choosing what one includes! The material itself comes from two books which I have found excellent in their different ways : Bratteli–Robinson [BrR1] *Operator Algebras and Quantum Statistical Mechanics I*, and Pedersen [Ped] *C^*–algebras and their Automorphism Groups*. The second book includes very beautiful mathematics, but may be *too* complete. The first one has greatly helped us by its useful selection of topics. For the results in this section, we also recommend Reed and Simon [ReS], and specially Parthasarathy's recent book [Par1], which is specially intended for quantum probability,.

Hilbert–Schmidt operators

1 We denote by $\mathcal{H}$ the basic Hilbert space. Given an orthonormal basis (e_n) and an operator $\mathbf{a}$, we put

$$\| \mathbf{a} \|_2 = \left(\sum\nolimits_n \| \mathbf{a}e_n \|^2\right)^{1/2} \le +\infty \,. \tag{1.1}$$

Apparently this depends on the basis, but let (e'_n) be a second o.n.b. ; we have

$$\| \mathbf{a} \|_2^2 = \sum\nolimits_{nm} |<\mathbf{a}e_n, e'_m>|^2 = \sum\nolimits_{nm} |<e_n, \mathbf{a}^* e'_m>|^2 = \| \mathbf{a}^* \|'^2_2$$

where the $'$ on the extreme right indicates that the "norm" is computed in the new basis. Taking first the two bases to be the same we get that $\| \mathbf{a} \|_2 = \| \mathbf{a}^* \|_2$, and then one sees that $\| \mathbf{a} \|_2$ does not depend on the choice of the basis. It is called the *Hilbert–Schmidt norm* of $\mathbf{a}$, and the space of all operators of finite HS norm is denoted by HS or by $\mathcal{L}^2$. In contrast to this, the space $\mathcal{L}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ will sometimes be denoted by $\mathcal{L}^\infty$ and its norm by $\| \;\|_\infty$. Since every unit vector can be included in some o.n.b., we have $\| \mathbf{a} \|_\infty \le \| \mathbf{a} \|_2$. According to (1.1) we have for every bounded operator $\mathbf{b}$

$$\| \mathbf{ba} \|_2^2 = \sum\nolimits_n \| \mathbf{ba}e_n \|^2 \le \| \mathbf{b} \|_\infty^2 \| \mathbf{a} \|_2^2 \tag{1.2}$$

and taking adjoints

$$\| \mathbf{ab} \|_2^2 \le \| \mathbf{b} \|_\infty^2 \| \mathbf{a} \|_2^2 \ . \tag{1.3}$$

Note in particular that $\| \mathbf{ab} \|_2 \le \| \mathbf{a} \|_2 \| \mathbf{b} \|_2$.

It is clear from (1.1) that a (hermitian) scalar product between HS operators can be defined by $<\mathbf{a}, \mathbf{b}>_{HS} = \sum_n <\mathbf{a}e_n, \mathbf{b}e_n>$. It is easily proved that the space *HS* is complete.

Trace class operators

2 Let first $\mathbf{a}$ be bounded and *positive,* and let $\mathbf{b} = \sqrt{\mathbf{a}}$ be its positive square root (if $\mathbf{a}$ has the spectral representation $\int_0^\infty t\, dE_t$, its square root is given by $\mathbf{b} = \int \sqrt{t}\, dE_t$). We have in any o.n. basis (e_n)

$$\sum_n < e_n, \mathbf{a}e_n > = \sum_n < \mathbf{b}e_n, \ \mathbf{b}e_n > = \| \mathbf{b} \|_2^2 \le +\infty \ . \tag{2.1}$$

Thus the left hand side does not depend on the basis : it is called the *trace* of $\mathbf{a}$ and denoted by $\mathrm{Tr}(\mathbf{a})$. For a positive operator, the trace (finite or not) is always defined. Since it does not depend on the basis, it is unitarily invariant ($\mathrm{Tr}(\mathbf{u}^*\mathbf{au}) = \mathrm{Tr}(\mathbf{a})$ if $\mathbf{u}$ is unitary).

Consider now a product $\mathbf{a} = \mathbf{bc}$ of two HS operators. We have

$$\sum_n |<e_n, \mathbf{a}e_n>| = \sum_n |<\mathbf{b}^* e_n, \mathbf{c}\, e_n>| \le \| \mathbf{b} \|_2 \| \mathbf{c} \|_2 < \infty \tag{2.2}$$

and

$$\sum_n < e_n, \mathbf{a}e_n > = \sum_n < \mathbf{b}^* e_n, \mathbf{c}\, e_n > = < \mathbf{b}^*, \mathbf{c} >_{HS} \ . \tag{2.3}$$

Since the right hand side does not depend on the basis, the same is true of the left hand side. On the other hand, the left hand side does not depend on the decomposition $\mathbf{a} = \mathbf{bc}$, so the right side does not depend on it either. Operators $\mathbf{a}$ which can be represented in this way as a product of two HS operators are called *trace class operators* (sometimes also *nuclear* operators), and the complex number (2.1) is denoted by $\mathrm{Tr}(\mathbf{a})$ and called the *trace* of a. One sees easily that, for positive operators, this definition of the trace is compatible with the preceding one. Note also that, given two arbitrary HS operators $\mathbf{b}$ and $\mathbf{c}$, their HS scalar product $<\mathbf{b}, \ \mathbf{c}>_{HS}$ is equal to $\mathrm{Tr}(\mathbf{b}^*\mathbf{c})$.

Intuitively speaking, HS operators correspond to "square integrable functions", and a product of two square integrable function is just an "integrable function", the trace corresponding to the integral. This is why the space of trace class operators is sometimes denoted by $\mathcal{L}^1$. In other contexts it may be denoted by $\mathcal{M}(\mathcal{H})$, a notation suggesting a space of bounded measures. The same situation occurs in classical probability theory on a discrete countable space like $\mathbb{N}$, on which all measures are absolutely continuous w.r.t. the counting measure. Indeed, non-commutative probability on $\mathcal{L}(\mathcal{H})$ is an extension of such a situation, the trace playing the role of the counting integral. More general σ-fields are represented in non-commutative probability by arbitrary von Neumann algebras, which offer much more variety than their commutative counterparts.

Given a bounded operator $\mathbf{a}$, we denote by $|\mathbf{a}|$ the (positive) square root of $\mathbf{a}^*\mathbf{a}$ — this is usually not the same as $\sqrt{\mathbf{a}\mathbf{a}^*}$, and this "absolute value" mapping has some pathological properties : for instance it is not subadditive. An elementary result called *the polar decomposition of bounded operators* asserts that $\mathbf{a} = \mathbf{u}|\mathbf{a}|$, $|\mathbf{a}| = \mathbf{u}^*\mathbf{a}$ where $\mathbf{u}$ is a unique *partial isometry, i.e.* is an isometry when restricted to $(\mathrm{Ker}\ \mathbf{u})^\perp$. We will not need the details, only the fact that $\mathbf{u}$ always has a norm ≤ 1, and is unitary if $\mathbf{a}$ is invertible. For all this, see Reed–Simon, theorem VI.10.

THEOREM. *The operator* $\mathbf{a}$ *is a product of two HS operators (i.e. belongs to the trace class) if and only if* $\mathrm{Tr}(|\mathbf{a}|)$ *is finite.*

PROOF. Let us assume $\mathrm{Tr}(|\mathbf{a}|) < \infty$, and put $\mathbf{b} = \sqrt{|\mathbf{a}|}$, a HS operator. Then $\mathbf{a} = \mathbf{u}|\mathbf{a}| = (\mathbf{u}\mathbf{b})\mathbf{b}$ is a product of two HS operators. Conversely, let $\mathbf{a} = \mathbf{h}\mathbf{k}$ be a product of two HS operators. Then $|\mathbf{a}| = \mathbf{u}^*\mathbf{a} = (\mathbf{u}^*\mathbf{h})\mathbf{k}$ and the same reasoning as (2.1) gives (the trace being meaningful since $|\mathbf{a}|$ is positive)

$$\mathrm{Tr}(|\mathbf{a}|) \leq \| \mathbf{u}^*\mathbf{h} \|_2 \| \mathbf{k} \|_2 \leq \| \mathbf{h} \|_2 \| \mathbf{k} \|_2$$

since $\|\mathbf{u}\|_\infty \leq 1$.

The same kind of proof leads to other useful consequences. Before we state them, we define the *trace norm* $\| \mathbf{a} \|_1$ of the operator $\mathbf{a}$ as $\mathrm{Tr}(|\mathbf{a}|)$. We shall see later that this is indeed a norm, under which $\mathcal{L}^1$ is complete.

a) *If* $\mathbf{a}$ *is a trace class operator, we have* $|\mathrm{Tr}(\mathbf{a})| \leq \|\mathbf{a}\|_1$.

Indeed, putting $\mathbf{b} = \sqrt{|\mathbf{a}|}$ as above, we have from the polar decomposition $\mathbf{a} = (\mathbf{u}\mathbf{b})\mathbf{b}$

$$|\mathrm{Tr}(\mathbf{a})| = |< \mathbf{b}^*\mathbf{u},\, \mathbf{b} >_{HS}| \leq \| \mathbf{b}^*\mathbf{u} \|_2 \| \mathbf{b} \|_2 \leq \| \mathbf{b} \|_2^2 = \| \mathbf{a} \|_1 \ .$$

b) *For* $\mathbf{a} \in \mathcal{L}^1$, $\mathbf{h} \in \mathcal{L}^\infty$, *we have* $\| \mathbf{a}\mathbf{h} \|_1 \leq \| \mathbf{a} \|_1 \| \mathbf{h} \|_\infty$.

Indeed, we have with the same notation $\mathbf{a} = \mathbf{u}\mathbf{b}\mathbf{b}$, $\mathbf{a}\mathbf{h} = (\mathbf{u}\mathbf{b})(\mathbf{b}\mathbf{h})$, then

$$\| \mathbf{a}\mathbf{h} \|_1 \leq \| \mathbf{u}\mathbf{b} \|_2 \| \mathbf{u}\mathbf{h} \|_2 \leq \| \mathbf{b} \|_2 \| \mathbf{b} \|_2 \| \mathbf{h} \|_\infty \ .$$

c) *If* $\mathbf{a} \in \mathcal{L}^1$ *we have* $\mathbf{a}^* \in \mathcal{L}^1$, $\| \mathbf{a} \|_1 = \| \mathbf{a}^* \|_1$.

Indeed $\mathbf{a} = \mathbf{u}|\mathbf{a}|$ gives $\mathbf{a}^* = |\mathbf{a}|\mathbf{u}^*$ and the preceding property implies $\| \mathbf{a}^* \|_1 \leq \| \mathbf{a} \|_1$, from which equality follows. Knowing this, one may take adjoints in b) and get the same property with $\mathbf{h}$ to the left of $\mathbf{a}$.

d) *For* $\mathbf{a} \in \mathcal{L}^1$, $\mathbf{h} \in \mathcal{L}^\infty$, *we have* $\mathrm{Tr}(\mathbf{a}\mathbf{h}) = \mathrm{Tr}(\mathbf{h}\mathbf{a})$.

This fundamental property has little to do with the above method of proof : if $\mathbf{h}$ is unitary, it reduces to the unitary invariance of the trace class and of the trace itself. The result extends to all bounded operators, since they are linear combinations of (four) unitaries.

The last property is a little less easy :

e) *If* $\mathbf{a} \in \mathcal{L}^1$, $\mathbf{b} \in \mathcal{L}^1$, *we have* $\mathbf{a} + \mathbf{b} \in \mathcal{L}^1$ *and* $\| \mathbf{a} + \mathbf{b} \|_1 \leq \| \mathbf{a} \|_1 + \| \mathbf{b} \|_1$.

To see this, write the three polar decompositions $\mathbf{a} = \mathbf{u}|\mathbf{a}|, \mathbf{b} = \mathbf{v}|\mathbf{b}|, \mathbf{a}+\mathbf{b} = \mathbf{w}|\mathbf{a}+\mathbf{b}|$. Then $|\mathbf{a} + \mathbf{b}| = \mathbf{w}^*(\mathbf{a} + \mathbf{b}) = \mathbf{w}^*\mathbf{u}^*|\mathbf{a}| + \mathbf{w}^*\mathbf{v}^*|\mathbf{b}|$. Let (e_n) be a finite o.n. system; we have

$$\sum\nolimits_n < e_n, |\mathbf{a}+\mathbf{b}| e_n > = \sum\nolimits_n < e_n, \mathbf{w}^*\mathbf{u}^*|\mathbf{a}| e_n > + \sum\nolimits_n < e_n, \mathbf{w}^*\mathbf{v}^*|\mathbf{b}| e_n > \leq \| \mathbf{a} \|_1 + \| \mathbf{b} \|_1 \ .$$

Then we let (e_n) increase to an orthonormal basis, etc.

EXAMPLE. Let $\mathbf{a}$ be a selfadjoint operator. Then it is easy to prove that $\mathbf{a}$ belongs to the trace class if and only if $\mathbf{a}$ has a discrete spectrum (λ_i), and $\sum_i |\lambda_i|$ is finite; then this sum is $\|\mathbf{a}\|_1$, and $\mathrm{Tr}(\mathbf{a}) = \sum_i \lambda_i$. It follows easily that the two operators $\mathbf{a}^+$, $\mathbf{a}^-$ belong to $\mathcal{L}^1$ if $\mathbf{a}$ does, and that $\|\mathbf{a}\|_1 = \|\mathbf{a}^+\|_1 + \|\mathbf{a}^-\|_1$, a result which corresponds to the Jordan decomposition of bounded measures in classical measure theory.

Duality properties

3 The results of this subsection are essential for the theory of von Neumann algebras, and have some pleasant probabilistic interpretations.

Let us denote by E_{xy} the operator of rank one

$$E_{xy}\, z = \langle y, z\rangle x \qquad (\,|x\rangle\langle y| \quad \text{in Dirac's notation.})$$

Then we have $(E_{xy})^* = E_{yx}$, $(E_{xy})^* E_{xy} = \|x\|^2 E_{yy}$, $\|E_{xy}\|_1 = \|x\|\|y\|$. The space $\mathcal{F}$ generated by these operators consists of all *operators of finite rank*. One can show that its closure in operator norm consists of all *compact operators*. Our aim is to prove

THEOREM. *The dual space of $\mathcal{F}$ is $\mathcal{L}^1$, and the dual of $\mathcal{L}^1$ is the space $\mathcal{L}^\infty$ of all bounded operators.*

The proof also makes explicit the duality functional between these spaces, in both cases the bilinear functional $(\mathbf{a}, \mathbf{b}) \longmapsto \mathrm{Tr}(\mathbf{ab})$). We leave it to the reader to check that the theorem remains true if all three spaces are restricted to their selfadjoint elements.

Note that the theorem implies that $\mathcal{L}^1$ is complete.

PROOF. 1) We remark first that for $\mathbf{a}$ positive we have $\mathrm{Tr}(\mathbf{a}) = \sup_{\mathbf{h}\in\mathcal{F}_1} \mathrm{Tr}(\mathbf{ah})$, where $\mathcal{F}_1$ denotes the unit ball of $\mathcal{F}$. To see this, if $\mathbf{a}$ has discrete spectrum, diagonalize it and choose for $\mathbf{h}$ a diagonal matrix whose diagonal coefficients are zeroes and finitely many ones, tending to the identity matrix. If $\mathbf{a}$ has some continuous spectrum, then the right side of the relation is $+\infty$, and on the other hand $\mathbf{a}$ dominates $c\mathbf{k}$ where $c > 0$ is a constant and $\mathbf{k}$ is the projector on some infinite dimensional subspace. Then taking $\mathbf{h}$ to be almost the identity matrix of this subspace we see that the right side is also equal to $+\infty$.

We extend this relation to arbitrary $\mathbf{a}$ using the polar decomposition $\mathbf{a} = \mathbf{u}|\mathbf{a}|$, $|\mathbf{a}| = \mathbf{u}^*\mathbf{a}$. Then we have from the above discussion

$$\begin{aligned} \|\mathbf{a}\|_1 &= \mathrm{Tr}(|\mathbf{a}|) = \sup_{\mathbf{k}\in\mathcal{F}_1} \mathrm{Tr}(|\mathbf{a}|\mathbf{k}) \\ &= \sup_{\mathbf{k}\in\mathcal{F}_1} \mathrm{Tr}(\mathbf{u}^*\mathbf{ak}) = \sup_{\mathbf{k}\in\mathcal{F}_1} \mathrm{Tr}(\mathbf{aku}^*) \end{aligned}$$

We now put $\mathbf{ku}^* = \mathbf{h}$, which belongs to $\mathcal{F}_1$, and we get $\mathrm{Tr}(\mathbf{a}) \le \sup_{\mathbf{h}\in\mathcal{F}_1} \mathrm{Tr}(\mathbf{ah})$; the reverse inequality is obvious.

2) Let φ be a continuous linear functional on $\mathcal{F}$. Then it is continuous for the topology induced by HS, which is *stronger* than the operator norm topology. Hence it can be written $\varphi(\cdot) = \langle\mathbf{a}, \cdot\rangle_{HS}$ for some HS operator $\mathbf{a}$. Using the preceding computation, one sees that $\|\mathbf{a}\|_1 = \|\varphi\|$, and in particular that $\mathbf{a}$ belongs to $\mathcal{L}^1$.

3) Let x, y be two normalized vectors. and let (e_n) be an o.n. basis whose first element is y. Then for every operator $\mathbf{h}$ we have $\mathbf{h}E_{xy}(e_n) = \mathbf{h}(x)$ for $n = 1$, 0 otherwise, hence $\mathrm{Tr}(\mathbf{h}E_{xy}) = <y, \mathbf{h}(x)>$. Since E_{xy} belongs to the unit ball of $\mathcal{L}^1$, we have

$$\|\mathbf{h}\|_\infty \leq \sup_{\|\mathbf{a}\|_1 \leq 1} \mathrm{Tr}(\mathbf{ha}) ,$$

whence the equality. If $\mathbf{h}$ is selfadjoint one can take $y = x$ in this argument, and therefore $\mathbf{a}$ can be taken to be selfadjoint.

4) Finally, let φ be a continuous linear functional on $\mathcal{L}^\infty$. Then $(y, x) \longmapsto \varphi(E_{xy})$ is a continuous hermitian bilinear functional, which can be written as $<y, \mathbf{h}x>$ for some bounded operator $\mathbf{h}$. The computation above shows that the operator norm of $\mathbf{h}$ is equal to the norm of φ. The two functionals $\varphi(\cdot)$ and $\mathrm{Tr}(\mathbf{h}\cdot)$ are equal on operators of finite rank, hence on all of $\mathcal{L}^1$. The proof is concluded.

Weak convergence properties

4 We are going now to study weak convergence properties for "measures" and "functions". The results concerning measures are pleasant, but not very important, and we do not prove them in detail.

We stop using boldface letters for operators.

We start with measures, recalling first Bourbaki's terminology for weak convergence on a locally compact space E. One says that a sequence (we leave it to the reader to rewrite the statements for filters, if he cares to) of probability measures (μ_n) converges *vaguely* to μ if $\mu_n(f)$ converges to $\mu(f)$ for every continuous function f with compact support. Then the limit is a positive measure with total mass ≤ 1. If μ is a true probability measure, then the sequence is said to converge *narrowly*, and then $\mu_n(f)$ converges to $\mu(f)$ for every continuous bounded function f. Convergence of sequences in the narrow topology implies *Prohorov's condition* : for every $\varepsilon > 0$ there exists a compact set K whose complement has measure $\leq \varepsilon$ for every measure μ_n. Conversely, every sequence μ_n which satisfies Prohorov's condition is relatively compact in the narrow topology. In fact, we do not need any sophisticated theory : quantum probability in the situation of this chapter is similar to measure theory on $\mathbb{N}$.

What corresponds to *vague* convergence for a sequence ρ_n of density matrices is the convergence of $\mathrm{Tr}(\rho_n a)$ to $\mathrm{Tr}(\rho a)$ for every operator a *of finite rank,* while *narrow* convergence holds if in addition $\mathrm{Tr}(\rho) = 1$. Since the closure in norm of the space of finite rank operators is the space of all compact operators, whose dual is $\mathcal{L}^1$, the unit ball of $\mathcal{L}^1$ is compact in the vague topology, just as in classical probability. If (ρ_n) converges narrowly to ρ, then $\rho_n(a)$ converges to $\rho(a)$ for every bounded operator a, just as in classical probability. *Sketch of proof* : we reduce to a self adjoint, then we approximate a in norm by operators with discrete spectrum : using the spectral representation of a this amounts to approximating uniformly the function t by a step function $f(t)$ on a compact interval. Then a can be diagonalized in some o.n.b. (e_n) and we have $\mathrm{Tr}(\rho_n a) = \sum_n \rho_{nn}\lambda_n$, where λ_n is the eigenvalue of a corresponding to the eigenvector e_n and ρ_{nn} is the n-th diagonal matrix element of ρ. Then we are reduced to a trivial problem of narrow convergence on $\mathbb{N}$.

"Prohorov's condition" takes the following form : for every $\varepsilon > 0$ there exists a projector P of finite rank such that $\|\rho_n - P\rho_n P\|_1 \le \varepsilon$ for all n. If (ρ_n) converges narrowly to ρ, then Prohorov's condition in this sense holds ([Dav], p. 291, lemma 4.3), and in fact ρ_n tends to ρ *in trace norm*. This is entirely similar to the fact that weak and strong convergence are the same *for sequences* in ℓ^1 (Dunford–Schwartz , *Linear Operators I*, p. 296, Cor. 14). This theorem is wrong for states on a general von Neumann algebra (see Dell'Antonio, [D'A]).

The "normal" topology for operators

5 The results in this subsection are fundamental for the theory of von Neumann algebras. They describe several weak topologies on the space $\mathcal{L}^\infty$ of all bounded operators on $\mathcal{H}$.

The best known among them are the *weak* and the *strong* topologies. As usual with locally convex topologies, they are defined by a family of seminorms $(p_\lambda)_{\lambda\in\Lambda}$, operators $(a_i)_{i\in I}$ converging to 0 (along some filter on I) if and only if for every λ the numbers $p_\lambda(a_i)$ converge to 0. In the case of the weak topology, Λ is the set of all pairs (x,y) of vectors, and $p_{x,y}(a) = |<y, ax>|$. In the case of the strong topology, Λ is the set of all vectors, and $p_x(a) = \|ax\|$.

There is a third basic topology, which arises from the fact that $\mathcal{L}^\infty$ is the dual of $\mathcal{L}^1$: in this case, Λ is the set of all trace class operators (or, what amounts to the same, the set of all density matrices ρ), and $p_\rho(a) = |\operatorname{Tr}(\rho a)|$. It turns out that it is the most important one, and that it has no satisfactory name. Its old name "ultraweak topology" is bad, since it seems to mean "still weaker than the weak topology", while it is *stronger* than the weak topology! Many people call it *σ-weak topology* (σ has the same meaning here as in *σ-finite*, the explanation being given below). My own tendency, given its importance, and the very frequent use of the adjective *normal* in the context of von Neumann algebras, to mean *continuous* in this topology, consists in calling it the *normal topology*.

The usual description of the normal topology is as follows : note that every trace class operator can be written as a product bc of two Hilbert–Schmidt operators, and that in an o.n.b. (e_n) one can write

$$\operatorname{Tr}(bca) = \operatorname{Tr}(cab) = \sum\nolimits_n <c^* e_n \, , ab(e_n)>$$

with $\sum_n \|x_n\|^2 < \infty$, $\sum_n \|y_n\|^2 < \infty$. Consider now a direct sum $\mathcal{K}$ of countably many copies of $\mathcal{H}$; an element of this space is a sequence $\mathbf{x} = (x_n)$ of elements of $\mathcal{H}$ such that $\|\mathbf{x}\|^2 = \sum_n \|x_n\|^2 < \infty$. On the other hand, associate with every bounded operator a on $\mathcal{H}$ the operator $\mathbf{a}$ on $\mathcal{K}$ which maps (x_n) into (ax_n). Then note that for every sequence $\mathbf{y}$ there is exactly one operator b which transforms (e_n) into (y_n). So the above displayed expression can be written $<\mathbf{x}, \mathbf{a}\mathbf{y}>$, and we see that *convergence of a in the normal topology of $\mathcal{H}$ amounts to the convergence of $\mathbf{a}$ in the usual weak topology of $\mathcal{K}$*.

One can define in the same way the "σ–strong topology", which is interesting but will not be considered here.

Since $\mathcal{L}^\infty$ is the dual of $\mathcal{L}^1$, its unit ball is compact in the normal topology, hence on the unit ball there cannot be a different weaker Hausdorff topology. Otherwise stated, as far as one restricts oneself to some norm bounded set of operators, there is no distinction between the normal and the weak topologies.

We end this section with a simple and well known result.

THEOREM. *Let φ be a linear functional on $\mathcal{L}^\infty(\mathcal{H})$, continuous for the strong topology of operators. Then φ is a finite linear combination of functionals of the type $a \longmapsto \langle y_k, ax_k \rangle$ (in particular, φ is also continuous for the weak topology).*

PROOF. By definition of the strong topology, there exist finitely many vectors x_i such that $\sup_i \| ax_i \| \leq 1 \Rightarrow |\varphi(a)| \leq 1$. Let p be the projector on the subspace $\mathcal{K}$ generated by these vectors. The relation $ap = 0$ implies $ax_i = 0$ for all i, hence $\varphi(a) = 0$, so we must have $\varphi(a) = \varphi(ap)$ for all a. Choose an o.n.b. (e_n) whose first k elements generate $\mathcal{K}$, and express $\varphi(a)$ as a linear combination of matrix elements of a in this basis, etc.

Since the dual space of $\mathcal{L}^\infty$ in the normal topology is $\mathcal{L}^1$, we see that the two topologies are rather different. On the other hand, a classical theorem of Banach (of which a very simple proof has been given by Mokobodzki : see for instance [DeM3], chap. X, n° 52), asserts that a linear functional on a dual Banach space is continuous in the weak* topology if and only if its restriction to the unit ball is continuous. Here, a linear functional on $\mathcal{L}^\infty$ is normally continuous (or, as stated briefly, is *normal*) if and only if its restriction to the unit ball is normally continuous. But on the unit ball the weak and normal topologies coincide. This will be useful as a convenient characterization of normal functionals without looking at the normal topology itself.

Tensor products of Hilbert spaces

6 The tensor product of two Hilbert spaces plays in quantum probability the role of the ordinary product of measure spaces in classical probability. This subsection is very important, and nearly trivial.

Given two Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$, we define their Hilbert space tensor product to consist of 1) a Hilbert space $\mathcal{C}$ and 2) a bilinear mapping $(f,g) \longmapsto f \otimes g$ from $\mathcal{A} \times \mathcal{B}$ to $\mathcal{C}$ such that

$$(6.1) \qquad \langle f \otimes g, h \otimes k \rangle_{\mathcal{C}} = \langle f, h \rangle_{\mathcal{A}} \langle g, k \rangle_{\mathcal{B}} .$$

On the other hand, the set of all vectors $f \otimes g$ should span $\mathcal{C}$.

From these two properties it is easy to deduce that, given two o.n. bases (f_α) of $\mathcal{A}$ and (g_β) of $\mathcal{B}$, the family $(f_\alpha \otimes g_\beta)$ is an o.n.b. of $\mathcal{C}$. It follows immediately that the space $\mathcal{C}$ and the bilinear mapping $\otimes$ are defined up to isomorphism. It is possible also to define uniquely a canonical tensor product, *i.e.* $\mathcal{A} \otimes \mathcal{B}$ and all the mappings involved are constructed from $\mathcal{A}$ and $\mathcal{B}$ in the language of set theory, but the need for such a precise definition is never felt.

What about existence? Assume $\mathcal{A}$ and $\mathcal{B}$ are given as concrete Hilbert spaces $L^2(E, \mathcal{E}, \lambda)$ and $L^2(F, \mathcal{F}, \mu)$ (for instance, o.n.b.'s (e_α) and (f_β) have been chosen, and λ, μ are the counting measures on the index sets). Then $\mathcal{C}$ is nothing but $L^2(E \times F, \lambda \otimes \mu)$, the product measure, $f \otimes g$ being interpreted as the function of

two variables $f(x)g(y)$ on $E \times F$ (which is quite often denoted $f \otimes g$ in a more general context). The verification of the two axioms is very easy. Besides that, the concrete case of L^2 spaces covers all practical situations.

Returning to abstract Hilbert spaces, it is convenient to have a name for the linear span (no closure operation) of all vectors $f \otimes g$, f and g ranging respectively over two subspaces $\mathcal{A}_\circ$ and $\mathcal{B}_\circ$ of $\mathcal{A}$ and $\mathcal{B}$: we call it the *algebraic tensor product of* $\mathcal{A}_\circ$ *and* $\mathcal{B}_\circ$. We introduce no specific notation for it (usually, one puts some mark like $\overline{\otimes}$ on the tensor sign).

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces, U and V be two continuous linear mappings from $\mathcal{A}$ to $\mathcal{H}$, $\mathcal{B}$ to $\mathcal{K}$. There exists a unique continuous mapping $W = U \otimes V$ from the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{H} \otimes \mathcal{K}$ such that

$$W(f \otimes g) = (Uf) \otimes (Vg)\,, \tag{6.2}$$

and we are going to prove that

$$\| W \| \leq \| U \| \, \| V \| \,. \tag{6.3}$$

Then it will extend by continuity to the full tensor product. To prove (6.3) we may proceed in two steps, assuming one of the two mappings to be identity — for instance $\mathcal{B} = \mathcal{K}$, $V = I$. Let (f_α), (g_β) be o.n. bases for $\mathcal{A}$ and $\mathcal{B}$, and let $\mathcal{A}_\circ$, $\mathcal{B}_\circ$ be their linear spans (without completion). Since the vectors $f_\alpha \otimes g_\beta$ are linearly independent, there is a unique operator W with domain $\mathcal{A}_\circ \otimes \mathcal{B}_\circ$ which maps $f_\alpha \otimes g_\beta$ to $U(f_\alpha) \otimes g_\beta$. Every vector in the domain can be written as a finite sum $z = \sum_\beta x_\beta \otimes g_\beta$, and its image is a sum $z' = Wz = \sum_\beta U(x_\beta) \otimes g_\beta$ of *orthogonal* vectors. Then we have

$$\| z' \|^2 = \sum \| U(x_\beta) \otimes g_\beta \|^2 \leq \| U \|^2 \sum \| x_\beta \|^2 \| g_\beta \|^2 = \| U \|^2 \| z \|^2 \,.$$

Relation (6.2) is proved on the domain, which is dense, and then the extension of W to an everywhere defined operator satisfying (6.2) and (6.3) is straightforward.

In classical probability, when we want to consider simultaneously two probabilistic objects, *i.e.* spaces $(E, \mathcal{E}, \mathbb{P})$ and $(F, \mathcal{F}, \mathbb{Q})$, we consider the product $(E \times F, \mathcal{E} \times \mathcal{F})$, and then a joint law on it which, if the two objects are physically unrelated, is the product law $\mathbb{P} \otimes \mathbb{Q}$ (classical probabilistic independence), and is some other law if there is a non–trivial correlation between them. The quantum probabilistic analogue consists in forming the tensor product $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ of the Hilbert spaces describing the two quantum objects, the state of the pair being the tensor product $\rho \otimes \sigma$ of the individual states if there is no interaction. This explains the basic character of the tensor product operation. On the other hand, we met in Chapter IV a new feature of quantum probability : systems of *indistinguishable objects* are not adequately described by the ordinary tensor product, but rather by subspaces of the tensor product subject to symmetry rules.

7 There are some relations between tensor products and HS operators. We describe them briefly.

First of all, we consider a Hilbert space $\mathcal{H}$ and its dual space $\mathcal{H}'$, *i.e.* the space of complex linear functionals on $\mathcal{H}$. Mapping a *ket* $x = |x\rangle$ to the corresponding *bra*

$x^* = < x \mid$ provides an ***antilinear*** 1–1 mapping from $\mathcal{H}$ onto $\mathcal{H}'$. Since $\mathcal{H}'$ is a space of complex functions on $\mathcal{H}$, the tensor product $u \otimes v$ of two elements of $\mathcal{H}'$ has a natural interpretation as the function $u(x)v(y)$ on $\mathcal{H} \times \mathcal{H}$, which is bilinear. If we denote by (e_α) an o.n.b. for $\mathcal{H}$, by e^α its dual basis, an o.n.b. for $\mathcal{H}' \otimes \mathcal{H}'$ is provided by the bilinear functionals $e^{\alpha\beta} = e^\alpha \otimes e^\beta$, and the elements of $\mathcal{H}' \otimes \mathcal{H}'$ are bilinear forms which may be expanded as $\sum c_{\alpha\beta} e^{\alpha\beta}$ with a square summable family of complex coefficients $c_{\alpha\beta}$. This is much smaller than the space of all continuous bilinear forms on $\mathcal{H} \times \mathcal{H}$, and is called the space of *Hilbert-Schmidt forms.* One may define similarly multiple tensor products, and Hilbert–Schmidt n–linear forms.

Why Hilbert-Schmidt ? Instead of considering $\mathcal{H}' \otimes \mathcal{H}'$ let us consider $\mathcal{H}' \otimes \mathcal{H}$, a space of bilinear functionals on $\mathcal{H} \times \mathcal{H}'$. Any operator A on $\mathcal{H}$ provides such a bilinear form, namely $(x, y') \longmapsto (y', Ax)$ — everything here is bilinear and the hermitian scalar product is not used — and the basis elements $e^\alpha \otimes e_\beta$ are read in this way as the operators $\mid e_\beta > < e_\alpha \mid$ (Dirac's notation) which constitute an o.n.b. for the space of Hilbert–Schmidt operators.

More generally, the tensor product $x^* \otimes y$ can be interpreted as the rank one operator $|y><x|$, and the algebraic tensor product $\mathcal{H}' \check{\otimes} \mathcal{H}$ as the space of all operators of finite rank.

Appendix 2

Conditioning and Kernels

Conditioning is one of the basic ideas of classical probability, and the greatest success of the Kolmogorov system of axioms was its inclusion of conditioning as a derived notion, without need of special axioms. Therefore classical probabilists will expect a discussion of conditioning in quantum probability. The answer is the report of a failure, deeply rooted into the physics of observation and measurement — a topic which has been, and still is, the subject of innumerable discussions, and which we avoid altogether because of our incompetence.

The results of this Appendix are not used elsewhere in these notes.

Conditioning : discrete case

1 The discussion in this and the following subsection is inspired from Davies' book [Dav], *Quantum Theory of Open Systems,* p. 15–17.

Consider first a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X taking values in some nice measurable space E. Then the space Ω is decomposed into the "slices" $\Omega_x = X^{-1}(x)$, $x \in E$, and the measure $\mathbb{P}$ can be disintegrated according to the observed value of X. Namely, for every $x \in E$, there exists a law $\mathbb{P}_x$ on Ω, carried by Ω_x, such that for $A \in \mathcal{F}$ (μ denoting the law of X under $\mathbb{P}$)

$$\mathbb{P}(A) = \int \mathbb{P}_x(A)\, \mu(dx) \ .$$

Intuitively speaking, if we observe that $X = x$, then the absolute law $\mathbb{P}$ is reduced to the conditional law $\mathbb{P}_x$. What about getting a similar disintegration in quantum probability?

We now denote by Ω, as we did previously, the basic Hilbert space, and by X a r.v. on Ω, taking values in E (a spectral measure over E). We begin with the case of a countable space E : then for each $i \in E$ we have an "event" (subspace) A_i and its "indicator" (spectral projector) P_i. Let Z denote a real valued random variable (selfadjoint operator), bounded for simplicity.

The easiest thing to describe is the decomposition of Ω in slices : these are simply the subspaces $A_i = \{X = i\}$. For continuous random variables, this will be non–trivial, and we deal with this problem in subsection 4.

In quantum physics, a measurement is a model for the concrete physical process of installing a macroscopic apparatus which filters the population of particles according to some property, here the value of X. When this is done (physically : when the power is turned on) the state of the system changes. If it was described by the density operator ρ, it is now described by

$$\widetilde{\rho} = \sum\nolimits_i P_i \rho P_i \ . \tag{1.1}$$

Note that $\widetilde{\rho}$ is a density operator, which commutes with all the spectral projectors P_i of X (we simply say : which commutes with X). Also note that, if ρ originally did commute with X, no change has occurred. This change of state has nothing to do with our looking at the result of the experiment. Indeed, *if we do,* that is if we filter the population which goes out of the apparatus and select those particles for which X takes the value i, a further change takes place, which this time is the same familiar one as in classical probability : $\widetilde{\rho}$ is replaced by the conditional density operator

$$\widetilde{\rho}_i = \frac{P_i \rho P_i}{\mathrm{Tr}(\rho P_i)} \ . \tag{1.2}$$

Starting from these assumptions about the effect of a measurement of X on the state of the system, we are going to compute the expectation of Z and "joint distributions".

First of all, the expectation of Z under the new law $\widetilde{\rho}$ is equal to

$$\begin{aligned}\widetilde{\mathbb{E}}[Z] = \mathrm{Tr}(\widetilde{\rho}Z) &= \sum\nolimits_i \mathrm{Tr}(P_i\rho P_i Z)\\ &= \sum\nolimits_i \mathrm{Tr}(\rho P_i Z P_i)\end{aligned}$$

(the property that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ has been used). This is not the same as the original expectation of Z — otherwise stated, contrary to the classical probability case, *conditioning changes expectation values.* The difference is

$$\begin{aligned}\mathbb{E}[Z] - \widetilde{\mathbb{E}}[Z] &= \sum\nolimits_{ik} \mathrm{Tr}(\rho P_i Z P_k) - \sum\nolimits_i \mathrm{Tr}(\rho P_i Z P_i)\\ &= \sum\nolimits_{i\neq k} \mathrm{Tr}(\rho P_i Z P_k)\end{aligned}$$

(note that the difference vanishes if either Z or ρ commutes with X). Instead of deciding that the state has changed (Schrödinger's picture) we might have decided that the random variable Z has been replaced by $\widetilde{Z} = \sum_i P_i Z P_i$, the state remaining unchanged (Heisenberg's picture). Though both points of view are equivalent for Hamiltonian evolutions, here the first point of view seems more satisfactory. Indeed, basic properties of Z like being integer valued (having a spectrum contained in $\mathbb{N}$, with the possible physical meaning of being the output of a counter) is not respected by the above operation on r.v.'s, while one has no objection to a shift from discrete spectrum to continuous spectrum in a change of density matrix.

Let us consider a second r.v. Y taking values in a countable space F (points denoted j, events B_j, spectral projectors Q_j), and let us compute a "joint distribution" for X and Y according to the preceding rules. If it has been observed that $X = i$, then the probability that $Y = j$ is $\mathrm{Tr}(\widetilde{\rho}_i Q_j) = \mathrm{Tr}(\rho P_i Q_j P_i)/\mathrm{Tr}(\rho P_i)$, and the probability that *first* the measure of X yields i and *then* the measure of Y yields j is

$$p_{ij} = \mathrm{Tr}(\rho P_i Q_j P_i)$$

These coefficients define a probability law, but they depend on the order in which the measurements are performed. Note also that the mapping $(i,j) \longmapsto P_i Q_j P_i$ does not define an observable on $E \times F$, since the selfadjoint operators $P_i Q_j P_i$ generally are not projectors (unless X and Y commute). This obviously calls for a generalization of the notion of observable (see subs. 4.1).

Conditioning : continuous case

2 We are going now to deal with the case of a continuous r.v. X, taking values in a measurable space $(E,\mathcal{E})$ with a countably generated σ-field. Consider an increasing family of finite σ-fields $\mathcal{E}_n$ whose union generates $\mathcal{E}$. For notational simplicity, let us assume that the first σ-field $\mathcal{E}_0$ is trivial, and that every atom H_i of $\mathcal{E}_n$ is divided in two atoms of $\mathcal{E}_{n+1}$, so that the atoms of $\mathcal{E}_n$ are indexed by the set $W(n)$ of dyadic words with n letters. Let X_n be the random variable X considered as a spectral measure on $(E,\mathcal{E}_n)$. Since X_n is a discrete random variable, the conditional expectation of Z given X_n is the following function η_n on E (I_{H_i} here is an indicator function in the usual sense, not a projector, and our conditional expectation is a r.v. in the usual sense)

$$\eta_n = \sum\nolimits_{i\in W(n)} \frac{\mathrm{Tr}(\rho P_i Z P_i)}{\mathrm{Tr}(\rho P_i)} I_{H_i} \ .$$

The analogy with the theory of derivation leads us to compute $\eta_n - \mathbb{E}\,[\eta_{n+1}|\,\mathcal{E}_n]$ (ordinary conditional expectation of the r.v. X, with respect to the law $\mathbb{P}$). To help intuition, let us set for every "event" (subspace) A with associated projector P, $\mathbb{E}\,[\,A\,;\,Z\,] = \mathrm{Tr}(\rho PZP)$. Then the function we want to compute is equal on H_i to

$$\frac{1}{\mathbb{P}(H_i)}\left(\mathbb{E}\,[\{X\in H_i\}\,;\,Z\,] - \mathbb{E}\,[\{X\in H_{i0}\}\,;\,Z\,] - \mathbb{E}\,[\{X\in H_{i1}\}\,;\,Z\,]\right)$$

and we have

$$\mathbb{E}\,[\,|\eta_n - \mathbb{E}\,[\eta_{n+1}|\,\mathcal{E}_n\,]\,|\,] = \sum\nolimits_{i\in W(n)} |\,\mathrm{Tr}(\rho P_{i0} Z P_{i1}) + \mathrm{Tr}(\rho P_{i1} Z P_{i0})|$$

Let H belong to $\mathcal{E}$ and P_H denote the projector on the subspace $\{X\in H\}$. Then the function

$$(H,K)\longmapsto \mathrm{Tr}(\rho P_H\, Z P_K\,) \tag{2.1}$$

is a *complex bimeasure* ν, and we will assume it has bounded variation, *i.e.* can be extended into a bounded complex measure on $E\times E$, still denoted by ν (we return to the discussion of ν at the end of this subsection). The above expectation then is bounded by

$$\sum\nolimits_{i\in W(n)} \left(|\nu|(H_{i0}\times H_{i1}) + |\nu|(H_{i1}\times H_{i0})\right)$$

and finally, summing over n, we get that

$$\sum\nolimits_n \mathbb{E}\,[\,|\eta_n - \mathbb{E}\,[\eta_{n+1}|\mathcal{E}_n]\,|\,] \le |\nu|(E\times E\setminus\Delta)$$

Δ denoting the diagonal. Thus the (ordinary) random variables η_n constitute a *quasimartingale*. Since we have $\mathbb{E}\,[\,|\eta_n|\,] = \sum_{i\in\omega(n)} |Tr(\rho P_i Z P_i)| \le \sum_i |\nu|(H_i\times H_i) \le |\nu|(E\times E)$, this quasimartingale is bounded in L^1, and finally η_n *converges* $\mathbb{P}$-*a.s.*.

Let us end with a remark on the bimeasure $\nu(H,K) = \mathrm{Tr}(P_H\, Z P_K\,)$. We have used it to estimate the change in expectation due to conditioning, that is an expression of

the following form, where (H_i) is some partition of E

$$\mathbb{E}[\sum_i P_{H_i} Z P_{H_i} - Z] = \mathbb{E}[\frac{1}{2}\sum_i (2P_{H_i} Z P_{H_i} - P_{H_i} Z - Z P_{H_i})]$$
$$= \mathbb{E}[\frac{1}{2}\sum_i [P_{H_i}, [Z, P_{H_i}]]] .$$

Hence instead of studying the complex bimeasure ν, we may as well study the real bimeasure $\theta(H,K) = \mathrm{Tr}\,(\rho\,[P_H, [Z, P_K]])$.

3 We illustrate the preceding computations, by the basic example of the *canonical pair* (studied in detail in Chapter 3). Explicitly, Ω is the space $L^2(\mathbb{R})$ (Lebesgue measure); E is the line, the initial partition is given by the dyadic integers, and then we proceed by cutting each interval in two equal halves; X is the identity mapping from Ω to $\mathbb{R}$, *i.e.* for a Borel set A of the line P_A is the operator of multiplication by I_A. For ρ we choose the pure state ε_ω corresponding to the "wave function" ω. Finally, the operator we choose for Z will not be a bounded selfadjoint operator, but rather the *unitary* operator $Zf(x) = f(x-u)$: since Z is a complex linear combination of two (commuting) selfadjoint operators, the preceding theory can be applied without problem. It is clear that Z is the worst possible kind of operator from the point of view of the "slicing" : instead of operating along the slices it interchanges them. We shall see later that $Z = e^{-iuY}$, where $Y = -i\mathrm{D}$ is the momentum operator of the canonical pair.

The bimeasure we have to consider in this case is

$$\nu(H,K) = \mathrm{Tr}\,(\rho P_H Z P_K) = \langle\omega, P_H Z P_K \omega\rangle$$
$$= \int \bar{\omega}(s)\, I_H(s)\, I_K(s-u)\,\omega(s-u)\, ds .$$

Let $\lambda(ds)$ be the complex measure on the line with density $\bar{\omega}(s)\,\omega(s-u)$; since ω is normalized, the total mass of λ is at most 1. Let n be the image of λ under the mapping $s \longmapsto (s, s+u)$; then $\nu(H,K) = n(H \times K)$, and therefore the basic assumption for the convergence of the conditional expectations is satisfied; we may forget the notation n and use ν for both objects, bimeasure and measure.

In the case we are considering, *the limit of* $\mathbb{E}[Z|\mathcal{E}_n]$ *is a.s. equal to* 0. Indeed, $\mathrm{Tr}(\rho P_i Z P_i) = 0$ when the partition is fine enough, simply because then the square $H_i \times H_i$ does not meet the line $y = x + u$ which carries ν. Otherwise stated, we have

$$\mathbb{E}[e^{-iuY}|X] = 0 \quad \text{if} \quad u \neq 0 \;, \; = 1 \quad \text{if} \quad u = 0 \;.$$

In weak convergence problems, this degenerate characteristic function is typical of cases where all the mass escapes to infinity.

Multiplicity theory

4 We are going now to extend the idea of "slicing" or "combing" the basic Hilbert space Ω to the case of a continuous r.v. X. This is also called "multiplicity theory", and applies just as well to *real* Hilbert spaces (for an application of the real case, see

Sém. Prob IX, LN. 465, p. 73–88, which also contains a proof of the theorem itself). A complete proof is also given in [Par1].

Note first that, a nice measurable space being isomorphic to a Borel subset of $\mathbb{R}$, we lose no generality by assuming that X is real valued. Hence X is associated with an orthogonal resolution of identity $(\mathcal{H}_t)$ with spectral projectors E_t. Let us call *martingale* any curve $x = x(\cdot)$ such that $x(s) = E_s x(t)$ for $s < t$, and denote by η the "bracket" of the martingale, *i.e.* the measure on $\mathbb{R}$ such that $\eta(]\,s,t\,]) = \|\,x(t) - x(s)\,\|^2$. Let us call *stable subspace of* Ω any closed subspace which is stable under all projectors E_t. An example of such a subspace is given by $S(x)$, the set of all "stochastic integrals" $\int f(s)\,dx(s)$ with $f \in L^2(\eta)$; its orthogonal space $S(x)^{\perp}$ is a stable subspace too.

The mapping $f \longmapsto \int f(s)\,dx(s)$ is an isomorphism from the Hilbert space $\mathcal{M}(\eta) = L^2(\mathbb{R},\eta)$, which we call the *model space,* onto $S(x)$; it is slightly more than that : the model space carries a natural spectral family $(\mathcal{I}_t)$, corresponding to functions supported by the half-line $]-\infty,t]$ (the selfadjoint operator it generates is multiplication by the function x), and $\mathcal{I}_t$ is carried by the isomorphism into the resolution of identity $\mathcal{H}_t \cap S(x)$, induced on $S(x)$ by our original spectral family. *Thus* Ω *contains a stable subspace which is a copy of the model.*

We now replace Ω by $\Omega_1 = S(x)^{\perp}$ and (if it is not reduced to 0) extract from it a second copy of the model, possibly with a different measure η_1. Iterating transfinitely this procedure, it is very intuitive that Ω *can be decomposed into a direct sum of copies of model spaces.* This intuition can be made rigorous by Zorn's lemma. Since Ω is always assumed to be separable, this direct sum decomposition is necessarily countable. Rearranging the indexes into a single sequence, denote by $S(x_n)$ the spaces and by η_n the corresponding measures, choose a measure θ such that every η_n is absolutely continuous w.r. to θ (a probability measure if you wish), and denote by h_n a density of η_n w.r. to θ.

Every $\omega \in \Omega$ can be represented uniquely in the form

$$\omega = \sum\nolimits_n \int f_n(s)\,dx_n(s) \quad \text{with} \quad \sum \int |\,f_n(s)\,|^2 \eta_n(ds) < \infty \ .$$

Associate with every $s \in \mathbb{R}$ the Hilbert space $\mathcal{F}_s$ consisting of the sequences (x_n) of complex numbers such that $\sum_n |x_n|^2 h_n(s) < \infty$ (for simplicity we assume $h_n(s) > 0$ for all n ; if this condition is not fulfilled, consider only those n such that $h_n(s) > 0$). Then for θ–a.e. s the sequence $(f_n(s))$ belongs to $\mathcal{F}_s$, and the above isomorphism realizes the "slicing" of Ω we were looking for. Such a "slicing" has an official name : in Hilbert space language, one says that Ω *is isomorphic to the continuous sum of the measurable family of Hilbert spaces* $\mathcal{F}_s$ *over the measure space* $(\mathbb{R}, \theta)$. We do not need to make here an axiomatic theory of continuous sums of Hilbert spaces, since the above description was entirely explicit. For a general discussion, see Dixmier [Dix], part II, Chapter 1.

Up to now, we have not risen much above the level of triviality. Things become more interesting when we try to get some kind of uniqueness. Here we shall explain the results, without even sketching a proof.

First of all, the "models" we use are not uniquely determined. If γ and η are two equivalent measures on $\mathbb{R}$, and j is a density of η w.r. to γ, the mapping $f \longmapsto f\sqrt{\gamma}$ is

an isomorphism of $\mathcal{M}(\eta)$ onto $\mathcal{M}(\gamma)$ which preserves the given resolutions of identity. Thus it is the *equivalence class* of η (also called the *spectral type* of the model) which matters. Next, if η is decomposed into a sum of two mutually singular measures λ and μ, the model space $\mathcal{M}(\eta)$ gets decomposed into a direct sum $\mathcal{M}(\lambda) \oplus \mathcal{M}(\mu)$, compatible with the given resolutions of identity. Hence if we want to add as many "models" as we can in a single operation, our interest is to choose a spectral type as strong as we can. Then the above construction can be refined: one starts with a measure η which has *maximal spectral type.* At the following step, when one restricts oneself to the orthogonal Ω_1 of the first model, one again chooses the maximal spectral type allowed in Ω_1, and so on. Then one can show that transfinite induction is unnecessary, and (more important) *the spectral types of* $\eta_1, \eta_2, \ldots$, *which become weaker and weaker, are uniquely determined.* This is the well known *spectral multiplicity theorem* (Hellinger-Hahn theorem). For a detailed proof, see [ReS], Chap. VII. On the other hand, the "slicing" itself (the decomposition of Ω into a continuous sum of Hilbert spaces) can be shown to be unique once the measure θ is chosen.

Finally, let us show briefly the relation of the above slicing with the definition of a complete observable, as given above in this chapter (§2, subs. 3) : X is complete if and only if one single copy of the "model" is sufficient to exhaust Ω, *i.e.* if the slices are one–dimensional. Given that operators which operate slice by slice commute with X, it is easy to prove that the observable X is complete if and only if all operators which commute to X are of the form J_f^X.

Transition kernels

The results in the following subsections have much theoretical importance, but are not used in the sequel. For a perusal, the important point is the definition of *completely positive mappings* in subsection 8.

5 Let us first recall the definition of a *(transition) kernel* in classical probability.

Consider two measurable spaces $(E,\mathcal{E})$ and $(F,\mathcal{F})$. Then a kernel K from E to F is a mapping which associates with every point u of E a probability measure $K(u,dv)$ on F, depending measurably on u — this means that for every bounded measurable function f on F, the function $Kf \;:\; u \longmapsto \int K(u,dv)\,f(v)$ is measurable on E. Thus K may be described as a mapping from (measurable) functions on F to (measurable) functions on E, which is positive, transforms 1 into 1, and satisfies a countable additivity property.

On the other hand, let $\mathcal{P}(E)$ be the convex set of probability measures on E, and similarly $\mathcal{P}(F)$. The kernel K defines an affine mapping from $\mathcal{P}(E)$ to $\mathcal{P}(F)$

$$\lambda \longmapsto \lambda K = \int \lambda(du)\,K(u,\cdot)$$

However, one cannot characterize simply, by a property like countable additivity, those affine mappings which arise in this way from transition kernels.

Every measurable mapping h from E to F defines a kernel H from E to F, such that $H(u,dv) = \varepsilon_{h(u)}(dv)$. Such a kernel is deterministic, in the following sense : if one imagines a kernel $K(u,dv)$ as the probability distribution of bullets fired on F from a

gun located at $u \in E$, then the gun corresponding to the kernel H has no spreading : all bullets from u fall at the point $h(u)$. This is reflected into the algebraic property that $H(fg) = H(f)H(g)$ for any two (bounded measurable) functions f, g on F.

6 We give now a first extension of the notion of kernel to a non-commutative setup, though we are not going to present a detailed theory. A very good reference for further study is [Dav]; another excellent reference (less easily available) is the set of notes [EvL] by Evans and Lewis.

We replace the space E by its quantum analogue, namely a Hilbert space Ω. On the other hand, we get two different generalizations of kernels by keeping the second space F classical, or turning it also into a Hilbert space.

We may call *(transition) kernel between* F (a measurable space) *and* Ω (a Hilbert space) a mapping K which associates with every measurable set $A \subset F$ a selfadjoint, bounded and positive operator $K(A)$ so that

$$K(F) = I \quad ; \quad K(\emptyset) = 0 \quad ; \quad K(\bigcup_n A_n) = \sum_n K(A_n) \quad \text{for disjoint} \quad A_n \ ,$$

convergence taking place in the strong topology. Rigorously speaking it should be called a kernel from Ω to F, but the mapping goes the other way, and the terminology is disturbing. Such a kernel is a true measure taking values in a locally convex space, and a detailed integration theory exists for such measures. In particular, there is no difficulty in defining $K(f)$ for f measurable and bounded on F, by the usual monotone class procedure starting with step functions. If $K(A)$ is a projector for every A, then we fall back on ordinary observables. Such objects have been called by several names : *positive operator valued measures, non orthogonal resolutions of the identity* (when $F = \mathbb{R}$), or even *observables* ([Dav], p. 36). For probabilists, the name "kernel" is natural.

This notion may turn out to be important in physics (covariant non orthogonal measures have been shown to exist in cases one lacks orthogonal ones). From a mathematical point of view, let us mention Theorem II.4.3 in Holevo's book [Hol1], which associates with each maximal symmetric densely defined operator a unique non orthogonal resolution of identity, which becomes orthogonal for a selfadjoint operator.

Kernels are more complicated than spectral measures in the sense that, in the latter case, orthogonality automatically implies that all operators J_A^X commute, while no commutativity is implied by the definition above. So there seems to be a wide gap in generality between orthogonal spectral measures and kernels. This is only apparent : one can prove that every kernel is the projection on Ω of some orthogonal spectral measure taking values in some larger Hilbert space (see [Dav], theorem 9.3.2, p. 142).

7 A second form of the definition of non commutative kernels arises if one also replaces the second measurable space by a Hilbert space $\mathcal{H}$. Let us denote by $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{H})$ the convex sets of probability laws (*i.e.* density matrices, positive operators of trace 1) on the two Hilbert spaces, and by $\mathcal{M}(\Omega)$, $\mathcal{M}(\mathcal{H})$ the corresponding spaces of trace class operators (the letter $\mathcal{M}$ of course suggests that we are dealing with *measures*). It is tempting to call a *transition kernel from* Ω *(Hilbert) to* $\mathcal{H}$ *(Hilbert)* any continuous linear mapping $\mathbf{K}$ from $\mathcal{M}(\Omega)$ to $\mathcal{M}(\mathcal{H})$ which preserves positivity and the value of

the trace[1] (hence carries $\mathcal{P}(\Omega)$ into $\mathcal{P}(\mathcal{H})$). The use of the word ***kernel*** to describe two different generalizations of the same idea does not seem too confusing, since the ranges of the mappings carry different structures, and we keep to it. The word *operation* has been used to denote this type of "kernel" : see [Dav], p. 17.

It turns out that this definition of kernels is incomplete (it will be completed in subsection 8), but let us accept it temporarily. We use everywhere a boldface $\mathbf{K}$ to indicate that a kernel is one level higher than an operator, and carries states to states instead of vectors to vectors.

Our definition corresponds to the mapping $\mu \longmapsto \mu K$ in classical probability theory, and we shall keep the notation $\rho\mathbf{K}$ to denote the action of $\mathbf{K}$ on states. What about $f \longmapsto Kf$? Since *vectors* in the basic Hilbert space correspond more or less to *points* in classical probability, and *operators* on Ω to *functions* in classical probability, we expect this mapping go from operators on $\mathcal{H}$ to operators on Ω, and this is the right idea : it is shown in Appendix 1 that the dual space of $\mathcal{M}(\Omega)$ is $\mathcal{L}(\Omega)$, the space of all bounded operators on Ω. So by transposition we get a positive continuous mapping from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\Omega)$, which maps I to I, and this is the kind of object we need. We use the notation $\mathbf{K}F$ as in the classical case (F being now an operator, not a vector). This mapping is a little more than norm continuous : it is continuous in the weak topology associated with the above duality, and this corresponds to the countable additivity of classical kernels — but we shall not insist on this point for the moment.

EXAMPLE. Let Ω be a probability space $L^2(E,\mathcal{E},\lambda)$ and let K be a classical transition kernel from E to some measurable space $(G,\mathcal{G})$. Let $\mu = \lambda K$ be the image measure, and $\mathcal{H}$ be the Hilbert space $L^2(\mu)$. Then K induces a *contraction* from $\mathcal{H}$ to Ω, for whick we keep the same notation K. If the classical kernel K was "deterministic" ($Kf = f \circ h$ for some measurable mapping $h \ : \ E \longmapsto G$), then read as a mapping from $\mathcal{H}$ to Ω, K becomes an *isometry.*

Which is now concretely the quantum probabilistic interpretation of K ? To define maps from bounded operators F on $\mathcal{H}$ to bounded operators on Ω, and from trace class operators ρ on Ω to trace class operators on $\mathcal{H}$, we put

$$\mathbf{K}F = KFK^* \quad ; \quad \rho\mathbf{K} = K^*\rho K \ .$$

The second mapping is clearly positive, and the fact that it decreases the trace is due to K being a contraction (this is proved in Appendix 1 for contractions of a Hilbert space into itself; the case of two Hilbert spaces makes little difference — even no difference at all if they are of the same dimension!). However, even if K was a Markov kernel in the classical sense, there is no reason why it should *preserve* the trace, or dually map $I_{\mathcal{H}}$ to I_{Ω}. On the other hand, the fact that K originally was a classical kernel has been completely forgotten : K could be any contraction from $\mathcal{H}$ to Ω.

Note that if ρ is a pure law ε_u, $u \in \Omega$, then $\rho\mathbf{K}$ is the measure (not necessarily a state) ε_v where $v = K^*u$. Quantum probabilistic kernels with this property deserve

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[1] Classical probability also uses submarkov transition kernels, which have total mass ≤ 1 instead $= 1$: this would correspond to the trace being decreased instead of preserved, leading to a (not specially important) definition of "quantum submarkov kernels".

a section in [Dav], p. 21–26, under the name of *pure operations*. You will find there a theorem (far from trivial) stating that, if a degenerate type is left aside, these kernels either are of the form described above, or are of the same type with a *conjugate linear* contraction. This implies a famous result of Wigner, according to which every automorphism of the state space $\mathcal{M}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is induced by an automorphism or antiautomorphism of $\mathcal{H}$ itself (see [Par1], Chapter 1, section 14).

Completely positive maps

8 Let us return to the general situation : we have described our definition of kernels above as temporary, and we will give now the true definition. We write the word "kernel" between inverted commas to denote its previous meaning.

Let us first recall the classical definition of a mapping of positive type $K(x,y)$ from $E\times E$ to $\mathbb{C}$, where E is any set : for every finite sequence $(x_i)_{i\leq n}$ in E, the hermitian form $\sum_{ij}\overline{z}_i K(x_i,x_j)\,z_j$ on $\mathbb{C}^n$ is positive. This can be extended to a mapping $K(x,y)$ taking values in the space $\mathcal{L}(\Omega)$ of all bounded operators on Ω : we demand that for any sequence of vectors $u_i\in\Omega$, or equivalently of operators $L_i\in\mathcal{L}(\Omega)$

$$\sum_{ij} <u_i,\, K(x_i,x_j)\,u_j> \geq 0\,, \qquad \text{or} \qquad \sum_{ij} L_i^* K(x_i,x_j)\,L_j \geq 0$$

— the second form remains meaningful if $\mathcal{L}(\mathcal{H})$ is replaced by a *–algebra. In particular, the operators $K(x,x)$ are positive. Returning now to a positive and trace preserving mapping $\mathbf{K} : \mathcal{L}(\mathcal{H}) \longmapsto \mathcal{L}(\Omega)$ we shall make it a part of our definition of a kernel (from Ω to $\mathcal{H}$) that $(A,B) \longmapsto \mathbf{K}(A^*B)$ *should be of positive type on* $\mathcal{L}(\mathcal{H})$. Then $\mathbf{K}$ is called a *completely positive mapping*. This extremely important property remains meaningful for linear mappings between *–algebras, and one can show that between commutative algebras it reduces to positivity.

The justification of complete positivity is the following : given a "kernel" $\mathbf{K}$ from Ω to $\mathcal{H}$ in the sense of subsection 7, it is natural to expect that, for any Hilbert space $\mathcal{F}$, $\mathbf{K}\otimes I$ should be a "kernel" from $\Omega\otimes\mathcal{F}$ to $\mathcal{H}\otimes\mathcal{F}$. It turns out that such a trivial demand is not automatically satisfied by our former "kernels", and is equivalent to $\mathbf{K}$ being completely positive. Roughly, it suffices to consider the case of $\mathcal{F}=\mathbb{C}^n$; then an element of $\mathcal{L}(\mathcal{H}\otimes\mathcal{F})$ is a matrix $A=(A_i^j)$ of operators on $\mathcal{H}$, which is mapped by $\mathbf{K}\otimes I$ into the matrix $(\mathbf{K}A_i^j)$. To check that this mapping preserves positivity, we write $A\geq 0$ in the form B^*B, and then A is a finite sum of positive matrices $A_i^j = X_j^* X_i$ with $X_i\in\mathcal{L}(\mathcal{H})$. Finally one is reduced to the positive type condition.

EXAMPLES. a) Let ρ be a representation of $\mathcal{L}(\mathcal{H})$ in some Hilbert space Ω (*i.e.* ρ maps $\mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\Omega)$, preserving the identity, products and adjoints). Then ρ is completely positive. Indeed,

$$\sum_{ij} L_i^*\rho(X_i^*X_j)\,L_j = Y^*Y \geq 0$$

where $Y=\sum_i \rho(X_i)\,L_i \in \mathcal{L}(\Omega)$.

b) If V is a bounded linear mapping from Ω to $\mathcal{H}$, then the mapping $A\longmapsto V^*AV$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\Omega)$ is completely positive. Indeed

$$\sum_{ij} <u_i,\, V^*(X_i^*X_j)\,Vu_j> \;=\; <y,y> \geq 0$$

where $Y = \sum_i \rho(X_i)\, L_i \in \mathcal{L}(\Omega)$.

b) If V is a bounded linear mapping from Ω to $\mathcal{H}$, then the mapping $A \longmapsto V^* AV$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\Omega)$ is completely positive. Indeed

$$\sum\nolimits_{ij} < u_i,\, V^*(X_i^* X_j)\, V u_j > \, = \, < y, y > \, \geq 0$$

where $y = \sum_i X_i V u_i$.

It turns out that, combining these two examples, one can construct every completely positive map $\mathbf{K}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\Omega)$. Namely, there exists a representation ρ of $\mathcal{L}(\mathcal{H})$ in some Hilbert space $\mathcal{K}$, a bounded linear map V from Ω to $\mathcal{K}$, such that $\mathbf{K} A = V^* \rho(A) V$. This is *Stinespring's theorem,* see [Dav] p. 137, or [Par1] p. 254.

9 In classical probability theory, Markov semi–groups and their generators are the starting point of the theory of Markov processes. In quantum probability the corresponding role is played *quantum dynamical semigroups, i.e.* semi–groups of (completely positive) trace preserving kernels, which represent irreversible evolutions of quantum systems. We refer the reader to the notes [AlL] by Alicki and Lendi. The structure of *norm continuous* quantum dynamical semigroups has long been known (Lindblad [Lin], Gorini et *al* [GKS], see [Par1] p. 257–273). Much progress has taken place recently on the strongly continuous case.

However, there is a very fundamental difference with classical probability : given a probability law λ on a measurable space E and a Markov kernel K from E to a second measurable space F, there exists on $E \times F$ a unique probability law μ such that

$$\mu(f \otimes g) = \int f(x) g(y)\, \mu(dx, dy) = \lambda(f K g)\ . \tag{9.1}$$

Calling X and Y the co-ordinate projections on $E \times F$, K gives us the conditional probability distribution of the pair (X, Y) given X. We have seen that conditional distributions generally do not exist in quantum probability. Eventhough the classical *disintegration* result does not carry out to quantum probability, one may wonder whether the above *integration* result does. Namely, given a kernel $\mathbf{K}$ from a Hilbert space Ω to a second Hilbert space $\mathcal{H}$ and a state λ on Ω, can one use the same procedure to construct a state on the Hilbert space $\Omega \otimes \mathcal{H}$?

If we try to use the analogue of (9.1), F and G now denoting projectors, or more generally bounded s.a. operators on Hilbert spaces Ω and $\mathcal{H}$ respectively, we get into the difficulty that the product $F\mathbf{K}G$ is not selfadjoint unless F and $\mathbf{K}G$ commute. The solution which at once comes to the mind consists in replacing this product by $\frac{1}{2}\,(F(\mathbf{K}G) + (\mathbf{K}G)\,F)$. However, this may give a real measure, but there is no reason for it to be positive. There is a deeper reason : if we want to make a selfadjoint operator out of two non-commuting s.a. operators F and G, why should one prefer $\frac{1}{2}\,(FG + GF)$ to $(\theta FG + \overline{\theta} GF)$, where θ is any complex number with $\Re e\, \theta = 1/2$? This kind of "gauge ambiguity" appears in the quantum Radon–Nikodym theorems and plays a basic role in the Tomita–Takesaki theory (see Appendix 4).

Thus the problem of constructing "quantum Markov processes" from transition kernels is more difficult in non-commutative than in classical probability.

Appendix 3

Two Events

This Appendix illustrates a strange feature of quantum probability : the "σ–field" generated by two non–commuting events contains uncountably many projectors. The results are borrowed from M.A. Rieffel and A. van Daele, [RvD1] (1977), and through them from Halmos [Hal] (1969). As one may guess from the title of [RvD1], the results are useful for the proof of the main theorem of Tomita–Takesaki, which will be given in Appendix 4.

The operators denoted here with lower case or greek letters $p, q, j, d, \gamma, \sigma \dots$ are denoted P, Q, J, D, S, C in the section on Tomita's theory.

1 Let $\mathcal{H}$ be a Hilbert space, complex or real (the applications to Tomita–Takesaki essentially use the *real* case) and let A and B be two "events" (= closed subspaces) and I_A, I_B be the corresponding projectors. These projectors leave the four subspaces $A \cap B$, $A \cap B^{\perp}$, $A^{\perp} \cap B$, $A^{\perp} \cap B^{\perp}$ invariant. On the sum $\mathcal{K}$ of these four subspaces, the projectors I_A and I_B commute, and nothing interesting happens. If $\mathcal{K} = \{0\}$ we say that the two subspaces are *in general position.* This property can be realized by restriction to the invariant space $\mathcal{K}^{\perp}$. Thus we assume

$$(1.1) \qquad A \cap B = A^{\perp} \cap B^{\perp} = \{0\} \, (= A \cap B^{\perp} = A^{\perp} \cap B) \, .$$

However, in the application to the Tomita–Takesaki theory only the first two conditions are realized, and we indicate carefully the places where "general position" assumptions come into play.

To simplify notation we set $I_A = p$, $I_B = q$. Then we put

$$(1.2) \qquad \sigma = p - q \quad ; \quad \gamma = p + q - I$$

These operators are bounded and selfadjoint, and we have the following relations (depending only on p and q being projectors : $p^2 = p$, $q^2 = q$: none of the conditions (1.1) is used)

$$(1.3) \qquad \sigma^2 + \gamma^2 = 1 \quad ; \quad \sigma\gamma + \gamma\sigma = 0$$

The notations σ and γ suggest a "sine" and a "cosine". The first relation implies that $<\sigma x, \sigma x> + <\gamma x, \gamma x> \, = \, <x, x>$, so the spectrum of each operator lies in the interval $[-1, 1]$, and we may write the spectral representations (valid also in the real case!)

$$(1.4) \qquad \sigma = \int_{-1}^{1} \lambda dE_\lambda \quad ; \quad \gamma = \int_{-1}^{1} \lambda dF_\lambda$$

The relation $\sigma x = 0$ means $px = qx$, which implies $px = 0 = qx$ since $A \cap B = \{0\}$; then $x \in A^{\perp} \cap B^{\perp}$ which in turn implies $x = \{0\}$. Otherwise stated, *σ is injective.* Similarly, one can prove that *$I \pm \gamma$ is injective.* Using the remaining relations (1.1), one may prove that γ, $I - \sigma$, $I + \sigma$ are injective. In the application to Tomita–Takesaki theory, only the left side of (1.1) is true, so these last three operators will not be injective. Note that the injectivity of σ means that the spectral measure dE_λ has no jump at 0, *i.e.* $E_{0-} = E_0$.

We now define

$$(1.5) \qquad j = \operatorname{sgn}(\sigma) = \int_{-1}^{1} \operatorname{sgn}(\lambda)\, dE_\lambda \quad ; \quad d = |\sigma| = \int_{-1}^{1} |\lambda|\, dE_\lambda$$

Since the spectral measure has no jump at 0, it is not necessary to define the sign of 0 and we have $j^2 = I$: j is a symmetry, which we call the *main symmetry.* On the other hand, d is self-adjoint positive. Since we have $d^2 = \sigma^2$, d is the only positive square root of $1 - \gamma^2$. Then it commutes with γ and, since it already commutes with σ, it commutes with *all the operators* we are considering. Finally, we have

$$djp = \sigma - p = (p - q)p = (I - q)(p - q) = (I - q)dj = d(I - q)j \ .$$

Since d is injective, this implies

$$(1.6) \qquad jp = (I - q)j \quad \text{whence taking adjoints} \quad jq = (I - p)j \ .$$

Adding these relations, we get

$$(1.7) \qquad j\gamma = -\gamma j \quad ; \quad j\sigma = \sigma j$$

(the second equality is not new, we recall it for symmetry). The fact that the spaces are in general position has not been fully used yet. If we do, operating on F_λ as we did on E_λ gives a second symmetry $k = \operatorname{sgn}(\gamma)$ and a second positive operator $e = |\gamma|$ such that

$$\gamma = ke = ek \ , \ e = |\gamma| = (I - \sigma^2)^{1/2} \ , \ k\sigma = -\sigma k \ , \ k\gamma = \gamma k$$

and e commutes with all other operators. We call k the *second symmetry.* It turns out that *the two symmetries k and j anticommute.* Indeed, k anticommutes with every odd function of σ, and in particular with $\operatorname{sgn}(\sigma) = j$.

Let us add a few words on the application of the above results to the Tomita-Takesaki theory : in this situation, $\mathcal{H}$ is a complex Hilbert space (also endowed with the *real* Hilbert space structure given by the real part of its scalar product), and the two subspaces A and B are *real,* and such that $B = iA$. Another way to state this is the relation $ip = qi$. Then the "cosine" operator γ is complex linear, but the "sine" operator σ and the main symmetry j are *conjugate linear.* We shall return to this situation in due time.

The σ–field generated by two events

2 Assuming that the two subspaces are in general position, denote by $\mathcal{M}$ the eigenspace $\{jx = x\}$ of the main symmetry. Then $\mathcal{M}^\perp$ is the eigenspace $\{jx = -x\}$, and it is not difficult to see that k is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}^\perp$. Note that since $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ it must have even or infinite dimension. If we identify $\mathcal{M}$ to $\mathcal{M}^\perp$ by means of k, we identify $\mathcal{H}$ with $\mathcal{M} \oplus \mathcal{M}$ (every vector in $\mathcal{H}$ having a unique representation as $x+ky$, $x, y \in \mathcal{M}$) and every bounded operator on $\mathcal{H}$ is represented by a $(2,2)$ matrix of operators on $\mathcal{M}$. In particular, $j(x+ky) = x-ky$, $k(x+ky) = y+kx$, and therefore

$$j = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad ; \quad k = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

It is now clear that $\mathcal{H} \approx \mathcal{M} \otimes \mathbb{C}^2$ with j corresponding to $I \otimes \sigma_z$ and K to $I \otimes \sigma_x$. If we assume our Hilbert space is complex and put $l = -ijk$ (where i denotes the operator of multiplication by the complex scalar i !) this operator is represented by the third Pauli matrix

$$l = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} .$$

The operator d commutes with j and k and leaves $\mathcal{M}$ invariant. Consider now the family $\mathcal{A}$ of all operators

$$tI + xk + yl + zj = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} ,$$

where x, y, z, t are not scalars, but are restrictions to $\mathcal{M}$ of functions of d. Then it is very easy to see that $\mathcal{A}$ is closed under multiplication and passage to the adjoint, and one can show that $\mathcal{A}$ is exactly the *von Neumann algebra generated by p and q*. So this family contains uncountably many projectors, *i.e.* events in the probabilistic sense.

An interesting by–product of the preceding discussion is the fact that two anticommuting symmetries j and k on $\mathcal{H}$ necessarily look like two of the Pauli matrices, and in fact (taking an o.n.b. of $\mathcal{M}$) the space decomposes into a direct sum of copies of $\mathbb{C}^2$ equipped with the Pauli matrices. On the other hand, consider a Hilbert space $\mathcal{H}$ and two mutually adjoint bounded operators b^+ and b^- such that $b^{+2} = b^{-2} = 0 = b^+b^- + b^-b^+$. Then it is easy to see that $b^+ + b^-$ and $i(b^+ - b^-)$ are anticommuting symmetries . Thus the Hilbert space splits into a countable sum of copies of $\mathbb{C}^2$ equipped with the standard creation and annihilation operators. This is the rather trivial analogue, for the canonical anticommutation relations, of the Stone-von Neumann uniqueness theorem for the canonical commutation relations.

Appendix 4

C^*-Algebras

This appendix contains a minimal course on C^* and von Neumann algebras, with emphasis on probabilistic topics, given independently of the course on Fock spaces. Von Neumann algebras play in non–commutative probability the role of measure theory in classical probability, though in practice (as in the classical case!) deep results on von Neumann algebras are rarely used, and a superficial knowledge of the language is all one needs to read the literature. Our presentation does not claim to be original, except possibly by the choice of the material it excludes.

§1. ELEMENTARY THEORY

Definition of C^*–algebras

The proofs in this section are very classical and beautiful, and can be traced back to Gelfand and Naimark, with several substantial later improvements. For details of presentation we are specially indebted to the (much more complete) books by Bratelli–Robinson and Pedersen. We assume that our reader has some knowledge of the elementary theory of Banach algebras, which is available in many first year functional analysis courses.

1 By a *C^*-algebra* we mean a complex algebra $\mathcal{A}$ with a norm $\|\cdot\|$, an involution * *and also a unit* I (the existence of an unit is not always assumed in standard courses, but we are supposed to do probability), which is complete with respect to its norm, and satisfies the basic axiom

$$\|a^*a\| = \|a\|^2 \ . \tag{1.1}$$

The most familiar *–algebra of elementary harmonic analysis, the convolution algebra $L^1(G)$ of a locally compact group, is not a C^*–algebra.

C^*–algebras are non–commutative analogues of the algebras $\mathcal{C}(K)$ of complex, continuous functions over a compact space K (whence the C) with the uniform norm, the complex conjugation as involution, and the function 1 as unit. Thus bounded linear functionals on C^*–algebras are the non–commutative analogues of (bounded) complex measures on a compact space K.

The relation $(ab)^* = b^*a^*$ implies that I^* is a unit, thus $\mathrm{I}^* = \mathrm{I}$, and (1.1) then implies $\|\mathrm{I}\| = 1$. On the other hand, to prove (1.1) is suffices to check that $\|a^*a\| \geq \|a\|^2$. Indeed, this property implies (since the inequality $\|ab\| \leq \|a\|\,\|b\|$ is

an axiom of normed algebras) $\| a^* \| \| a \| \geq \| a \|^2$, hence $\| a^* \| \geq \| a \|$, hence equality. Finally $\| a^* a \| \leq \| a^* \| \| a \| = \| a \|^2$, and equality obtains.

This allows us to give the fundamental example of a non-commutative C^*-algebra : *the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space,* with the operator norm as $\| \cdot \|$, the adjoint mapping as involution, and the operator I as unit. To see this, we remark that for an operator A

$$\| A \|^2 = \sup_{\| x \| \leq 1} < Ax, Ax > = \sup_{\| x \| \leq 1} < A^* Ax, x > \leq \| A^* A \| .$$

One can show that every C^*-algebra is isomorphic to a closed subalgebra of some $\mathcal{L}(\mathcal{H})$.

By analogy with the case of $\mathcal{L}(\mathcal{H})$, one says that a in some C^*-algebra $\mathcal{A}$ is *selfadjoint* or *hermitian* if $a = a^*$, is *normal* if a and a^* commute, and is *unitary* if $aa^* = a^* a = I$. The word *selfadjoint* is so often used that we abbreviate it into *s.a.*.

Application of spectral theory

2 Let us recall a few facts from the elementary theory of Banach algebras. First of all, the *spectrum* $\mathrm{Sp}(a)$ of an element a of a Banach algebra $\mathcal{A}$ with unit I is the set of all $\lambda \in \mathbb{C}$ such that $\lambda \mathrm{I} - a$ is not invertible in $\mathcal{A}$. It is a non–empty compact set of the complex plane, contained in the disk with radius $\| a \|$, but the *spectral radius* $\rho(a)$ *of* a, *i.e.* the radius of the smallest disk containing $\mathrm{Sp}(a)$, may be smaller than $\| a \|$. The complement of $\mathrm{Sp}(a)$ is called the *resolvent set* $\mathrm{Res}(a)$. The spectrum, and hence the spectral radius, are defined without reference to the norm. On the other hand, the spectral radius is given explicitly by

$$\rho(a) = \limsup_n \| a^n \|^{1/n} . \tag{2.1}$$

We use many times the *spectral mapping theorem,* which in its simplest form asserts that if $F(z)$ is an entire function $\sum_n c_n z^n$ on $\mathbb{C}$, and $F(a)$ denotes the sum $\sum_n c_n a^n \in \mathcal{A}$, then $\mathrm{Sp}(F(a))$ is the image $F(\mathrm{Sp}(a))$. This result is true more generally for functions which are not holomorphic in the whole plane, but only on a neighbourhood of the spectrum. From this theorem we only need here the following particular case

(2.2) *If a is invertible, then* $\mathrm{Sp}(a^{-1}) = (\mathrm{Sp}(a))^{-1}$.

Another property we will use is the following

(2.3) $\mathrm{Sp}(ab)$ *and* $\mathrm{Sp}(ba)$ *differ at most by* $\{0\}$.

The proof goes as follows : assume $\lambda \mathrm{I} - ba$ has an inverse c. Then the easy equality

$$(\lambda \mathrm{I} - ab)(\mathrm{I} + acb) = (\mathrm{I} + acb)(\lambda \mathrm{I} - ab) = \lambda \mathrm{I}$$

implies, for $\lambda \neq 0$, that $\lambda \mathrm{I} - ab$ is invertible too.

3 The following properties are specific to C^* algebras. They are very simple and have important consequences.

a) *If a is normal, we have $\rho(a) = \| a \|$.* This follows from the computation (2.1) of $\rho(a)$, and the following consequence of (1.1) (note that $b = a^*a$ is s.a.)

$$\| a^{2^n} \|^2 = \| a^{*\,2^n} a^{2^n} \| = \| (a^*a)^{2^n} \| = \| (a^*a)^{2^{n-1}} \|^2 = \ldots = \| a \|^{2^{n+1}} .$$

As a corollary, note that a Banach algebra $\mathcal{A}$ has at most one C^* norm, given by $\| a \| = (\rho(a^*a))^{1/2}$. Also, if $\mathcal{A}$ and $\mathcal{B}$ are two C^* algebras, f a morphism from $\mathcal{A}$ to $\mathcal{B}$ (*i.e.* f preserves the algebraic operations, involutions and units), it is clear that $f(a)$ has a smaller spectral radius than a, hence also a smaller norm.

b) *The spectrum of a unitary element a is contained in the unit circle.* Indeed, the fact that $a^*a = \mathrm{I}$ implies $\rho(a^*a) = 1$. Therefore the spectrum of a lies in the unit disk. The same applies to a^{-1}, and using (2.2) we find their spectra are on the unit circle.

c) *A s.a. element a has a real spectrum, and at least one of the points $\pm\| a \|$ belongs to* $\mathrm{Sp}(a)$. One first remarks that e^{ia} is unitary, hence has its spectrum on the unit circle. The first statement then follows from the spectral mapping theorem. On the other hand, $\mathrm{Sp}(a)$ is compact, and therefore contains a point λ such that $|\lambda| = \rho(a)$. Since the spectrum is real and $\rho(a) = \| a \|$, we have $\lambda = \pm\| a \|$.

Positive elements

4 We adopt the following definition of a positive element a in the C^* algebra $\mathcal{A}$: *a is s.a. and its spectrum is contained in $\mathbb{R}_+$.* For instance, since every s.a. element has a real spectrum, its square is positive by the spectral mapping theorem. In subsection 7 we prove that positive elements are exactly those of the form b^*b. We start with a few elementary results which have extremely clever and elegant proofs.

a) *If a finite sum of positive elements is equal to 0, each of them is 0.* It is sufficient to deal with a sum $a + b = 0$. Since $\mathrm{Sp}(a) = -\mathrm{Sp}(b)$ are in $\mathbb{R}_+$ we must have $\mathrm{Sp}(a) = \{0\}$, hence $\rho(a) = 0 = \| a \|$.

b) *Let a be s.a. with $\| a \| \leq 1$. Then $a \geq 0 \iff \| \mathrm{I} - a \| \leq 1$.* Indeed, if $a \geq 0$ and $\| a \| \leq 1$ then $\mathrm{Sp}(\mathrm{I} - a) \subset [0,1]$, hence $\rho(\mathrm{I} - a) = \| \mathrm{I} - a \| \leq 1$. Conversely, from $\| a \| \leq 1$ and $\| \mathrm{I} - a \| \leq 1$ we deduce that $\mathrm{Sp}(a)$ lies in the intersection of the real axis, and of the two disks of radius 1 with centers at 0 and 1. That is, in the interval $[0,1] \subset \mathbb{R}_+$ and a is positive.

Let $\mathcal{A}_+$ be the set of positive elements. It is clearly a cone in $\mathcal{A}$. The preceding property easily implies that its intersection with the unit ball is convex and closed, from which it easily follows that *$\mathcal{A}_+$ is closed, and closed under sums* (*i.e.* is a closed convex cone)

The following result is extremely important :

c) *Every positive element a has a unique square root b; it belongs to the closed algebra generated by a* (it is not necessary to add the unit element).

We may assume $\| a \| \leq 1$, and then $\alpha = \mathrm{I} - a$ has norm ≤ 1. We recall that $(1 - z)^{1/2} = 1 - \sum_{n \geq 1} c_n z^n$ with positive coefficients c_n of sum 1. We define $b = \mathrm{I} - \sum_n c_n \alpha^n = \sum_n c_n(\mathrm{I} - (\mathrm{I} - a)^n)$. Then it is very easy to see that b is a positive square root of a. It appears as a limit of polynomials in a without constant term, whence the last statement.

Before we prove uniqueness, we note that *a product of two commuting positive elements* a, a' *is positive.* Indeed, their (just constructed) square roots b, b' commute as limits of polynomials in a, a', hence bb' is s.a. with square aa'.

To prove uniqueness, consider any positive c such that $c^2 = a$. Then $a = c^2$ belongs to the closed algebra generated by c, and then the same is true for the square root b considered above. Thus all three elements commute and we have $0 = (b^2 - c^2)(b - c) = (b - c)^2(b + c)$. Now $(b - c)^2 b$ and $(b - c)^2 c$ are positive from the preceding remark and their sum is 0. By a) above, they are themselves equal to 0, and so is their difference $(b - c)^3$. This implies $\rho(b - c) = 0$, and finally $\| b - c \| = 0$.

Symbolic calculus for s.a. elements

5 Symbolic calculus in an algebra $\mathcal{A}$ consists in giving a reasonable meaning to the symbol $F(u)$, where u ranges over some class of elements of $\mathcal{A}$ and F over some algebra of functions on $\mathbb{C}$. Notations like e^a or $\sqrt{a}$ are examples of symbolic calculus in algebras. It is clear that polynomials operate on any algebra with unit, and entire functions on every Banach algebra; more generally, the symbolic calculus in Banach algebras defines $F(u)$ whenever F is analytic in a neighbourhood of $\mathrm{Sp}(u)$. We are going to show that *continuous functions on* $\mathbb{R}$ *operate on s.a. elements of* C^* *algebras.* Later, we will see that in von Neumann algebras this can be extended to Borel bounded functions.

Let u be a s.a. element of a C^*-algebra, with (compact) spectrum $\mathrm{Sp}(u) = K$. Let $\mathcal{P}$ be the *-algebra of all complex polynomials on $\mathbb{R}$, with the usual complex conjugation as involution. Then the mapping $P \longmapsto P(u)$ is a morphism from $\mathcal{P}$ to the subalgebra of $\mathcal{A}$ generated by u and I (a *-subalgebra since u is s.a.). By the spectral mapping theorem, $\rho(P(u))$ (which is equal to $\| P(u) \|$ since this element is normal) is exactly the uniform norm of P on $\mathrm{Sp}(u) = K$. Polynomials being dense in $\mathcal{C}(K)$ by Weierstrass' theorem, we extend the mapping to $\mathcal{C}(K)$ by continuity, to define an *isomorphism* between $\mathcal{C}(K)$, and the closed *-algebra generated by u and I. *This isomorphism is the continuous symbolic calculus.* Essentially the same reasoning can be extended to normal u instead of s.a., starting from polynomials $P(z, \overline{z})$ on $\mathcal{C}$, and more generally one may define continuous functions of several *commuting* s.a. or normal elements.

6 Let us give some applications of the symbolic calculus.

a) The decomposition $f = f^+ - f^-$ of a continuous function gives rise to a decomposition $u = u^+ - u^-$ of any s.a. element of $\mathcal{A}$ into a difference of two (commuting) positive elements. Also we may define $|u|$ and extend to the closed real algebra generated by u the order properties of continuous functions.

b) Given a *state* μ of $\mathcal{A}$ (see §2 below), μ induces a state of the closed subalgebra generated by u, which translates into a positive linear functional of mass 1 on $\mathcal{C}(\mathrm{Sp}(u))$, called the *law of* u *in the state* μ. More generally, one could define in this way the joint law of a finite number of commuting s.a. elements of $\mathcal{A}$.

c) Let u be s.a. (or more generally, normal) and invertible in $\mathcal{A}$. Then its spectrum K does not contain 0, and the function $1/x$ thus belongs to $\mathcal{C}(K)$. Therefore u^{-1} belongs to the closed algebra generated by u. More generally, for v not necessarily s.a.,

v is invertible if and only if $u = v^*v$ is invertible, and therefore if v is invertible in $\mathcal{A}$, it is invertible in any C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing v. It follows that $\mathrm{Sp}(v)$ also is the same in $\mathcal{A}$ and $\mathcal{B}$.

d) Every element a in a C^*-algebra is a linear combination of (four) unitaries. We first decompose a into $p + iq$ with $p = (a + a^*)/2$, $q = i(a - a^*)/2$ selfadjoint. Then we remark that if the norms of p, q are ≤ 1, the elements $p \pm i\sqrt{\mathrm{I} - p^2}$, $q \pm i\sqrt{\mathrm{I} - q^2}$ are unitary.

Characterization of positive elements

7 Every positive element in a C^*-algebra $\mathcal{A}$ has a s.a. square root, and therefore can be written in the form a^*a. Conversely, in the operator algebra $\mathcal{L}(\mathcal{H})$ it is trivial that A^*A is positive in the operator sense, hence has a positive spectrum, and finally satisfies the algebraic definition of positivity. So it was suspected from the beginning that in an arbitrary C^*-algebra every element of the form a^*a is positive, but Gelfand and Naimark could not prove it : it was finally proved by Kelley and Vaught.

We put $b = a^*a$. Since b is s.a., we may decompose it into $b^+ - b^-$ and we want to prove that $b^- = 0$. We put $ab^- = s + it$ with s, t s.a. and define $u = (ab^-)^*(ab^-)$, $v = (ab^-)(ab^-)^*$. We make two remarks.

1) $u = (ab^-)^*(ab^-) = (b^-)^*a^*ab^- = b^-bb^- = (-b^-)^3$ is s.a. ≤ 0. On the other hand $u = (s - it)(s + it) = s^2 + t^2 + i(st - ts)$, thus $i(st - ts) \leq 0$.

2) Then $v = (ab^-)(ab^-)^* = s^2 + t^2 - i(st - ts)$ is s.a. ≥ 0 as a sum of two such elements.

According to (2.3), u and v have the same spectrum, except possibly the value 0. Since $u \leq 0$ and $v \geq 0$, this spectrum is $\{0\}$. Since u, v are s.a., this implies $u = v = 0$. Then the relation $u = -(b^-)^3$ implies $\rho(b^-) = 0$, and $b^- = 0$ since b^- is s.a..

Non-commutative inequalities

8 Many inequalities that are used everyday in commutative life become non-commutative traps. For instance, it is not true that $0 \leq a \leq b \Longrightarrow a^2 \leq b^2$. If we try to define $|a|$ as $(a^*a)^{1/2}$ it is not true that $|a + b| \leq |a| + |b|$, etc. Thus it is worth making a list of true inequalities.

a) Given one single positive element a, we have $a \leq \| a \| \mathrm{I}$, and $a^2 \leq \| a \| a$: this amounts to the positivity of the polynomials $\| a \| - x$ and $\| a \| x - x^2$ on $\mathrm{Sp}(a)$.

b) $a \geq 0$ implies $c^*ac \geq 0$ for arbitrary c (write a as b^*b). By difference, $a \geq b \Longrightarrow c^*ac \geq c^*bc$.

c) $a \geq b \geq 0 \Longrightarrow \| a \| \geq \| b \|$. Indeed, $\| a \| \mathrm{I} \geq a \geq b$, hence $\mathrm{Sp}(b) \subset [0, \| a \|]$.

d) $a \geq b \geq 0 \Longrightarrow (\lambda\mathrm{I} + a)^{-1} \leq (\lambda\mathrm{I} + b)^{-1}$ for $\lambda \geq 0$. This is less trivial : one starts from $0 \leq \lambda\mathrm{I} + b \leq \lambda\mathrm{I} + a$, then applies b) to get

$$\mathrm{I} \leq (\lambda\mathrm{I} + b)^{-1/2}(\lambda\mathrm{I} + a)(\lambda\mathrm{I} + b)^{-1/2} .$$

Then the function x^{-1} reverses inequalities on positive s.a. operators, and we deduce

$$\mathrm{I} \geq (\lambda\mathrm{I} + b)^{1/2}(\lambda\mathrm{I} + a)^{-1}(\lambda\mathrm{I} + b)^{1/2} ,$$

and we apply again b) to get the final result.

e) The preceding result gives an example of increasing functions $F(x)$ on $\mathbb{R}_+$ such that $0 \le a \le b \Longrightarrow F(a) \le F(b)$, namely the function $1-(\lambda+x)^{-1} = x/(x+\lambda)$ for $\lambda \ge 0$. We get other examples by integration w.r.t. a positive measure $\mu(d\lambda)$, the most important of which are the powers x^α for $0 < \alpha < 1$, and in particular the function $\sqrt{x}$.

§2. STATES ON C^*–ALGEBRAS

1 We gave in Chapter I the general definition of a probability law, or state, on a complex unital *–algebra $\mathcal{A}$. Let μ be a linear functional on a C^*–algebra are no exception, such that $\mu(a^*a) \ge 0$ for arbitrary a. According to subsection 7, this means exactly that μ is positive on positive elements, hence real on s.a. elements. Since every element of $\mathcal{A}$ can be written as $a = p+iq$, $a^* = p-iq$, we see that $\mu(a^*) = \overline{\mu(a)}$, and one of the general axioms becomes unnecessary.

The same familiar reasoning that leads to the Schwarz inequality gives here

$$|\,\mu(b^*a)\,|^2 \le \mu(b^*b)\,\mu(a^*a)\,. \tag{1.1}$$

In particular, we have $|\,\mu(a)\,|^2 \le \mu(a^*a)\,\mu(\mathrm{I})$. On the other hand, we have $a^*a \le \|\,a\,\|^2\mathrm{I}$, and since $\mu(\mathrm{I}) = 1$ we find that $|\,\mu(a)\,| \le \|\,a\,\|$ — otherwise stated, μ is bounded and has norm 1.

Conversely, a bounded linear functional on $\mathcal{A}$ such that $\mu(\mathrm{I}) = \|\,\mu\,\| = 1$ is a state. Consider indeed a s.a. positive element a with spectrum K; define a linear functional on $\mathcal{C}(K)$ by $m(f) = \mu(f(a))$. Since μ is bounded, m is a complex measure, of norm $\|\,m\,\| \le \|\,\mu\,\| = \mu(1) = m(1)$. Then m is positive from standard measure theory, and we have $\mu(a) = m(x) \ge 0$.

C^* algebra have many states : let us prove that *for every s.a.* $a \in \mathcal{A}$, *there exists a state* μ *such that* $|\,\mu(a)\,| = \|\,a\,\|$. We put $K = \mathrm{Sp}(a)$, $\|\,a\,\| = \alpha$, and recall (§1, subs.2, c)) that at least one of the two points $\pm\alpha$ belongs to K ; for definiteness we assume α does. The linear functional $f(a) \longmapsto f(\alpha)$ on the C^*-algebra generated by a is bounded with norm 1, and by the Hahn–Banach theorem it can be extended to a linear functional μ of norm 1 on $\mathcal{A}$. The equality $\mu(1) = 1 = \|\,\mu\,\|$ implies that μ is a state on $\mathcal{A}$ such that $\mu(a) = \alpha$.

REMARK. The set of all bounded linear functionals on $\mathcal{A}$ is a Banach space, the unit ball of which is compact in its weak* topology. The subset of all positive linear functionals is weakly closed and hence compact $\mathcal{A}$, and finally the set of all states itself is a weakly compact convex set. According to the Krein–Milman theorem, it is the closed convex hull of the set of its extreme points, also called *pure states.* For instance, the state $\mu(A) = \,<x, Ax>$ of the algebra $\mathcal{L}(\mathcal{H})$, associated with an unit vector x, can be shown to be pure in this sense. It is not difficult to show that the state μ is the preceding proof can be chosen to be pure.

The space of all states of a C^*-algebra has received considerable attention, specially because of the (unsuccessful) attempts to classify all states of the CCR and CAR algebras.

Representations and the GNS theorem

2 A *representation* of a C^*-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is a morphism φ from $\mathcal{A}$ into the C^*-algebra $\mathcal{L}(\mathcal{H})$. Then every unit vector $x \in \mathcal{H}$ gives rise to a state $\mu(a) = <x, \varphi(a)\,x>$ of $\mathcal{A}$ (generally not a pure one). The celebrated theorem of Gelfand–Naimark–Segal, which we now prove, asserts that every state of a C^*-algebra $\mathcal{A}$ arises in this way — and pure states arise from *irreducible* representations, an important point we do not need here.

From an analyst's point of view, the main point of the GNS theorem is to show that a C^*-algebra has many (irreducible) representations. For us, its interest lies in the construction itself, which is a basic step in the integration theory w.r.t. a state.

The proof is very easy. For $a, b \in \mathcal{A}$ one defines $<b, a> = \mu(b^*a)$. Then $\mathcal{A}$ becomes a prehilbert space. Let $\mathcal{H}$ denote the associated Hilbert space, which means that first we neglect the space $\mathcal{N}$ of all $a \in \mathcal{A}$ such that $\mu(a^*a) = 0$, and then complete $\mathcal{A}/\mathcal{N}$. We denote by $\widetilde{a}$, for a few lines only, the class $mod\,\mathcal{N}$ of $a \in \mathcal{A}$.

We have $\mu((ba)^*ba) = \mu(a^*(b^*b)a)$, and since $b^*b \leq \|b\|^2 I$ this is bounded by $\|b\|^2 \mu(a^*a)$. Thus the operator ℓ_b of left multiplication by b is bounded on $\mathcal{H}$. We have $\ell_u \ell_v = \ell_{uv}$, etc. so that $\ell_{\bullet}$ is a representation of $\mathcal{A}$ in $\mathcal{H}$. Denoting by $\mathbf{1}$ the vector in $\mathcal{H}$ corresponding to I, we have $\mu(a) = <\mathbf{1}, \ell_a \mathbf{1}>$.

The relation $aI = a$ for all $a \in \mathcal{A}$ implies that $\ell_a \mathbf{1} = \widetilde{a}$. The set of all $\ell_a \mathbf{1}$ is thus *dense* in $\mathcal{H}$: $\mathbf{1}$ is said to be a *cyclic vector* for the representation, and the representation itself is said to be *cyclic*.

Assume some other representation φ is given on a Hilbert space $\mathcal{K}$, with a cyclic vector x such that $\mu(a) = <x, \varphi(a)x>$ for $a \in \mathcal{A}$. Then the mapping $a \longmapsto \varphi(a)x$ is an isometry from the prehilbert space $\mathcal{A}$ into $\mathcal{K}$, with a dense image. Then it can be transferred to $\mathcal{A}/\mathcal{N}$, extended to $\mathcal{H}$ by continuity, and becomes an isomorphism between $\mathcal{H}$ and $\mathcal{K}$. Thus the pair $(\mathcal{H}, \mathbf{1})$ is independent, up to isomorphism, of the particular way it was constructed, and we may speak loosely of *the* GNS representation associated with the state μ.

3 COMMENTS. a) Assume we are in the commutative case, and $\mathcal{A}$ is one of the three usual C^*-algebras of measure theory : the algebra $\mathcal{C}(K)$ of continuous functions on a compact set K (with an arbitrary state μ), the algebra $\mathcal{B}(\mathcal{E})$ of all bounded functions for a σ-field $\mathcal{E}$ (in this case, to develop integration theory we need a countable additivity assumption on μ), and the algebra $L^\infty(\mu)$ (this one is a von Neumann algebra, of which μ is a "normal state"). In all three cases, $\mathcal{H}$ is the space $L^2(\mu)$, on which $\mathcal{A}$ acts by multiplication. Thus it seems that $\mathcal{H}$ is a kind of non commutative L^2 space. It is indeed, but the GNS construction leads only to a "left" L^2 space, on which right multiplication generally defines unbounded operators. We return to this subject later.

b) From the GNS construction, one can understand how every C^*-algebra can be realized as a C^*-algebra of operators on some Hilbert space — a very large one, possibly the direct sum of the Hilbert spaces associated with all irreducible representations of the algebra. We do not insist on this point.

c) The state μ is said to be *faithful* if

$$\forall a \in \mathcal{A} \quad (\mu(a^*a) = 0) \Longrightarrow (a = 0)\,. \tag{3.1}$$

This is equivalent to saying that for $a \geq 0$, $\mu(a) = 0 \Longrightarrow a = 0$. From the point of view of the representation, we have $\mu(a^*a) = \|\ell_a \mathbf{1}\|$, and (3.1) means that $\ell_a \mathbf{1} = 0 \Longrightarrow a = 0$: the vector $\mathbf{1}$ then is said to be *separating.* Note that the state being faithful is a stronger property than the GNS representation being faithful $((\ell_a = 0) \Longrightarrow (a = 0))$.

In classical measure theory, μ is a faithful state on $\mathcal{C}(K)$ if the support of μ is the whole of K ; μ is practically never faithful on the algebra of Borel bounded functions, and is always (by construction) faithful on $L^\infty(\mu)$. In the non-commutative case, we cannot generally turn a state into a faithful one by taking a quotient subalgebra, *because* $\mathcal{N}$ *usually is only a left ideal, not a two sided one.* This difficulty does not occur for *tracial states*, defined by the property $\mu(ab) = \mu(ba)$.

As a conclusion, while in classical measure theory all states on $\mathcal{C}(K)$ give rise to countably additive measures on the Borel field of K with an excellent integration theory, it seems that not all states on non-commutative C^*-algebras $\mathcal{A}$ are "good". Traces certainly are good states, as well as states which are normal and faithful on some von Neumann algebra containing $\mathcal{A}$. Up to now, the most efficient definition of a class of "good" states has been the so called *KMS condition.* We state it in the next section, subsection 10.

4 EXAMPLES. As in Chapter II, consider the probability space Ω generated by N independent Bernoulli random variables x_k, and the corresponding "toy Fock space" $\Gamma = L^2(\Omega)$ with basis x_A (A ranging over the subsets of $\{1, \ldots, N\}$). We turn Γ into a Clifford algebra with the product

$$x_A x_B = (-1)^{n(A,B)} x_{A \Delta B} \,.$$

and we embed Γ into $\mathcal{L}(\Gamma)$, each element being interpreted as the corresponding left multiplication operator. Then the vacuum state on $\mathcal{L}(\Gamma)$

$$\mu(U) = <\mathbf{1}\,,\, U\mathbf{1}>$$

induces a state on the Clifford algebra, completely defined by

$$\mu(x_A) = 0 \quad \text{if } A \neq \emptyset, \quad \mu(\mathbf{1}) = 1\,.$$

Then we have $\mu(x_A x_B) = (-1)^{n(A,A)}$ if $A \neq \emptyset$, 0 otherwise, and *this state is tracial.*

The C^*-algebra generated by all creation and annihilation operators $a_i^\pm$ is the whole of $\mathcal{L}(\Gamma)$. The vacuum state μ is read on the basis $a_A^+ a_B^-$ of normally ordered products as $\mu(a_A^+ a_B^-) = 0$ unless $A = B = \emptyset$, $\mu(I) = 1$. It is not tracial, since for $A \neq \emptyset$ $\mu(a_A^+ a_A^-) = 0$, $\mu(a_A^- a_A^+) \neq 0$ — anyhow, $\mathcal{L}(\Gamma)$ has no other tracial state than the standard one $U \longmapsto 2^{-N}\,\mathrm{Tr}(U)$. It is also not faithful (in particular, annihilation operators kill the vacuum).

5 We mentioned above several times the construction of a quotient algebra $\mathcal{A}/\mathcal{J}$, where $\mathcal{J}$ is a two sided ideal. This raises a few interesting problems. For instance, whether the quotient algebra will be a C^*-algebra. We mention the classical answers, mostly due to Segal. It is not necessary to study this subsection at a first reading.

Let first $\mathcal{J}$ be a left ideal in $\mathcal{A}$, not necessarily closed or stable under the involution. We assume $\mathcal{J} \neq \mathcal{A}$, or equivalently $I \notin \mathcal{J}$. Let $\mathcal{J}_+$ denote the set of all positive

elements in $\mathcal{J}$. We have seen in §1, subs. 8, d) that the mapping $j \longmapsto e_j = j(I+j)^{-1}$ on $\mathcal{J}_+$ is increasing. It maps $\mathcal{J}_+$ into the "interval $[0,\mathrm{I}]$" in $\mathcal{J}$. The symbol $\lim_j$ is understood as a limit along the directed set $\mathcal{J}_+$.

The following important result explains why e_j is called an *approximate unit* for $\mathcal{J}$.

THEOREM. *For $a \in \mathcal{J}$ we have* $\lim_j \| a - ae_j \| = 0$.

PROOF. For $n \in \mathbb{N}$ and $j \geq na^*a$, we have (§1, 8 b))

$$(a - e_j)^*(a - ae_j) = (\mathrm{I} - e_j)\, a^*a\, (\mathrm{I} - e_j) \leq \frac{1}{n}(\mathrm{I} - e_j)\, j\, (\mathrm{I} - e_j) \;.$$

We must show that the norm of the right side tends to 0, and taking n large it suffices to find a universal bound for the norm of $j(I-e_j)^2$. Using symbolic calculus this reduces to the trivial remark that $x/(1+x)^2$ is bounded for $x \in \mathbb{R}$. □

APPLICATIONS. a) This reasoning does not really use the fact that $a \in \mathcal{J}$: since $\mathcal{J}$ is a left ideal ae_j belongs to $\mathcal{J}$ for $a \in \mathcal{A}$: it applies whenever a^*a is dominated by some element of $\mathcal{J}_+$, with the consequence that a then belongs to the closure of $\mathcal{J}$.

b) If $a = \lim ae_j$, we have $a^* = \lim e_j a^*$. Thus if $\mathcal{J}$ is a closed two sided ideal, $\mathcal{J}$ is also stable under the involution.

c) Let $\mathcal{J}$ be a closed two sided ideal, and $\widetilde{a}$ denote the class $mod\,\mathcal{J}$ of $a \in \mathcal{A}$. The general definition of $\| \widetilde{a} \|$ in the Banach space $\mathcal{A}/\mathcal{J}$ is $\inf_{b \in \mathcal{J}} \| a + b \|$, and therefore we have $\| a - ae_j \| \geq \| \widetilde{a} \|$. Let us prove that

$$\| \widetilde{a} \| = \lim_j \| a - ae_j \| \;. \tag{5.1}$$

Since $\| \mathrm{I} - e_j \| \leq 1$ we have

$$\| a + b \| \geq \| (a + b)(\mathrm{I} - e_j) \| = \| (a - ae_j) + (b - be_j) \|$$

For $b \in \mathcal{J}$, $b - be_j$ tends to 0 in norm, and therefore $\| a + b \| \geq \limsup_j \| a - ae_j \|$, implying (5.1).

d) It follows that $\mathcal{A}/\mathcal{J}$ *is a C^*-algebra* :

$$\begin{aligned} \| \widetilde{a} \|^2 &= \lim_j \| a - ae_j \|^2 = \lim_j \| (a^* - e_j a^*)(a - ae_j) \| \\ &= \lim_j \| (\mathrm{I} - e_j)(a^*a + b)(\mathrm{I} - e_j) \| \quad \text{for every } b \in \mathcal{J} \\ &\leq \| a^*a + b \| \quad \text{for every } b \in \mathcal{J} \end{aligned}$$

Taking an $\inf_b$ we have that $\| \widetilde{a} \|^2 \leq \| \widetilde{a}^*\widetilde{a} \|^2$, hence equality (§1, subs. 1).

e) An interesting consequence : let f be a morphism from a C^*-algebra $\mathcal{A}$ into a second one $\mathcal{B}$, and let $\mathcal{J}$ be its kernel. We already know that f is continuous (norm decreasing), hence $\mathcal{J}$ is closed. Then let $\widetilde{f}$ be the algebraic isomorphism from the quotient C^*-algebra $\mathcal{A}/\mathcal{J}$ into $\mathcal{B}$; from subs. 3 b), $\widetilde{f}$ is also a norm isomorphism, *and therefore its image $f(\mathcal{A})$ is closed in $\mathcal{B}$.*

§3. Von NEUMANN ALGEBRAS

This section is a modest one : more than ever its proofs are borrowed from expository books and papers, and we have left aside the harder results (and specially all of the classification theory).

Weak topologies for operators

1 Though von Neumann (vN) algebras can be defined as a class of abstract C^*-algebras, all the vN–algebras we discuss below will be concrete *-algebras of operators on some Hilbert space $\mathcal{H}$. As operator algebras, they are characterized by the property of being closed under the weak operator topologies instead of the uniform topology, and our first task consists in making this precise.

The most striking characterization of vN algebras as abstract C^*-algebras is possibly that of Sakai : a C^*-algebra $\mathcal{A}$ is a vN algebra if and only if the underlying Banach space is a dual. Also, vN algebras are characterized by the fact that bounded increasing families of s.a. elements possess upper bounds. On the other hand, Pedersen has studied natural classes of algebras satisfying weaker properties, like that of admitting a Borel symbolic calculus for individual s.a. elements, or of admitting upper bounds of bounded increasing *sequences* of s.a. elements. He has shown that, in all practical cases, they turn out to be the same as vN algebras.

There are three really useful weak topologies for bounded operators. The first one is the *weak operator topology* : $a_i \to a$ iff $< y, a_i x > \to < y, ax >$ for fixed x, y — here and below, the use of i rather than n means that convergence is meant for filters or directed sets, not just for sequences. The second is the *strong operator topology* : $a_i x \to ax$ in norm for fixed x. However, the most important for us is the third one, usually called the σ-*weak* or *ultraweak* topology. We have seen in Appendix 1 that the space $\mathcal{L} = \mathcal{L}^\infty$ of all bounded operators is the dual of the space $\mathcal{L}^1$ of trace class operators, and the ultraweak topology is the weak topology on $\mathcal{L}$ relative to this duality. Otherwise stated, convergence of a_i to a means that $\mathrm{Tr}(a_i b) \to \mathrm{Tr}(ab)$ for every $b \in \mathcal{L}^1$. To emphasize the importance of this topology, and also because states which are continuous in this topology are called *normal states*, we call it the *normal topology* (see App. 1, subs. 5).

More precisely, the word *normal* usually refers to a kind of order continuity, described later on, which will be proved to be equivalent to continuity in the "normal" topology (see subs. 6). Thus our language doesn't create any dangerous confusion.

It is good to keep in mind the following facts from Appendix 1 :

a) The normal topology is stronger than the weak topology, and not comparable to the strong one. The strong topology behaves reasonably well w.r.t. products : if $a_i \to a$ and $b_i \to b$ strongly, $\| b_i \|$ (or $\| a_i \|$) remaining bounded, then $a_i b_i \to ab$ and $b_i a_i \to ba$ strongly. On the other hand, it behaves badly w.r.t. adjoints, while if $a_i \to a$ weakly, then $a_i^* \to a^*$ weakly.

b) The dual of $\mathcal{L}(\mathcal{H})$ is the same for the weak and the strong topologies, and consists of all linear functionals $f(a) = \sum_j < y_j, a x_j >$ where x_j, y_j are (finitely many) elements

of $\mathcal{H}$. The dual for the normal topology consists of all linear functionals $f(a) = \mathrm{Tr}(aw)$ where w is a trace class operator.

c) One can give a simple characterization of the normal linear functionals on $\mathcal{L}(\mathcal{H})$, *i.e.* those which are continuous in the normal topology : *a linear functional μ on $\mathcal{L}(\mathcal{H})$ is normal iff its restriction to the unit ball is continuous in the weak topology.*

As an application, since every weakly or strongly convergent *sequence* is norm bounded, it also converges in the normal topology.

Von Neumann algebras

2 Since the weak and strong topologies on $\mathcal{L}(\mathcal{H})$ have the same dual, the Hahn–Banach theorem implies that the weakly and strongly closed subspaces of $\mathcal{L}(\mathcal{H})$ are the same.

DEFINITION *A (concrete) von Neumann algebra (vNa) is a *-subalgebra of $\mathcal{L}(\mathcal{H})$, containing I, closed in the strong or weak topology.*

The *commutant* J' of a subset $J \subset \mathcal{L}(\mathcal{H})$ is the set of all bounded operators a such that $ab = ba$ for every $b \in J$. It is clear that J' is an algebra, closed in the strong topology, and that the *bicommutant* J'' of J contains J. If J is stable under the involution, so is J' (which therefore is a vNa), and so is J''.

Here is the first, and most important, result in this theory.

VON NEUMANN'S BICOMMUTANT THEOREM. *Let $\mathcal{A}$ be a sub-*-algebra of $\mathcal{L}(\mathcal{H})$ containing I. Then the bicommutant $\mathcal{A}''$ is exactly the strong (= weak) closure of $\mathcal{A}$. In particular, $\mathcal{A}$ is a von Neumann algebra iff it is equal to its bicommutant.*

Note that it is not quite obvious that the strong closure of an algebra is an algebra!

PROOF. It is clear that the strong closure of $\mathcal{A}$ is contained in $\mathcal{A}''$. We must prove conversely that, for every $a \in \mathcal{A}''$, every $\varepsilon > 0$ and every finite family $x_1, \ldots, x_n$ of elements of $\mathcal{H}$, there exists $b \in \mathcal{A}$ such that $\| bx_i - ax_i \| \leq \varepsilon$ for all i.

We begin with the case of one single vector x. Let K the closed subspace in $\mathcal{H}$ generated by all bx, $b \in \mathcal{A}$; K is closed under the action of $\mathcal{A}$, and the same is true for $K^\perp$ (if $< y, bx > = 0$ for every $b \in \mathcal{A}$, we also have for $c \in \mathcal{A}$ $< cy, bx > = < y, c^* bx > = 0$). This means that the orthogonal projection p on K commutes with every $b \in \mathcal{A}$; otherwise stated, $p \in \mathcal{A}'$. Then $a \in \mathcal{A}''$ commutes with p, so that $ax \in K$. This means there exist $a_n \in \mathcal{A}$ such that $a_n x \to ax$.

Extension to n vectors : let $\widehat{\mathcal{H}}$ be the direct sum of n copies of $\mathcal{H}$. A linear operator on $\widehat{\mathcal{H}}$ can be considered as a (n, n) matrix of operators on $\mathcal{H}$, and we call $\widehat{\mathcal{A}}$ the algebra consisting of all operators $\widehat{a}$ repeating $a \in \mathcal{A}$ along the diagonal ($\widehat{a}(y_1 \oplus \ldots \oplus y_n) = ay_1 \oplus \ldots \oplus ay_n$). It is easy to see that $\widehat{\mathcal{A}}$ is a vNa whose commutant consists of all matrices with coefficients in $\mathcal{A}'$, and whose bicommutant consists of all diagonals $\widehat{a}$ with $a \in \mathcal{A}''$. Then the approximation result for n vectors $(x_1, \ldots, x_n)$ in $\mathcal{A}$ follows from the preceding result applied to $x_1 \oplus \ldots \oplus x_n$.

COROLLARY. *The von Neumann algebra generated by a subset J of $\mathcal{L}(\mathcal{H})$ is $(J \cup J^*)''$.*

3 We give other useful corollaries as comments, rather than formal statements (the reader is referred also to the beautiful exposition of Nelson [Nel2]). The general idea is, that every "intrinsic" construction of operator theory with an uniquely defined result,

when performed on elements of a vNa $\mathcal{A}$, also leads to elements of $\mathcal{A}$. We illustrate this by an example.

First of all, let us say that an operator with domain $\mathcal{D}$, possibly unbounded, is *affiliated* to the vNa $\mathcal{A}$ if $\mathcal{D}$ is stable under every $b \in \mathcal{A}'$ and $abx = bax$ for $x \in \mathcal{D}$ (thus a bounded everywhere defined operator is affiliated to $\mathcal{A}$ iff it belongs to $\mathcal{A}$: this is a restatement of von Neumann's theorem).

Consider now an unbounded selfadjoint operator S on $\mathcal{H}$, and its spectral decomposition $S = \int_{\mathbb{R}} \lambda \, dE_\lambda$. This decomposition is unique. More precisely, if $u : \mathcal{H} \to \mathcal{K}$ is an isomorphism between Hilbert spaces, and T is the operator uSu^{-1} on $\mathcal{K}$, then the spectral projections of T are given by $F_t = uE_tu^{-1}$. Using this trivial remark, we prove by "abstract nonsense" that *if S is affiliated to a vNa $\mathcal{A}$, then its spectral projections E_t belong to $\mathcal{A}$.*

Indeed, it suffices to prove that for every $b \in \mathcal{A}'$ we have $E_tb = bE_t$. Since b is a linear combination of unitaries within $\mathcal{A}'$, we may assume $b = u$ is unitary. Then we apply the preceding remark with $\mathcal{K} = \mathcal{H}$, $T = S$, and get that $E_t = F_t = uE_tu^{-1}$, the desired result.

More generally, this applies to the so called *polar decomposition* of every closed operator affiliated with $\mathcal{A}$.

Kaplansky's density theorem

4 Let $\mathcal{A}$ be a *-algebra of operators. Since strang convergence without boundedness is not very useful, von Neumann's theorem is not powerful enough as an approximation result of elements in $\mathcal{A}''$ by elements of $\mathcal{A}$. The Kaplansky theorem settles completely this problem, and shows that elements in $\mathcal{A}''$ of one given kind can be approximated boundedly by elements of the same kind from $\mathcal{A}$.

THEOREM. *For every element a from the unit ball of $\mathcal{A}''$, there exists a filter a_i on the unit ball of $\mathcal{A}$ (not necessarily a sequence) that converges strongly to a. If a is s.a., positive, unitary, the a_i may be chosen with the same property.*

PROOF (from Pedersen [Ped]). Let f be a real valued, continuous function on $\mathbb{R}$, so that we may define $f(a)$ for every s.a. operator a. We say that f is *strong* if the mapping $f(\cdot)$ is continuous in the strong operator topology. Obvious examples of strong functions are $f(t)=1$, $f(t)=t$. The set $\mathcal{S}$ of all strong functions is a linear space, closed in the uniform topology. On the other hand, the product of two elements of $\mathcal{S}$, one of which is bounded, belongs to $\mathcal{S}$. The main remark is

LEMMA. *Every continuous and bounded function on $\mathbb{R}$ is strong.*

We first put $h(t) = 2t/(1+t^2)$ and prove that it is strong. Let a, b be s.a. and put $A = (1+a^2)^{-1}$, $B = (1+b^2)^{-1}$. Then we compute as follows the difference $h(b) - h(a)$ (forgetting the factor 2)

$$Bb - Aa = B\,\left[b(1+a^2) - (1+b^2)a\right]\,A = B(b-a)A + Bb(a-b)aA\,.$$

If $b_i \to a$, B_i and B_ib_i remain bounded in norm, and therefore $h(b_i) \to h(a)$.

This bounded function being strong, we find (by multiplication with the strong function t) that $t^2/(1+t^2)$ by t is strong, and so is $1/(1+t^2)$ by difference. Applying

the Stone–Weierstrass theorem on $\overline{\mathbb{R}}$ we find that all bounded continuous functions on $\overline{\mathbb{R}}$ are strong. Then, if f is only continuous and bounded on $\mathbb{R}$, $tf(t)/(1+t^2)$ tends to 0 at infinity and hence is strong and bounded. Then writing

$$f(t) = \frac{f(t)}{1+t^2} + t\,\frac{tf(t)}{1+t^2}$$

we find that f itself is strong. This applies in particular to $\cos t$, $\sin t$, and we have everything we need to prove the theorem.

S.a. operators. Let $b \in \mathcal{A}''$ be s.a. with norm ≤ 1. Since b belongs to the weak closure of $\mathcal{A}$ and the involution is continuous in the weak topology, it belongs to the weak closure of $\mathcal{A}_{sa}$. This set being convex, b belongs in fact to its strong closure. We choose selfadjoint $b_i \in \mathcal{A}_{sa}$ converging strongly to b, and we deduce from the lemma that $h(b_i) \to h(b)$ strongly. And now $h(b_i)$ belongs to the unit ball of $\mathcal{A}_{sa}$, while every element a in the unit ball of $\mathcal{A}''_{sa}$ can be represented as $h(b)$, since h induces a homeomorphism of $[-1,1]$ to itself.

The same reasoning gives a little more : if a is positive so is b. Then we also have $h^+(b_i) \to h^+(b) = a$, and these approximating operators are positive too.

Unitaries. If $u \in \mathcal{A}''$ is unitary, it has a spectral representation as $\int_{-\pi}^{\pi} e^{it} dE_t$ with spectral projections $E_t \in \mathcal{A}''$. The s.a. operator $a = \int_{-\pi}^{\pi} t\, dE_t$ can be approximated strongly by s.a. operators $a_j \in \mathcal{A}$, and then the unitaries e^{ia_j} converge strongly to u.

Arbitrary operators of norm ≤ 1. Consider $a \in \mathcal{A}''$ of norm ≤ 1. On $\mathcal{H} \oplus \mathcal{H}$ let the vNa $\widehat{\mathcal{A}}$ consist of all matrices $\begin{pmatrix} j & k \\ l & m \end{pmatrix}$ with $j,k,l,m \in \mathcal{A}$. Then $\widehat{\mathcal{A}}'$ consists of the matrices $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ with $p \in \mathcal{A}'$, and the operator $\begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$ is s.a. with norm ≤ 1 and belongs to the bicommutant $\widehat{\mathcal{A}}''$. From the result above it can be approximated by operators $\begin{pmatrix} r & s^* \\ s & t \end{pmatrix}$ of norm ≤ 1, with $r,s,t \in \mathcal{A}$. Then s has norm ≤ 1 and converges strongly to a, and the proof gives an additional result : the approximants $s \to a$ can be chosen so that $s^* \to a^*$.

REMARK. Assume $\mathcal{A}$ was closed in the "normal" topology. We have just seen that the unit ball B of $\mathcal{A}''$ is the strong or weak closure of the unit ball of $\mathcal{A}$. But on the unit ball of $\mathcal{L}(\mathcal{H})$ the weak and normal topologies coincide, and therefore B is contained in $\mathcal{A}$. Thus $\mathcal{A} = \mathcal{A}''$ and therefore $\mathcal{A}$ is a vNa.

The predual of a von Neumann algebra

5 The Banach space $\mathcal{L}(\mathcal{H})$ is the dual of $\mathcal{T}(\mathcal{H})$, the space of trace class operators. Therefore every subspace $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ which is closed for the normal topology is itself a dual, that of the quotient Banach space $\mathcal{T}/\mathcal{A}^{\perp}$. In the case $\mathcal{A}$ is a vNa, this space is called the *predual of* $\mathcal{A}$ and denoted by $\mathcal{A}_*$. The predual can be described as the space of all complex linear functionals on $\mathcal{A}$ which are continuous in the normal topology. We call them *normal linear functionals* below.

According to the Hahn–Banach theorem, every normal functional μ on $\mathcal{A}$ can be extended to a normal functional on $\mathcal{L}(\mathcal{H})$, and therefore can be represented as

$\mu(a) = \mathrm{Tr}(aw)$ where w is a trace class operator. In particular, decomposing w we see that every normal linear functional is a complex linear combination of four normal laws (states). In the case of laws, the following lemma (which is not essential for the sequel) gives a more precise description

LEMMA. *Every normal law μ on $\mathcal{A}$ can be extended as a normal law on $\mathcal{L}(\mathcal{H})$, hence the trace class operator w can be taken positive.*

PROOF. For simplicity, we assume that $\mathcal{H}$ is separable. Since $\mu(a) = \overline{\mu}(a^*)$, μ is also associated with the trace class operator w^*, and Replacing w by $(w+w^*)/2$ we may assume w is s.a.. Then we choose an o.n.b. (e_n) such that $we_n = \lambda_n e_n$ with $\sum_n |\lambda_n| < \infty$. Consider the direct sum $\widehat{\mathcal{H}}$ of countably many copies $\mathcal{H}_n$ of $\mathcal{H}$, on which $\mathcal{A}$ operates diagonally ($\widehat{a}(\sum_n x_n) = \sum_n ax_n$). Let $\mathbf{1} = c\sum_n \sqrt{|\lambda_n|}\, e_n \in \widehat{\mathcal{H}}$, where c is a normalization constant, and ν be the corresponding state $\nu(a) = \sum_n < \mathbf{1}, \widehat{a}\mathbf{1} >$. We have

$$\mu(a) = \sum_n \lambda_n < e_n, ae_n > , \qquad \nu(a) = c^2 \sum_n \sqrt{|\lambda_n|} < e_n, ae_n > .$$

Let $\mathcal{K}$ be the closed (stable) subspace generated by $\mathcal{A}\mathbf{1}$ in $\widehat{\mathcal{H}}$. Since λ_n is bounded μ is dominated by a scalar multiple of ν, and therefore (by a simple lemma proved in subs. 8 below) is of the form $< T\mathbf{1}, \widehat{a}\mathbf{1} >$ where T is a bounded positive operator on $\mathcal{K}$ which *commutes* with the representation. Then putting $j = \sqrt{T}\,\mathbf{1}$ we have $\mu(a) = < j, \widehat{a}j >$. Taking $a = I$ note that j is an unit vector. In the last expression we may replace a by $b \in \mathcal{L}(\mathcal{H})$ operating diagonally on $\widehat{\mathcal{H}}$, and we have found an extension of μ as a law on $\mathcal{L}(\mathcal{H})$.

6 In this subsection, we prove that for a positive linear functional on $\mathcal{A}$, being normal is equivalent to being *order continuous,* an intrinsic property, in the sense that it can be defined within $\mathcal{A}$, independently of the way it acts on the Hilbert space $\mathcal{H}$. As a consequence, the predual $\mathcal{A}_*$ and the normal topology itself are intrinsic.

In the literature, properties of a concrete vNa $\mathcal{A}$ of operators on $\mathcal{H}$ which depend explicitly on $\mathcal{H}$ are often called *spatial* properties.

Let (a_i) be a family of positive elements of $\mathcal{A}$, which is filtering to the right and norm bounded. Then for $x \in \mathcal{H}$ the limit $\lim_i < x, a_i x >$ exists, and by polarization also $\lim_i < y, a_i x >$. Therefore $\lim_i a_i = a$ exists in the weak topology, and of course belongs to $\mathcal{A}$. Since $\| a_i \|$ is bounded, the convergence also takes place in the normal topology. It is also strong, as shown by the relation

$$\| (a - a_i)x \|^2 \le \| a - a_i \| < x, (a - a_i)x > \to 0$$

(the inequality $b^2 \le \| b \| b$ for $b \ge 0$ is used here). On the other hand, a can be described without reference to $\mathcal{H}$ as the l.u.b. of the family (a_i) in $\mathcal{A}$.

Let μ be a positive linear functional on $\mathcal{A}$. If $a = \sup_i a_i$ implies $\mu(a) = \sup_i \mu(a_i)$ we say that μ is *order continuous.* Since order convergence implies normal convergence, every normal functional is also order continous. We are going to prove the converse.

We assume that μ is order continuous. We associate with every positive element $b \in \mathcal{A}$ a (complex!) linear functional $\mu_b(a) = \mu(ab)$ and if μ_b is normal we say that b is *regular.* Thus our aim is to prove that I is regular. We achieve this in three steps.

1) Let (b_i) a norm bounded, increasing family of regular elements. We prove its upper bound b is regular.

Since the predual $\mathcal{A}_*$ is a Banach space, it suffices to prove that μ_{b_i} converges in norm to μ_b. For a in the unit ball of $\mathcal{A}$ we have

$$\begin{aligned} |\mu(a(b-b_i)| &= |\mu(a(b-b_i)^{1/2}(b-b_i)^{1/2})| \\ &\le \mu(a(b-b_i)\,a^*)^{1/2}\mu(b-b_i)^{1/2} \qquad \text{(Schwarz)} \end{aligned}$$

The first factor is bounded independently of a, the second one tends to 0 since μ is order continuous.

2) Let $m \in \mathcal{A}$ be positive and $\neq 0$. Then there exists a regular element $q \neq 0$ dominated by m.

We choose a vector x such that $<x, mx> \, > \mu(m)$ and choose p positive and $\le m$, such that $<x, px> \, \le \mu(p)$. Since μ is order continuous, Zorn's lemma allows us to choose a maximal p. Let us prove that $q = m - p$ fulfills our conditions. First of all, it is positive, dominated by m, and the choice of x forbids that $q = 0$. The maximality of p implies that for every positive $u \le q$ we have $\mu(u) \le \, <x, ux>$, and this is extended to the case $u \le Cq$ by homogeneity.

We now prove that μ_q is strongly continuous, hence also weakly continuous, and finally normal. For $a \in \mathcal{A}$ we have $|\mu(aq)| \le \mu(I)^{1/2}\mu(qa^*aq)^{1/2}$. On the other hand, $qa^*aq \le \|a\|^2 q^2 \le \|a\|^2 \|q\|\, q$, a constant times q. Therefore

$$\mu(qa^*aq) \le \, <x, qa^*aq\, x> \, = \|aqx\|^2 .$$

So finally we have $\mu_q(a) \le \mu(I)^{1/2}\|aqx\|$, whence the strong continuity of μ_q.

3) Zorn's lemma and 1) allow us to choose a maximal regular element b such that $0 \le b \le I$. Using 2) the only possibility is $b = I$, and we are finished.

About integration theory

7 The standard theory of non-commutative integration is developed for *faithful normal laws (states) on a von Neumann algebra.* However, as usual with integration theory, the problem is to start from a functional on a small space of functions and to extend it to a larger space. We are going to discuss (rather superficially) this problem.

First of all, we consider a C^*-algebra denoted by $\mathcal{A}^{\circ}$, with a probability law μ. The GNS construction provides us with a Hilbert space $\mathcal{H}$ on which $\mathcal{A}^{\circ}$ acts (we simply write ax for the action of $a \in \mathcal{A}^{\circ}$ on $x \in \mathcal{H}$), and a cyclic vector $\mathbf{1}$ such that $\mu(a) = \, <\mathbf{1}, a\mathbf{1}>$. Since the kernel $\mathcal{N}$ of the GNS representation is a two sided ideal and the quotient $\mathcal{A}^{\circ}/\mathcal{N}$ is a C^*-algebra, we do not lose generality by assuming the GNS representation is 1–1. Thus $\mathcal{A}^{\circ}$ appears as a norm closed subalgebra of $\mathcal{L}(\mathcal{H})$. Let us denote by $\mathcal{A}$ its bicommutant, a von Neumann algebra containing $\mathcal{A}^{\circ}$, and to which μ trivially extends as the (normal) law $\mu(a) = \, <\mathbf{1}, a\mathbf{1}>$.

What have we achieved from the point of view of integration theory ? Almost nothing. Think of the commutative case. Let $\mathcal{F}^{\circ}$ be a boolean algebra of subsets of some set E, and let μ be a finitely additive law on $\mathcal{F}^{\circ}$. The space of complex elementary functions (linear combinations of finitely many indicator of sets) is a unital *-algebra, and its

completion in the uniform norm is a commutative C^*-algebra $\mathcal{A}^\circ$. If we perform all the construction above, we get a normal law on some huge commutative vNa containing $\mathcal{A}^\circ$, and the essential assumption of integration theory, namely the countable additivity of μ, is completely irrelevant!

Besides that, to develop a rich integration theory, it seems necessary to use a faithful law, which cannot be achieved simply (as we pointed out in §2 subs. 3) by a further passage to the quotient. To demand that the state be faithful on $\mathcal{A}^\circ$ is not sufficient to make sure it will be faithful on $\mathcal{A}$. However, the following easy lemma sets a necessary condition :

LEMMA. *Assume the law* μ *is faithful, and* $(a_i)_{i\in I}$ *is a norm bounded family such that* $\lim_i \mu(a_i^* a_i) = 0$ *(along some filter on* I*). Then we also have* $\lim_i \mu(a_i b) = 0$ *for* $b \in \mathcal{A}$.

PROOF. We use the language of the GNS representation. It is sufficient to prove that $\mu(a_i b) \to 0$ along every ultrafilter on the index set I, finer than the given filter. Since the operators a_i are bounded in norm, they have a limit a in the weak topology of operators (even in the normal one), and $a_i \mathbf{1} \to a\mathbf{1}$ weakly. The condition $\mu(a_i^* a_i) \to 0$ means that $\lim_i a_i \mathbf{1} = 0$ in the strong sense, and therefore $a\mathbf{1} = 0$. Since the vector $\mathbf{1}$ is separating, this implies $a = 0$. On the other hand $\mu(a_i b) = < \mathbf{1}, a_i b\mathbf{1} >$ tends to $< \mathbf{1}, ab\mathbf{1} > = 0$.

The same conclusion would be true under the hypothesis that the norm bounded family a_i converges weakly to 0 : this will be useful in subsection 9.

Knowing this, we assume that the probability law μ *on the* C^**-algebra* $\mathcal{A}^\circ$ satisfies the property

$$(7.1) \qquad \| a_i \| \text{ bounded, } \lim_i \mu(a_i^* a_i) = 0 \text{ implies } \lim_i \mu(a_i b) = 0 \text{ for every } b \in \mathcal{A}^\circ .$$

(It would be sufficient to assume this for sequences : the easy proof is left to the reader). In particular, $(\mu(a^*a) = 0) \Longrightarrow (\mu(ab) = 0)$ for every b, and therefore the set of all "left negligible elements" is the same as the kernel $\mathcal{N}$ of the GNS representation. Passing to the quotient, we may assume that the GNS representation is faithful, and imbed $\mathcal{A}^\circ$ in $\mathcal{L}(\mathcal{H})$. On the other hand, the property (7.1) is preserved when we pass to the quotient.

Let us prove now that *the cyclic vector* $\mathbf{1}$ *is separating for the von Neumann algebra* $\mathcal{A}$, *the bicommutant of* $\mathcal{A}^\circ$.

PROOF. Consider $a \in \mathcal{A}$ such that $a\mathbf{1} = 0$. According to Kaplansky's theorem, there exists a norm bounded family of elements a_i of $\mathcal{A}^\circ$ that converges strongly to a, and in particular $\lim_i \| a_i b\mathbf{1} \| = 0$ for every $b \in \mathcal{A}^\circ$. Taking $b = I$ we have $\mu(a_i^* a_i) \to 0$, and the same for $c^* a_i$ ($c \in \mathcal{A}^\circ$ since left multiplications are bounded. Then we deduce from (7.1) that

$$0 = \lim_i \mu(c^* a_i b) = \lim_i < c\mathbf{1}, a_i b\mathbf{1} > = < c\mathbf{1}, ab\mathbf{1} > .$$

Since b, c are arbitrary and $\mathbf{1}$ is cyclic, we have $a = 0$.

From now on, we use the notation $[\![a]\!]$ *for the operator norm, and keep the notation* $\| a \|$ *for the norm* $\mu(a^*a)^{1/2}$.

8 In the commutative case, a space $L^\infty(\mu)$ can be interpreted as the space of essentially bounded functions, and as a space of measures possessing bounded densities

with respect to μ. In the non-commutative case, if we think of $\mathcal{A}^{\circ}$ as consisting of "continuous functions", the first interpretation of L^{∞} becomes the von Neumann algebra $\mathcal{A}$, and we are going to show that the second interpretation leads, when $\mathcal{A}$ has a cyclic and separating vector $\mathbf{1}$, to the *commutant of* $\mathcal{A}$.

A positive linear functional π on $\mathcal{A}$ may be said to have a bounded density if it is dominated by a constant c times μ. Then we have for $a, b \in \mathcal{A}$

$$(8.1) \qquad |\,\pi(b^* a)\,| \leq \pi(b^* b)^{1/2}\, \pi(a^* a)^{1/2} \leq c\, \|\, b \,\|\, \|\, a \,\| \ .$$

Forgetting the middle, the inequality is meaningful for a *complex* linear functional π. If it is satisfied, we say that π is a *complex measure with bounded density.* Let $\mathcal{D}$ be the dense subspace $\mathcal{A}\mathbf{1}$ of $\mathcal{H}$ (in the GNS construction, this is nothing but another name for $\mathcal{A}$ itself). Since $\mathbf{1}$ is separating we may define on $\mathcal{D}$ $p(b\mathbf{1}, a\mathbf{1}) = \pi(b^* a)$, a bounded sesquilinear form, which can be extended to $\mathcal{H}$. Given any unitary $u \in \mathcal{A}$ we have $p(ub\mathbf{1}, ua\mathbf{1}) = p(b\mathbf{1}, a\mathbf{1})$, and by continuity this can be extended to $p(uy, ux) = p(y, x)$ for $x, y \in \mathcal{H}$. With the bounded form p on $\mathcal{H} \times \mathcal{H}$ we associate the unique bounded operator α on $\mathcal{H}$ such that $p(y, x) = \, < \alpha y, x >$. Then the unitary invariance of p means that α commutes with every unitary $u \in \mathcal{A}$ *and thus belongs to* $\mathcal{A}'$. Otherwise stated, the commutant $\mathcal{A}'$ of $\mathcal{A}$ appears as a space of measures on $\mathcal{A}$, given by the formula

$$(8.2) \qquad \pi(a) = p(\mathbf{1}, a\mathbf{1}) = \, < \alpha\mathbf{1}, a\mathbf{1} > .$$

Conversely, any linear functional on $\mathcal{A}$ of the form (8.2) satisfies (8.1).

REMARK. The name of "measures with bounded density" for functionals of the form $\pi(a) = \mu(\alpha a)$ associated with $\alpha \in \mathcal{A}'$ is reasonable. First assume $\alpha \geq 0$. Then π is positive and for $a \in \mathcal{A}_+$ we have

$$0 \leq \pi(a) = \|\, \sqrt{\alpha}\sqrt{a}\,\mathbf{1}\,\|^2 \leq \|\, \sqrt{\alpha}\,\|^2\, \|\, \sqrt{a}\,\mathbf{1}\,\|^2 = \|\, \sqrt{\alpha}\,\|^2\, \mu(a) \ .$$

Thus π is dominated by a constant times μ. On the other hand, every element in $\mathcal{A}'$ can be written as $p + iq$ using s.a. elements, each of which can be decomposed into two positive elements of $\mathcal{A}'$ to which the preceding reasoning applies.

The symmetry between $\mathcal{A}$ and $\mathcal{A}'$ is very clear in the following lemma (which can easily be improved to show that $\mathbf{1}$ is cyclic (separating) for $\mathcal{A}$ iff it is separating (cyclic) for $\mathcal{A}'$).

LEMMA. *If the vector* $\mathbf{1}$ *is cyclic and separating for* $\mathcal{A}$, *it has the same properties with respect to* $\mathcal{A}'$.

PROOF. 1) Let $\mathcal{K}$ be the closed subspace generated by $\mathcal{A}'\mathbf{1}$. The projection on $\mathcal{K}^{\perp}$ commutes with $\mathcal{A}'$ (hence belongs to $\mathcal{A}$) and kills $\mathbf{1}$. Therefore $\mathcal{K}^{\perp} = \{0\}$ and $\mathbf{1}$ is cyclic for $\mathcal{A}'$.

2) Consider $\alpha \in \mathcal{A}'$ such that $\alpha\mathbf{1} = 0$. Then for $a \in \mathcal{A}$ we have $\alpha a\mathbf{1} = a\alpha\mathbf{1} = 0$, hence $\alpha = 0$ on the dense subspace $\mathcal{D}$, thus $\alpha = 0$ and we see that $\mathbf{1}$ is separating for $\mathcal{A}'$.

The linear Radon–Nikodym theorem

9 As we have just seen, measures absolutely continuous w.r.t. μ are naturally defined by "densities" belonging to $\mathcal{A}'$. But if we insist to have densities in $\mathcal{A}$, it is unreasonable to define the "measure with (s.a.) density m" by a formula like $\pi(a) = \mu(ma)$, since it would give a complex result for non commuting s.a. operators. We have a choice between non–linear formulas like $\pi(a) = \mu(\sqrt{m}\, a \sqrt{m})$ (for $m \geq 0$, leading to a positive linear functional) and linear ones like

$$\pi(a) = \frac{1}{2}\, \mu(ma + am) \ , \tag{9.1}$$

which give a real functional on s.a. operators if the "density" m is s.a., but usually not a positive functional for $m \geq 0$. The possibility of representing "measures" by such a formula is an important lemma for the Tomita–Takesaki theory.

However, if we want to construct a s.a. operator from a pair of s.a. operators a, m, we are not restricted to (9.1) : we can use any linear combination of the form $kam + \overline{k}ma$ where k is complex. For this reason, we put

$$\pi_{m,k}(a) = \frac{1}{2}\, \mu(kam + \overline{k}ma) \tag{9.2}$$

(written simply as π if there is no ambiguity). Then we can state *Sakai's linear Radon–Nikodym theorem* as follows.

THEOREM. *Let φ be a positive linear functional on $\mathcal{A}$ dominated by μ. For every k such that $\Re e(k) > 0$ there exists a unique positive element $m \in \mathcal{A}$ such that $\varphi = \pi_{m,k}$, and we have $m \leq \Re e(k)^{-1} I$.*

PROOF. *Uniqueness.* Let us assume that $\mu(kam + \overline{k}ma) = 0$ for $a \in \mathcal{A}_{sa}$. Since $\Re e(k) \neq 0$ the case $a = m$ gives $\mu(m^2) = 0$. Since m is s.a. and μ is faithful this implies $m = 0$.

Existence. We may assume that $\Re e(k) = 1$. The set M of all $m \in \mathcal{A}_{sa}$ such that $0 \leq m \leq I$ is compact and convex in the normal topology. For $m \in M$ the linear functional π_m from (9.2) on $\mathcal{A}$ is normal; indeed, if a family (a_i) from the unit ball of $\mathcal{A}$ is weakly convergent to 0, the same is true for ma_i and $a_i m$, and therefore $\pi_m(a_i) \to 0$. Also, the 1–1 mapping $m \longmapsto \pi_m$ is continuous from M to $\mathcal{A}_*$: this amounts to saying that if a family (m_i) from the unit ball of $\mathcal{A}$ converges weakly to 0, then $\mu(m_i a)$ and $\mu(am_i)$ tend to 0, and this was mentioned as a remark following the lemma in subsection 7.

Let the real predual consist of those normal functionals on $\mathcal{A}$ which are real valued on $\mathcal{A}_{sa}$. It is easy to see that it is a real Banach space with dual $\mathcal{A}_{sa}$, and we may embed M in it, via the preceding 1-1 mapping, as a convex, weakly compact subset. We want to prove that $\varphi \in M$. According to the Hahn-Banach theorem, this can be reduced to the following property

$$\text{for every } a \in \mathcal{A}_{sa} \text{ such that } < M, a > \leq 1 \text{ we have } < \varphi, a > \leq 1 .$$

The condition $< M, a > \leq 1$ means that $\frac{1}{2}\mu(kam + \overline{k}ma) \leq 1$ for every $m \in \mathcal{A}_{sa}$ such that $0 \leq m \leq I$. We decompose a into $a^+ - a^-$ and take for m the projection

$I_{]0,\infty[}(a)$. Then $ma = am = a^+$, and the condition implies that $\frac{1}{2}(k+\overline{k})\mu(a^+) \leq 1$. Since we assumed that $\Re e(k) = 1$ we have $\mu(a^+) \leq 1$. On the other hand, φ is a positive linear functional dominated by μ, and therefore $\varphi(a) \leq \varphi(a^+) \leq \mu(a^+) \leq 1$.

The KMS condition

10 Let $\mathcal{H}$ be a Hilbert space, and let H be a "Hamiltonian" : a s.a. operator, generally unbounded and positive (or at least bounded from below) generating a unitary group $U_t = e^{itH}$ on $\mathcal{H}$, and a group of automorphisms of the C^*-algebra $\mathcal{L}(\mathcal{H})$

$$\eta_t(a) = e^{itH}\, a\, e^{-itH} .$$

One of the problems in statistical mechanics is the construction and the study of *equilibrium states* for this evolution. Von Neumann suggested that the natural equilibrium state at (absolute) temperature T is given by

$$\mu(a) = \mathrm{Tr}(aw) \quad \text{with } w = Ce^{-H/\kappa T}$$

where κ would be Boltzmann's constant if we had been using the correct physical notations and units. The coefficient $1/\kappa T$ is frequently denoted by β. However, this definition is meaningful only if $e^{-\beta H}$ is a trace class operator, a condition which is not generally satisfied for large quantum systems. For instance, it is satisfied by the number operator of a finite dimensional harmonic oscillator, but not by the number operator on boson Fock space.

The KMS (Kubo–Martin–Schwinger) condition is a far reaching generalization of the preceding idea, which consists in forgetting about traces, and retaining only the fact that $e^{-\beta H}$ is an analytic extension of e^{itH} to purely imaginary values of t which is well behaved w.r.t. μ. From our point of view, the most remarkable feature of the KMS property is the natural way, completely independent of its roots in statistical mechanics, in which it appears in the Tomita–Takesaki theory. It seems to be a very basic idea indeed!

Given an automorphism group (η_t) of a C^*-algebra $\mathcal{A}$, we say that $a \in \mathcal{A}$ is *entire* if the function $\eta_t a$ on the real axis is extendable as an $\mathcal{A}$-valued entire function on $\mathbb{C}$. Every strongly continuous automorphism group of a C^*-algebra has a dense set of entire elements (see subsection 11).

DEFINITION. *A law μ on the C^*-algebra $\mathcal{A}$ satisfies the KMS condition at β w.r.t. the automorphism group (η_t) if there exists a dense set of entire elements a such that we have, for every $b \in \mathcal{A}$ and t real*

$$\mu((\eta_t a)\, b) = \mu(b(\eta_{t+i\beta}\, a)) . \tag{10.1}$$

One can always assume that $\beta = 1$, replacing if necessary η_t by $\eta_{t/\beta}$ (this applies also to $\beta < 0$).

One can prove that, whenever $e^{-\beta H}$ is a trace class operator, the state defined above satisfies the KMS property at β.

There is another version of the KMS condition, which has the advantage of applying to arbitrary elements instead of entire ones. We say that a function is *holomorphic on a*

closed strip of the complex plane if it is holomorphic in the open strip and continuous on the closure.

THEOREM. *The KMS condition is equivalent to the following one : for arbitrary $a, b \in \mathcal{A}$, there exists a bounded function f, holomorphic on the horizontal unit strip $\{0 \leq \Im m(z) \leq 1\}$, such that*

$$f(t+i0) = \mu(b(\eta_t a)) \quad , \quad f(t+i1) = \mu((\eta_t a)\, b)\ . \tag{10.2}$$

PROOF. Assume the KMS condition holds. We may approximate a by a sequence (a_n) of entire elements satisfying (10.1), and we set $f_n(z) = \mu(b(\eta_z a_n))$, an entire function. We prove that f_n converges uniformly on the horizontal unit strip, its limit being then the function f. Since the analytic functions $\eta_{t+z} a_n$ and $\eta_t \eta_z a_n$ (for real t) are equal for $z \in \mathbb{R}$, they are equal, and

$$|\, \mu(b(\eta_{t+is} a_n))\,| \leq \| b \| \, \| \eta_{t+is} a_n \| = \| b \| \, \| \eta_{is} a_n \|$$

is bounded in every horizontal strip, and therefore satisfies the maximum principle on horizontal strips. On the other hand

$$|\, \mu(b\eta_t(a_n - a_m))\,| \leq \| b \| \, \| a_n - a_m \|$$

and using KMS

$$|\mu(b\eta_{t+i}(a_n - a_m))| = |\mu(\eta_t(a_n - a_m)\, b)| \leq \| b \| \, \| a_n - a_m \| \ .$$

Therefore f_n converges uniformly.

Conversely, assume (10.2) holds and consider an entire element a. The function $f(z) - \mu(b(\eta_z a))$ is holomorphic in the open unit strip and its limit on the real axis is 0. By the reflection principle, it is extendable to the open strip $\{-1 < \Im m(z) < 1\}$, equal to 0 on the real axis, hence equal to 0. Therefore we also have $f(t+i) = \mu(b(\eta_{t+i} a))$. On the other hand, by (10.2) we have $f(t+i) = \mu((\eta_t a)\, b)$, and (10.1) is established. Note that it has been proved for an arbitrary entire vector a.

We draw two consequences from the KMS condition. The first one is the *invariance* of μ. Let a be an entire element. We apply KMS with $b = I$. We have just remarked that the function $\mu_z(a)$ is bounded on horizontal strips. On the other hand, KMS implies that it is periodic with period i. An entire bounded function being constant, μ is invariant.

The second consequence is the regularity property (7.1) :

$$b_j \text{ bounded, } \lim_j \mu(b_j^* b_j) = 0 \text{ implies } \forall a \in \mathcal{A} \ \lim_j \mu(b_j a) = 0.$$

We may assume a is an entire vector, and pass to the limit. On the other hand by KMS $\mu(b_j a) = \mu((\eta_{-i} a)\, b)$, and we are reduced to the trivial case of left multiplication by a fixed element.

11 Finally, we mention the result on entire vectors we used above. Let (U_t) be any strongly continuous group of contractions on a Banach space B. For every $b \in B$ set $b_n = c_n \int e^{-ns^2} U_s b\, ds$, where c_n is a normalization constant. Then we have

$$U_t b_n = c_n e^{-nt^2} \int_{-\infty}^{\infty} e^{2nst-s^2} U_s b\, ds$$

and it is trivial to replace t by z in this formula. On the other hand, strong continuity of the semigroup implies that for every b we have $b_n \to b$ in norm. We remark that this function is norm bounded in *vertical* strips of finite width.

§4. THE TOMITA–TAKESAKI THEORY

Nearly all papers on non–commutative integration begin with the statement "let $\mathcal{A}$ be a von Neumann algebra with a faithful normal state..." In the language of GNS representations, the statement becomes " Let $\mathcal{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, with a cyclic and separating vector...". This is also the starting point for the Tomita–Takesaki theorem.

Though the theory is extremely beautiful, it does not apply to our main example, that of the vacuum state on the CCR algebra acting on simple Fock space, which isn't a separating state. It applies to the vacuum state of non–Fock representations.

We did our best in the following sections to justify this starting point : μ being a law on a C^*–algebra $\mathcal{A}^\circ$, we know from the GNS theorem that it can be interpreted as the law associated with a cyclic vector $\mathbf{1}$ in a Hilbert space $\mathcal{H}$ on which $\mathcal{A}^\circ$ operates. Then we showed that a weak regularity assumption on μ allows to reduce to the case $\mathbf{1}$ is also separating for the vNa $\mathcal{A}$ generated by $\mathcal{A}^\circ$.

We follow the approach of Rieffel-van Daele [RvD1], which avoids almost completely the techniques of unbounded closable operators. The beginning of their paper has already been given in Appendix 3. It should be mentioned that the same methods lead to a simple proof ([RvD2]) of the commutation theorem $(\mathcal{A} \otimes \mathcal{B})' = \mathcal{A}' \otimes \mathcal{B}'$ for tensor products of von Neumann algebras. There is also a Tomita–Takesaki theorem for *weights* on vN algebras, which are the analogue of σ–finite measures. This result does not concern us here.

Before we start , let us describe the aim of this section. In classical integration theory, bounded measures absolutely continuous w.r.t. the law μ on $L^2(\mu)$ are represented as scalar products

$$\pi(f) = \int \overline{p}(x)\, f(x)\, \mu(dx)\ .$$

for $p \in L^1$, and in particular measures with bounded densities are represented by means of elements of L^∞. Thus we have an antilinear 1–1 mapping $p \longmapsto \overline{p}$ from functions to measures with bounded densities. In the non commutative setup, this becomes an antilinear 1–1 mapping between the vNa $\mathcal{A}$ and its commutant $\mathcal{A}'$, called the *modular conjugation* J.

This conjugation then gives a general meaning to the preceding formula, at least in the case of a positive measure π absolutely continuous w.r.t. μ (translation : a positive element of the predual of $\mathcal{A}$). Its positive density in L^1 is interpreted as a product $q\overline{q}$ for some $q \in L^2$, *i.e.* a scalar product $\pi(f) = \langle q, fq \rangle$. A similar representation holds in the non commutative case, but we will only state this result, which is too long to prove.

Finally, when one has defined conveniently L^1, L^2 and L^∞, the way to L^p is open by interpolation and duality. We do not include the theory here.

The main operators

We will use the notations and results of the Appendix on "two events", beginning with the definition of the real subspaces of $\mathcal{H}$ to which it will be applied.

1 We consider $\mathcal{H}$ as a real Hilbert space, with the scalar product $(x,y) = \Re e \langle x, y \rangle$. We introduce the real subspaces

$$A_0 = \mathcal{A}_{sa}\mathbf{1}\ ,\ A'_0 = \mathcal{A}'_{sa}\mathbf{1}\ ,\ B_0 = iA_0\ ,\ B'_0 = iB_0\ .$$

We denote the corresponding closures with the same letters, omitting the index $_0$. Since $\mathbf{1}$ is cyclic for $\mathcal{A}$, $A_0 + B_0$ (and therefore $A + B$) is dense in $\mathcal{H}$. Similarly, $A' + B'$ is dense.

The following lemma translates a little of the algebra into a geometric property.

LEMMA. *We have* $A^\perp = B'$, *and similarly* $(A')^\perp = B$.

PROOF. Consider $a \in \mathcal{A}_{sa}$, $\alpha \in \mathcal{A}'_{sa}$. Since the product of two commuting s.a. operators is s.a., $\langle a\mathbf{1}, i\alpha\mathbf{1} \rangle$ is purely imaginary, and this is translated as the orthogonality of A and B'.

To say that the orthogonal of A is B' and no larger amounts to saying that the only $x \in \mathcal{H}$ orthogonal to A and B' is $x = 0$. As often, this will be proved by a matrix trick.

We have $a \in \mathcal{A}$ act on $\mathcal{H}\oplus\mathcal{H}$ as a matrix $\widehat{a} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. We put also $\widehat{x} = \begin{pmatrix} \mathbf{1} \\ x \end{pmatrix}$. Let j be the (complex) projection on the subspace K generated by all vectors $\widehat{a}\widehat{x}$ $(a \in \mathcal{A})$; since it commutes with each $\widehat{a}$, it can be written as $j = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}$ with $\alpha, \beta \in \mathcal{A}'_{sa}$ (more precisely, $0 \le \alpha \le 1$), and $\gamma \in \mathcal{A}'$. From the definition of x we deduce three simple properties.

1) *for* $a \in \mathcal{A}_{sa}$, $\langle x, a\mathbf{1} \rangle$ *is purely imaginary.* Otherwise stated

$$\langle x, a\mathbf{1} \rangle = \overline{\langle a\mathbf{1}, x \rangle} = -\langle a\mathbf{1}, x \rangle = -\langle \mathbf{1}, ax \rangle\ .$$

The equality between the extreme members is a $\mathbb{C}$–linear property, and therefore extends from $\mathcal{A}_{sa}$ to $\mathcal{A}$. Then it is interpreted as the complex orthogonality of $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ x \end{pmatrix}$ and $\begin{pmatrix} x \\ \mathbf{1} \end{pmatrix}$, or as the property $j\begin{pmatrix} x \\ \mathbf{1} \end{pmatrix} = 0$. In particular

$$(i) \qquad \alpha x + \gamma\mathbf{1} = 0\,, \quad \text{implying } \langle x, \alpha x \rangle = -\langle x, \gamma\mathbf{1} \rangle\,.$$

2) *For* $\varepsilon \in \mathcal{A}'_{sa}$, $<x, \varepsilon\mathbf{1}>$ *is real.* As above, we deduce

$$<x, \varepsilon\mathbf{1}> = \overline{<\varepsilon\mathbf{1}, x>} = <\varepsilon\mathbf{1}, x> = <\mathbf{1}, \varepsilon x> .$$

The equality between the extreme members is a $\mathbb{C}$-linear property, and therefore extends to $\varepsilon \in \mathcal{A}'$. Taking then $\varepsilon = \gamma$ we have

$$(ii) \qquad -<x, \gamma\mathbf{1}> = -<\mathbf{1}, \gamma x> .$$

3) *Since* $I \in \mathcal{A}$, $\widehat{x}$ *belongs to the subspace* K, *therefore* $j\widehat{x} = \widehat{x}$. In particular

$$(iii) \qquad \alpha\mathbf{1} + \gamma x = \mathbf{1}, \quad \text{implying } -<\mathbf{1}, \gamma x> = -<\mathbf{1}, \mathbf{1} - \alpha\mathbf{1}>.$$

The relations *(i)–(iii)* taken together imply $<x, \alpha x> = -<\mathbf{1}, \mathbf{1} - \alpha\mathbf{1}>$, an equality between a positive and a negative number. Therefore $<x, \alpha x> = 0$, implying $\sqrt{\alpha}x = 0$, then $\alpha x = 0$. Similarly the second relation implies $(I - \alpha)\mathbf{1} = 0$. Since $\mathbf{1}$ is separating for $\mathcal{A}'$ we have $\alpha = I$, and finally $x = 0$.

2 In this subsection we define the modular symmetry and the modular group. We refer to Appendix **3** for some details (note that we use capital letters here for the main operators)

The real subspaces A, B are not in "general position", since $A \cap B^{\perp} = A \cap A'$ contains $c\mathbf{1}$ for every c belonging to the center of $\mathcal{A}$, and similarly $A^{\perp} \cap B$ contains $ic\mathbf{1}$. We have

$$(2.1) \qquad A \cap B = \{0\} = A^{\perp} \cap B^{\perp} .$$

The projections on A, B being denoted by P, Q, we put

$$S = P - Q \quad ; \quad C = P + Q - I .$$

These operators are symmetric and bounded, such that $S^2 + C^2 = 1$, $SC + CS = 0$, and their spectra are contained in $[-1, 1]$. The operator S being injective, we may define $J = \operatorname{sgn}(S)$, which will be called the *modular symmetry.* We also put $D = |S|$, which commutes with P, Q, S, C, J. On the other hand, we have

$$(2.2) \qquad JP = (I - Q)J , \qquad JQ = (I - P)J .$$

The operator of multiplication by i satisfies $iP = Qi$, so that C and D commute with i, *i.e.* are complex linear) while S and J *anticommute with* i (are conjugate linear).

3 The operators $I \pm C$ are injective too, and we define the *modular operator* Δ by the following formula

$$(3.1) \qquad \Delta = \frac{1 - C}{1 + C} = \int_{-1}^{1} \frac{1 - \lambda}{1 + \lambda}\, dF_{\lambda} .$$

It is selfadjoint, positive, injective, generally unbounded. Its square root occurs often and will be denoted by ε. The domain of Δ contains the range of $I + C$. For instance, we have $P\mathbf{1} = \mathbf{1}$, $Q\mathbf{1} = 0$, hence $(I + C)\mathbf{1} = \mathbf{1}$ and $\Delta\mathbf{1} = \mathbf{1}$.

The *modular group* is the unitary group

$$\Delta^{it} = (I - C)^{it}(I + C)^{-it} . \tag{3.2}$$

Note that the right side does not involve unbounded operators. These operators commute with C and D, but they have the fundamental property of also *commuting with* J, hence also with $JD = S$, with P and Q. Otherwise stated, they preserve the two subspaces A and B, and define automorphisms of the whole geometric situation. On the other hand, the really deep result of the Tomita theory is the fact that they preserve, not only the closed subspaces A and B, but also A_0, B_0, A'_0, B'_0.

PROOF. It suffices to prove that $J(I \pm C)^{it} J = (I \mp C)^{-it}$. Introducing the spectral decompositions U_λ and V_λ of $I - C$ and $I + C$ respectively, this amounts to

$$J(\int \lambda^{it}\, dU_\lambda)\, J = \int \lambda^{-it}\, dV_\lambda \,.$$

Since J and C anticommute, we have $J(I - C) = (I + C)\, J$, and therefore $JU_\lambda J = V_\lambda$. The result then follows from the fact that J is conjugate linear.

If we exchange the vNa $\mathcal{A}$ with its commutant $\mathcal{A}'$, the space A gets exchanged with $A' = B^\perp$ (this is almost the only place where we need the precise result of subsection 1). Thus $P' = I - Q$, $Q' = I - P$, $S' = S$, $J' = J$, $C' = -C$, and $\Delta' = \Delta^{-1}$. This remark will be used at the very end of the proof.

Interpretation of the adjoint

4 To simplify notation, we often identify $a \in \mathcal{A}$ with the vector $a\mathbf{1} \in \mathcal{H}$. Then the space $A_0 + B_0$ is identified with the vNa $\mathcal{A}$ itself. Every $x \in \mathcal{H}$ can be interpreted as a (generally unbounded) operator $Op_x : \alpha\mathbf{1} \longmapsto \alpha x$ with domain $\mathcal{D} = \mathcal{A}'\mathbf{1}$.

The following result is very important, because it sets a bridge between the preceding (*i.e.* Rieffel and van Daele's) definition of J and ε, and the classical presentation of the Tomita–Takesaki theory, in which $J\varepsilon$ would appear as the polar decomposition of the (unbounded, closable) operator $a + ib \longmapsto a - ib$ for $a, b \in \mathcal{A}_{sa}$, which describes the operation $*$.

THEOREM *For* $a, b \in \mathcal{A}_{sa}$, *the vector* $a + ib$ *belongs to* Dom(ε) *and we have*

$$J\varepsilon(a + ib) = a - ib \qquad (\varepsilon = \Delta^{1/2}) . \tag{4.1}$$

PROOF. We remark that, for $a \in \mathcal{A}_{sa}$

$$2a = 2Pa = (I + C + S)\, a = (I + C + (I - C^2)^{1/2} J)\, a = (I + C)^{1/2} x$$

with $x = (I + C)^{1/2} a + (I - C)^{1/2} Ja$, so that a belongs to the range of $(I + C)^{1/2}$, and therefore to the domain of $\varepsilon = (I - C)^{1/2}/(I + C)^{1/2}$. Then we have

$$2\varepsilon a = (I - C^2)^{1/2} a + (I - C)\, Ja \doteq Da + (I - S)\, Ja$$

and applying J we have $2J\varepsilon a = JDa + (I + C)\, a = 2a$.

Since C, ε are complex linear and J conjugate linear, this implies $J\varepsilon ib = -b$ for $b \in \mathcal{A}_{sa}$, and (4.1) follows. □

The modular property

5 The second important result is the so called "modular property" of the unitary group Δ^{it} relative to the real subspace A. It will be interpreted later as a KMS condition. Recall that a function holomorphic in a closed strip of the complex plane is a function holomorphic in the open strip and continuous in the closure. For clarity, we do not identify a and $a\mathbf{1}$, etc. in the statement of the theorem (we do in the proof).

THEOREM *Given arbitrary $a, b \in A_{sa}$, there exists a bounded holomorphic function $f(z)$ in the closed strip $0 \leq \Im m\, z \leq 1$, and such that for t real*

$$f(t+i0) = <b\mathbf{1}, \Delta^{-it}a\mathbf{1}> \ , \qquad f(t+i1) = <\Delta^{-it}a\mathbf{1}\ , \ b\mathbf{1}>$$

PROOF. We try to extend the relation

$$\Delta^{-it} = (I-C)^{-it}(I+C)^{it}$$

to complex values of t. Since $I \pm C$ is a positive operator, $(I-C)^{-iz}$ can be defined as a bounded operator for $\Im m\, z \geq 0$ and $(I+C)^{iz}$ for $\Im m\, z \leq 0$: there is no global extension outside the real axis. However, we saw at the beginning of the preceding proof that $a \in A$ can be expressed as $(I+C)^{1/2}u$. Then we put

$$\Delta^{-iz}a = F(z) = (I-C)^{-iz}(I+C)^{iz+1/2}u \ ,$$

which is well defined, holomorphic and bounded in the closed strip $0 \leq \Im m\, z \leq 1/2$. For $z = t + i0$ we have $<b, F(t)> = <b, \Delta^{-it}a>$. *We are going to prove that $<b, F(t+1/2)>$ is real.* By the Schwarz reflection principle, this function will then be extendable to the closed strip $0 \leq \Im m\, z \leq 1$, and assume conjugate values at corresponding points $t+i0$, $t+i1$ of the boundary.

We start from the relation

$$F(t+i/2) = (I-C)^{-it+1/2}(I+C)^{it}u = \Delta^{-it}(I-C)^{1/2}u \ .$$

On the other hand, $a = (I+C)^{1/2}u$, hence $(I-C)^{1/2}u = Ja$, and

$$<b, F(t+i/2)> = <b, \Delta^{-it}Ja> = <b, Jc> \quad \text{with } c = \Delta^{-it}a \in A.$$

Now Jc belongs to $(iA)^{\perp}$ (the real orthogonal space of iA), meaning that $<ib, Jc>$ is purely imaginary, or $<b\,, Jc>$ is real. □

Rieffel and van Daele prove that the modular group is the only strongly continuous unitary group which leaves A invariant and satisfies the above property.

Using the linear Radon–Nikodym theorem

6 All the preceding discussion was about the closed real subspaces A, A'. We now discuss the von Neumann algabras $\mathcal{A}$, $\mathcal{A}'$ themselves, and proceed to the deeper results, translating first the Sakai linear R–N theorem, whose statement we recall. Given any k such that $\Re e\, k > 0$, and any $\beta \in \mathcal{A}'_{sa}$ (β is a "measure", whence the greek letter) there exists a unique $b \in \mathcal{A}_{sa}$ (a "function") such that for $a \in \mathcal{A}$

$$<\beta\mathbf{1}, a\mathbf{1}> = <\mathbf{1}\ , \ (kab + \overline{k}ba)\mathbf{1}> \tag{6.1}$$

(we keep writing explicitly $\mathbf{1}$ here and below, for clarity). For each k, this extends to a conjugate linear mapping from $\mathcal{A}'$ to $\mathcal{A}$, which for the moment has no name except $\beta \longmapsto b$. We are going to express (6.1) with the help of the operators C and S, as follows

LEMMA. *For $k = 1/2$, we have*

$$P\beta\mathbf{1} = b\mathbf{1}\ , \quad Q\beta\mathbf{1} = 0 \quad \textit{hence } S\beta\mathbf{1} = b\mathbf{1}\,. \tag{6.2}$$

and in the general case

$$S\beta S = k(I+C)\,b(I-C) + \overline{k}(I-C)\,b(I+C) \tag{6.3}$$

PROOF. The case $k = 1/2$ goes as follows :

$$<\beta\mathbf{1}\,,\,a\mathbf{1}> = \frac{1}{2}<\mathbf{1}\,,\,(ab+ba)\,\mathbf{1}> =$$
$$\frac{1}{2}(<a\mathbf{1}\,,\,b\mathbf{1}> + <b\mathbf{1}\,,\,a\mathbf{1}>) = \Re e<a\mathbf{1}\,,\,b\mathbf{1}> = (a\mathbf{1}, b\mathbf{1})\ .$$

Since a is arbitrary, this means $b\mathbf{1} = P\beta\mathbf{1}$. On the other hand, $\mathcal{A}'_{sa}\mathbf{1}$ is orthogonal to $B = iA$, hence $Q\beta\mathbf{1} = 0$.

We now consider a general k. We replace in (6.1) a by ca ($a, b, c \in \mathcal{A}_{sa}$, but $ca \in \mathcal{A}$ only), so that

$$<c\beta\mathbf{1}\,,\,a\mathbf{1}> = k<\mathbf{1}\,,\,cab\mathbf{1}> + \overline{k}<\mathbf{1}\,,\,bca\mathbf{1}> = k<c\mathbf{1}\,,\,ab\mathbf{1}> + \overline{k}<cb\mathbf{1}\,,\,a\mathbf{1}>\ .$$

Let λ, μ be arbitrary in $\mathcal{A}'_{sa}$, and ℓ, m be the corresponding elements of $\mathcal{A}$ through the preceding construction with $k = 1/2$. In the preceding formula we replace c by ℓ and a by m

$$<\ell\beta\mathbf{1}\,,\,m\mathbf{1}> = k<\ell\mathbf{1}\,,\,mb\mathbf{1}> + \overline{k}<\ell b\mathbf{1}\,,\,m\mathbf{1}>\ .$$

Since $S\lambda\mathbf{1} = \ell\mathbf{1}$, $S\mu\mathbf{1} = m\mathbf{1}$, and ℓ commutes with β, this is rewritten

$$<\beta S\lambda\mathbf{1}\,,\,S\mu\mathbf{1}> = k<S\lambda\mathbf{1}\,,\,mb\mathbf{1}> + \overline{k}<\ell b\mathbf{1}\,,\,S\mu\mathbf{1}>\ .$$

Since S is real selfadjoint and complex conjugate linear, it satisfies the relation $<u, Sv> = <Su, v>^-$. Therefore we can move the second S in the left member to the "bra" side, the result being $<S\beta S\lambda\mathbf{1}\,,\,\mu\mathbf{1}>^-$.

In the right member, we use the result from subsection 4 :

$$mb\mathbf{1} = J\varepsilon(mb)^*\mathbf{1} = J\varepsilon bm\mathbf{1} = J\varepsilon bS\mu\mathbf{1}\ , \qquad \text{similarly} \quad \ell b\mathbf{1} = J\varepsilon bS\lambda\mathbf{1}.$$

Then $<S\lambda\mathbf{1}, mb\mathbf{1}> = <S\lambda\mathbf{1}, J\varepsilon bS\mu\mathbf{1}> = <J\varepsilon S\lambda\mathbf{1}, bS\mu\mathbf{1}>^-$, $J\varepsilon = \varepsilon J$ being real self-adjoint and complex conjugate linear. On the other hand, $J\varepsilon S = I - C$. Doing the same on the second term we get

$$<S\beta S\lambda\mathbf{1}\,,\,\mu\mathbf{1}>^- = k<(I-C)\,\lambda\mathbf{1}\,,\,bS\mu\mathbf{1}>^- + \overline{k}<bS\lambda\mathbf{1}\,,\,(I-C)\,\mu\mathbf{1}>^-\ .$$

On the other hand, $\lambda\mathbf{1}$, $\mu\mathbf{1}$ belong to the kernel of Q, and the relations $S = P - Q$, $I + C = P + Q$ allow us to replace S by $I + C$. Taking out the conjugation sign, and using the fact that C and $S\beta S$ are complex selfadjoint, we get

$$<\lambda\mathbf{1}\,,\,S\beta S\mu\mathbf{1}> = \overline{k}<\lambda\mathbf{1}\,,\,(I-C)\,b(I+C)\,\mu\mathbf{1}> + k<\lambda\mathbf{1}\,,\,(I+C)\,b(I-C)\,\mu\mathbf{1}>\,.$$

This is complex linear in μ, and therefore can be extended from $\mathcal{A}'_{sa}$ to $\mathcal{A}'$, and similarly in λ. Since $\mathbf{1}$ is cyclic for $\mathcal{A}'$, we can write this as the operator relation (6.1).

7 We now make explicit the dependence on k, putting $k = e^{i\theta/2}$ — the condition $\Re e(k) > 0$ then becomes $|\theta| < \pi$. We keep $\beta \in \mathcal{A}'_{sa}$ fixed, and denote by b_θ the element of $\mathcal{A}_{sa}$ corresponding to β for the given value of θ. It is really amazing that b_θ can be explicitly computed by the integral formula

$$b_\theta = \int_{\mathbb{R}} \Delta^{it} J\beta J \Delta^{-it}\, h_\theta(t)\,dt\ , \qquad h_\theta(t) = \frac{e^{-\theta t}}{2\mathrm{Ch}(\pi t)}\ . \tag{7.1}$$

PROOF. We consider two analytic vectors $x, y \in \mathcal{H}$ for the unitary group Δ^{it}, as constructed in §2, subs.11. Then $\Delta^z x$, $\Delta^z y$ are entire functions of z, bounded in every vertical strip of finite width, and the same is true for the scalar function

$$f(z) = <\Delta^{\overline{z}}x\,,\,DbD\Delta^{-z}y>\,.$$

We recall that $D = |S|$ commutes with C and S, and $S = JD$ (see Appendix 3). We have $\Delta^{1/2}D = I - C$, $\Delta^{-1/2}D = I + C$. Then

$$f(\tfrac{1}{2}+it) = <x\,,\,\Delta^{-it}\Delta^{1/2}x\,,\,DbD\Delta^{-1/2}\Delta^{-it}y> = <x\,,\,\Delta^{it}(I-C)\,b(I+C)\,\Delta^{-it}y>\,.$$

Similarly

$$f(-\tfrac{1}{2}+it) = <x\,,\,\Delta^{it}(I+C)\,b(I-C)\,\Delta^{-it}y>\ .$$

Then using (6.3) we get the basic formula ($k = e^{i\theta/2}$)

$$k\,f(\tfrac{1}{2}+it) + \overline{k}\,f(-\tfrac{1}{2}+it) = <x\,,\,\Delta^{it}S\beta S\,\Delta^{-it}y>\ .$$

We now apply the following Cauchy integral formula for a bounded holomorphic function $g(z)$ in the closed vertical strip $|\,\Re e(z)\,| \leq 1/2$

$$g(0) = \int \big(g(\tfrac{1}{2}+it) + g(-\tfrac{1}{2}+it)\big)\,\frac{dt}{2\,\mathrm{Ch}(\pi t)}\ .$$

We replace $g(z)$ by $e^{iz\theta}f(z)$ with $\theta < \pi$, and get

$$f(0) = \int \big(\,kf(\tfrac{1}{2}+it) + \overline{k}\,f(-\tfrac{1}{2}+it)\big)\,h_\theta(t)\,dt\ ,$$

or explicitly

$$<x, DbDy> = <x\,,\,\big(\int \Delta^{it}S\beta S\Delta^{-it}\,h_\theta(t)\,dt\,\big)\,y>\ .$$

This equality can be extended by density to arbitrary x, y. Since D is selfadjoint and commutes with J, Δ we may write

$$< Dx, bDy > = < Dx, \Big(\int \Delta^{it} J\beta J\Delta^{-it} h_\theta(t)\, dt \Big) Dy > ,$$

and (7.1) follows, the range of D being dense.

Before we proceed to the main results, we need one more remark :

(7.2) *For* $\beta \in \mathcal{A}'$ *we have* $\Delta^{it} J\beta J\Delta^{-it} \in \mathcal{A}$.

It suffices to prove this operator commutes with every $\alpha \in \mathcal{A}'$. We put $g(t) = < x\,,\ [\Delta^{it} J\beta J\Delta^{-it}\,,\ \alpha]\, y >$. From (7.1) we deduce $\int g(t)\, h_\theta(t)\, dt = < x\,,\ [b_\theta\,,\ \alpha]\, y > = 0$. Since θ is arbitrary one may deduce that $g(t) = 0$ a.e., and then everywhere.

The main theorems

8 We first introduce some notation. The modular symmetry J operates on vectors on $\mathcal{H}$; given a bounded operator $a \in \mathcal{L}(\mathcal{H})$ we define $ja = JaJ$. This is a conjugation on $\mathcal{L}(\mathcal{H})$: it is conjugate linear, but doesn't reverse the order of products. Similarly, we extend the modular group as a group of automorphisms on $\mathcal{L}(\mathcal{H})$, putting $\delta_t a = \Delta^{it} a \Delta^{-it}$. The fact that Δ^{it} leaves $\mathbf{1}$ fixed implies that the law μ is invariant (this is made more precise by the KMS condition in theorem 3).

THEOREM 1. *The conjugation j exchanges $\mathcal{A}$ and $\mathcal{A}'$.*

PROOF. If in (7.2) we take $t = 0$, we find that $j(\mathcal{A}') \subset \mathcal{A}$. Exchanging the roles of $\mathcal{A}$ and $\mathcal{A}'$ doesn't change the modular symmetry J, and therefore $j(\mathcal{A}) \subset \mathcal{A}'$. Then we have $\mathcal{A} = j(j(\mathcal{A})) \subset j(\mathcal{A}')$ and equality follows.

COMMENT. We had seen previously that J maps the closed space A onto A'. Now we have the much more precise result that it maps A_0 onto A'_0, without taking closures. Also, ja is the unique $\alpha \in \mathcal{A}'$ such that $\alpha\mathbf{1} = Ja\mathbf{1}$.

THEOREM 2. *The group (δ_t) preserves $\mathcal{A}$ and $\mathcal{A}'$.*

PROOF. The fact that $\mathcal{A}$ is preserved follows from (7.2) and theorem 1, since $J\beta J$ is an arbitrary element of $\mathcal{A}$. Then we exchange the roles of $\mathcal{A}$ and $\mathcal{A}'$.

The statement in the next theorem is not exactly the same as that of the KMS condition (10.2) in the preceding section : the boundary lines of the strip are exchanged.

THEOREM 3. *Given arbitrary $a, b \in \mathcal{A}$, there exists a bounded holomorphic function $f(z)$ in the closed strip $0 \le \Im m(z) \le 1$ such that*

$$f(t + i0) = \mu((\delta_t a)\, b)\ ,\ \ f(t + i1) = \mu(b\delta_t a)\ . \tag{8.1}$$

PROOF. It is sufficient to prove this result when a, b are selfadjoint. Then the result follows from the modular property on vectors, if we remark that ($\mathbf{1}$ being invariant by $\Delta^{\pm it}$)

$$\mu(b\delta_t a) = < \mathbf{1}, b\Delta^{it} a\Delta^{-it}\mathbf{1} > = < b\mathbf{1}, \Delta^{it} a\mathbf{1} >$$
$$\mu((\delta_t a)\, b) = \mu(a\delta_{-t} b) = < \mathbf{1}\,,\ a\Delta^{-it} b\Delta^{it}\mathbf{1} > = < \Delta^{it} a\mathbf{1}\,,\ b\mathbf{1} > .$$

One can prove that this property characterizes the automorphism group (δ_t).

Additional results

9 The Tomita–Takesaki theorem is not an end in itself, but the gateway to new developments. The end of the academic year also put an end to the author's efforts to give examples and self-contained proofs, and the work could not be resumed. We only give a sketch without proofs of important results due to Araki, Connes, Haagerup. They are available in book form in [BrR1].

Everything we are doing depends on the choice of the faithful normal state μ (or the separating cyclic vector $\mathbf{1}$ in the GNS Hilbert space $\mathcal{H}$). It turns out at the end, as in the classical case, that replacing μ by an equivalent state produces leaves the situation unchanged up to well controlled isomorphisms.

We take the bold step of representing by $\overline{x}$ the vector Jx for $x \in \mathcal{H}$, and by $\overline{a}$ the operator $j(c) = JcJ$ for $c \in \mathcal{L}(\mathcal{H})$, returning to the "$J$" notation whenever clarity demands it. Note that $\overline{a}b = b\overline{a}$ for $a, b \in \mathcal{A}$. For $x \in \mathcal{H}$, $c \in \mathcal{L}(\mathcal{H})$, we have $\overline{(cx)} = \overline{c}\,\overline{x}$; indeed, $(JcJ)\,Jx = J(cx)$. Similarly, for $c, d \in \mathcal{L}(\mathcal{H})$, $\overline{(cd)} = \overline{c}\,\overline{d}$ and $\overline{c\overline{d}} = \overline{c}\,d$.

The *positive cone* $\mathcal{P}$ in the Hilbert space $\mathcal{H}$ is by definition the closure of the set of vectors $a\overline{a}\,\mathbf{1}$ for $a \in \mathcal{A}$. Since $\mathbf{1}$ is invariant by J, the same is true for every element of $\mathcal{P}$ (all elements of $\mathcal{P}$ are "real"). Also, for $a, b \in \mathcal{A}$ we have $b\overline{b}\,a\overline{a} = (ba)\,\overline{(ba)}$, therefore $\mathcal{P}$ is stable under $b\overline{b}$.

We now give a list of properties of the positive cone

1) $\mathcal{P}$ is the closure of $\Delta^{1/4}(\mathcal{A}_+\mathbf{1})$, and therefore is a convex cone.

2) $\mathcal{P}$ is stable under the modular group Δ^{it}.

3) $\mathcal{P}$ is pointed, i.e. $(-\mathcal{P}) \cap \mathcal{P} = \{0\}$, and is self-dual (i.e. a form $< x, \cdot >$ is positive on $\mathcal{P}$ if and only if $x \in \mathcal{P}$).

4) Every "real" vector x (i.e. $Jx = x$) is a difference of two elements of $\mathcal{P}$, which can be uniquely chosen so as to be orthogonal.

Now, we describe what happens when we change states, assuming first the new state is a pure state $\nu(a) = <\omega, a\omega>$ with $\omega \in \mathcal{P}$ (we will see later that this is not a restriction).

5) The state ν is faithful if and only if ω is a cyclic vector for $\mathcal{A}$ operating on $\mathcal{H}$.

6) Assuming this is true, we have $J_\omega = J$, $\mathcal{P}_\omega = \mathcal{P}$.

Finally, we give the main result, which describes all states on $\mathcal{A}$ which are absolutely continuous with respect to μ.

THEOREM. *Every positive normal linear functional φ on $\mathcal{A}$ can be uniquely represented as*

$$\varphi(a) = <\omega, a\omega> \quad \text{with } \omega \in \mathcal{P}.$$

Formally, this says that every positive element of L^1 is something like the square of a positive element of L^2.

10 EXAMPLES. 1) The trivial case from the point of view of the T–T theory is that of a tracial state μ. The tracial property can be written

$$< a\mathbf{1}, b\mathbf{1} > \doteq \mu(a^*b) = \mu(ba^*) = \overline{< b\mathbf{1}, a\mathbf{1} >}\ .$$

Then property (8.1) holds with $\delta_t = I$, and taking for granted the uniqueness property mentioned after Theorem 3, we see that the modular group is trivial. According to (4.1), since $\varepsilon = I$ the modular symmetry J must be given by

$$Ja\mathbf{1} = a^*\mathbf{1} \quad \text{for } a \in \mathcal{A}.$$

Indeed, this is a well defined operator on $\mathcal{A}\mathbf{1}$, and we have $< Ja\mathbf{1}, Jb\mathbf{1} > = < b\mathbf{1}, a\mathbf{1} >$ because the state is tracial, so that J extends to a well defined conjugation on $\mathcal{H}$. To check that $\overline{a} = JaJ$ belongs to $\mathcal{A}'$, it is sufficient to prove that for arbitrary $b, c, d \in \mathcal{A}$,

$$< c\mathbf{1}, \overline{a}b\, d\mathbf{1} > = < c\mathbf{1}, b\overline{a}\, d\mathbf{1} > .$$

On the left hand side we replace $\overline{a}bd\mathbf{1} = JaJbd\mathbf{1}$ by $Jad^*b^*\mathbf{1}$ and then by $bda^*\mathbf{1}$. Similarly on the right hand side we replace $b\overline{a}d\mathbf{1}$ by $bda^*\mathbf{1}$, and the result is the same.

2) The second case which can be more or less explicitly handled is that of $\mathcal{A} = \mathcal{L}(\mathcal{K})$ with a state of the form $\mu(a) = \mathrm{Tr}(aw)$. Then w is faithful if and only if w is injective. The modular group is given by $\delta_t(a) = w^{it}aw^{-it}$, because this group of automorphisms satisfies the KMS condition of Theorem 3. Indeed, if we put for $a, b \in \mathcal{A}$

$$f(z) = \mathrm{Tr}(w\, w^{iz} aw^{-iz} b)$$

it is easy to see that $f(z)$ is holomorphic and bounded in the unit horizontal strip, with $f(t + i0) = \mu(\delta_t(a)\, b)$ and $f(t + i1) = \mu(b\delta_t(a))$.

3) Let us mention a striking application of the T–T theory, due to Takesaki : given a von Neumann algebra $\mathcal{A}$ with a faithful normal law μ and a von Neumann subalgebra $\mathcal{B}$, a conditional expectation relative to $\mathcal{B}$ exists if and only if $\mathcal{B}$ is stable by the modular automorphism group of $\mathcal{A}$ w.r.t. μ. On the subject of conditional expectations, we advise the reader to consult the two papers [AcC] by Accardi and Cecchini given in the references.

These lectures on von Neumann algebras stopped here, just when things started to be really interesting, and no chance was given to resume work and discuss L^p spaces, etc.. Unfortunately, most results are still scattered in journals, and no exposition for probabilists seems to exist.

Appendix 5

Local Times and Fock Space

This Appendix is devoted to applications of Fock space to the theory of local times. We first present *Dynkin's formula,* ([Dyn2]) which shows how expectations relative to the stochastic process of local times L_∞^x of a symmetric Markov process, indexed by the points of the state space, can be computed by means of an auxiliary Gaussian process whose covariance is the potential density of the Markov process. Though this theorem does not mention explicitly Fock space, it was suggested to Dynkin by Symanzik's program for the construction of interacting quantum fields. Then we mention without proof a remarkable application to the continuity of local times, discovered by Marcus and Rosen. We continue with the "supersymmetric" version of Dynkin's theorem, due to Le Jan, and mention (again without proof) its application to the self-intersection theory of two-dimensional Brownian motion. Thus local times and self-intersection form the probabilistic background to present a variety of results on Fock space (symmetric, antisymmetric, and mixed) which might belong as well to Chapter V. There is an incredible amount of literature on the antisymmetric case, of which I have read very little : so please consider this chapter as an mere introduction to the subject.

Some parts of this Appendix depend on the theory of Markov processes. As they concern mostly the probabilistic motivations, and we presume our reader is interested principally in the non-commutative methods, we have chosen to omit technical details altogether. Everything can be found in the original papers, and the beautiful Marcus-Rosen article also contains information on Markov processes for non-specialists.

§1. Dynkin's formula

Symmetric Markov processes

1 Let E be a state space with a σ-finite measure η. We consider on this space a transient (sub)Markov semigroup (P_t) whose potential kernel G has a potential density $g(x,y)$ w.r.t. η. Explicitly, for $f \geq 0$ on E

$$(1.1) \qquad \int_0^\infty \int_E P_t(x,dy)\, f(y)\, dt = \int G(x,dy)\, f(y) = \int g(x,y)\, f(y)\, \eta(dy)\,.$$

Then $g(\cdot,y)$ may be chosen to be excessive ($=$ positive superharmonic) for every y, and we may define potentials of positive measures, $G\mu(x) = \int g(x,y)\, \mu(dy)$. We are mostly interested in the case of a *symmetric* density g. The case of non-transient processes

like low-dimensional Brownian motion can be reduced to the transient case, replacing the semigroup (P_t) by $e^{-ct}P_t$ (which amounts to killing the process at an independent exponential time).

On the sample space Ω of all right continuous mappings from $\mathbb{R}_+$ to E with lifetime ζ, and left limits in E up to the lifetime, we define the measures $\mathbb{P}^\theta$, under which (X_t) (the co-ordinate process) is Markov with semigroup (P_t) and initial measure θ. This requires some assumptions on E and (P_t), which we do not care to describe. When $\theta = \varepsilon_x$, one writes simply $\mathbb{P}^x, \mathbb{E}^x$, and for every positive r.v. h on Ω, $\mathbb{E}^\bullet[h]$ is a function on E.

We will also need to use "conditioned" measures, $\mathbb{P}^{\theta/k}$ where k is an excessive function; the corresponding initial measure is θ, and the semigroup is replaced by

$$P_t^{/k}(x,dy) = \frac{P_t(x,dy)\,k(y)}{k(x)} \quad . \tag{1.2}$$

The corresponding potential $G^{/k}(x,dy)$ is $G(x,dy)\,k(y)/k(x)$, and the potential density w.r.t. η has the same form[1]. It would take us too far to explain how conditioning establishes a symmetry between initial and final specifications on the process, but this will be visible on the last formula (1.9).

A *random field* is a linear mapping $f \longmapsto \Phi(f)$ from some class of functions to random variables. We shall be interested here in the *occupation field*, which maps f to $A(f)_\infty = \int_0^\infty f(X_s)\,ds$ — this is always meaningful for positive functions, and transience means that it is meaningful for bounded signed functions of suitably restricted support.

Let f be a function such that $|f|$ has a bounded potential, let us put $A = A(f)$ and define, for n a positive integer,

$$h^{(n)} = \mathbb{E}^\bullet\left[A_\infty^n\right] \; . \tag{1.3}$$

In particular, $h = h^{(1)}$ is the potential $\mathbb{E}^\bullet[A_\infty] = Gf$. Then we have

$$\begin{aligned} h^{(n)} &= \mathbb{E}^\bullet\Big[\int_0^\infty -d(A_\infty - A_s)^n\Big] = \mathbb{E}^\bullet\Big[\int_0^\infty n(A_\infty - A_s)^{n-1}\,dA_s\Big] \\ &= \mathbb{E}^\bullet\Big[\int_0^\infty n\,\mathbb{E}\left[(A_\infty - A_s)^{n-1}\,|\,\mathcal{F}_s\right]\,dA_s\Big] = \mathbb{E}^\bullet\Big[\int_0^\infty n\,h^{(n-1)}(X_s)\,dA_s\Big] \\ &= G(h^{(n-1)}f) \; . \end{aligned}$$

An easy induction then gives the formula

$$h^{(n)}(x) = n!\int g(x,y_1)f(y_1)\eta(dy_1)\,g(y_1,y_2)\cdots g(y_{n-1},y_n)\;f(y_n)\eta(dy_n) \; . \tag{1.4}$$

We put $\mu = f\cdot\eta$ and integrate w.r.t. an initial measure θ, adding an integration to the left

$$\mathbb{E}^\theta\left[(A_\infty)^n\right] = n!\int g(x,y_1)\,g(y_1,y_2)\dots g(y_{n-1},y_n)\;\theta(dx)\,\mu(dy_1)\dots\mu(dy_n) \; . \tag{1.5}$$

[1] Symmetry is lost, but could be recovered using $k^2\eta$ as reference measure.

To add similarly an integration on the right, we apply the preceding formula to the conditioned semigroup $P_t^{/k}$ with $k = G\chi$, assumed for simplicity to be finite and strictly positive. Then we have, still denoting by μ the measure $f\cdot\eta$

$$(1.6)\qquad \mathbb{E}^{k\theta/k}\left[(A_\infty)^n\right] = \\ n!\int g(x,y_1)\,g(y_1,y_2)\cdots g(y_{n-1},y_n)\,g(y_n,z)\;\;\theta(dx)\,\mu(dy_1)\ldots\mu(dy_n)\,\chi(dz)\,.$$

In the computation, the first $1/k(x)$ of $g^{/k}$ simplifies with the k of $k\theta$, the intermediate $k(y_i)$'s collapse to 1, and the last remaining $k(y_n)$ is made explicit as the integral w.r.t. χ on the right.

Now the symmetry comes into play. We define the symmetric bilinear form on measures (first positive ones)

$$(1.7)\qquad e(\mu,\nu) = \int g(x,y)\,\mu(dx)\,\nu(dy)\,.$$

We say that μ has *finite energy* if $e(|\mu|,|\mu|) < \infty$. For instance, all positive bounded measures with bounded potential have finite energy. One can prove that the space of all measures of finite energy with the bilinear form (1.7) is a prehilbert space $\mathcal{H}$ (usually it is not complete : there exist "distributions of finite energy" which are not measures). On the other hand, with every measure μ of finite energy one can associate a continuous additive functional $A_t(\mu)$ (or A_t^μ for the sake of typesetting) reducing to $A_t(f)$ when $\mu = f\cdot\eta$, and we still have

$$(1.8)\qquad \mathbb{E}^{k\theta/k}\left[(A_\infty(\mu))^n\right] = \\ n!\int g(x,y_1)\,g(y_1,y_2)\cdots g(y_{n-1},y_n)\,g(y_n,z)\;\;\theta(dx)\,\mu(dy_1)\ldots\mu(dy_n)\,\chi(dz)\,.$$

Some dust has been swept under the rugs here : there was no problem about defining the same $A(f)$ for the two semigroups (P_t) and $(P_t^{/k})$, because we had an explicit expression for it, but as far as $A(\mu)$ is concerned it requires some work. Take us on faith that it can be done.

Finally, we "polarize" this relation to compute the expectation $\mathbb{E}^{k\theta/k}\left[A_\infty^{\mu_1}\ldots A_\infty^{\mu_n}\right]$ (called by Dynkin "the n-point function of the occupation field") as a sum over all permutations σ of $\{1,\ldots,n\}$

$$(1.9)\qquad \sum_\sigma \int g(x,y_1)\,g(y_1,y_2)\cdots g(y_n,z)\;\theta(dx)\;\mu_{\sigma(1)}(dy_1)\cdots\mu_{\sigma(n)}(dy_n)\;\chi(dz)\,.$$

This is the main probabilistic formula, which will be interpreted using the combinatorics of Gaussian random variables. Note that it is symmetric in the measures μ_i, even if the potential kernel is not symmetric.

A Gaussian process

2 For simplicity, we reduce the prehilbert space $\mathcal{H}$ to the space of all signed measures which can be written as differences of two positive bounded measures with bounded potentials — such measures clearly are of finite energy. Though we will need it only much later, let us give an interpretation of formula (1.9) on this space : we denote by α_i the operator on $\mathcal{H}$ which maps a measure λ to $(G\lambda)\mu_i$ (a bounded measure with bounded potential since $G\lambda$ is bounded), and rewrite (1.9) as

$$\sum_{\sigma} (\theta\,,\, \alpha_{\sigma(1)}\alpha_2 \ldots \alpha_{\sigma(n)}\chi)\,. \tag{2.1}$$

On the same probability space as (X_t) and independently of it, we construct a Gaussian (real, centered) linear process Y_μ, indexed by $\mu \in \mathcal{H}$, with covariance

$$< Y_\mu, Y_\nu > = \int \mu(dx)\, g(x,y)\, \nu(dy)\,. \tag{2.2}$$

We claim that, given a function $F \geq 0$ on $\mathbb{R}^n$, we have *Dynkin's formula*

$$\mathbb{E}^{k\theta/k}\left[F(\,A_1 + (Y_1^2/2), \ldots, A_n + (Y_n^2/2))\,\right] = \mathbb{E}\left[F(Y_1^2/2, \ldots, Y_n^2/2)\, Y_0 Y_{n+1}\right]. \tag{2.3}$$

Note that θ and χ have been moved inside the Gaussian field expectation on the right hand side.

We follow the exposition of Marcus and Rosen, which is Dynkin's proof without Feynman diagrams. The first step consists in considering the case of a function of n variables which is a product of coordinates, $F(x_1, \ldots, x_n) = x_1 \ldots x_n$. Since the measures μ_i are not assumed to be different, (2.3) will follow for polynomials. Then a routine path will lead us to the general case via complex exponentials, Fourier transforms and a monotone class theorem — this extension will not be detailed, see Dynkin [Dyn2] or Marcus–Rosen [MaR].

Given a set E with an even number $2p$ of elements, a *pairing* π of E is a partition of E into p sets with two elements — one of which is arbitrarily chosen so that we call them $\{\alpha, \beta\}$. We may also consider π as the mapping $\pi(\alpha) = \beta$, $\pi(\beta) = \alpha$, an involution without fixed point. The expectation of a product of an even number $m = 2p$ of (centered) gaussian random variables ξ_i is given by a sum over all pairings π of the set $\{1, \ldots, 2p\}$ (see formula (4.3) in Chapter IV, subs. 3)

$$\sum_{\pi} \prod_{\alpha} \mathbb{E}\left[\xi_\alpha \xi_\beta\right]\,. \tag{2.4}$$

We apply (2.4) to the following family of $2n+2$ gaussian r.v.'s : for $i = 1, \ldots, n$ we define $x_i = Y_i$, $y_i = Y_i$, and the r.v.'s Y_θ, Y_χ which play a special role are denoted by x_0 and x_{n+1}. Given a pairing π, we separate the indexes i in two classes : "closed" indexes such that x_i is paired to y_i, and "open" indexes i, such that x_i is paired to some x_j or y_j with $j \neq i$, necessarily open. Note that $i = 0, n+1$ is open since there is no y_i, and if $i \neq 0, n+1$ y_i is paired to some x_k or y_k with $k \neq i$. Then we consider the corresponding product in (2.5) : first we have factors corresponding to "closed" indexes, $E\left[x_i y_i\right] = \mathbb{E}\left[Y_i^2\right]$. Next we start from the open index 0 ; $\pi(x_0)$ is some z_j, where

j is open and z_j is either x_j or y_j (unless $j=n+1$ in which case the only choice is $z_j=x_j$). Then $\pi(z_j)$ is some z_k with k open, *etc.* and we iterate until we end at x_{n+1}. This open chain of random variables contributes to the product a factor

$$\mathbb{E}[Y_0Y_{j_1}]\,\mathbb{E}[Y_{j_1}Y_{j_2}]\,\ldots\mathbb{E}[Y_{j_k}Y_{n+1}]\ .$$

On the other hand, this chain does not necessarily exhaust the "open" indexes (think of the case x_0 is paired directly with x_{n+1}). Then starting with an "open" index that does not belong to the chain, we may isolate a new chain, this time a closed one, or cycle

$$\mathbb{E}[Y_{k_1}Y_{k_2}]\,\mathbb{E}[Y_{k_2}Y_{k_3}]\,\ldots\mathbb{E}[Y_{k_n}Y_{k_1}]\quad;$$

one or more such closed chains may be necessary to exhaust the product. Calling the number of expectation signs the *length* of the chain (open or closed), we see that the case of "closed indexes" appears as that of cycles of minimal length 1.

Thus we are reorganizing the right hand side of (2.3) as a sum of products of expectations over one open and several closed chains of indexes, and we must take into account the numbers of summands and the powers of 2. If we exchange in all possible ways the names x_i, y_i given to each "open" r.v. Y_i, $i=1,\ldots,n$, we get all the pairings leading to the same decomposition into chains. Assuming the lengths of the chains are m_0 for the open chain, $m_1,\ldots,m_k$ for the cycles, with $m_0+\ldots+m_k=n+2$, the number of such pairings corresponding to the given decomposition in chains is $2^{m_0-1}\ldots2^{m_k-1}=2^{n+1-k}$. On the other hand, the right side of (4) has a factor of 2^{-n}. Therefore, we end with a factor of 2^{k-1}, k being *the number of cycles.*

We would have a similar computation for a product containing only squares, *i.e.* without the factor Y_0, Y_{n+1} : the open chain linking Y_0 to Y_{n+1} would disappear, and we would get a sum of products over cycles only.

We now turn to the left hand side of (2.3) : we expand the product into a sum of terms $2^{-j}\mathbb{E}[Y_{m_1}^2\ldots Y_{m_j}^2]\,\mathbb{E}[A_{i_1}\cdots A_{i_{n-j}}]$ where the two sets of indexes partition $\{1,\ldots,n\}$. We have just computed the first expectation as a sum of products over cycles, and formula (1.9) tells exactly that the second expectation contributes the open chain term.

3 Marcus and Rosen use Dynkin's formula to attack the problem of continuity of local times : assuming the potential density is finite and continuous (as in the case of one dimensional Brownian motion killed at an exponential time), point masses are measures of finite energy, and the corresponding additive functionals are the *local times* (L_t^x). On the other hand, we have a Gaussian process Y_x indexed by points of the state space. Then Marcus and Rosen prove that the local times can be chosen to be jointly continuous in (t,x) *if and only if* the Gaussian process has a continuous version. The relationship can be made more precise, local continuity or boundedness properties being equivalent for the two processes. This is proved directly, without referring to the powerful theory of regularity of Gaussian processes — but as soon as the result is known, the knowledge of Gaussian processes can be transferred to local times. We can only refer the reader to the article [MaR], which is beautifully written for non-specialists.

§2 Le Jan's "supersymmetric" approach

Taking his inspiration from Dynkin's theorem and from a paper by Luttinger, Le Jan developed a method leading to an isomorphism theorem of a simpler algebraic structure than Dynkin's. This theorem is "supersymmetric" in the general sense that it mixes commuting and anticommuting variables. That is, functions as the main object of analysis are replaced by something like non homogeneous differential forms of arbitrary degrees. Starting from a Hilbert space $\mathcal{H}$, one works on the tensor product of the symmetric Fock space over $\mathcal{H}$, providing the coefficients of the differential forms, and the antisymmetric Fock space providing the differential elements (the example of standard differential forms on a manifold, in which the coefficients are smooth functions while the differential elements belong to a finite dimensional exterior algebra, suggests that two different Hilbert spaces should be involved here). However, this is not quite sufficient : Le Jan's theorem requires analogues of a *complex* Brownian motion Fock space, *i.e.* , the symmetric and antisymmetric components are built over a Hilbert space which has already been "doubled" . We are going first to investigate this structure.

The construction will be rather long, as we will first construct leisurely the required Fock spaces, adding several interesting results (many of which are borrowed from papers by J. Kupsch), supplementing Chapter V on complex Brownian motion.

Computations on complex Brownian motion (symmetric case)

1 We first recall a few facts about complex Brownian motion, mentioned in Chapter V, §1. The corresponding L^2 space, generated by multiple integrals of all orders (m,n) w.r. t. two real Brownian motions X,Y, is isomorphic to the Fock space over $\mathcal{G}=L^2(\mathbb{R}_+)\oplus L^2(\mathbb{R}_+)$. It is more elegant to generate it by multiple integrals w.r.t. two conjugate complex Brownian motions $Z,\overline{Z}$. The first chaos of the ordinary Fock space over $\mathcal{G}$ contains two kinds of stochastic integrals associated with a given $u\in L^2(\mathbb{R}_+)$

$$Z_u=\int u_s dZ_s \quad , \quad \overline{Z}_u=\int u_s d\overline{Z}_s \,. \tag{1.1}$$

both of them being linear in u. The conjugation maps Z_u into $\overline{Z}_{\overline{u}}$, exchanging the two kinds of integrals. The Wiener product of random variables is related to the Wick product by

$$Z_u Z_v = Z_u : Z_v \quad , \quad \overline{Z}_u\overline{Z}_v=\overline{Z}_u:\overline{Z}_v \quad , \quad Z_u\overline{Z}_v=Z_u:\overline{Z}_v+(u,v)\mathbf{1}\,. \tag{1.2}$$

It is convenient to formulate things in a more algebraic way : we have a complex Hilbert space $\mathcal{G}$ with a conjugation, and two complex subspaces $\mathcal{H}$ and $\mathcal{H}'$ exchanged by the conjugation, such that $\mathcal{G}=\mathcal{H}\oplus\mathcal{H}'$. On $\mathcal{G}$ we have a bilinear scalar product $(u,v)=<\overline{u},v>$, and this bilinear form is equal to 0 on $\mathcal{H}$ and $\mathcal{H}'$ — they are *(maximal) isotropic subspaces*. Thus the Fock space structure over $\mathcal{G}$ is enriched in two ways : by the conjugation, which leads to the bilinear form (u,v) (and to the notion of a Wiener product), and by the choice of the pair $\mathcal{H},\mathcal{H}'$.

Here is an important example of such a situation : we take any complex Hilbert space $\mathcal{H}$, denote by $\mathcal{H}'$ its dual space — the pairing is written $v'(u)=(v',u)$ — and define

$\mathcal{G} = \mathcal{H} \oplus \mathcal{H}'$. There is a canonical antilinear mapping $u \longmapsto u^*$ from $\mathcal{H}$ to $\mathcal{H}'$ (if u is a *ket*, u^* is the corresponding *bra*), and it is very easy to extend it to a conjugation of $\mathcal{G}$. Then every element of $\mathcal{G}$ can be written as $u + v^*$ with $u, v \in \mathcal{H}$, and we have

$$(u + v^*, w + z^*) = < v, w > + < z, u > . \tag{1.3}$$

An element of the incomplete (boson) Fock space over $\mathcal{G}$ is a finite sum of homogeneous elements of order (m, n), themselves linear combinations of elements of the form

$$u_1 \circ \ldots \circ u_m \circ v'_1 \circ \ldots \circ v'_n \tag{1.4}$$

with $u_i \in \mathcal{H}$, $v'_i \in \mathcal{H}'$. Another way of writing this is

$$u_1 \circ \ldots \circ u_m \circ v^*_1 \circ \ldots \circ v^*_n \tag{1.4'}$$

where this time all u_i, v_i belong to $\mathcal{H}$, and * is the standard map from $\mathcal{H}$ to $\mathcal{H}'$.

If a conjugation is given on $\mathcal{H}$, we also have a canonical scalar product (bilinear) $(v, u) = < \overline{v}, u >$, and a canonical linear mapping from $\mathcal{H}$ to $\mathcal{H}'$, denoted $v \longmapsto v'$. Then we may split each v' in (1.4) into a $v \in \mathcal{H}$ and a $'$. This is essentially what we do when we use the probabilistic notation, a third way of writing the generic vectors :

$$Z_{u_1} : \ldots : Z_{u_m} : \overline{Z}_{v_1} : \ldots : \overline{Z}_{v_n} \tag{1.4''}$$

again with $u_i, v_i \in \mathcal{H} = L^2(\mathbb{R}_+)$, with its standard conjugation. The symbols $\circ$ and $:$ have the same meaning : generally, we use the first when dealing with abstract vectors, and the second one for random variables — a mere matter of taste.

The form (1.4″) can be rewritten as a multiple Ito integral

$$\int u_1(s_1) \ldots u_m(s_m)\, v_1(t_1) \ldots v_n(t_n)\, dZ_{s_1} \ldots dZ_{s_m}\, d\overline{Z}_{t_1} \ldots d\overline{Z}_{t_n} , \tag{1.5}$$

which is incorrectly written : through a suitable symmetrization of the integrand, (1.5) should be given the form

$$\frac{1}{m!n!} \int f(s_1, \ldots, s_m ; t_1, \ldots, t_n)\, dZ_{s_1} \ldots dZ_{s_m}\, d\overline{Z}_{t_1} \ldots d\overline{Z}_{t_n} . \tag{1.6}$$

extended to $\mathbb{R}_+^m \times \mathbb{R}_+^n$, f being a symmetric function in the variables s_i, t_i separately. In our case, because of the factorials in front of the integral, f is equal to a sum over permutations

$$\sum_{\sigma,\tau} u_{\sigma(1)}(s_1) \ldots u_{\sigma(m)}(s_m)\, v_{\tau(1)}(t_1) \ldots v_{\tau(n)}(t_n) . \tag{1.7}$$

2 Instead of the Wick product (1.4″), let us compute the *Wiener* product (usually written without a product sign; if necessary we use a dot)

$$Z_{u_1} \ldots Z_{u_m}\, \overline{Z}_{v_1} \ldots \overline{Z}_{v_n} \tag{2.1}$$

According to the complex Brownian motion form of the multiplication formula (Chapter V, §1, formula (2.4)) the Wiener product (2.1) has a component in every chaos of order $(m-p, n-p)$. For $p=0$ we have the Wick product

$$Z_{u_1} : \ldots : Z_{u_m} : \overline{Z}_{v_1} : \ldots : \overline{Z}_{v_n} \ ,$$

for $p=1$, we have a sum of Wick products

$$(2.2) \qquad (v_j, u_i)\, Z_{u_1} : \ldots \widehat{Z_{u_i}} \ldots : Z_{u_m} : \overline{Z}_{v_1} : \ldots \widehat{\overline{Z}_{v_j}} \ldots : \overline{Z}_{v_n} \ ,$$

where the terms with a hat are omitted. Such a term is called the *contraction* of the indices s_i and t_j in the Wick product. Similarly, the term in the chaos of order $(m-2, n-2)$ involves all contractions of two pairs of indexes, etc.

If we had computed a Wiener product in the form (1.4′), in algebraic notation

$$u_1 \cdot \ldots \cdot u_m \cdot v_1^* \cdot \ldots \cdot v_n^* \ ,$$

then the contractions would have involved the *hermitian* scalar product $< v_j, u_i >$.

The expectation of (2.1) is equal to 0 for $m \neq n$, and for $m=n$ it is given by

$$(2.3) \qquad \mathbb{E}\,[\, Z_{u_1} \ldots Z_{u_m} \overline{Z}_{v_1} \ldots \overline{Z}_{v_n} \,] \ = \mathrm{per}\,(v_j, u_i)$$

where *per* denotes a *permanent.* We may translate this into algebraic language, as a linear functional on the incomplete Fock space, given in the representation (1.4) by

$$(2.4) \qquad \mathbf{Ex}\,[\, u_1 \circ \ldots \circ u_m \circ v_1' \circ \ldots \circ v_m' \,] \ = \mathrm{per}\,(v_j', u_i) \ .$$

(the notation **Ex** should recall the word "expectation" without suggesting an "exponential"). In the representation (1.4′) we have

$$(2.5) \qquad \mathbf{Ex}\,[\, u_1 \circ \ldots \circ u_m \circ v_1^* \circ \ldots \circ v_m^* \,] \ = \mathrm{per}\!< v_j, u_i > \ .$$

3 The following two subsections are devoted to comments on the preceding definitions. They are very close to considerations in Slowikowski [Slo1] and Nielsen [Nie].

The Wiener and Wick products can be described in the same manner : consider a (pre)Hilbert space $\mathcal{H}$ with a bilinear functional $\xi(u,v)$. We are going to define on the incomplete Fock space $\Gamma_0(\mathcal{H})$ an associative product admitting $\mathbf{1}$ as unit element, such that on the first chaos

$$(3.1) \qquad u \cdot v = u \circ v + \xi(u,v)\, \mathbf{1} \ .$$

Note the analogy with Clifford algebra. This product is commutative if ξ is symmetric, and satisfies in general an analogue of the CCR

$$u \cdot v - v \cdot u = (\xi(u,v) - \xi(v,u))\, \mathbf{1} \ .$$

We must supplement the rule (3.1) by a prescription for higher order products of vectors belonging to the first chaos

$$(3.2) \qquad \begin{aligned} (u_1 \circ \ldots \circ u_m) \cdot v &= u_1 \circ \ldots \circ u_m \circ v + \textstyle\sum_i \xi(u_i, v)\, u_1 \circ \ldots \widehat{u_i} \ldots \circ u_m \\ u \cdot (v_1 \circ \ldots \circ v_n) &= u \circ v_1 \circ \ldots \circ v_n + \textstyle\sum_i \xi(u, v_i)\, v_1 \circ \ldots \widehat{v_i} \ldots \circ v_n \end{aligned} \ ,$$

the hat on a vector indicating that it is omitted. Then an easy induction shows that the product $(u_1 \circ \ldots \circ u_m)\cdot(v_1 \circ \ldots \circ v_n)$ is well defined, with a term in each chaos of order $m+n-2p$ involving all possible p-fold contractions. The Wick (symmetric) product corresponds to $\xi=0$; the Wiener product needs a conjugation and corresponds to $\xi(u,v)=<\overline{u},v>$.

We have, $u_\circ^m$ and $v_\circ^n$ denoting symmetric (Wick) powers,

$$u_\circ^m \cdot v_\circ^n = \sum_{k\le m,n} k!\binom{m}{k}\binom{n}{k}\,\xi^k(u,v)\,u_\circ^{m-k}\circ v_\circ^{n-k} \;. \tag{3.3}$$

Note the analogy with the multiplication formula for Hermite polynomials. We deduce a product formula for exponential vectors

$$\mathcal{E}(u)\cdot\mathcal{E}(v) = e^{\xi(u,v)}\mathcal{E}(u+v) \;, \tag{3.4}$$

on which associativity is seen to depend on the identity

$$\xi(u,v+w)+\xi(v,w) = \xi(u,v)+\xi(u+v,w) \;, \tag{3.5}$$

trivially satisfied in our case. Another relation with Hermite polynomial occurs if we express the ξ-power u_ξ^m using Wick powers, and conversely (the index u_ξ or $u_\circ$ indicates in which sense the polynomial operates on u)

$$u_\circ^m = H_m(u_\xi,\xi(u,u)) \quad , \quad u_\xi^m = H_m(u_\circ,-\xi(u,u)) \;, \tag{3.5}$$

where the generalized Hermite polynomials are defined by

$$e^{tu-at^2/2} = \sum\nolimits_m \frac{t^m}{m!}\,H_m(u,a) \;, \tag{3.5'}$$

and we have in particular, as in the Wiener product case

$$e_\xi^u = e^{\xi(u,u)/2}\mathcal{E}(u) \;. \tag{3.6}$$

from which a generalized Weyl relation follows

$$\exp_\xi^u \exp_\xi^v = e^{(\xi(u,v)-\xi(v,u))/2}\exp_\xi(u+v) \;.$$

Let us give at least one interesting non-commutative example. We return to the case of $\mathcal{G}=\mathcal{H}\oplus\mathcal{H}'$, with elements represented as $u+v'$, and we identify u with the creation operator a_u^+, and v' with the annihilation operator $a_{v'}^-$ (indexed by an element of $\mathcal{H}'$: see VI.1.4). Then, the *operator multiplication* can be described as an ξ-product involving the non-symmetric bilinear form

$$\xi(u+v^*,w+z^*) = <v,w> \;, \tag{3.7}$$

"half" the symmetric bilinear form (1.3) that leads to the Wiener product. For instance

$$(a_u^+ + a_{v'}^-)\,(a_w^+ + a_{z'}^-) = (a_u^+ + a_{v'}^-):(a_w^+ + a_{z'}^-) + (v',w)\,\mathbf{1} \;. \tag{3.8}$$

One easily recovers the well known fact that the (commutative) Wick product of a string of creation and annihilation operators is equal to its normally ordered operator product. On the other hand, for non normally ordered strings the operator product is reduced to the Wick product using the CCR.

For instance, the operator exponential e^{u+v^*} is given by (3.6), and can be seen to act on coherent vectors as follows, (interpreting the Wick product as a normally ordered product, and then performing the computation)

$$\begin{aligned} e^{u+v^*}\mathcal{E}(h) &= e^{<v,u>/2} e^u e^{v^*} \mathcal{E}(h) \\ &= e^{<v,h>+<v,u>/2}\mathcal{E}(u+h) \end{aligned}$$

which is the formula we gave in Chapter IV for the action of Weyl operators on exponential vectors.

4 We now discuss a topic of probabilistic interest, that of *Stratonovich integrals.* This is close to what [Slo1] and [Nie] call "ultracoherence" , with a different language. We again proceed leisurely, including topics which are not necessary for our main purpose. On the other hand, many details have been omitted : see the papers of Hu–Meyer [HuM], Johnson–Kallianpur [JoK], and a partial exposition in [DMM].

First of all, we consider for a while a *real* Hilbert space $\mathcal{H}$, to forget the distinction between the hermitian and bilinear scalar products. As usual, a generic element of Fock space is a sum $f = \sum_n f_n/n!$, f_n being a symmetric element of $\mathcal{H}^{\otimes n}$ and the sum $\sum_n \| f_n \|^2/n!$ being convergent. We call $F = (f_n)$ the *representing sequence* of f, and use the notation $f = I(F)$, I for "Ito" since when $\mathcal{H} = L^2(\mathbb{R}_+)$, F is the sequence of coefficients in the Wiener–Ito expansion of the r.v. f. Many interesting spaces of sequences (with varying assumptions of growth or convergence) can be interpreted as spaces of "test-functions" or of "generalized random variables" (see the references [Kré], [KrR]).

Given a (continuous) bilinear functional ξ on $\mathcal{H}$, we wish to define a mapping Tr_ξ from $\mathcal{H}^{\otimes m}$ to $\mathcal{H}^{\otimes (m-2)}$, which in the case of $\mathcal{H} = L^2(\mathbb{R}_+)$, $\xi(u,v) = (u,v)$ (the bilinear scalar product), should be the trace

$$\mathrm{Tr}\; f_m(s_1,\ldots,s_{m-2}) = \int f_m(s_1,\ldots,s_{m-2},s,s)\,ds \tag{4.1}$$

(for $m = 0,1$ we define $\mathrm{Tr}\; f_m = 0$). If ξ is given by a kernel $a(s,t)$, we will have

$$\mathrm{Tr}\; f_m(s_1,\ldots,s_{m-2}) = \int f_m(s_1,\ldots,s_{m-2},s,t)\,a(s,t)\,dsdt\ . \tag{4.2}$$

To give an algebraic definition, we begin with the incomplete tensor power $\mathcal{H}^m$ and define

$$\mathrm{Tr}_\xi(u_1\otimes\ldots\otimes u_m) = \xi(u_{m-1},u_m)\,u_1\otimes\ldots\otimes u_{m-2}\ . \tag{4.3}$$

Since we will apply this definition to symmetric tensors, which indexes are contracted is irrelevant, and we made an arbitrary choice. This definition is meaningful for any bounded bilinear functional ξ, and it can be extended to the completed m-th tensor power if ξ belongs to the Hilbert-Schmidt class.

We now consider Tr_ξ as a mapping from sequences $F = (f_n)$ to sequences $\mathrm{Tr}_\xi F = ((\mathrm{Tr}(f_{n+2}))$. For instance, given an element u from the first chaos, the sequence $F = (u^{\otimes n})$ is the representing sequence for $\mathcal{E}(u)$, and we have $\mathrm{Tr}_\xi(F) = \xi(u,u)\,F$.

At least on the space of finite representing sequences, we may define the iterates Tr_ξ^k and put $T_\xi = e^{(1/2)\mathrm{Tr}_\xi} = \sum_n \mathrm{Tr}_\xi^n /2^n\, n!$. On the example above, we have $T_\xi(F) = e^{(1/2)\xi(u,u)}F$. It is clear on this example that, if we carry the operator T_ξ to Fock space without changing notation, what we get is

$$T_\xi \mathcal{E}(u) = e^{(1/2)\xi(u,u)}\mathcal{E}(u) = \exp_\xi(u)\ , \tag{4.4}$$

the exponential of u for the ξ-product. If ξ is symmetric, we have $\exp_\xi(u+v) = \exp_\xi(u)\exp_\xi(v)$, and this suggests T_ξ is a homomorphism from the incomplete Fock space with the Wick (=symmetric) product onto itself with the ξ-product. It turns out that this result is true (see [HuM] for the case of the Wiener product) — and it can't be true in the non-symmetric case, since the Wick product is commutative.

Given a sequence F, and assuming $I(T_\xi(F))$ is meaningful, it is denoted by $S_\xi(F)$. In the case of $L^2(\mathbb{R}_+)$ and the usual scalar product, $S(F)$ is called a *Stratonovich* chaos expansion. An intuitive description can be given as follows : an Ito multiple integral and a Stratonovich integral of the same symmetric function $f(s_1,\ldots,s_m)$ differ by the contribution of the diagonals, which is 0 in the case of the Ito integral, and computed in the case of the Stratonovich integral according to the rule $dX_s^2 = ds$.

Let us now discuss the most important case for the theory of local times : that of complex Browniam motion. The chaos coefficients are separately symmetric functions, and the "diagonals" which contribute to the Stratonovich integral are the diagonals $\{s_i = t_j\}$, the trace being computed according to the rules

$$dZ_s^2 = d\overline{Z}_s^2 = 0 \quad , \quad dZ_s\, d\overline{Z}_s = d\overline{Z}_s\, dZ_s = ds\ .$$

More generally, there exist contracted integrals using a function $\xi(s,t)$, not necessarily symmetric. The trace in the complex case involves the contraction of one single s with one single t, and the Tr_ξ operator applied to the representing sequence $F = (u^{\otimes m}v^{\otimes n})$ of an exponential vector multiplies it by $\xi(u,v)$. The operator T_ξ (Stratonovich integral) has no coefficient $1/2$: $S_\xi(F) = I(e^{\mathrm{Tr}_\xi}F)$.

The linear functional $\mathbf{Ex}$ on the space of sequences, *i.e.* the ordinary (vacuum) expectation of the Stratonovich integral, is generalized as follows (we write the formula in the complex case)

$$\mathbf{Ex}_\xi\,[F] = \sum_m \frac{1}{m!}\,\mathrm{Tr}_\xi^m\, f_{mm}\ . \tag{4.5}$$

5 The following computation is an essential step in the proof of Le Jan's result. Let G be an element of the chaos of order $(1,1)$

$$G = Z_{u_1}\circ \overline{Z}_{v_1} + \ldots + Z_{u_n}\circ \overline{Z}_{v_n} \tag{5.1}$$

In algebraic language, the vectors v_i would rather be written as $v_i' \in \mathcal{H}'$, and G would be $u_1 \circ v_1' + \ldots + u_n \circ v_n'$. We also associate with G the corresponding Wiener product

$$\widehat{G} = Z_{u_1}\overline{Z}_{v_1} + \ldots + Z_{u_n}\overline{Z}_{v_n} \tag{5.2}$$

which is the "Stratonovich" version of (5.1). Similarly, the Wiener exponential $e^{-\varepsilon\widehat{G}}$ is the "Stratonovich" version of the Wick exponential $e_{\circ}^{-\varepsilon G}$, and therefore we have

$$\mathbb{E}[e^{-\varepsilon\widehat{G}}] = \mathbf{Ex}[e_{\circ}^{-\varepsilon G}], \tag{5.3}$$

which will be seen to exist for ε close to 0. We are going to compute this quantity.

Lemma 1. *Let A be the square matrix (v_j, u_i). Then we have*

$$\mathbb{E}[e^{-\varepsilon\widehat{G}}] = \frac{1}{\det(I+\varepsilon A)}. \tag{5.4}$$

Here is a useful extension : *for ε small enough, we have*

$$\mathbb{E}[Z_{a_1}\overline{Z}_{b_1}\ldots Z_{a_m}\overline{Z}_{b_m}e^{-\varepsilon\widehat{G}}] = \frac{\mathrm{per}(b_j,(I+\varepsilon C)^{-1}a_i)}{\det(I+\varepsilon A)}. \tag{5.5}$$

where C is the matrix (b_j, a_i). We give two proofs of this formula, a combinatorial one and a more probabilistic one (only sketched) — indeed, this is nothing but a complex Gaussian computation.

We work with the Wiener product and $\mathbb{E}$ instead of the Wick product and $\mathbf{Ex}$. We expand the exponential series (5.2)

$$\mathbb{E}[e^{-\varepsilon\widehat{G}}] = 1 + \sum_k \frac{(-\varepsilon)^k}{k}\mathbb{E}[(Z_{u_1}\overline{Z}_{v_1} + \ldots + Z_{u_n}\overline{Z}_{v_n})^k] \tag{5.6}$$

$$= 1 + \sum_k \frac{(-\varepsilon)^k}{k}\sum_{\tau\in F_k}\mathbb{E}[Z_{u_{\tau(1)}}\overline{Z}_{v_{\tau(1)}} + \ldots + Z_{u_{\tau(k)}}\overline{Z}_{v_{\tau(k)}}] \tag{5.7}$$

where F_k is the set of all mappings from $\{1,\ldots,k\}$ to $\{1,\ldots,n\}$. Before we compute (5.7), we make a remark : using the basic formula (2.3) for the expectation of a product, we know it can be developed into a permanent of scalar products (v_j, u_i). Therefore, the expectation does not change if we replace each v_i by its projection on the subspace K generated by the vectors u_j. On the other hand, $\widehat{G}$ depends only on the finite rank operator $\alpha = v_1' \otimes u_1 + \ldots v_n' \otimes u_n$ with range K, and from the above we may assume the v_j 's also belong to K. Then we may assume that the u_i constitute an orthogonal basis of K, in which the matrix of α is triangular, and the determinant we are interested in is the product of the diagonal elements $(1+\varepsilon\lambda_i)$ of $I+\varepsilon\alpha$.

We now start computing (5.7) : let $i_1 < \ldots < i_p$ be the range of the mapping τ ; since τ is not necessarily injective, each point i_j has a multiplicity k_j with $k_1+\ldots+k_p=k$. All the mappings τ with the same range and multiplicities contribute the same term to the sum,and their number is $k!/k_1!\ldots k_p!$. Therefore we rewrite (5.7) as

$$1+\sum_k(-\varepsilon)^k \sum_{\substack{k_1+\ldots+k_p=k\\ i_1<\ldots<i_p}} \mathbb{E}[(Z_{u_{i_1}}\overline{Z}_{v_{i_1}})^{k_1}\ldots(Z_{u_{i_p}}\overline{Z}_{v_{i_1}})^{k_p}]/k_1!\ldots k_p! \tag{5.8}$$

We rewrite the expectation as

$$\mathbb{E}\,[Z_{f_1}\overline{Z}_{h_1}\dots Z_{f_k}\overline{Z}_{h_k}]$$

with $f_j = u_{i_1}$, $h_j = v_{i_1}$ for $1 \le j \le k_1$, u_{i_2}, v_{i_2} for $k_1 < j \le k_1+k_2$, etc, and formula (2.3) expands it as a sum, over all permutations of $\{1,\dots,k\}$, of products $\prod_j (h_j, f_{\sigma(j)})$. The matrix A being triangular, σ does not contribute unless it preserves the intervals $(1,k_1)$, $(k_1+1,k_1+k_2)\dots$; the contribution of such a permutation is $(v_{i_1},u_{i_1})^{k_1}\dots(v_{i_p},u_{i_p})^{k_p}$, and the number of such permutations is $k_1!\dots k_p!$. Therefore the expectation (5.6) has the value

$$1+\sum_k(-\varepsilon)^k \sum_{\substack{k_1+\dots+k_p=k\\ i_1<\dots<i_p}} \lambda_{i_1}^{k_1}\dots\lambda_{i_p}^{k_p} = (1+\varepsilon\lambda_1)^{-1}\dots(1+\varepsilon\lambda_n)^{-1}$$

for ε small enough. The theorem is proved. The extension is not difficult : assuming the t_i's and ε are small enough, we have from the preceding result

$$\mathbb{E}\,[\exp(-\varepsilon\widehat{G}+t_1 Z_{a_1}\overline{Z}_{b_1}+\dots+t_m Z_{a_m}\overline{Z}_{b_m})] = 1/\det(I+\varepsilon A-U)\;,$$

where $U = \sum_i t_i b_i' \otimes a_i$. Putting $C = I+\varepsilon A$ and assuming C is invertible, this determinant can be written $\det(C)\det(I-C^{-1}U)$. On the other hand, $C^{-1}U$ is the finite rank operator $\sum_i t_i b_i' \otimes c_i$, with $c_i = C^{-1}a_i$, and if the t_i are small enough, we may use again the main result to compute $1/\det(I-C^{-1}U)$. Therefore

$$\mathbb{E}\,[\exp(-\varepsilon\widehat{G}+\textstyle\sum_i t_i Z_{a_i}\overline{Z}_{b_i})\,] = \mathbb{E}\,[\exp(\textstyle\sum_i t_i Z_{a_i}\overline{Z}_{c_i})\,]\,/\det(C)$$

We identify the coefficients of $t_1\dots t_m$ on both sides and get the required formula.

There is a similar computation for the "ξ–product" defined above, if ξ is symmetric : A is now the matrix $\xi(v_j,u_i)$ and C the matrix $\xi(b_j,a_i)$.

Let us now sketch a proof using Gaussian integrals. It has the advantage of expressing the Wick exponential itself as a standard (Wiener) exponential, instead of merely computing its expectation. We start with the elementary Fock space over $\mathbb{C}$ or $\mathbb{C}^2$, *i.e.* the L^2 space generated by a standard Gaussian variable X (real) or Z (complex). We recall the classical formula (for $c>0$)

$$e^{cx^2/2} = \frac{1}{\sqrt{c}}\int e^{xu}\,e^{-u^2/2c}\,đu \tag{5.9}$$

where $đu$ is the "Plancherel measure." From the interpretation of $\exp_{:}(uX)$ as $e^{uX-u^2/2}$ we deduce that

$$\exp_{:}(cX^2/2) = \frac{1}{\sqrt{1+c}}\,e^{\frac{c}{1+c}X^2/2} \quad , \quad e^{aX^2/2} = \sqrt{1-a}\,\exp_{:}(\frac{a}{1-a}X^2/2)\;. \tag{5.10}$$

What about the ξ–product? Since the basic Hilbert space $\mathcal{H}$ is simply $\mathbb{C}$, let us put $\xi(1,1)=q$ and assume for the moment that $q>0$. Then the ξ–product is the Wiener product w.r.t. a Gaussian law of variance q, and the Wick exponential $\exp_{:}(ux)$ takes

the form $e^{ux-qu^2/2}$. This has a combinatorial meaning which can be extended to all values of q. The same reasoning as above then gives us

$$(5.11)\qquad \exp_{:}(cX^2/2) = \frac{1}{\sqrt{1+qc}}\exp_{\xi}(\frac{c}{1+qc}X^2/2)\ .$$

The computation is similar, but simpler, in the complex case : we have $\exp_{:}(uZ+v\overline{Z}) = e^{uZ+v\overline{Z}-uv}$, and (returning to the representation $Z=(X+iY)/\sqrt{2}$ as an intermediate step) we deduce

$$(5.12)\qquad \exp_{:}(cZ\overline{Z}) = \frac{1}{1+c}\,e^{\frac{c}{1+c}Z\overline{Z}}\ .$$

This elementary formula can now be used to compute Wick exponentials on the complex Brownian motion Fock space, for elements of the second chaos of the form

$$G = Z_{u_1}:\overline{Z}_{v'_1} + \cdots + Z_{u_n}:\overline{Z}_{v'_n}$$

(vectors and linear functionals are identified via the bilinear scalar product). To perform this, one remarks that G depends only on the operator $A=\sum_i u_i\otimes v'_i$, and then one reduces A to a diagonal or at least triangular form : we give no details.

Antisymmetric computations

For a complete exposition of the rich algebra underlying the informal presentation we give here, we refer to the papers by Helmstetter in the references.

6 We start with a translation of the elementary Clifford algebra of Chapter II. There we had a finite dimensional Hilbert space $\mathcal{H}$ with a basis x_i, orthonormal for the bilinear scalar product $(\ ,\ )$; we built the corresponding basis (x_A) for the (symmetric or) antisymmetric (toy) Fock space, and the Clifford multiplication, which we now denote explicitly by $\diamond$, was defined by the table $x_A\diamond x_B = (-1)^{n(A,B)}x_{A\Delta B}$. Now the antisymmetric Fock space is nothing but the exterior algebra over $\mathcal{H}$, and for $A=\{i_1<\ldots<i_n\}$ x_A is nothing but the exterior product $x_{i_1}\wedge\ldots\wedge x_{i_n}$. Thus we have two different multiplications on the same space, and we want to put into algebraic language the relation between them, as we did for Wick and Wiener. The main element of structure is the bilinear scalar product, which we denote by $\xi(u,v)$ instead of (u,v).

First of all, $\mathbf{1}$ is the unit for both products. Then for elements of the first chaos, we have

$$(6.1)\qquad u\diamond v = u\wedge v + \xi(u,v)\,\mathbf{1}\ ,$$

implying the CAR $\{u,v\}_{\diamond} = 2\xi(u,v)\mathbf{1}$.

We have a "multiplication formula for stochastic integrals" which expresses

$$(u_1\wedge\ldots\wedge u_m)\diamond(v_1\wedge\ldots\wedge v_n)$$

as the sum of terms in all the chaos of order $(m-p,n-p)$ involving p contractions. Let us write the simplest non trivial case :

$$(6.2)\qquad u\diamond(v_1\wedge\ldots\wedge v_n) = \\ u\wedge v_1\wedge\ldots\wedge v_n + \sum_i(-1)^{i+1}\xi(u,v_i)v_1\wedge\ldots\wedge\widehat{v}_i\wedge\ldots\wedge v_n\ ,$$

omitting the vector with a hat. We leave it to the reader to write down the general formula. These prescriptions can be checked directly on the multiplication table, when each u_i or v_j is a basis element x_k, and then extended to arbitrary elements of the first chaos by linearity. On the other hand, the associativity of the multiplication formula we get is clear, as we are computing within an algebra known to be associative! Since any non-degenerate symmetric bilinear form can be interpreted as a scalar product, we get in this way a general associativity result. Once it is known for non-degenerate bilinear forms it extends to degenerate ones. Finally, the result extends at once to infinite-dimensional spaces.

REMARK. In the symmetric case, it is clear on formula (3.4) that the standard annihilation operators $a^-_{u'}$ act as derivations on all ξ-products. Here we have the same result for fermion operators $b^-_{u'}$, with the only difference that the derivations are graded ones, *i.e.* $b^-_{u'}(x\diamond y)=(b^-_{u'}x)\diamond y+\sigma(x)\diamond b^-_{u'}(y)$, where σ is the parity automorphism.

Now comes a crucial remark for Le Jan's result : what we have done *does not require the symmetry of* ξ — though of course the proof of associativity by reduction to toy Fock space does. *We postpone the general proof of associativity to the last subsection of this Appendix.* The symmetric form 2ξ must be replaced in the CAR by $\eta(u,v)=\xi(u,v)+\xi(v,u)$: thus all algebras with the same η are isomorphic Clifford algebras, but the way they are concretely realized on the antisymmetric Fock space depends on the choice of ξ. In particular, if ξ is antisymmetric, what we get is a new *Grassmann algebra* structure[1] on Fock space, which we distinguish from the usual one by means of the sign $\diamond$.

We take now a linear space $\mathcal{H}$ with a bilinear form ξ, and denote by $\mathcal{H}'$ a second copy of $\mathcal{H}$ — the $'$ may thus have two meanings, either of suggesting duality when a scalar product is given, or of a mark distinguishing elements of the second summand, and the confusion between these meanings is harmless, or even useful. We define $\mathcal{G}=\mathcal{H}\oplus\mathcal{H}'$, and extend ξ as a bilinear form on $\mathcal{G}$ under which the two summands are totally isotropic spaces :

$$\xi(u+v',w+z')=\xi(u,z)\pm\xi(w,v)\,. \tag{6.3}$$

If we choose the $+$ $(-)$ sign, we get a symmetric (antisymmetric) form. The generators we use for the (incomplete) antisymmetric Fock space over $\mathcal{G}$ are "normally ordered" ones, $u_1\wedge\ldots\wedge u_m\wedge v'_1\wedge\ldots\wedge v'_n$. Applying the preceding remark, we define multiplications on this double Fock space, with the following prescriptions : strings of u's or v's are multiplied as in the exterior algebra without correction; the computation of products of mixed strings are expressed as modified exterior products involving contractions. For instance

$$u\diamond(u_1\wedge\ldots\wedge u_m\wedge v'_1\wedge\ldots\wedge v'_n)= \tag{6.4}$$
$$u\wedge u_1\wedge\ldots\wedge u_m\wedge v'_1\wedge\ldots\wedge v_n+\sum\nolimits_i(-1)^{m+i+1}\xi(u,v_i)\,u_1\wedge\ldots\wedge u_m\wedge v'_1\wedge\ldots\widehat{v'_i}\ldots\wedge v'_n$$

$$v'\diamond(u_1\wedge\ldots\wedge u_m\wedge v'_1\wedge\ldots\wedge v'_n)= \tag{6.4'}$$
$$(-1)^m u_1\wedge\ldots\wedge u_m\wedge v'\wedge v'_1\wedge\ldots\wedge v'_n\mp\sum\nolimits_i(-1)^{m+i}\,\xi(u,v_i)\,u_1\wedge\ldots\widehat{u_i}\ldots\wedge u_m\wedge v'_1\wedge\ldots\wedge v'_n$$

[1] This was misunderstood in the report [Mey4], which contains a confusion between this "new Grassmann" product and a Clifford product.

(the sign $\mp$ is $-$ for Clifford, $+$ for "new Grassmann").

If $\mathcal{H}=L^2(\mathbb{R}_+)$ and ξ is the standard scalar product, one may also use a probabilistic notation : we take an antisymmetric double Fock space and denote by P and Q the corresponding two complex Brownian motions (which anticommute). Then we read u as $\int u(s)\,dP_s$ and v' as $\int v(s)\,dQ_s$. We also read

$$u_1\wedge\ldots\wedge u_m\wedge v_1'\wedge\ldots\wedge v_n' \quad \text{as} \quad P_{u_1}\wedge\ldots\wedge P_{u_m}\wedge Q_{v_1}\wedge\ldots\wedge Q_{v_n}\,.$$

The notation P,Q comes from symplectic geometry : it has nothing to do with the two real Brownian motions on simple Fock space, which satisfied the CCR : we are here in the CAR realm. In both cases we have for all s,t that dP_s anticommutes with dP_t, dQ_s with dQ_t without restriction, and dP_s anticommutes with dQ_t for $s\neq t$. For the Clifford product, we have

$$dP_t\diamond dQ_t = dt = dQ_t\wedge dP_t\,, \tag{6.5}$$

and therefore $\{dP_t,dQ_t\}=2dt$, while for the "new Grassmann" product, we have

$$dP_t\diamond dQ_t = I\,dt = -dQ_t\diamond dP_t\,, \tag{6.6}$$

and thus dP_s anticommutes with dQ_t without restriction.

We concentrate our interest on the "new Grassmann" product $\diamond$, and leave aside the Clifford one.

The ξ-products associated with non-symmetric forms have another interesting application : Le Jan [LeJ3] used them to extend Dynkin's method to non-symmetric Markov processes.

One can define on the incomplete double Fock space a linear functional $\mathbf{Ex}$ similar to that of symmetric Fock space, and such that

$$\begin{aligned}\mathbf{Ex}\,[u_1\wedge\ldots\wedge u_m\wedge v_1'\wedge\ldots\wedge v_n'] &= \mathbb{E}\,[u_1\diamond\ldots\diamond u_m\diamond v_1'\diamond\ldots\diamond v_n'] = \\ &\mathbb{E}\,[(u_1\wedge\ldots\wedge u_m)\diamond(v_1'\wedge\ldots\wedge v_n')]\,.\end{aligned}$$

Intuitively speaking, $\mathbf{Ex}$ is the expectation of an antisymmetric Stratonovich integral, which, however, has not been systematically developed in the antisymmetric case : a "trace" operator should be introduced and studied to define $\mathbf{Ex}$ for classes of coefficients belonging to the completed L^2 spaces.

This linear functional is given by the formula

$$\mathbf{Ex}\,[P_{u_1}\wedge\ldots\wedge P_{u_m}\wedge Q_{v_1}\wedge\ldots\wedge Q_{v_n}] = \rho(m)\det((u_i,v_j)) \quad \text{if } m=n, \tag{6.7}$$

and 0 if $m\neq n$.

7 We are going now to prove the antisymmetric analogue of (5.4). We consider an element of the second chaos of the form

$$G = P_{u_1}\wedge Q_{v_1}+\ldots+P_{u_n}\wedge Q_{v_n}\,, \tag{7.1}$$

a finite sum; $u_1\wedge v_1'+\cdots+u_n\wedge v_n'$ if you prefer algebraic notation. We associate with it a similar "new Grassman" object

$$\widehat{G} = P_{u_1}\diamond Q_{v_1}+\ldots+P_{u_n}\diamond Q_{v_n}\,. \tag{7.2}$$

The corresponding exponential series are a finite sums, and no convergence problems arise. The result is the following :

LEMMA 2. *Denoting by A the matrix (v'_j, u_i)*, we have

$$\mathbf{Ex}\,[e^G_\wedge] = \mathbb{E}\,[e^{\widehat{G}}_\diamond] = \det(I+A)\,. \tag{7.3}$$

As in the symmetric case there is a useful extension :

$$\mathbf{Ex}\,[P_{a_1} \wedge Q_{b_1} \ldots P_{a_m} \wedge Q_{b_m} e^G_\wedge] = \det(b_j, (I+C)^{-1}a_i)\,\det(I+A)\,, \tag{7.4}$$

where C is the matrix (b_j, a_i) and $I+C$ is assumed to be invertible.

PROOF. We use the standard Grassmann product and $\mathbf{Ex}$. We first expand

$$e^G_\wedge = 1 + \sum_k \frac{1}{k!} \sum_{\tau \in I_k} P_{u_{\tau(1)}} \wedge Q_{v_{\tau(1)}} \wedge \ldots \wedge P_{u_{\tau(k)}} \wedge Q_{v_{\tau(k)}}$$

where I_k is the set of all injective mappings from $\{1,\ldots,k\}$ to $\{1,\ldots,n\}$. Moving around a block $P_u Q_v$ produces an even number of inversions and does not change the product; therefore we may reduce I_k to the set J_k of strictly increasing mappings, cancelling the factor $1/k!$. Now we take all the P_u's to the left and get

$$1 + \sum_k \sum_{\tau \in J_k} \rho(k)\, P_{u_{\tau(1)}} \wedge \ldots \wedge P_{u_{\tau(k)}} \wedge Q_{v_{\tau(1)}} \wedge \ldots \wedge Q_{v_{\tau(k)}}\,.$$

Then we apply $\mathbf{Ex}$ and the factor $\rho(k)$ disappears : we get (putting $a_{ji} = (v_j, u_i)$)

$$1 + \sum_k \sum_{\tau \in J_k} \det((v_{\tau(j)}, u_{\tau(i)})) = 1 + \sum_{B \subset \{1,\ldots,n\}} \det{}_{i,j \in B}\, a_{ij}\,.$$

We must show this is the same as

$$\det(I+A) = \sum_\sigma \varepsilon_\sigma \prod_i (\delta_{i\sigma(i)} + a_{i\sigma(i)})\,.$$

We expand the product on the right. Each index i may contribute either a factor $a_{i\sigma(i)}$ or a factor $\delta_{i\sigma(i)}$. Let B the the set of indices of the first kind. If the product is different from 0, σ must leave the indices of the second kind fixed, and therefore arises from a permutation of B, which has the same signature. We then finish the proof by rearranging the sum as a sum over B. The extension is proved as in the symmetric case.

Supersymmetric Fock space

8 We now consider the *supersymmetric* Fock space, *i.e.* the tensor product of the previously considered symmetric and antisymmetric Fock spaces, contructed over $\mathcal{H}\oplus\mathcal{H}'$ — with $\mathcal{H} = L^2(\mathbb{R}_+)$ in the probabilistic interpretation, two complex Brownian motions being thus involved, or four real ones. The classical analogue is the space of all differential forms on a symplectic manifold. We may provide this space with several different products. Those we are interested in are the "Wick" product (Wick $\otimes$ Grassmann) which

we denote by $*$, and the "Wiener" product (Wiener$\otimes$new Grassmann) which we denote by $\diamond$ (sometimes omitted!) as its antisymmetric part.

Consider an operator α of finite rank on $\mathcal{H}$: it can be written (non-uniquely) as

$$\alpha(\cdot) = \sum_i v_i'(\cdot)\, u_i \tag{8.1}$$

with $v_i' \in \mathcal{H}'$. Then we associate with it an element of the supersymmetric Fock space (which depends only on α, not on the representation)

$$\lambda(\alpha) = \sum_i (Z_{u_i}\overline{Z}_{v_i} - P_{u_i} \diamond Q_{v_i}) = \sum_i (Z_{u_i} : \overline{Z}_{v_i} - P_{u_i} \wedge Q_{v_i})\ . \tag{8.2}$$

We have used the probabilistic interpretation, where v_i' becomes v_i. The "supersymmetric miracle" which happens with these elements is the following : if we multiply (5.4) and (7.3), the determinants on the numerator and the denominator cancel, and therefore we have, for ε small enough, the exponential being a Wiener one

$$\mathbb{E}\,[\exp_\diamond(\lambda(\alpha)]\ = 1\ . \tag{8.3}$$

Now we insert in front of each term in the sum (8.2) a coefficient ε_i (assumed to be small) and differentiate with respect to all of them. We find that given any finite family of operators of finite rank

$$\mathbb{E}\,[\lambda(\alpha_1)\ldots\lambda(\alpha_n)]\ = 0\ . \tag{8.4}$$

If instead of applying (5.4) and (7.3) we apply their extensions, we find that

$$\mathbb{E}\,[Z_a\overline{Z}_b\, e^{\sum_i \varepsilon_i (Z_{u_i}\overline{Z}_{v_i} - P_{u_i} \diamond Q_{v_i})}]\ = (b, (I - (\sum_i t_i v_i' \otimes u_i)^{-1} a)$$

and then if we identify the coefficients of $\varepsilon_1 \ldots \varepsilon_n$ on both sides, we get, putting $\alpha_i = v_i' \otimes u_i$ (an operator of rank 1)

$$\mathbb{E}\,[\, Z_a\overline{Z}_b \lambda(\alpha_1)\ldots\lambda(\alpha_n)]\ = \sum_\sigma \big(b\,,\, \alpha_{\sigma(1)}\ldots\alpha_{\sigma(n)} a\,\big) \tag{8.5}$$

By linearity, we may extend this to operators α_i of finite rank. The formula can be extended to Hilbert-Schmidt operators, but still this is a serious restriction.

The right side of (8.5) is the same as (2.1) in section 1, which itself was a rewriting of (1.9), "the n-point function of the occupation field." This has now become the expectation of one single product on supersymmetric Fock space, a simpler result than we could achieve in §1. May be our reader has forgotten the concrete application we had in view, assuming local times do exist : the computation, for F a polynomial in n variables, of the expectation $\mathbb{E}^{k\theta/k}\,[\,F(L_\infty^{x_1} \ldots L_\infty^{x_n})\,]$!

Properties of the supersymmetric Wiener product

9 We are going to prove a few useful algebraic results concerning the supersymmetric Wiener product. First, let us compute the norm of $\lambda(\alpha)$ in Fock space. Let α and $\widetilde{\alpha}$ be two finite rank operators. Then we have

$$< \lambda(\alpha)\,,\, \lambda(\widetilde{\alpha}) > \,=\, 2< \alpha, \widetilde{\alpha} >_{HS}\ , \tag{9.1}$$

where the scalar product on the right is the Hilbert-Schmidt scalar product between operators. To see this, we write

$$\lambda(\alpha) = \sum\nolimits_i Z_{u_i} : \overline{Z} v_i - P_{u_i} \wedge Q_{v_i}$$

and similarly for $\widetilde{\alpha}$ with vectors $\widetilde{u}_i, \widetilde{v}_i$ (it does not restrict generality to assume the index sets are the same). We understand these elements as random variables in the second chaos of a double complex Brownian motion Fock space. Then the $Z, \overline{Z}$ and P, Q parts decouple, and contribute the same quantity (hence the factor 2), which is

$$< \sum\nolimits_i \int u_i(s)\, v_i(t)\, dZ_s d\overline{Z}_t\ ,\ \sum\nolimits_i \int \widetilde{u}_i(s)\, \widetilde{v}_i(t)\, dZ_s d\overline{Z}_t >$$

which is easily interpreted as a H–S scalar product.

We now describe a second "supersymmetric miracle" . We consider an arbitrary vector u in $\mathcal{H}$ with $(u,u) = q$. We deduce from formula (5.12) that, for s small enough

$$\exp_{:}(s\, Z_u \overline{Z}_u) = \frac{1}{1+sq}\ \exp\Big(\frac{sq}{1+sq}\, Z\overline{Z}\Big)\ . \tag{9.2}$$

On the other hand, we have

$$\begin{aligned} \exp_{\wedge}(-s P_u \wedge Q_u) &= 1 - s P_u \wedge Q_u = 1 - s(P_u \diamond Q_u - q) \\ &= (1+sq)\Big(1 - \frac{sq}{1+sq}\, P \diamond Q\Big) = (1+sq)\ \exp_{\diamond}\Big(-\frac{sq}{1+sq}\, P \diamond Q\Big) \end{aligned} \tag{9.3}$$

Multiplying (9.2) and (9.3) we get

$$\exp_{*}(s\lambda(u \otimes u')) = \exp_{\diamond}\Big(\frac{sq}{1+sq}\, \lambda(u \otimes u')\Big)\ . \tag{9.4}$$

We recall the generating function for a family of Laguerre polynomials

$$\exp\, \frac{tx}{1+t} = \sum\nolimits_n P_n(x)\, t^n \quad \text{with} \quad P_n(x) = (-1)^n L_n^{(-1)}(x)\ ,$$

and then expanding the exponential gives an expression for supersymmetric "Wick" powers using "Wiener" powers

$$\lambda(u \otimes u')^n_* = n!\, q^n\, P_n(\lambda(u \otimes u')/q)_{\diamond}\ . \tag{9.5}$$

10 After we used the theory of local times as our motivation to get into twenty pages of algebra, it may be disappointing that we devote little space to probabilistic applications.

However, we cannot hide the fact that, for these applications, the best methods are purely probabilistic : a good way to dive into the literature is to look at the volumes XIX to XXI of the *Séminaire de Probabilités*, and to look therein at the papers by Dynkin, Le Gall, Le Jan, Rosen and Yor (and their references). On the other hand, the relations with quantum field theory are described in a series of papers by Dynkin. Here, we merely mention the problems.

Recall that our basic (pre)Hilbert space $\mathcal{H}$ is a space of signed measures with bounded potential, with the energy scalar product. With every measure μ of finite energy, we associated an operator a_μ on the space

$$a_\mu(\nu) = (G\nu)\,\mu\ .$$

If the process has local times, *i.e.* if point measures have finite energy, and if the support of μ is a finite set, then a_μ has finite rank, and we may associate with it an element $\lambda(\mu)$ of the second chaos as explained before, and apply all the previous algebraic computations. In particular, we may associate with every point x an element $Z_x\overline{Z}_x$ of the second symmetric chaos, and an element $\lambda_x = Z_x\overline{Z}_x - P_xQ_x$ of the second supersymmetric chaos. Computations involving the system of local times may be reduced to algebraic computations on these vectors (see Le Jan's paper [LeJ1] for details). Note that λ_x has expectation 0 : its definition implicitly contains a "renormalization" .

Assume now that local do not exist. Then we may think of $Z_x\overline{Z}_x$ and λ_x as "generalized random variables" and wonder whether they belong to a true "generalized field," *i.e.* whether we may define something like $\int \lambda_x f(x)\,dx$ for a smooth function with compact support f, and more generally $\int \lambda_x^{*n} f(x)\,dx$ (Wick power). The symmetric part of this integral will then be interpreted as a "renormalized power" of the occupation field. The method to achieve this is the following : we replace the point measure ε_x by a regularization $\varepsilon_x(t)$, λ_x by the corresponding $\lambda_x(t)$, and use the algebra to check whether the corresponding integrals converge in L^2 as $t \to 0$. It turns out that the computation is not too difficult, and that convergence takes place in dimension 2, but not in higher dimensions (the divergence of $g(x,y)$ on the diagonal is logarithmic in dimension 2). For details, we refer to [LeJ2]. Unfortunately, this description of the renormalization problem has reduced it to its abstract core, stripping it of its probabilistic interest : relations with multiple points of Brownian motion, asymptotic behaviour of the volume of the Wiener sausage, etc.

Proof of associativity of ξ-products

11 We are going to prove that the computations involving contractions we performed around (6.2) define an associative product. The following proof is adapted from Le Jan.

We may reduce the infinite dimensional case to a finite dimensional one : consider a space $\mathcal{H}$ of finite dimension N with a scalar product (u,v). We give ourselves a bilinear form $\xi(u,v) = (Au,v)$ on $\mathcal{H}$, and put $\eta(u,v) = \xi(u,v) + \xi(v,u)$. We will assume that ξ and η are non-degenerate. This is not a serious restriction : non-degeneracy holds for $\xi(u,v) + t(u,v)$ except for finitely many values of t, and the associativity result may be extended by continuity to $t=0$.

On the antisymmetric Fock space Φ over $\mathcal{H}$, we denote by b_u^+, b_u^- the standard fermion creation and annihilation operators, satisfying the CAR $\{b_u^+, b_v^-\} = (u,v)$.

Then we put

$$R_u = b_u^+ + b_{Au}^- \quad , \quad S_v = b_{Bv}^+ - b_v^- \ , \tag{.1}$$

where B satisfies $(Au, Bv) = (u, v)$. Knowing how $b_u^{\pm}$ operates on exterior products, all amounts to proving this : *there exists an associative product* $\diamond$ *on* Φ, *with* $\mathbf{1}$ *as unit, such that for* $u \in \mathcal{H}$ *and* $x \in \Phi$ *we have* $u \diamond x = R_u x$.

It is easy to see that

$$\{R_u, R_v\} = (u, Av) + (Au, v) \quad , \quad \{S_u, S_v\} = -(u, Bv) - (Bu, v) \quad , \quad \{Ru, Sv\} = 0 \ .$$

Therefore, the operators R_u, S_v taken together generate a Clifford algebra over the space $\mathcal{H} \oplus \mathcal{H}$ of even dimension $2N$, relative to a non-degenerate quadratic form. Applying the fundamental uniqueness result on Clifford algebras (II.5.5) we see that the operators $R_u S_v$ generate an algebra of dimension 2^{2N}, which is therefore the full algebra $\mathcal{L}(\Phi)$. On the other hand, the two algebras $\mathcal{R}$ and $\mathcal{S}$ generated by the R_u and S_v separately are Clifford algebras of dimension 2^N.

Let Φ^+ (Φ^-) be the even (odd) subspaces of Φ; let $\mathcal{R}^+$ ($\mathcal{S}^+$) and $\mathcal{R}^-$ ($\mathcal{S}^-$) be the even (odd) subspaces of $\mathcal{R}$ and $\mathcal{S}$. Note that R_u and S_u are odd operators in the usual sense, and therefore $\mathcal{R}^+, \mathcal{S}^+$ preserve parity in Φ while $\mathcal{R}^-, \mathcal{S}^-$ change it. We denote below by $R^{\pm}, S^{\pm}$ generic elements of $\mathcal{R}^{\pm}, \mathcal{S}^{\pm}$.

Let $A \in \mathcal{R}$ such that $A\mathbf{1} = 0$; decompose it into $A^+ + A^-$. Since $A^{\pm}\mathbf{1}$ belongs to $\Phi^{\pm}$ we must have $A^{\pm}\mathbf{1} = 0$. Then applying R^{ε} for $\varepsilon = \pm$ and using the appropriate commutation relation we find that $A^{\pm}R^{\varepsilon}\mathbf{1} = 0$. Then applying S^{δ} for $\delta = \pm$ and using the appropriate commutation relation with $A^{\pm}R^{\varepsilon}$ we find that $A^{\pm}R^{\varepsilon}S^{\delta}\mathbf{1} = 0$. Putting everything together we find that $ARS\mathbf{1} = 0$ for arbitrary $R \in \mathcal{R}$, $S \in \mathcal{S}$. On the other hand, the algebra $\mathcal{RS}$ is the full algebra, and therefore $A = 0$. The mapping $A \longmapsto A\mathbf{1}$ from $\mathcal{R}$ to Φ being injective, a dimension argument shows that it is surjective. Therefore we may use it to identify Φ with $\mathcal{R}$, and define on Φ an associative product $\diamond$, with $\mathbf{1}$ as unit, such that, if x, y are elements of Φ, and R_x, R_y are the unique elements of $\mathcal{R}$ such that $x = R_x\mathbf{1}$, $y = R_y\mathbf{1}$, we have $x \diamond y = R_x R_y \mathbf{1}$.

In particular, if u, v belong to $\mathcal{H}$, we have

$$u \diamond v = R_u R_v \mathbf{1} = (b_u^+ + b_{Au}^-)\, v = u \wedge v + (Au, v) \ .$$

More generally, we have for $x \in \Phi$ $u \diamond x = R_u R_x \mathbf{1} = R_u x$, the main property we wanted to prove.

REMARK. In Helmstetter's notes [Hel2] (or p. 35 of the more easily accessible paper [Hel1]), one can find an explicit formula for the ξ-product, from which a direct and general proof of associativity can be deduced, including the degenerate cases. We will give now this formula.

Let $\mathcal{E}$ be a finite dimensional space $\mathcal{E}$ with dual $\mathcal{E}'$. We use greek letters for elements of $\mathcal{E}'$ and its exterior algebra. We recall there is a mapping $(\alpha, x) \longmapsto \alpha \lhd x$ from $\bigwedge(\mathcal{E}') \times \bigwedge(\mathcal{E})$ to $\bigwedge(\mathcal{E})$, called the *interior product,* which is a generalized annihilation operator :

1) For $\alpha \in \mathcal{E}'$, $x \in \bigwedge(\mathcal{E})$, we have $\alpha \lhd x = b_\alpha^-(x)$, with the additional convention that $\mathbf{1} \lhd x = x$.

2) $(\alpha \wedge \beta) \triangleleft x = \alpha \triangleleft (\beta \triangleleft x)$.

This product acts as a graded derivation :

3) $\alpha \triangleleft (x \wedge y) = (\alpha \triangleleft x) \wedge y + \sigma(x) \wedge (\alpha \triangleleft y)$, σ denoting the parity automorphism.

More generally, Helmstetter ([Hel2] p. 25) shows that we may replace $\wedge$ by $\diamond$, an arbitrary ξ-product on E.

We take $\mathcal{E} = \mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ is finite dimensional with dual $\mathcal{H}'$. To distinguish the second component we use a $\widetilde{\ }$ since we cannot use the $'$ as before. We identify the exterior algebra $\bigwedge(\mathcal{E})$ with $\bigwedge(\mathcal{H}) \otimes \bigwedge(\mathcal{H})$: this means we use antisymmetry to express elements of $\bigwedge(\mathcal{E})$

$$(u_1 + \widetilde{v}_1) \wedge \ldots \wedge (u_p + \widetilde{v}_p)$$

as linear combinations of normally ordered products

$$(u_{i_1} \wedge \ldots \wedge u_{i_m}) \otimes (\widetilde{v}_{j_1} \wedge \ldots \wedge \widetilde{v}_{j_n})$$

with $m+n = p$. The dual $\bigwedge(\mathcal{E}')$ then gets identified with $\bigwedge(\mathcal{H}') \otimes \bigwedge(\mathcal{H}')$. On the other hand, a bilinear form ξ on $\mathcal{H}$ defines an antisymmetric bilinear form on $\mathcal{H} \oplus \mathcal{H}$

$$\xi(u + \widetilde{v}, w + \widetilde{z}) = \xi(u, z) - \xi(w, v)$$

and therefore $\mathcal{H} \otimes \mathcal{H}$ is imbedded into $\bigwedge_2(\mathcal{E}')$, and has an exterior exponential in $\bigwedge(\mathcal{E}')$. Finally, there is a natural mapping μ from $\mathcal{E}$ to $\mathcal{H}$ given by $\mu(u + \widetilde{v}) = u + v$, which extends to the exterior algebras by $\mu(h \wedge k) = \mu(h) \wedge \mu(k)$ for $h, k \in \bigwedge(\mathcal{E})$; for example, given $x, y \in \bigwedge(\mathcal{H})$ we simply have $\mu(x \otimes \widetilde{y}) = x \wedge y$.

After all these preliminaries, we can give the closed formula for the ξ-product on $\bigwedge(\mathcal{H})$: given $x, y \in \bigwedge(\mathcal{H})$, $x \otimes y$ belongs to $\bigwedge(\mathcal{E})$, and we take

$$x \diamond y = \mu(\exp_{\wedge}(\xi) \triangleleft (x \otimes \widetilde{y}))$$

For instance, if $x, y \in \mathcal{H}$, the only terms in the exponential that contribute are $\mathbf{1} + \xi$, and we get

$$\mathbf{1} \triangleleft (x \otimes \widetilde{y})) = x \otimes \widetilde{y} \quad , \quad \xi \triangleleft (x \otimes \widetilde{y}) = \xi(x, y)\, \mathbf{1} \ ,$$

and applying μ we have $x \diamond y = x \wedge y + \xi(x, y)\,\mathbf{1}$ as it should be. This is clearly the analogue of the trace formula for Stratonovich integrals.

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