Appendix A
The (Affine) B.G. Property for Simplicial Sheaves

A.1 Some Recollections on the B.G. Property

We make free use of some notions and results from [15, 59].

Definition A.1. Let $\mathcal{B}$ be a presheaf of simplicial sets on Sm_k.

1) [15] We say that it satisfies the *B.G.-property in the Zariski topology* if for each $X \in Sm_k$ and each open covering of X by two open subschemes U and V the following diagram of simplicial sets is homotopy cartesian:

$$\begin{array}{ccc} \mathcal{B}(X) & \to & \mathcal{B}(V) \\ \downarrow & & \downarrow \\ \mathcal{B}(U) & \to & \mathcal{B}(U \cap V) \end{array}$$

2) We say that it satisfies the $\mathbb{A}^1$-*B.G. property in the Zariski topology* if $\mathcal{B}$ satisfies the B.G. property in the Zariski topology and if moreover, for any $X \in Sm_k$ the map

$$\mathcal{B}(X) \to \mathcal{B}(X \times \mathbb{A}^1)$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is a weak equivalence.

The notion in (1) was introduced by Brown and Gersten in [15]. One of their main Theorem is:

Theorem A.2. *[15] A morphism $\mathcal{X} \to \mathcal{Y}$ of presheaves of simplicial sets on a Zariski site (for instance on Sm_k) which satisfy both the B.G. property in the Zariski topology and which is a local simplicial weak equivalence in the Zariski topology*[1] *induces, for any $U \in Sm_k$ a weak equivalence $\mathcal{X}(U) \to \mathcal{Y}(U)$ of simplicial sets.*

[1]That is, induces a weak equivalence on each Zariski stalk.

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As an application they endowed the category of sheaves of simplicial sheaves on some space with a model category structure (which is called the B.G. structure). One can do the same for the category $\Delta^{op}Shv(Sm_k)_{Zar}$ of simplicial sheaves of sets on Sm_k in the Zariski topology. We will denote by R_{Zar} a chosen fibrant resolution functor. As a consequence of their result, one obtains:

Lemma A.3. *Let $\mathcal{B}$ be a simplicial presheaf of sets on Sm_k which satisfies the B.G. property in the Zariski topology. Then the canonical morphism of simplicial presheaves of sets $\mathcal{B} \to R_{Zar}(a_{Zar}(\mathcal{B}))$ induces for any $X \in Sm_k$ a weak equivalence of simplicial sets*

$$\mathcal{B}(X) \to R_{Zar}(a_{Zar}(\mathcal{B}))(X)$$

Here a_{Zar} denotes the sheafification functor in the Zariski topology.

Indeed both terms satisfy the B.G. property in the Zariski topology and the morphism clearly induces a weak equivalence on Zariski stalks.

Remark A.4. This result allows one to compute maps in the associated simplicial homotopy category $\mathcal{H}_s(Sm_k)_{Zar}$ from any $X \in Sm_k$ to $a_{Zar}(\mathcal{B})$: indeed $\pi_0(R_{Zar}(a_{Zar}(\mathcal{B}))(X)$ is by the very definition of the model category structure equal to the set $Hom_{\mathcal{H}_s((Sm_k)_{Zar})}(X, a_{Zar}(\mathcal{B}))$.

Definition A.5. Let $\mathcal{B}$ be a presheaf of simplicial sets on Sm_k.

1) [59] We say that $\mathcal{B}$ satisfies the *B.G.-property in the Nisnevich topology* if and only if for any distinguished square[2] in Sm_k of the form

$$\begin{array}{ccc} W & \subset & V \\ \downarrow & & \downarrow \\ U & \subset & X \end{array}$$

the diagram of simplicial sets

$$\begin{array}{ccc} \mathcal{B}(X) & \to & \mathcal{B}(V) \\ \downarrow & & \downarrow \\ \mathcal{B}(U) & \to & \mathcal{B}(W) \end{array}$$

is homotopy cartesian.
2) We say it satisfies the $\mathbb{A}^1$-*B.G. property in the Nisnevich topology* if it satisfies the B.G. property in the Nisnevich topology and if moreover, for any $X \in Sm_k$ the map

[2] In the sense of [59, Definition 1.3 p.96].

$$\mathcal{B}(X) \to \mathcal{B}(X \times \mathbb{A}^1)$$

induced by the projection $X \times \mathbb{A}^1 \to X$ is a weak equivalence.

One of the technical result in [59] involving the B.G. property in the Nisnevich topology is quite analogous to the original result of Brown and Gersten: a morphism $\mathcal{X} \to \mathcal{Y}$ of presheaves of simplicial sets on Sm_k which satisfy both the B.G. property in the Nisnevich topology and which is a local simplicial weak equivalence in the Nisnevich topology[3] induces, for any $U \in Sm_k$ a weak equivalence $\mathcal{X}(U) \to \mathcal{Y}(U)$ of simplicial sets.

Remark A.6. Let $\mathcal{X}$ be a simplicial presheaf which satisfies the $\mathbb{A}^1$-B.G. property in the Nisnevich topology. Denote by $a_{Nis}(\mathcal{X})$ its sheafification in the Nisnevich topology. Then it is $\mathbb{A}^1$-local in the sense of [59, Definition 2.1 p. 106]. Indeed take a simplicially fibrant resolution $a_{Nis}(\mathcal{X}) \to \mathcal{Y}$; by the previous result just quoted (see also [59, Proposition 1.16 p. 100]) then for any $U \in Sm_k$ the morphism $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a simplicial weak equivalence. As a consequence for any $U \in Sm_k$ the morphism

$$\mathcal{Y}(U) \to \mathcal{Y}(U \times \mathbb{A}^1) = \mathcal{Y}^{\mathbb{A}^1}(U)$$

is a weak equivalence. This implies at once that the morphism $\mathcal{Y} \to \mathcal{Y}^{\mathbb{A}^1}$ is a simplicial weak equivalence. Thus $\mathcal{Y}$ is $\mathbb{A}^1$-local and so is $\mathcal{X}$.

From a technical point of view we will be interested in slightly weaker conditions, only involving affine smooth k-schemes.

Definition A.7. Let $\mathcal{B}$ be a presheaf of simplicial sets on Sm_k.

1) We say that $\mathcal{B}$ satisfies the *affine B.G. property in the Zariski topology* if for any smooth k-algebra A and any coprime elements f and g of A the diagram
$$\begin{array}{ccc} \mathcal{B}(Spec(A)) & \to & \mathcal{B}(Spec(A_f)) \\ \downarrow & & \downarrow \\ \mathcal{B}(Spec(A_g)) & \to & \mathcal{B}(Spec(A_{f.g})) \end{array}$$
is homotopy cartesian.
2) We say that $\mathcal{B}$ satisfies the *affine B.G. property in the Nisnevich topology* if for any smooth k-algebra A, any étale A-algebra $A \to B$ and $f \in A$ such that $A/f \to B/f$ is an isomorphism, the diagram
$$\begin{array}{ccc} \mathcal{B}(Spec(A)) & \to & \mathcal{B}(Spec(A_f)) \\ \downarrow & & \downarrow \\ \mathcal{B}(Spec(B)) & \to & \mathcal{B}(Spec(B_f)) \end{array}$$
is homotopy cartesian.

[3] That is, induces a weak equivalence on each Nisnevich stalk.

3) We say that $\mathcal{B}$ satisfies the *affine* $\mathbb{A}^1$*-invariance property* if for any smooth k-algebra A the morphism

$$\mathcal{B}(Spec(A)) \to \mathcal{B}(Spec(A) \times \mathbb{A}^1)$$

induced by the projection $Spec(A) \times \mathbb{A}^1 \to Spec(A)$, is a weak equivalence.

Observe that the affine B.G. property in the Nisnevich topology implies the affine B.G. property in the Zariski topology.

A.2 The Affine Replacement of a Simplicial Presheaf

For X a smooth k-scheme, we denote by Sm_k^{af}/X the category of smooth affine k-schemes over X that is to say the category whose objects are morphism of smooth k-schemes $Y \to X$ with Y affine, with the obvious notion of morphisms.

Let $\mathcal{B}$ be a presheaf of simplicial sets on Sm_k. For any $X \in Sm_k$ we denote by $\mathcal{B}^{af}$ the presheaf of simplicial sets on Sm_k defined for $X \in Sm_k$ by the formula:

$$\mathcal{B}^{af}(X) := holim_{(Y\to X)\in Sm_k^{af}/X} Ex(\mathcal{B}(Y))$$

where Ex denotes a fixed choice of a functorial fibrant resolution in the category of simplicial sets (see [59] for instance). We call it the affinisation of $\mathcal{B}$.

We observe that by definition [13], this is indeed a presheaf of simplicial sets on Sm_k and moreover that there is a morphism of presheaf of simplicial sets on Sm_k:

$$\mathcal{B} \to \mathcal{B}^{af}$$

Lemma A.8. *The previous morphism induces for each affine smooth k-scheme X a weak equivalence*

$$\mathcal{B}(X) \to \mathcal{B}^{af}(X)$$

Proof. Because X is affine, the category Sm_k^{af}/X admits a final object and is thus contractible. By [13], the morphism is thus a weak equivalence. □

In particular this morphism of simplicial presheaves is a local weak equivalence (i.e induces a weak equivalence on each stalk in the Zariski topology—or any topology with affine local point).

Theorem A.9. *Let $\mathcal{B}$ be a presheaf of simplicial sets on Sm_k. Assume that $\mathcal{B}$ satisfies the affine B.G. property and the affine $\mathbb{A}^1$-invariance property.*

Then the affine replacement $\mathcal{B}^{af}$ satisfies the $\mathbb{A}^1$-B.G. property in the Zariski topology.

Proof. The proof follows an idea of Weibel [84] in the same way as the proof of [49, Théorème 3.1.6 p. 37]. The details are left to the reader. □

An immediate consequence is the following affine version of the result of Brown–Gersten (Lemma A.3) which thus says that a presheaf of simplicial sets on Sm_k which satisfies the affine B.G. property in the Zariski topology and the affine $\mathbb{A}^1$-invariance property computes the "right thing" for affine smooth k-schemes. Observe however that we use the $\mathbb{A}^1$-invariance property which is not used in the classical case.

Lemma A.10. *Let $\mathcal{B}$ be a simplicial presheaf of sets on Sm_k which satisfies the affine B.G. property in the Zariski topology and the affine $\mathbb{A}^1$-invariance. Then the canonical morphism of simplicial presheaves of sets $\mathcal{B} \to R_{Zar}(a_{Zar}(\mathcal{B}))$ induces for any affine $X \in Sm_k$ a weak equivalence of simplicial sets*

$$\mathcal{B}(X) \to R_{Zar}(a_{Zar}(\mathcal{B}))(X)$$

Proof. We use the following commutative square of simplicial presheaves of sets

$$\begin{array}{ccc} \mathcal{B} & \to & R_{Zar}(a_{Zar}(\mathcal{B})) \\ \downarrow & & \downarrow \\ \mathcal{B}^{af} & \to & R_{Zar}(a_{Zar}(\mathcal{B}^{af}) \end{array}$$

The left vertical morphism induces a weak equivalence on sections on affine smooth k-schemes by Lemma A.8. The right vertical morphism induces a weak equivalence on sections on any smooth k-schemes because $a_{Zar}(\mathcal{B}) \to a_{Zar}(\mathcal{B}^{af})$ is a local weak equivalence in the Zariski topology. The bottom horizontal morphism induces a weak equivalence on sections on any smooth k-schemes by Theorem A.2 because it is a local weak equivalence in the Zariski topology between presheaves with the B.G. property in the Zariski topology. For the latter one it is clear and for the first one it is Theorem A.9. This gives the result. □

A.3 The Affine B.G. Property in the Nisnevich Topology

Slightly more difficult will be the analogue in the Nisnevich topology of Theorem A.9. In fact we do not know how to prove the analogue. We will have to assume that $\mathcal{G}$ is a presheaf of simplicial groups.

Theorem A.11. *Let $\mathcal{G}$ be a simplicial presheaf of groups on Sm_k. Assume that it satisfies the affine B.G. property in the Nisnevich topology as well as the affine $\mathbb{A}^1$-invariance property.*

Then the affine replacement $\mathcal{G}^{af}$ satisfies the $\mathbb{A}^1$-B.G. property in the Nisnevich topology.

We observe the following immediate consequence which is proven in the same way as Lemma A.10:

Corollary A.12. *Let $\mathcal{G}$ be a simplicial presheaf of groups on Sm_k satisfying the assumption of the previous theorem. Then its associated sheaf in the Nisnevich topology $a_{Nis}(\mathcal{G})$ is $\mathbb{A}^1$-local and moreover for any smooth affine k-scheme U the map*

$$\mathcal{G}(U) \to R_{Nis}(a_{Nis}(\mathcal{G}))(U)$$

is a weak equivalence.

Remark A.13. Using [59, Theorem 1.66 p. 70] gives a resolution functor R_{Nis} (or R_{Zar}) which commutes to finite products. In particular it takes group object to group objects and in the statement of the corollary we may assume the morphism is a morphism of simplicial presheaves of groups.

Proof of the Theorem A.11. From Theorem A.9 we already know that the affine replacement $\mathcal{G}^{af}$ satisfies the $\mathbb{A}^1$-B.G. property in the Zariski topology. Moreover by Lemma A.8 it also still satisfies the affine Nisnevich property. Finally, $\mathcal{G}^{af}$ is a simplicial presheaf of groups. Thus it suffices to prove Theorem A.14 below. □

Theorem A.14. *Let $\mathcal{G}$ be a simplicial presheaf of groups on Sm_k. Assume that it satisfies the $\mathbb{A}^1$-B.G. property in the Zariski topology as well as the affine B.G. property in the Nisnevich topology. Then it satisfies the B.G. property in the Nisnevich topology.*

Remark A.15. The assumption that $\mathcal{G}$ satisfies the $\mathbb{A}^1$-invariance is crucial in our argument. We do not know whether the statement of the previous Theorem holds if we only assume $\mathcal{G}$ satisfies the B.G. property in the Zariski topology as well as the affine B.G. property in the Nisnevich topology.

To prove the theorem we need some preliminaries. We will prove the following crucial lemma:

Lemma A.16. *For any distinguished square of the form*

$$\begin{array}{ccc} W & \subset & V \\ \downarrow & & \downarrow \\ U & \subset & X \end{array}$$

for which the closed complement $Z := X - U$ with its reduced induced structure is k-smooth, the diagram

$$\begin{array}{ccc} \mathcal{G}(X) & \to & G(V) \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \to & \mathcal{G}(W) \end{array}$$

is homotopy cartesian.

We postpone the proof to the end of the section. We now prove how to deduce Theorem A.14 from this statement. We recall the following facts from [13]:

Lemma A.17. *1) For any commutative diagram of simplicial sets of the form*

$$\begin{array}{ccccc} \mathcal{B}_1 & \to & \mathcal{B}_2 & \to & \mathcal{B}_3 \\ \downarrow & (1) & \downarrow & (2) & \downarrow \\ \mathcal{C}_1 & \to & \mathcal{C}_2 & \to & \mathcal{C}_3 \end{array} \tag{A.1}$$

if the diagram (1) and (2) are homotopy cartesian the diagram

$$\begin{array}{ccc} \mathcal{B}_1 & \to & \mathcal{B}_3 \\ \downarrow & (3) & \downarrow \\ \mathcal{C}_1 & \to & \mathcal{C}_3 \end{array}$$

is homotopy cartesian.

2) Consider a functor $\mathcal{F} : \mathcal{I} \to \mathcal{S}quares$ from a small category $\mathcal{I}$ to the category of commutative squares of simplicial sets. Suppose that for any $i \in \mathcal{I}$, the square $F(i)$ is homotopy cartesian. Then the square of simplicial sets $holim_{\mathcal{I}}\mathcal{F}$ is still homotopy cartesian.

Proof of the Theorem A.14. We wish to prove that for any distinguished square of the form

$$\begin{array}{ccc} W & \subset & V \\ \downarrow & & \downarrow \\ U & \subset & X \end{array}$$

the diagram of simplicial groups

$$\begin{array}{ccc} \mathcal{G}(X) & \to & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \to & \mathcal{G}(W) \end{array} \tag{A.2}$$

is homotopy cartesian.

Denote by Z the complement closed subset of U in X endowed with its reduced induced structure. As k is perfect there exists a flag of increasing closed subschemes

$$\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_d = Z$$

with each $Z_{i+1} - Z_i$ smooth over k. Define the corresponding decreasing flag of open subsets

$$X = U_0 \supset U_1 \supset \cdots \supset U_d = U$$

by setting $U_i := X - Z_i$. Observe that $U_{i+1} = U_i - (Z_{i+1} - Z_i)$ with $Z_{i+1} - Z_i$ closed in U_i and k-smooth. For each i we thus have an elementary distinguished square (with obvious notations)

$$\begin{array}{ccc} V_{i+1} & \subset & V_i \\ \downarrow & & \downarrow \\ U_{i+1} & \subset & U_i \end{array} \tag{A.3}$$

and we know from Lemma A.16 that the associated commutative square

$$\begin{array}{ccc} \mathcal{G}(U_i) & \to & \mathcal{G}(U_{i+1}) \\ \downarrow & & \downarrow \\ \mathcal{G}(V_i) & \to & \mathcal{G}(V_{i+1}) \end{array}$$

is homotopy cartesian. By Lemma A.17, and an easy induction, this implies that the square

$$\begin{array}{ccc} \mathcal{G}(X) & \to & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(V) & \to & \mathcal{G}(W) \end{array}$$

is homotopy cartesian. The theorem is proven. □

Proof of the Lemma A.16. We will use the notations from [59, p. 115] concerning deformation to the normal cone. The morphism denoted there by $\tilde{g}_{X,Z}$ induces a morphism of simplicial sets denoted by $\mathcal{G}(\tilde{g}_{X,Z})$:

$$\mathcal{G}(B(X,Z) - f(Z \times \mathbb{A}^1))/^h\mathcal{G}(B(X,Z)) \to \mathcal{G}(X-Z)/^h\mathcal{G}(X) \tag{A.4}$$

and the morphism denoted by $\alpha_{X,Z}$ in *loc. cit.* induces a morphism of simplicial sets denoted by $\mathcal{G}(\alpha_{X,Z})$

$$\mathcal{G}(B(X,Z) - f(Z \times \mathbb{A}^1))/^h\mathcal{G}(B(X,Z)) \to \mathcal{G}(N^*_{X,Z})/\mathcal{G}(N_{X,Z}) \tag{A.5}$$

Here $N_{X,Z}$ denotes the normal bundle of Z in Y and $N^*_{X,Z}$ the complement of the zero section.

We introduce two assumptions depending on an integer $d \geq 1$:

(A1)(d) The statement of the Lemma A.16 is true if the codimension of Z in X is $\leq d$;

(A2)(d) For any closed immersion $Z \hookrightarrow X$, of codimension $\leq d$, between smooth k-schemes the two maps of simplicial sets

$$\mathcal{G}(\tilde{g}_{X,Z}) : \mathcal{G}(B(X,Z) - f(Z \times \mathbb{A}^1))/^h\mathcal{G}(B(X,Z)) \to \mathcal{G}(X-Z)/^h\mathcal{G}(X)$$

and

$$\mathcal{G}(\alpha_{X,Z}) : \mathcal{G}(B(X,Z) - f(Z \times \mathbb{A}^1))/^h\mathcal{G}(B(X,Z)) \to \mathcal{G}(N^*_{X,Z})/\mathcal{G}(N_{X,Z})$$

are very weak equivalences.

We recall that a very weak equivalence is a map of simplicial sets that induces a weak equivalence to a sum of some of the connected components of the target. Observe that a composition of very weak equivalences is still a very weak equivalence.

We now make the following observation. Given an open subset $\Omega \subset X$, we may form an other commutative square

$$\begin{array}{ccc} W_\Omega & \subset & V_\Omega \\ \downarrow & & \downarrow \\ U_\Omega & \subset & \Omega \end{array} \tag{A.6}$$

with $U_\Omega = U \cap \Omega$, and V_Ω (resp. W_Ω) being the inverse image of Ω (resp. of U_Ω through $V \to X$. This diagram is obviously still distinguished. As a consequence, because $\mathcal{G}$ satisfies the B.G. property in the Zariski topology and by Lemma A.17, to check the fact that the Lemma for a given diagram, it suffices to check it for the diagrams of the form (A.6) where the Ω run over an open covering of X as well as the intersections between the members of the covering. Thus to prove any of our two assumptions, me may choose X as small as we want around a given point in Z.

This implies using the techniques from [59, p. 115] that for any d:

$$(\mathbf{A1})(\mathbf{d}) \Leftrightarrow (\mathbf{A2})(\mathbf{d})$$

We left the details to the reader.

We also recall from Lemma 9.22 that a commutative square .

We will prove these equivalent properties for any $\mathcal{B}$ and any $d \geq 1$ by induction on d. We observe that **(A1)(1)** holds: by the affine B.G. property in the Nisnevich topology it holds for $X = Spec(A)$ affine k-smooth and $Z = Spec(A/f)$ a closed subscheme of X smooth over k. Let us assume $d \geq 2$ and that both **(A1)(d-1)** and **(A2)(d-1)** hold for any presheaf of groups satisfying the assumptions of the Lemma. We now want to prove $(\mathbf{A1})(\mathbf{d})$.

Assume

$$\begin{array}{ccc} W & \subset & V \\ \downarrow & & \downarrow \\ U & \subset & X \end{array}$$

is a distinguished square in Sm_k, with $Z := (X - U)_{red}$ k-smooth and of codimension d. From our above localization principe we may assume that X is affine and that $Z \hookrightarrow X$ is a regular closed immersion defined by a regular sequence $(x_1, \ldots, x_d)$ of regular functions on X such that moreover each closed k-subscheme of X of the form $\{x_1 = 0, \ldots, x_i = 0\}$, for $i \in \{1, \ldots, d\}$, is still smooth over k, in particular Z itself is smooth. Let us set $Y = X/(x_1 = 0, \ldots, x_{d-1} = 0)$; Y is also smooth over k and we have a diagram of closed immersion of the form: $Z \hookrightarrow Y \hookrightarrow X$, with Y of codimension $d-1$ in X and Z of codimension 1 in Y.

Denote by $Y' \subset V$ the pull back of Y through $V \to X$. To prove that the diagram of simplicial groups of the Lemma is homotopy cartesian we know from Lemma 9.22 that it suffices to prove that

$$\mathcal{G}(U)/^h\mathcal{G}(X) \to \mathcal{G}(W)/^h\mathcal{G}(V) \tag{A.7}$$

is a very weak equivalence.

We observe that for any diagram of simplicial groups of the form

$$\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$$

the obvious diagram of homotopy quotients

$$\mathcal{G}_2/^h\mathcal{G}_1 \to \mathcal{G}_3/^h\mathcal{G}_1 \to \mathcal{G}_3/^h\mathcal{G}_2$$

is a homotopy fibration sequence between pointed simplicial sets.

Taking this into account, we get a commutative diagram in which the lines are homotopy fiber sequences of pointed simplicial sets

$$\begin{array}{ccccc}
\mathcal{G}(X-Z)/^h\mathcal{G}(X) & \to & \mathcal{G}(X-Y)/^h\mathcal{G}(X) & \to & \mathcal{G}(X-Y)/^h\mathcal{G}(X-Z) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{G}(V-Z)/^h\mathcal{G}(V) & \to & \mathcal{G}(V-Y')/^h\mathcal{G}(V) & \to & \mathcal{G}(X-Y)/^h\mathcal{G}(X-Z)
\end{array}$$

To prove that (A.7) is a very weak equivalence, it is thus sufficient to prove that the square of simplicial sets on the right

$$\begin{array}{ccc}
\mathcal{G}(X-Y)/^h\mathcal{G}(X) & \to & \mathcal{G}(X-Y)/^h\mathcal{G}(X-Z) \\
\downarrow & & \downarrow \\
\mathcal{G}(V-Y')/^h\mathcal{G}(V) & \to & \mathcal{G}(X-Y)/^h\mathcal{G}(X-Z)
\end{array} \tag{A.8}$$

is homotopy cartesian. In fact this implies slightly more, because we prove that (A.7) is a weak equivalence.

Now to prove that statement, observe that each closed immersion $Y \hookrightarrow X$, $Y' \hookrightarrow V$ and thus $Y - Z \hookrightarrow X - Z$ and $Y' - Z \hookrightarrow V - Z$ are of codimension $\leq d-1$. Moreover there is a "closed immersion" of the distinguished square

$$\begin{array}{ccc}
Y' - Z & \to & Y' \\
\downarrow & & \downarrow \\
Y - Z & \to & Y
\end{array}$$

into the distinguished square

$$\begin{array}{ccc}
V - Z & \to & V \\
\downarrow & & \downarrow \\
X - Z & \to & X
\end{array}$$

As each of the morphisms in these diagrams are étale the functoriality of the deformation to the normal bundle discussed in [59, p.117] together with our inductive assumption **(A2)(d-1)** implies that the morphisms of the type $\tilde{g}$ and α induce in this case an explicit weak equivalence[4] between the diagram (A.8) and the diagram:

$$\begin{array}{ccc} \mathcal{G}(E(N_{X,Y})^*)/^h\mathcal{G}(E(N_{X,Y})) & \to & \mathcal{G}(E(N^*_{X-Z,Y-Z}))/^h\mathcal{G}(E(N_{X-Z,Y-Z})) \\ \downarrow & & \downarrow \\ \mathcal{G}(E(N_{V,Y'})^*)/^h\mathcal{G}(E(N_{V,Y'})) & \to & \mathcal{G}(E(N^*_{V-Z,Y'-Z}))/^h\mathcal{G}(E(N_{V-Z,Y'-Z})) \end{array}$$

But of course each of the normal bundles are trivialized in a compatible way (using the fixed regular sequence) and it is not hard, using finally the $\mathbb{A}^1$-invariance of $\mathcal{G}$ and the computations from [59, p. 112], to show that the last diagram is weakly equivalent to (with "obvious notations")

$$\begin{array}{ccc} \Omega_s^{d-1}(\mathcal{G}((\mathbb{G}_m)^{\wedge d} \wedge (Y_+))) & \to & \Omega_s^{d-1}(\mathcal{G}((\mathbb{G}_m)^{\wedge d} \wedge ((Y-Z)_+))) \\ \downarrow & & \downarrow \\ \Omega_s^{d-1}(\mathcal{G}((\mathbb{G}_m)^{\wedge d} \wedge (Y'_+))) & \to & \Omega_s^{d-1}(\mathcal{G}((\mathbb{G}_m)^{\wedge d} \wedge ((Y'-Z)_+))) \end{array}$$

But the presheaf $Y \mapsto \Omega_s^{d-1}(\mathcal{G}((\mathbb{G}_m)^{\wedge d} \wedge (Y_+)))$ also satisfies the assumptions of the Lemma. And moreover the commutative square

$$\begin{array}{ccc} Y'-Z & \to & Y' \\ \downarrow & & \downarrow \\ Y-Z & \to & Y \end{array}$$

is distinguished and $Z \hookrightarrow Y$ of codimension 1. Thus by the property **(A1)(1)** this is homotopy cartesian. □

A.4 A Technical Result

We start with the following lemma:

Lemma A.18. *Let $\mathcal{X}$ be a simplicial presheaf of pointed sets on Sm_k. Assume the associated simplicial sheaf $a_{Nis}(\mathcal{X})$ in the Nisnevich topology is $\mathbb{A}^1$-local. Assume also that the associated sheaf in the Zariski topology to the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$ is trivial (Thus $\mathcal{X}$ is also 0-connected in the Nisnevich topology). Assume further that $\mathcal{X}$ satisfies the B.G. property in the Zariski topology. Then it satisfies the B.G. property in the Nisnevich topology.*

[4] This means a zig-zag—in fact only two morphisms in two different directions—of morphisms of diagrams whose morphisms are weak equivalences at each corner.

Proof. Let $\mathcal{X} \to R_{Nis}(a_{Nis}(\mathcal{X}))$ denotes a simplicially fibrant resolution of $a_{Nis}(\mathcal{X})$ in the sense of [34, 59]. Then $R_{Nis}(a_{Nis}(\mathcal{X}))$ is of course still $\mathbb{A}^1$-local and satisfies the Brown–Gersten property in the Nisnevich topology. It is thus sufficient to prove that

$$\mathcal{X} \to R_{Nis}(a_{Nis}(\mathcal{X}))$$

induces a local weak equivalence of presheaves in the Zariski topology, because as both sides satisfy the Brown–Gersten property in the Zariski topology, it will induce a termwise weak-equivalence of simplicial presheaves by the main result of [15], thus proving the result.

To prove the above morphism is a local weak equivalence in the Zariski topology it is sufficient to prove that for any localization X_x of some smooth k-variety X at some point $x \in X$ the map of simplicial sets

$$\mathcal{X}(X_x) \to R_{Nis}(a_{Nis}(\mathcal{X}))(X_x)$$

is a weak equivalence.

To do this we use Corollaries 6.2 and 6.9 which assert that under, our assumptions, for $n \geq 1$ the n-homotopy sheaf associated to the presheaf $U \mapsto \pi_n(\mathcal{X}(U))$ in the Nisnevich topology is strongly $\mathbb{A}^1$-invariant, and is also the Zariski sheaf associated to this presheaf (from the proof. As this is also true by assumption for $n = 0$ as everything is assumed to be trivial, this implies[5] that $\mathcal{X} \to R_{Nis}(a_{Nis}(\mathcal{X}))(X_x)$ induces a weak equivalence on sections over any smooth local k-scheme. $\square$

Here is our main application in this section:

Theorem A.19. *Let $\mathcal{B}$ be a pointed presheaf of Kan simplicial sets on Sm_k. Assume $\mathcal{B}$ satisfies the affine B.G. property in the Zariski topology, the affine $\mathbb{A}^1$-invariance property and that the presheaf of (simplicial) loop spaces $\Omega^1_s(\mathcal{B})$ satisfies the affine B.G. property in the Nisnevich topology. Assume further that the sheaf associated to the presheaf $\pi_0(\mathcal{B})$ is trivial in the Zariski topology (and thus also in the Nisnevich topology) and that the associated sheaf to $\pi_1(\mathcal{B}) = \pi_0(\Omega^1_s(\mathcal{B}))$ in the Zariski topology is the same as the one associated in the Nisnevich topology and is a strongly $\mathbb{A}^1$-invariant sheaf of groups.*

Then $a_{Nis}(\mathcal{B})$ is $\mathbb{A}^1$-local and for any smooth affine k-scheme U the morphism

$$\mathcal{B}(U) \to R(a_{Nis}(\mathcal{B}))(U)$$

is a weak equivalence of simplicial sets.

[5] Using comparison of the Postnikov towers in both the Zariski and the Nisnevich topology.

Remark A.20. The main example of application we have in mind is of course $\mathcal{B} = \underline{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{G}r_n)$, $n \neq 2$, which is actually a simplicial sheaf in the Nisnevich topology; see the proof of Theorem 8.1. □

Proof. By our assumptions the presheaf of simplicial sets $\mathcal{B}$ satisfy the hypotheses of Theorem A.9. Thus its affine replacement $\mathcal{B}^{af}$ satisfies the B.G. property in the Zariski topology and is also $\mathbb{A}^1$-invariant.

Now consider the presheaf of (simplicial) loop spaces $\Omega_s^1(\mathcal{B}^{af})$; we claim it is nothing but the affine replacement of the presheaf $\Omega_s^1(\mathcal{B})$. This comes from the commutation between loops space and homotopy limits [13]. Moreover using Kan's construction, we can find an equivalent[6] presheaf of simplicial groups. By our hypothesis it satisfies the assumptions of Theorem A.11 and thus satisfies the B.G. property in the Nisnevich topology and is $\mathbb{A}^1$-invariant.

We claim that the presheaf $\mathcal{B}^{af}$ satisfies the assumptions of Lemma A.18. Indeed, from what we have just said the sheaf $a_{Nis}(\mathcal{B}^{af})$ is $\emptyset$-connected and its loop space $\Omega_s^1(a_{Nis}(\mathcal{B}^{af}))$ satisfies the $\mathbb{A}^1$-B.G. property in the Nisnevich topology: it is thus $\mathbb{A}^1$-local by Remark A.6. Now by assumption, $a_{Nis}(\mathcal{B}^{af})$ is 0-connected, and its π_1 is a strongly $\mathbb{A}^1$-invariant sheaf of groups. By our results from Chap. 6 we conclude that $a_{Nis}(\mathcal{B}^{af})$ is $\mathbb{A}^1$-local.

Now by Lemma A.18, we conclude that $\mathcal{B}^{af}$ satisfies the B.G property in the Nisnevich topology and moreover that $a_{Nis}(\mathcal{B}^{af})$ is $\mathbb{A}^1$-local. As

$$a_{Nis}(\mathcal{B}) \to a_{Nis}(\mathcal{B}^{af})$$

is a weak equivalence, the Theorem is proven. □

[6]Meaning together with a zig-zag of morphisms of presheaves of pointed simplicial sets, each morphism in the zig-zag being a global weak equivalence of presheaves.

Appendix B
Recollection on Obstruction Theory

What usually refers to obstruction theory is the following situation. Given a diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ X & \to & B \end{array}$$

in some category, can we find a sequence of "obstructions" whose triviality guarantees the existence of a morphism $X \to E$ which makes the obvious triangle commutative?

The main examples come from homotopy theory, in a reasonable closed model category [66] which has an appropriate notion of "truncated t-structure", in other words, in which objects admit a "reasonable" Postnikov tower. We will not try to formalize this further, in this appendix we will only recall[1] how the theory works in the homotopy category of simplicial sheaves on a site T (with enough points) which is of finite type in the sense of [59, Definition 1.31 p. 58]. This is the case for the sites Sm_k either in the Zariski or Nisnevich topology as it follows from Theorem 1.37 p. 60 from *loc. cit.*. Let us fix such a site T of finite type.

B.1 The Postnikov Tower of a Morphism

Given a morphism of simplicial sheaves of sets of the form $f : \mathcal{E} \to \mathcal{B}$ which is also a simplicial fibration,[2] we introduce the tower of simplicial sheaves

$$\mathcal{E} \to \cdots \to P^{(n)}(f) \to \cdots \to P^{(1)}(f) \to \mathcal{B}$$

[1] Actually we didn't find any reference for this.

[2] Any morphism in the simplicial homotopy category $\mathcal{H}_s(T)$ of simplicial sheaves of sets on T, see [59, Sect. 2.1 p. 46] for a recollection, can be represented up to an isomorphism in $\mathcal{H}_s(T)$ by a simplicial fibration [34, 59].

F. Morel, $\mathbb{A}^1$-*Algebraic Topology over a Field*, Lecture Notes in Mathematics 2052, DOI 10.1007/978-3-642-29514-0,

such that $P^{(n)}(f)$ is the associated simplicial sheaf to the presheaf $U \mapsto P^{(n)}(\mathcal{E}(U) \to \mathcal{B}(U))$; the latter is the usual Postnikov section construction associated to the Kan fibration, as f is assumed to be a simplicial fibration, $\mathcal{E}(U) \to \mathcal{B}(U)$ described for instance in [44]. When $\mathcal{B} = *$ is a point this is the construction described in [59, p. 57].

This tower is always a tower of local fibrations, and evaluating at each point x of the site T gives exactly the Postnikov tower of the stalk $x^*(f)$ of the morphism f at x. As a consequence, each morphism $P^{(n+1)}(f) \to P^{(n)}(f)$ is n-connected in the following sense:

Definition B.1. Let $n \geq -1$ be an integer. A morphism of simplicial sheaves of sets $\mathcal{E} \to \mathcal{B}$ is said to be *n-connected* if and only if for any points x of the site, the map of simplicial sets $\mathcal{E}_x \to \mathcal{B}_x$ is n-connected in the classical sense.

We simply recall that a map simplicial sets $f : E \to B$ is n-connected in the classical sense means that $\pi_0(E) \to \pi_0(B)$ is onto and that given any base point $y \in E$ the morphism

$$\pi_i(E; y) \to \pi_i(B; f(y))$$

is an epimorphism for $i = n + 1$ and isomorphism for $i \leq n$. When B is 0-connected with a base point $x \in B$, this is equivalent to the homotopy fiber Γ_x of $E \to B$ at x being an n-connected space: $\pi_i(\Gamma_x)$ trivial for $i \leq n$.

We have the following easy observation:

Lemma B.2. *Let $f : \mathcal{E} \to \mathcal{B}$ be a morphism of simplicial sheaves of sets. If $\mathcal{B}$ is 0-connected and $\mathcal{E}$ pointed, then f is n-connected if and only if the (homotopy) fiber $\mathcal{F} = f^{-1}(*)$ is an n-connected simplicial sheaf of sets.*

We will use the following Lemma which is a generalization of [59, Corollary 1.41 p.61].

Lemma B.3. *Assume that $\mathcal{E} \to \mathcal{B}$ is simplicial fibration between simplicially fibrant simplicial sheaves. Assume it is n-connected. Given an object $X \in T$ of cohomological dimension $\leq d$ for some integer $d \geq 0$ then the map of simplicial sets $\mathcal{E}(X) \to \mathcal{B}(X)$ is $(n - d)$-connected.*

Proof. cf [59, p. 60 and 61]. □

Remark B.4. 1) For X of cohomological dimension $\leq (n + 1)$, the lemma implies in that $\mathcal{E}(X) \to \mathcal{B}(X)$ is (-1)-connected. This means that map is surjective on the π_0 or in other words that any $\mathcal{H}_s(T)$-morphism $X \to \mathcal{B}$ can be lifted to a morphism $X \to \mathcal{E}$. If moreover X is of cohomological dimension $\leq n$, the map

$$Hom_{\mathcal{H}_s(T)}(X, \mathcal{E}) \to Hom_{\mathcal{H}_s(T)}(X, \mathcal{B})$$

is a bijection.

2) In fact for $f : \mathcal{E} \to \mathcal{B}$, the morphisms $P^{(n+1)}(f) \to P^{(n)}(f)$ in the Postnikov tower of f are more than n-connected: the fiber has "its homotopy sheaves concentrated in dimension $n+1$".

3) The morphisms $\mathcal{E} \to P^{(n)}(f)$ are also n-connected. A consequence of the previous Lemma is the property that the obvious morphism

$$\mathcal{E} \to holim_n P^{(n)}(f)$$

is a simplicial weak equivalence. □

Corollary B.5. *Given an object $X \in T$ of cohomological dimension $\leq d$ for some integer $d \geq 0$ then the map*

$$Hom_{\mathcal{H}_s(T)}(X, \mathcal{E}) \to Hom_{\mathcal{H}_s(T)}(X, P^{(n)}(f))$$

is surjective for $d \leq n+1$ and bijective for $d \leq n$.

B.2 Twisted Eilenberg–MacLane Objects

Let G be a sheaf of groups on T. Given a sheaf of G-modules M and an integer $n \geq 2$ we define a simplicial sheaf of sets $K^G(M; n)$ in the following way. Take the model of Eilenberg–MacLane simplicial sheaf $K(M, n)$ of type (M, n) constructed in [59, p. 56] for instance. This construction being functorial, the simplicial sheaf $K(M, n)$ is endowed with a canonical action of the sheaf of groups G. Let $E(G)$ the weakly contractible simplicial sheaf of sets on which G acts "freely" so that the quotient $E(G)/G$ is the classifying object $B(G)$ (see [59, p. 128]). Then we set

$$K^G(M, n) := E(G) \times_G K(M, n)$$

This simplicial sheaf is 0-connected and pointed, its π_1 is canonically isomorphic to G and there is an obvious morphism $K^G(M, n) \to B(G)$, which is a local fibration and whose fiber is $K(M, n)$. Observe that the base point of $K(M, n)$ provides a canonical section to $K^G(M, n) \to B(G)$.

We consider a simplicial fibration $f : \mathcal{E} \to \mathcal{B}$. We assume $\mathcal{E}$ is pointed, $\mathcal{E}$ and $\mathcal{B}$ are 0-connected and that the induced morphism on the π_1 is an isomorphism (we point $\mathcal{B}$ by the image through f of the base point of $\mathcal{E}$). We simply denote by G the sheaf of groups $\pi_1(\mathcal{E}) = \pi_1(\mathcal{B})$. Observe that for $n \geq 2$, the sheaf $\pi_n(f) := \pi_n(P^{(n)}(f))$ is thus in a canonical way a G-module, because $G = \pi_1(P^{(n)}(f))$. The following is the basic technical lemma needed to the obstruction theory we will use.

Lemma B.6. *We keep the previous assumptions and notations. Then* $P^{(0)}(f)$ *is weakly equivalent to the point,* $P^{(1)}(f)$ *is weakly equivalent to* $B(G)$*, and for each* $n \geq 2$ *there exists a canonical morphism in* $\mathcal{H}_{s,\bullet}(T)/B(G)$

$$P^{(n-1)}(f) \to K^G(\pi_n(f), n+1)$$

such that the square

$$\begin{array}{ccc} P^{(n)}(f) & \to & B(G) \\ \downarrow & & \downarrow \\ P^{(n-1)}(f) & \to & K^G(\pi_n(f), n+1) \end{array} \tag{B.1}$$

is a homotopy cartesian square.

Proof. The first statement is clear. The second statement follows at once from part (1) of Lemma B.7 below. We now observe that from the axioms of closed model categories, we can always obtain a commutative square of the form

$$\begin{array}{ccc} P^{(n)}(f) & \to & E' \\ \downarrow & & \downarrow \\ P^{(n-1)}(f) & \to & B(G) \end{array}$$

where $E' \to B(G)$ is a weak equivalence and $P^{(n)}(f) \to E'$ an inclusion (a cofibration). Form the amalgamate sum of the diagram

$$\begin{array}{ccc} P^{(n)}(f) & \to & E' \\ \downarrow & & \\ P^{(n-1)}(f) & & \end{array}$$

and call it E''. Of course we obtain a commutative diagram of simplicial sheaves of sets over $B(G)$:

$$\begin{array}{ccc} P^{(n)}(f) & \to & E' \\ \downarrow & & \downarrow \\ P^{(n-1)}(f) & \to & E'' \end{array}$$

It is clear now that $\pi_1(E'') = G$, that $\pi_i(E'') = 0$ for $i \in \{2, n\}$ and that $\pi_{n+1}(E'') = \pi_n(\mathcal{B})$: these facts follow from the observation that the cone of $(P^{(n)}(f) \to P^{(n-1)}(f))$ is by definition equal to the cone of $E' \to E''$, and the fact that $E' \cong B(G)$. From part (2) of Lemma B.7 below we get a canonical pointed morphism (in $\mathcal{H}_{s,\bullet}(T)$): $E'' \to K^G(\pi_n(\mathcal{B}), n+1)$ over $B(G)$. The composition

$$P^{(n-1)}(f) \to E'' \to K^G(\pi_n(\mathcal{B}), n+1)$$

has the required property: this follows using points of the site because it is known in classical algebraic topology. □

Lemma B.7. *Let $\mathcal{X}$ be a pointed 0-connected simplicial sheaf of sets, and denote simply by G its π_1-sheaf.*

1) There exists a canonical morphism in $\mathcal{H}_{s,\bullet}(T)$

$$\mathcal{X} \to B(G)$$

which induces the identity on π_1 (and thus a weak equivalence $P^{(1)}(\mathcal{X}) \cong B(G)$). Moreover this morphism induces for any sheaf of groups H a map:

$$Hom_{\mathcal{H}_{s,\bullet}(T)}(\mathcal{X}, B(H)) \to Hom(G, H)$$

which is a bijection. Here the right hand side means the set of morphisms of sheaves of groups and the map is evaluation at the π_1.

2) Assume that the $\mathcal{H}_{s,\bullet}(T)$-morphism $\mathcal{X} \to B(G)$ has a section $s : B(G) \to \mathcal{X}$ in $\mathcal{H}_{s,\bullet}(T)$ which we fix, and that $n \geq 2$ is an integer such that $\pi_i(\mathcal{X}) = 0$ for $1 < i < n$. Then there exists a canonical morphism in $\mathcal{H}_{s,\bullet}(T)$

$$\mathcal{X} \to K^G(\pi_n(\mathcal{X}), n)$$

which induces the identity morphism on π_i for $i \leq n$ and which is compatible in the obvious sense to both the projection to $B(G)$ and the section from $B(G)$. In particular, it induces a weak equivalence $P^{(n)}(\mathcal{X}) \cong K^G(\pi_n(\mathcal{X}), n)$).

Proof. First recall that for $n \geq 1$ and for an $(n-1)$-reduced simplicial set[3] L there exists a natural map $L \to K(\pi_n(L), n)$, the base point being the canonical one. This comes from the definition of $K(M, n)$: a morphism $X \to K(M, n)$ (for M a group for $n = 1$, an abelian group for $n \geq 2$) is the same thing as a an n-cocycle of the normalized cochain complex $C^*_N(X; M)$; see [44] for instance.

We also remind that for a simplicial set L with base point ℓ_0, one denotes by $L^{(n)} \subset L$ the sub-simplicial set consisting in dimension q of the simplexes of L whose n-skeleton is constant equal to the base point ℓ_0. When L is an n-connected Kan simplicial set, the inclusion $L^{(n)} \subset L$ is a weak-equivalence (See also [44] for instance). These facts at once generalize to locally fibrant pointed simplicial sheaves in an obvious way.

1) Having that in mind, let us denote by $\mathcal{X}^{(0)} \subset \mathcal{X}$ the sub-simplicial sheaf associated to the presheaf $U \mapsto (\mathcal{X}(U))^{(0)}$. The inclusion

$$\mathcal{X}^{(0)} \subset \mathcal{X}$$

[3] $(n-1)$-reduced means with only one i-simplex for $i \leq (n-1)$.

is thus a simplicial weak equivalence. Moreover, we have for each U a canonical map of pointed simplicial sets $(\mathcal{X}(U))^{(0)} \to B(\pi_1((\mathcal{X}(U))^{(0)}))$. Sheafification of this morphism of simplicial presheaves defines a morphism of simplicial sheaves $\mathcal{X}^{(0)} \to B(\pi_1(\mathcal{X})) = B(G)$. The diagram

$$\begin{array}{l} \mathcal{X}^{(0)} \to B(G) \\ \downarrow \\ \mathcal{X} \end{array}$$

in which the vertical morphism is a simplicial weak equivalence defines a $\mathcal{H}_{s,\bullet}(T)$-morphism $\mathcal{X} \to B(G)$. The assertion on the π_1 is clear. To check it has the second property we have to check surjectivity and injectivity. Given a morphism of sheaves of groups $G \to H$, it defines by the functoriality of the construction $G \mapsto BG$, a morphism of pointed simplicial sheaves of sets $BG \to BH$. Composition with the $\mathcal{H}_{s,\bullet}(T)$-morphism $\mathcal{X} \to BG$ just constructed proves surjectivity. Let's prove injectivity. Take two morphisms α_1 and α_2: $\mathcal{X} \to B(H)$ in $\mathcal{H}_{s,\bullet}(T)$. By the pointed version of [59, Proposition 1.13 p. 52] we may represent each of these morphisms by a diagram of pointed simplicial sheaves of sets

$$\begin{array}{l} \mathcal{Y}_{\alpha_i} \to B(H) \\ \downarrow \\ \mathcal{X} \end{array}$$

where $\mathcal{Y}_i \to \mathcal{X}$ is a (pointed) hypercovering. Taking the fiber product, we may further assume that $\mathcal{Y}_{\alpha_1} = \mathcal{Y}_{\alpha_2} = \mathcal{Y}$. We may further assume that $\mathcal{X}$ is locally fibrant (or in fact fibrant if we wish). As a consequence $\mathcal{Y}$ can also be assumed locally fibrant (and pointed) because $\mathcal{Y} \to \mathcal{X}$ is a trivial local fibration. From what we already saw, $\mathcal{Y}^{(0)} \to \mathcal{Y}$ is a simplicial weak equivalence. Now the diagram

$$\begin{array}{l} \mathcal{Y}^{(0)} \rightrightarrows B(H) \\ \downarrow \\ \mathcal{X} \end{array}$$

factors through

$$\begin{array}{l} \mathcal{Y}^{(0)} \to BG \rightrightarrows BH \\ \downarrow \\ \mathcal{X} \end{array}$$

by the functoriality of the Postnikov tower and the fact that for $\mathcal{Y}$ 0-reduced $P^{(1)}(\mathcal{Y}) = B\pi_1(\mathcal{Y})$. As a morphism of simplicial sheaves of the form $BG \to BH$ is always of the form $B(\rho : G \to H)$ for $\rho = \pi_1(BG \to BH)$, we get injectivity.

For the point (2) we proceed as follows. Let us denote by G the sheaf $\pi_1(\mathcal{X})$. We may assume that the morphism $f : \mathcal{X} \to B(G)$ (given by 1) is a simplicial fibration. We factor it as $\mathcal{X} \to P^{(n)}(f) \to B(G)$, so that $P^{(n)}(f) \to B(G)$ is a local fibration. We may clearly reduce to the case $\mathcal{X} \cong P^{(n)}(f)$ and as $B(G)$ is locally fibrant, we may assume that $\mathcal{X}$ is also locally fibrant. Moreover, again by the pointed version of [59, Proposition 1.13 p. 52], we may represent the section s by an actual diagram of pointed simplicial sheaves of sets

$$\begin{array}{ll} \mathcal{Y} & \to \mathcal{X} \\ \downarrow & \\ BG & \end{array}$$

in which $\mathcal{Y} \to BG$ is a pointed trivial local fibration. Let us denote by $\tilde{\mathcal{X}}$ the fiber product $EG \times_{BG} \mathcal{X}$ (see [59, p. 128] for an explicit definition of the simplicially weakly contractible simplicial sheaf EG as well as the morphism $EG \to BG$), by $\tilde{\mathcal{Y}}$ the fiber product $EG \times_{BG} \mathcal{Y}$ and by $\tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ the induced morphism of G-simplicial sheave of sets. As $\mathcal{Y} \to BG$ is a simplicial weak equivalence, the induced map $\tilde{\mathcal{Y}} \to EG$ is a G-equivariant simplicial weak equivalence. Thus $\tilde{\mathcal{Y}}$ is weakly contractible. Denote by $\mathcal{C}$ the cone of the morphism of simplicial sheaves of sets

$$\tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$$

by which we means the amalgamate sum $\tilde{\mathcal{X}} \coprod_{\tilde{\mathcal{Y}}} C(\tilde{\mathcal{Y}})$, where $C(\tilde{\mathcal{Y}}) := (\tilde{\mathcal{Y}} \times \delta^1)/(\tilde{\mathcal{Y}} \times \{0\})$. By the very construction, the morphism of simplicial sheaves of sets

$$\tilde{\mathcal{X}} \to \mathcal{C}$$

is a G-equivariant morphism, a simplicial weak equivalence, and moreover, $\mathcal{C}$ is pointed as a G-object. Consider the resolution functor $Ex^{\mathcal{G}}$ constructed in [59, Theorem 1.66 p. 69]. As it commutes to finite products by *loc. cit.*, then $Ex^{\mathcal{G}}(\mathcal{C})$ is clearly endowed with a G-action and the simplicial weak equivalence $\mathcal{C} \to Ex^{\mathcal{G}}(\mathcal{C})$ is G-equivariant. As $\mathcal{C}$ is pointed as a G-object, so is $Ex^{\mathcal{G}}(\mathcal{C})$. As the latter is simplicially fibrant, the morphism $Ex^{\mathcal{G}}(\mathcal{C})^{(n-1)} \to Ex^{\mathcal{G}}(\mathcal{C})$ is a G-equivariant pointed weak equivalence. Observe that $\pi_n(Ex^{\mathcal{G}}(\mathcal{C})^{(n-1)}) = \pi_n(\mathcal{X})$. As $Ex^{\mathcal{G}}(\mathcal{C})^{(n-1)}$ is an $(n-1)$-reduced simplicial sheaf of sets, from what we have recall above, there exists a natural G-equivariant map

$$Ex^{\mathcal{G}}(\mathcal{C})^{(n-1)} \to K(\pi_n(\mathcal{X}), n)$$

obtained by sheafification of the classical one. Now perform the Borel construction $EG \times_G (-)$ and remember the definition $K^G(\pi_n(\mathcal{X}), n) := EG \times_G (K(\pi_n(\mathcal{X}), n))$ of twisted Eilenberg–MacLane objects, to produce a diagram of simplicial sheaves of sets

$$
\begin{array}{ccc}
 & & EG \times_G (Ex^{\mathcal{G}}(\mathcal{C})^{(n-1)} \to K^G(\pi_n(\mathcal{X}), n) \\
 & & \downarrow \\
EG \times_G \tilde{\mathcal{X}} & \to & EG \times_G Ex^{\mathcal{G}}(\mathcal{C}) \\
\downarrow & & \\
\mathcal{X} & &
\end{array}
$$

in which all the vertical morphisms are simplicial weak equivalences. This clearly defines the $\mathcal{H}_{s,\bullet}(T)$-morphism we are seeking, because by construction using points of the site it is compatible to the classical construction. □

Remark B.8. In fact with a little bit more work, one can prove that the canonical morphism in $\mathcal{H}_{s,\bullet}(T)$

$$\mathcal{X} \to K^G(\pi_n(\mathcal{X}), n)$$

given by point (2) of the previous lemma is the unique one with these properties. □

B.3 The Obstruction Theory We Need

We can now explain the obstruction theory we will use. Let us fix a diagram in the category of simplicial sheaves of sets of the form

$$
\begin{array}{ccc}
 & & \mathcal{E} \\
 & & \downarrow \\
X & \to & \mathcal{B}
\end{array}
\tag{B.2}
$$

with $X \in T$. Our aim is to give, under some assumptions, both a criterium for the existence and/or uniqueness of a morphism in $\mathcal{H}_s(T)$

$$X \to \mathcal{E}$$

which makes the triangle

$$
\begin{array}{ccc}
 & & \mathcal{E} \\
 & \nearrow & \downarrow \\
X & \to & \mathcal{B}
\end{array}
$$

commutative in $\mathcal{H}_s(T)$. We may clearly assume that $f : \mathcal{E} \to \mathcal{B}$ is a simplicial fibration.

We make the following assumptions:

1. $\mathcal{B}$ is 0-connected and pointed.
2. The morphism $\mathcal{E} \to \mathcal{B}$ is $(n-2)$-connected, or in other words (by Lemma B.2) the homotopy fiber of $\mathcal{E} \to \mathcal{B}$ is $(n-2)$-connected.

We observe that $P^{(i)}(f) \to \mathcal{B}$ is a weak equivalence for $i \leq n-2$ and that from Lemma B.6 there exists a canonical homotopy cartesian square in $\mathcal{H}_{s,\bullet}(T)$:

$$\begin{array}{ccc} P^{(n-1)}(f) \to & & B(G) \\ \downarrow & & \downarrow \\ \mathcal{B} = P^{(n-2)}(f) \to & & K^G(\pi_{n-1}(f), n) \end{array} \tag{B.3}$$

Given $X \in T$ the previous homotopy cartesian square gives a surjection

$$Hom_{\mathcal{H}_s(T)}(X, P^{(n-1)}(f)) \twoheadrightarrow Hom_{\mathcal{H}_s(T)}(X, \mathcal{B}) \times_{Hom_{\mathcal{H}_s(T)}(X, K^G(\pi_{n-1}(f), n))} Hom_{\mathcal{H}_s(T)}(X, BG)$$

Obstruction to lifting. By Corollary B.5 the map

$$Hom_{\mathcal{H}_s(T)}(X, \mathcal{E}) \to Hom_{\mathcal{H}_s(T)}(X, P^{(n-1)}(f))$$

is a surjection for all X of cohomological dimension $\leq n$.

We thus obtained the following:

Theorem B.9. *Keeping the previous assumptions on $f : \mathcal{E} \to \mathcal{B}$, for any X of cohomological dimension $\leq n$ the map*

$$Hom_{\mathcal{H}_s(T)}(X, \mathcal{E}) \to Hom_{\mathcal{H}_s(T)}(X, \mathcal{B}) \times_{Hom_{\mathcal{H}_s(T)}(X, K^G(\pi_{n-1}(f), n))} Hom_{\mathcal{H}_s(T)}(X, BG) \tag{B.4}$$

is surjective.

We deduce the following obstruction theory:

Corollary B.10. *Keeping the previous assumptions on $f : \mathcal{E} \to \mathcal{B}$, for any X of cohomological dimension $\leq n$ and any morphism $g : X \to \mathcal{B}$ in $\mathcal{H}_s(T)$, there exists a morphism $h : X \to \mathcal{E}$ which lifts g in $\mathcal{H}_s(T)$ if and only if the composition $X \to \mathcal{B} \to K^G(\pi_{n-1}(f), n))$ lifts through $BG \to K^G(\pi_{n-1}(f), n))$.*

Now for any sheaf of G-modules M, for any $\lambda \in Hom_{\mathcal{H}_s(T)}(X, BG) \cong H^1(X; G)$ let us consider the subset

$$E^n_\lambda(X; M) \subset Hom_{\mathcal{H}_s(T)}(X, K^G(M, n))$$

of elements $X \to K^G(M, n)$ whose composition to BG gives λ. This set is pointed by $X \to BG$ composed with canonical section $BG \to K^G(M, n)$.

Given $g : X \to \mathcal{B}$ in $\mathcal{H}_s(T)$ one gets by composition with $\mathcal{B} \to BG$ a morphism

$$\lambda_g : X \to BG$$

Clearly a reformulation of the corollary is to say that the element $e(g) \in E^n_{\lambda_g}(X;\pi_{n-1}(f))$ is the obstruction to the lifting of g:

$(g$ lifts through $\mathcal{E} \to \mathcal{B}) \Leftrightarrow (e(g)$ is the base point in $E^n_{\lambda_g}(X;\pi_{n-1}(f)))$

On the "Kernel" of the Map $Hom_{\mathcal{H}_s(T)}(X,\mathcal{E}) \to Hom_{\mathcal{H}_s(T)}(X,\mathcal{B})$

By "Kernel" of that pointed map, we mean the subset K^n of $Hom_{\mathcal{H}_s(T)}(X,\mathcal{E})$ consisting of morphisms whose composition with $f : \mathcal{E} \to \mathcal{B}$ is trivial (the base point of $Hom_{\mathcal{H}_s(T)}(X,\mathcal{B})$).

We want to study this kernel in the critical case. By Corollary B.5 and our assumptions, the map

$$Hom_{\mathcal{H}_s(T)}(X,\mathcal{E}) \to Hom_{\mathcal{H}_s(T)}(X,\mathcal{B})$$

is a surjection for all X of cohomological dimension $\leq n-1$ and a bijection for all X of cohomological dimension $\leq n-2$. By the critical case we mean when the cohomological dimension is $\leq n-1$ so that the map is surjective. We just want to use the pointed simplicial fibration sequence

$$\Gamma \to \mathcal{E} \to \mathcal{B}$$

where Γ is the (homotopy) fiber at the base point, and which is $(n-2)$-connected by assumption. We will use the natural action up to homotopy of the h-group $\Omega^1_s(\mathcal{B})$ on the fiber Γ. For any X we obtain a exact sequence of pointed sets and groups

$$Hom_{\mathcal{H}_s(T)}(X,\Omega^1_s(\mathcal{B})) \to Hom_{\mathcal{H}_s(T)}(X,\Gamma) \to K^n \to *$$

The left hand side is indeed a group which acts on the middle set, and exactness means that K^n is the orbit set.

For X of cohomological dimension $\leq n-1$ we can express in a simpler way the middle term; by Corollary B.5 the maps

$$\begin{aligned} Hom_{\mathcal{H}_s(T)}(X,\Gamma) &\to Hom_{\mathcal{H}_s(T)}(X,P^{(n-1)}(\Gamma)) \\ &\to Hom_{\mathcal{H}_s(T)}(X,K(\pi_{n-1}(f),n-1)) \end{aligned}$$

are all bijective (the latter one uses point 2) of Lemma B.7). But has it is well known, the right hand side is an abelian group isomorphic to $H^{n-1}(X;\pi_{n-1}(f))$. At the end we get an exact sequence as above of the form

$$Hom_{\mathcal{H}_s(T)}(X,\Omega^1_s(\mathcal{B})) \to H^{n-1}(X;\pi_{n-1}(f)) \to K^n \to *$$

Remark B.11. Beware that in general the action on the left is not given by a homomorphism: take the universal situation, in the category of simplicial sets, that is to say the fibration

$$K(M,n) \to BG \to K^G(M,n)$$

In that case the loop space in question is $\Omega^1_s(K^G(M,n))$ which is easily seen to be equivalent to the semi-direct product $G \ltimes K(M,n-1)$ and the action of $\Omega^1_s(K^G(M,n)) = G \ltimes K(M,n-1)$ on $K(M,n-1)$ is the standard action of a semi-direct product $G \ltimes N$ onto a G-module N. This action can't induce in general a group homomorphism. □

Cohomological Interpretation of the Obstruction Sets $E^n_\lambda(X;M)$

The pointed sets of the form $E^n_\lambda(X;M)$ have a natural cohomological interpretation, and in particular are abelian groups, for each fixed λ. We end this appendix by explaining this fact.

First by [59, Proposition 1.15 p. 130], there is a canonical bijection

$$Hom_{\mathcal{H}_s(T)}(X, BG) \cong H^1(X;G)$$

identifying morphism $X \to BG$ and isomorphism classes of G-torsors over X. Thus our λ corresponds to an isomorphism class of G-torsors over X. Pick up one G-torsor $Y \to X$ in the class of λ. Consider the sheaf of sets M_λ on T obtained as

$$M_\lambda := {}_G\backslash(Y \times M)$$

The quotient being computed in the category of sheaves of sets on T. The obvious morphism $M_\lambda \to {}_G\backslash Y = X$ defines it as a sheaf of sets on X. It is called the sheaf obtained by twisting M by λ. Our aim is to prove that this sheaf is in a canonical way an abelian sheaf on X and that for each $n \geq 2$ the pointed set $E^n_\lambda(X;M)$ is canonically in bijection with the n-th cohomology group

$$H^n_{T/X}(X; M_\lambda)$$

Remark B.12. 1) Given λ and M, all these constructions only depend on the choice of a representative $Y \to X$ of λ.

2) If λ is the trivial G-torsor θ, the result is quite clear and in fact $E^n_\theta(X;M)$ is canonically in bijection with $H^n_{T/X}(X;M_\theta) = H^n_T(X;M)$.

The following two lemmas are easy to prove.

Lemma B.13. *Given a sheaf of groups G on T, a G-torsor Y over the final sheaf, and a G-sheaf of sets M the canonical morphism $Y \times M \to Y \times (Y \times_G M)$ is an isomorphism of sheaves on T.*

Lemma B.14. *Given a sheaf of groups G on T, a G-torsor Y over the point and two G-sheaves of sets M and N, the canonical morphism of sheaves on T: $Y \times M \times N \to (Y \times_G M) \times (Y \times_G N)$ induces an isomorphism*

$$Y \times_G (M \times N) \to (Y \times_G M) \times (Y \times_G N)$$

As a consequence, if M is a sheaf of G-module, the sheaf $Y \times_G M$ has a canonical structure of sheaf of abelian groups.

Lemma B.15. *Keeping the obvious notations, the X-sheaf M_λ admits a canonical structure of abelian X-sheaf. Let us denote by $K(M_\lambda, n)$ the usual simplicial Eilenberg–MacLane object in the category of sheaves over X. Then there exists a canonical isomorphism of simplicial sheaves over X of the form*

$$Y \times_G K(M, n) \cong K(M_\lambda, n)$$

Proof. We only use the explicit definition of $K(-, n)$ [44] which show that in each degree $K(M_\lambda, n)$ is a product (over X) of copies of M_λ. The conclusion follows from the two previous lemmas. □

For any $X \in T$, for any integer ≥ 0, for any sheaf of G-modules M and for any $\lambda \in Hom_{\mathcal{H}_s(T)}(X, BG) = H^1(X; G)$ we now describe a natural map of pointed sets

$$H^n(X; M_\lambda) \to E^n_\lambda(X; M) \tag{B.5}$$

We will use Verdier's formula to compute the left hand side; see [14] or also [59, Proposition 1.13 p. 52 and Proposition 1.26 p. 57], from which we freely use the notation and results.

Let $\mathcal{U} \to X$ be a hypercovering (a local trivial fibration) and $\mathcal{U} \to K(M_\lambda, n)$ a morphism of simplicial sheaves over X which represent a class $\alpha \in H^n(X; M_\lambda)$. From [59, Lemma 1.12 p.128], we may assume (up to taking a refinement of $\mathcal{U}$) that there exists a morphism of simplicial sheaves $\mathcal{U} \to BG$ such that the Pull-back of the G-torsor $EG \to BG$ is isomorphic to the G-torsor $Y \times_X \mathcal{U} \to \mathcal{U}$.

Now from Lemmas B.13 and B.14, there exists a canonical isomorphism of simplicial sheaves (over X)

$$Y \times_X K(M_\lambda, n) = Y \times K(M, n)$$

Beware there is no X in index on the right: this comes from the fact that we apply the Lemmas to the X-sheaf of abelian groups $M|xX \to X$. Our isomorphism then follows from the tautological one $Y \times K(M, n) = Y \times_X K(M \times X \to X, n)$. The previous isomorphism is moreover G-equivariant.

We thus obtain a G-equivariant morphism

$$Y \times_X \mathcal{U} \to Y \times_X K(M_\lambda, n) = Y \times K(M, n)$$

By assumption on $\mathcal{U}$ there exists a cartesian square

$$\begin{array}{ccc} Y \times \mathcal{U} & \to & EG \\ \downarrow & & \downarrow \\ \mathcal{U} & \to & BG \end{array}$$

the top horizontal one being G-equivariant. We now consider the G equivariant morphism

$$Y \times \mathcal{U} \to EG \times K(M, n)$$

and pass to the quotient by G to get a morphism os simplicial sheaves:

$$\mathcal{U} \to K^G(M, n)$$

As the composition $\mathcal{U} \to K^G(M, n) \to BG$ represents λ its associated class (by Verdier's formula [59, Proposition 1.13 p. 52]) in $Hom_{\mathcal{H}_s(T)}(X, K^G(M, n))$ actually lies in $E^n_\lambda(X; M)$. It is not hard to see that this class only depends on the class of α so that we have indeed constructed the expected map (B.5).

We are now ready to prove the following result:

Theorem B.16. *The map just constructed:*

$$H^n(X; M_\lambda) \to E^n_\lambda(X; M)$$

is a bijection.

Proof. We will indicate a way to construct a map the other way and will let it to the reader to check both maps are inverse to each other.

Take an element $\beta \in E^n_\lambda(X; M)$. That is to say a morphism in $\mathcal{H}_s(T)$: $X \to K^G(M, n)$ inducing λ.

Because $K^G(M, n)$ is locally fibrant, by the Verdier formula already used, we may represent β by an actual morphism

$$\mathcal{U} \to K^G(M, n)$$

where $\mathcal{U} \to X$ is an hypercovering. Denote by $\tilde{\mathcal{U}}$ the fiber product $EG \times_{BG} \mathcal{U}$ through $\mathcal{U} \to K^G(M, n) \to BG$. because the square

$$\begin{array}{ccc} EG \times K(M, n) & \to & EG \\ \downarrow & & \downarrow \\ K^G(M, n) & \to & BG \end{array}$$

is cartesian, there is a canonical G-equivariant morphism

$$\tilde{\mathcal{U}} \to EG \times K(M,n) \to K(M,n)$$

By the assumption the G-torsor $\tilde{\mathcal{U}} \to \mathcal{U}$ is also the pull-back of $Y \to X$ because β induces λ (use also [59, Proposition 1.15 p. 130]). This means that $\tilde{\mathcal{U}}$ is isomorphic to $Y \times_X \mathcal{U}$. We thus get a G equivariant morphism $\tilde{\mathcal{U}} \to Y$. We now claim that the G equivariant morphism product of the two previous morphisms

$$\tilde{\mathcal{U}} \to Y \times K(M,n)$$

induces after passing to quotient by G a morphism over X

$$\mathcal{U} \to K(M_\lambda, n)$$

We claim that the class of this map in $H^n(X; M_\lambda)$ only depends on β and that the map

$$E^n_\lambda(X; M) \to H^n(X; M_\lambda)$$

is the inverse to the map of the Theorem. $\square$

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