Appendix A
Excursion to Logic: Some Remarks on the Metamathematics of Minimal Internal Set Theory

A.1 An Alternative Road to Minimal Internal Set Theory

When we introduced Minimal Internal Set Theory in Chap. 1, we have tacitly assumed that most readers of this book will find it more intuitive to conceive of Minimal Internal Set Theory as an axiom system which describes an *extended universe*. Some readers, however, might be more comfortable with the idea that an "appropriate" mathematical model of the real numbers should contain infinitesimals and infinitely large numbers anyway. On this account, it would be more intuitive to simply *extend the language* of conventional mathematics by a new predicate, e.g. "…is a standard natural number", and impose additional axioms regulating the use of this predicate—in order to allow for a consistent and fruitful use of infinitesimals.

Of course, the choice of the axioms requires care, as the resulting axiom system should be consistent,[1] simple and powerful enough to permit a productive use of these axioms for infinitesimal calculus. In order to motivate our choice of an axiom system (which is inherited from Nelson's *Radically Elementary Probability Theory* [60]), we could have pointed to the relatively well-known fact that the Peano axiomatization of the natural numbers does not characterize the set of natural numbers completely.[2] For example, any model of the Peano axioms can be elementarily embedded as a proper subset into some other model of the Peano axioms. This observation already suffices to motivate the consistency of axiom systems with a modified principle of mathematical induction for the standard natural numbers. The axiom system in Nelson's *Radically elementary probability theory* [60] is exactly of such a kind.

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[1] At least relative to the consistency of conventional mathematics, which because of Gödel's second incompleteness theorem [28] admits no consistency proof.

[2] More precisely, indeed, by Gödel's first incompleteness theorem [28], no extension of the Peano axioms could provide such a unique characterization up to isomorphism.

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On this syntactic account of Minimal Internal Set Theory, the presentation of the axiom system only needs to be prefaced (as in Nelson's monograph [60]) by observing that the language of conventional mathematics does not use the word "standard", whence one may without hesitation introduce a new unary predicate for natural numbers with that name, i.e. a predicate of the form ". . . is a standard natural number". Having thus extended the language of conventional mathematics,[3] all that is left to do is to specify rules that govern the use of that predicate.

On this account, one may note[4] that the introduction of Minimal Internal Set Theory did not *per se* involve the addition of any new mathematical objects (be it atoms or sets). One may take the view that the universe of mathematical objects has remained the same, and only the language has been extended—by adjoining a new predicate which allows us to distinguish between standard natural numbers and nonstandard natural numbers. As one can gather from the axioms of Minimal Internal Set Theory and the fact (provable by External Induction, see below) that any nonstandard natural number is greater than every standard natural number, the correct interpretation of "standard" is "not extremely large".

Readers with an interest in the foundations of mathematics will observe that (i) the axiom system **minIST** would obviously be inconsistent if the Peano axioms characterized the natural numbers completely, and (ii) conversely, the incompleteness of the Peano axioms readily suggests that the axiom system **minIST** is consistent. In any case, it can be rigorously shown that **minIST** only proves those internal formulae can already be proved in conventional mathematics: **minIST** is a conservative extension of conventional mathematics and thus—in light of *ex falso quodlibet*—(relatively) consistent. The reason for the conservativity of **minIST** lies in the fact that it can be seen as a subsystem of Nelson's [59] (cf. Nelson [60, Appendix, p. 80]) which itself is a conservative extension of conventional mathematics.

We close this section with a few more technical comments on the axioms of **minIST**. First, the term "conventional mathematics" in the first axiom of Minimal Internal Set Theory is, of course, context-dependent; at present, most mathematicians would understand the term "conventional mathematics" to refer to Zermelo–Fraenkel set theory plus the Axiom of Choice (**ZFC**). In the following, we will side with the majority and view Minimal Internal Set Theory as an extension of **ZFC** by definition. We note, however, that radically elementary probability theory and radically elementary stochastic analysis certainly do not use **ZFC** to its fullest strength. Therefore, they might continue to be acceptable even when the consistency of **ZFC** should some day be subject to considerable doubt. (Edward Nelson for instance is less than convinced that Peano Arithmetic is consistent [63].)

[3] In fact, as Cantor, Frege, Russell, Whitehead and others had shown by the early 1900s, all of conventional mathematics may be reduced to set theory, so "the language of conventional mathematics" comes down to all that can be expressed with the $\in$-relation.

[4] This applies especially to those readers who already have come into loose contact with nonstandard analysis.

The additional axioms beyond **ZFC** are theorems of Nelson's [59] Internal Set Theory (cf. Nelson [60, Appendix, p. 80]), which itself is a conservative extension of **ZFC** (Nelson [59, Theorem 8.8, in part due to William C. Powell]) and thus consistent relative to **ZFC**. Hence, **minIST**$^+$ also is consistent relative to **ZFC** and every internal theorem of **minIST**$^+$ can also be proved in **ZFC**. It might be possible to develop Nelson's [60] radically elementary probability theory, at least partially, even when one replaces **ZFC** in our definitions of **minIST**$^+$ or **minIST** (or the even weaker system **minIST**$^-$ of Appendix A) by a weaker set-theoretic axiom system. This would be an interesting question for future research.[5]

A.2 A Simple Relative Consistency Proof for a Substantial Subsystem of minIST

Nelson [60, Appendix, p. 80] has shown, invoking the saturation principle of Internal Set Theory (cf. Nelson [61]), that the axioms of **minIST**$^+$ follow from **IST**, and since Nelson has also shown that **IST** is a conservative extension of **ZFC** [59, Theorem 8.8, in part due to William C. Powell], it follows that so is **minIST**$^+$.

The proof of the fact that **IST** is a conservative extension of **ZFC**, however, is a sophisticated argument using so-called adequate ultrapowers and ultralimits. For pedagogical reasons, one would wish to find a simple proof at least for the consistency of some subsystem of **minIST**$^+$ in which a substantial part of radically elementary probability theory can be developed. This is what we will now aim at. Consider the subsystem, henceforth denoted **minIST**$^-$, of **minIST** which one obtains through replacing the External Induction principle by the following two axioms:

- (**Unlimitedness of nonstandard numbers**) If $n \in \mathbf{N}$ is nonstandard, then $n > k$ for all standard $k \in \mathbf{N}$.
- (**Standard Induction**) Let $A(n)$ be a formula which is of the form $Q_1^{\mathbf{st}} v_1 \ldots Q_m^{\mathbf{st}} v_m \varphi(p_1, \ldots, p_\ell, v_1, \ldots, v_m, n)$, wherein $Q_i^{\mathbf{st}} v_i$ is a quantification either of the form "for all standard $v_i \in \mathbf{N}$" (abbreviated $\forall^{\mathrm{st}} v_i$) or of the form "there exists a standard $v_i \in \mathbf{N}$" (abbreviated $\exists^{\mathrm{st}} v_i$), $p_1, \ldots, p_\ell$ are standard natural numbers and φ is a formula of set theory with $\ell + m + 1$ free variables (and no parameters). Assume that $A(0)$ holds and that $A(n)$ entails $A(n+1)$ for all standard n. Then $A(n)$ readily holds for all standard n.

Note that the most important proof principle of **minIST**, viz. the underspill/overspill principle (Remark 1.1), still holds in **minIST**$^-$. For example, in order to prove that there is no set which consists only of the standard natural numbers,

[5] In fact, Henson and Keisler [30] have shown that adding nonstandard elements to certain relatively weak axiom systems of set theory may result in a stronger, i.e. non-conservative extension of the original weak axiom system.

we only have to remark that if there were such a set, one could prove by internal induction (i.e. induction in $\mathbf{N}$, which of course holds in **minIST**$^{-}$ as it extends **ZFC**) that this set is the whole of $\mathbf{N}$, contradicting the existence of nonstandard natural numbers (which continues to hold in **minIST**$^{-}$).

Furthermore, the External Induction principle can be replaced by the Standard Induction principle in proving a number of basic results of radically elementary mathematics. We give some examples for results which Nelson [60, p. 17] proves with the External Induction principle and which can also be proved in **minIST**$^{-}$ through the Standard Induction principle.

Lemma A.1 (minIST^{-}).

(1) If m and n are standard natural numbers, then so is $m + n$.
(2) If m and n are standard natural numbers, then so is mn.
(3) If n is a standard natural number and $a > 0$ is limited, then a^n is limited.
(4) For all $n \in \mathbf{N}$, n is standard if and only if it is limited.
(5) If x is infinitesimal and y is limited, then xy is infinitesimal.
(6) If $x \simeq y$ and $y \simeq z$, then $x \simeq z$.
(7) Let $n \in \mathbf{N}$ be standard and $(x_i)_{i<n}, (y_i)_{i<n} \in \mathbf{R}^n$. If $x_i \simeq y_i$ for all $i < n$, then $\sum_{i<n} x_i \simeq \sum_{i<n} y_i$.

Proof. (1) Let $m \in \mathbf{N}$ be standard. An inspection of the definition of ordinal addition and the proof of the ordinal recursion theorem shows that there exists a formula of set theory, denoted $\psi_+(m, n, k)$, whose only parameters are m, n, k and such that for all $m, n, k \in \mathbf{N}$,

$$m + n = k \Leftrightarrow \psi_+(m, n, k)\,.$$

Let us hence apply Standard Induction to the formula

$$\exists^{\mathrm{st}} k \qquad m + n = k.$$

The base step of the induction is tautological. For the induction step, it suffices to remark that if $m + n$ is standard (induction hypothesis), then $m + n + 1$ is standard.

(2) Let $m \in \mathbf{N}$ be standard. Again, an inspection of the definition of ordinal addition and the proof of the ordinal recursion theorem shows that there exists a formula of set theory $\psi_\times(m, n, k)$ whose only parameters are m, n, k and such that for all $m, n, k \in \mathbf{N}$,

$$mn = k \Leftrightarrow \psi_\times(m, n, k)\,.$$

We apply Standard Induction to the formula

$$\exists^{\mathrm{st}} k \qquad mn = k.$$

Base step: $m0 = 0$ is standard. Induction step: Suppose mn is standard (induction hypothesis). Then $m(n+1) = mn + n$ is the sum of two standard numbers and thus itself standard by part 1 of the present lemma.

(3) Of course, $a^n > 0$. Since a is limited, there exists some standard $m \in \mathbf{N}$ such that $a < m$. It is enough to verify the formula

$$\exists^{\mathrm{st}} k \qquad m^n = k.$$

An inspection of the definition of ordinal addition and the proof of the ordinal recursion theorem shows that there exists a formula of set theory $\psi_{\exp}(m, n, k)$ whose only parameters are m, n, k and such that for all $m, n, k \in \mathbf{N}$,

$$m^n = k \Leftrightarrow \psi_{\exp}(m, n, k).$$

Hence, we may apply Standard Induction to prove that $\exists^{\mathrm{st}} k \quad m^n = k$. Base step: $m^0 = 1$ is standard. Induction step: Suppose there is a standard k such that $m^n = k$ (induction hypothesis). Then $m^{n+1} = km$, which is the product of two standard numbers and thus itself standard by part 2 of the present lemma.

(4) If n is standard, then obviously limited (by the trivial estimate $n \leq n$). The converse follows from the unlimitedness of nonstandard numbers, an axiom of **minIST**$^-$.

(5) Fix a standard $m \in \mathbf{N}$. We have to prove $|xy| \leq 1/m$. Choose a standard $n \in \mathbf{N}$ such that $|y| \leq n$. By part 2 of the present lemma, mn is standard, whence

$$|xy| = |x|\,|y| \leq \frac{1}{mn} n \leq \frac{1}{m}.$$

(6) Fix a standard $m \in \mathbf{N}$. We have to prove $|x - z| \leq 1/m$. By part 2 of the present lemma, $2m$ is standard (as $2 = 0 + 1 + 1$ is standard), whence

$$|x - z| \leq |x - y| + |y - z| \leq \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}.$$

(7) Fix a standard $m \in \mathbf{N}$. We need to prove $\left|\sum_{i<n}(x_i - y_i)\right| \leq \frac{1}{m}$. However, mn is standard (by part 2 of the present lemma), so

$$\left|\sum_{i<n}(x_i - y_i)\right| \leq \sum_{i<n}|x_i - y_i| \leq \sum_{i<n}\frac{1}{mn} = \frac{1}{m}.$$

□

An advantage of **minIST** over **minIST**$^-$ is that its axioms are simpler and shorter to formulate; what speaks for **minIST**$^-$ is that it admits a short proof of its relative consistency.

Theorem A.2. *The axiom system* ***minIST***$^-$ *is a conservative extension of* ***ZFC***.

As an immediate corollary, **minIST**$^-$ is consistent relative to **ZFC**.

Proof. Let ψ be a formula of set theory which is not provable in **ZFC**. We shall construct a model $^*V$ of **minIST**$^-$ in which ψ fails. By the compactness theorem, let V be a set-size, transitive model of **ZFC**, called *ground model*, which models $\neg\psi$. Let $\mathbf{N}^V$ be the set of natural numbers as recognized by V, and let $\in^V$ denote the element-relation as recognized by V. Let I be an infinite set and let $\mathcal{U}$ be a non-principal ultrafilter on I. Consider the ultrapower $^*V = V^I/\mathcal{U}$, into which V can be canonically embedded, through $^* : v \mapsto [(v)_{i\in I}]_{\mathcal{U}}$. By Łoś's theorem, this is an elementary embedding: $^* : V \prec {}^*V$.

Let $^*\mathbf{N}$ be the set of natural numbers as recognized by $^*V$, and let $^*\!\in^V$ denote the element-relation as recognized by $^*V$. Call an element n of $^*\mathbf{N}$ *standard* (denoted $\mathbf{st}(n)$) if and only if it is of the form $^*n_0$ for some $n_0 \in \mathbf{N}^V$.

We now have to prove that $\left({}^*V, {}^*\!\in^V, \mathbf{st}\right)$ is a model of **minIST**$^-$ and of $\neg\psi$. Indeed, $^*V$ is a model of **ZFC** and of $\neg\psi$ since $V \prec {}^*V$. Moreover,

$$0^{*V} = \varnothing^{*V} = {}^*\varnothing = {}^*0$$

and for all $n_0 \in \mathbf{N}^V$, one has

$${}^*n_0 {}^*\!+ {}^*1 = {}^*(n_0 + 1).$$

Therefore, 0^{*V} is standard and for every standard n, $n {}^*\!+1$ is standard, too.

Consider next some $k \in {}^*\mathbf{N}$ with $k {}^*\!\leq n$ for some standard $n = {}^*n_0$. Let $k = [(k_i)_{i\in I}]_{\mathcal{U}}$, then $\{i \in I \ : \ k_i \leq n_0\} \in \mathcal{U}$ by Łoś's Theorem. Since $\mathcal{U}$ is non-principal and $\{i \in I \ : \ k_i \leq n_0\} = \bigcup_{j=0}^{n_0} \{i \in I \ : \ k_i = j\}$ for some finite number n_0, we must have $\{i \in I \ : \ k_i = j_0\} \in \mathcal{U}$ for some $j_0 \leq n_0$. But then $k = {}^*j_0$, whence k is standard.

Finally, we prove the Standard Induction principle in $^*V$. Let

$$A(n) = Q_1^{\mathbf{st}} v_1 \ldots Q_m^{\mathbf{st}} v_m \, \varphi\left({}^*p_1, \ldots, {}^*p_\ell, v_1, \ldots, v_m, n\right),$$

wherein $p_1, \ldots, p_\ell \in \mathbf{N}^V$ and φ is a formula of set theory without parameters, and define

$$A^V(n) = Q_1 v_1 \ldots Q_m v_m \, \varphi\left(p_1, \ldots, p_\ell, v_1, \ldots, v_m, n\right).$$

Inductively in m (the number of external quantifiers in A) one can prove that for every $n_0 \in \mathbf{N}^V$,

$$\left({}^*V, {}^*\!\in^V, \mathbf{st}\right) \models A({}^*n_0) \Leftrightarrow (V, \in^V) \models A^V(n_0). \tag{A.34}$$

(The base step of the induction uses that $V \prec {}^*V$.) Therefore, the assumptions in the Standard Induction principle mean that $A^V(0)$ and $A^V(n) \Rightarrow A^V(n+1)$ hold for every $n \in \mathbf{N}^V$, whence $A^V(n)$ must hold for all $n \in \mathbf{N}^V$ (by induction in V). Therefore, again by equivalence (A.34) we have that $A({}^*n)$ holds for all $n \in \mathbf{N}^V$ and thus $A(n)$ holds for all standard n. $\square$

A.3 Definable Models for (Minimal) Nonstandard Analysis

The consistency proofs for **IST** (cf. Nelson [59]) or Robinsonian nonstandard analysis (cf. Robinson [67])—and also our simple consistency proof for **minIST**$^{-}$—use ultrapower constructions and thus rest on the existence of non-principal ultrafilters, typically obtained from Zorn's Lemma. This, however, does not mean that the Axiom of Choice is an indispensable ingredient of these consistency proofs, since the ultrafilter existence theorem is in fact strictly weaker than the Axiom of Choice (cf. Halpern and Levy [29] and Banaschewski [7] for a discussion of the strength of the ultrafilter existence theorem).

Based on a technique developed by Kanovei and Shelah [40], Kanovei and Reeken [39] have shown that a slightly stronger set-theoretic axiom system than **ZFC** implies the existence of definable models of **IST** and thus of **minIST**$^{+}$. The definable nonstandard enlargement constructed in [31, 32] is obviously a model of a significant subsystem of **minIST**$^{-}$, viz. the subsystem obtained by removing those set-theoretic axiom scheme instances which do not hold for superstructures (such as **Extensionality** for atoms, in this case the reals). Moreover, by applying Kanovei and Shelah's [40] technique one can produce a countably saturated, definable, ultrapower-like extension of a set universe. In a similar manner as in the proof of our consistency result (Theorem A.2) one can then verify that this definable structure is a model of **minIST**$^{-}$.

Appendix B
Robinsonian vs. Minimal Nonstandard Analysis

The point of this book was to present a different approach to stochastic analysis, one that—for the sake of accessibility to mathematics undergraduates and students of other disciplines—avoids the use of measure theory and functional analysis which the classical approach requires and instead invokes a small axiom system, which might just be dubbed *minimal nonstandard analysis*,[1] but is a fragment of Internal Set Theory and thus called Minimal Internal Set Theory. Contrary to this intention, Robinsonian [67] nonstandard analysis has the express purpose to be just an additional tool in the hands of any research mathematician, so that any "nonstandard arguments" should yield standard theorems. For instance, the seminal result of nonstandard probability theory is the "conversion from nonstandard to standard measure spaces" [51] now known as the Loeb construction in honor of its inventor (or discoverer, depending on one's belief or disbelief in mathematical Platonism), Professor Peter A. Loeb.

From a more technical perspective, the two approaches also differ substantially: Internal Set Theory extends the *syntax* of conventional mathematics and views, say, the set of natural numbers as containing some (hitherto unclassified) nonstandard numbers—this is also the point of view taken in Nelson's *Radically Elementary Probability Theory* [60], where Minimal Internal Set Theory is derived from.

Robinsonian nonstandard analysis, however, operates *semantically*: It starts from (what may be seen as) a model of a modified fragment of Zermelo–Fraenkel set theory (with the real numbers as atoms or urelements) which is just sufficient for analysis in its broad sense—a *superstructure* over the real numbers. This is then extended to a *nonstandard universe*, viz. a superstructure over an extended set of real numbers, the *hyperreal numbers* (which is a real ordered field including infinitesimals and unlimited numbers), which also contains an extended set of natural numbers (including unlimited numbers), called *hypernatural numbers*.

[1] This term was suggested by Nelson in a more recent paper [62].

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The extension is constructed in such a way that (among other properties) the canonical embedding is well-behaved with respect to the $\in$-relation.[2] Images of elements of the original superstructure under the canonical embedding are called *standard*, elements of standard sets are called *internal*, all other sets are called *external*.

As our motivation of Minimal Internal Set Theory in Chap. 1 already suggests, one does not need to view Minimal Internal Set Theory merely as a fragment of Internal Set Theory. Instead, Minimal Internal Set Theory can also be linked to Robinsonian nonstandard analysis relatively easily—for instance, by noting that the nonstandard universe can be viewed as a model of **minIST**: If one takes (i) the set of hypernatural numbers to be the interpretation of the constant **N** in the language of **minIST** and (ii) the class of all those hypernatural numbers which were already present in the original superstructure to be the interpretation of the predicate "…is a standard natural number" in the language of **minIST**, then the axioms of **minIST** are satisfied, and the internal sets of the superstructure are just those sets which can be defined by internal formulae (possibly with parameters) in **minIST**.

This last observation permits a new reading of the present work from the perspective of Robinsonian nonstandard analysis: The content of this book is an analysis, frequently using external formulae, of certain internal sets which intuitively[3] correspond to objects of conventional stochastic analysis. In many instances, the results of Robinsonian nonstandard analysis applied to probability theory in general and to stochastic analysis in particular (cf. e.g. Loeb [51], Anderson [4], Lindstrøm [45–48], Keisler [41], Hoover and Perkins [37, 38], Stroyan and Bayod [74], Capiński and Cutland [21–23] as well as Albeverio et al. [3] or Osswald and Y. Sun [65] and the references therein) imply that the corresponding conventional ("standard") objects of stochastic analysis can be viewed as the *standard part* of our (internal) objects in a deep, well-defined, rigorous and topologically meaningful sense: Our external notions usually correspond to the so-called S-notions of Robinsonian nonstandard analysis; for example, our definition of continuity for trajectories is known as S-continuity in the Robinsonian framework, our notion of integrability is known as S-integrability, etc.

When the present book is viewed in this light, one finds that (1) the event-wise standard part (in the topology of the real line) of any of our probability measures is—by a celebrated theorem of Loeb's [51]—always a probability measure in the conventional sense, (2) the standard part of a Wiener walk (with respect to a natural path-space topology) is—by virtue of Anderson's [4] results—a Wiener process in the sense of conventional probability theory, (3) the right standard part of our Lévy

[2] The usual method to achieve this is to define the field of hyperreals as the ultrapower of the reals with respect to a non-principal ultrafilter, and then to use some kind of $\in$-recursion in order to embed the superstructure over the reals into the superstructure over the hyperreals. The result is also known as a *bounded ultrapower* construction, cf. e.g. Albeverio et al. [3, Sect. 1.2].

[3] And, as Nelson [60, Appendix] has shown for the objects of his radically elementary probability theory, even in a formal, rigorous sense.

processes (again with respect to a natural path-space topology) is—as we know through Lindstrøm's work [49]—a Lévy process as the term is used in conventional probability theory.

A systematic, historically as well as philosophically informed comparison of Robinsonian nonstandard analysis and (subsystems of) Internal Set Theory would be beyond the scope of this book and can be found in other works such as the monographs by Kusraev and Kutateladze [42] and, in particular, Vakil [75]. Any graduate student with an interest in mathematical logic (in particular, model theory) as well as in stochastic analysis should feel encouraged to study Robinsonian nonstandard probability theory and its very interesting applications by the authors cited above, their co-authors, and many others. Hopefully the brief explanations in this section will make the transition from radically elementary stochastic analysis to stochastic nonstandard analysis in the Robinson–Loeb–Anderson setting—and to standard stochastic analysis—a little bit easier. (The mere possibility of such a transition on the basis of radically elementary stochastic analysis also is an advantage over a rival infinitesimal approach to the theory of continuous-time stochastic processes due to Benci et al. [9].)

In any case, the present book shows how to formulate an accessible, yet rigorous introduction to stochastic calculus with infinitesimals that does not require acquaintance with model theory, measure theory or functional analysis.

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