Appendix A
Deconvolution Under the No-Slip Condition and the Loss of Regularity

Most of the estimates for filtering and deconvolution errors have been based on periodic boundary conditions. In this appendix we survey what can be proven for Dirichlet boundary conditions. Let Ω be a bounded, regular domain in $\mathbb{R}^d, d = 1, 2, 3$ with C^{k+2} boundary and $0 < \delta \leq 1$ a small parameter. Consider the elliptic-elliptic singular perturbation problem

$$-\delta^2 \triangle u + u = f, \text{ in } \Omega, \tag{A.1}$$

$$u = 0, \text{ on } \partial\Omega. \tag{A.2}$$

Let $H^k = H^k(\Omega)$ denote the Sobolev space of all functions with derivatives of order $\leq k$ in $L^2(\Omega) = H^0$ with associated norm $||\cdot||_k$ and semi-norm $|\cdot|_k$. If $k = 0$ we drop the subscript 0 in the norm and write simply $||\cdot||$. The Sobolev space $H_0^1(\Omega)$ is $H_0^1 := \{v \in H^1 : v = 0 \text{ on } \partial\Omega\}$. For (A.1), we assume

$$f \in H^k(\Omega) \bigcap H_0^1(\Omega). \tag{A.3}$$

In particular, we stress that *this implies the important condition*

$$f = 0 \text{ on } \partial\Omega. \tag{A.4}$$

This condition precludes simple boundary layers in u but does not imply higher derivatives of u are free of layers. From (A.1), it also implies that $\triangle u = 0$ on $\partial\Omega$.

The shift theorem, e.g., [GT], implies that the solution of (A.1)-(A.2) satisfies

$$u \in H^{k+2}(\Omega) \bigcap H_0^1(\Omega), \text{ and } ||u||_{k+2} \leq C(\delta)||f||_k,$$

where $C(\delta) \to \infty$ as $\delta \to 0$. Herein we investigate the question of *uniform in* δ regularity theorem: whether there is a $C = C(k, \Omega)$, independent of δ, such that the solution of (A.1)-(A.2) under (A.3) satisfies

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$$||u||_k \leq C||f||_k. \tag{A.5}$$

This simple question has turned out to be more delicate than it appeared at first. We prove the following in Sect. A.1.

Theorem 66. *Suppose* $f \in H^2(\Omega) \bigcap H_0^1(\Omega)$. *Then*

$$||u||_l \leq C||f||_l, \text{ for } l = 0, 1, 2. \tag{A.6}$$

If $f \in H^4(\Omega) \bigcap H_0^1(\Omega), \triangle f \in H_0^1(\Omega)$. *Then*

$$||u||_l \leq C||f||_l, \text{ for } l = 0, 1, 2, 3, 4. \tag{A.7}$$

In general, suppose $f \in H^{2k}(\Omega) \bigcap H_0^1(\Omega), \triangle^j f \in H_0^1(\Omega), j = 1, \cdots, k-1$. *Then for* $l = 1, \cdots, 2k$

$$||u||_l \leq C||f||_l. \tag{A.8}$$

The assumption that powers of $\triangle f$ vanish on the boundary can be weakened in appearance to read that normal derivatives of the same order vanish on the boundary, Sect. A.2. However, examples in Sect. A.3 show that an extra condition is necessary. Without it we have the following (Sect. A.2).

Corollary 67. *Suppose* $f \in H^3(\Omega) \bigcap H_0^1(\Omega)$. *Then*

$$||u||_3 \leq C(||f||_3 + \delta^{-1}||f||_2)$$

It seems likely that all the results herein are in the published literature, explicitly or implicitly, somewhere (e.g., Lions [L73]). Simple examples in $1d$ show that Theorem 66 cannot be true if (A.4) does not hold and that local versions of Theorem 66 cannot hold as well.

A.1 Regularity by Direct Estimation of Derivatives

We consider first the cases where results may be proven by a direct argument. The estimates in Lemma 68 likely appear in every paper on regularity of (A.1)-(A.2). The estimates in Lemma 69 appear in Tartar [T93]. All constants are uniform in δ.

Lemma 68. *Under* (A.3) *we have*

$$\delta^2||\triangle u|| + \delta||\nabla u|| + ||u|| \leq C||f||, \tag{A.9}$$

$$\delta||\triangle u|| + ||\nabla u|| \leq C||\nabla f||. \tag{A.10}$$

Proof. Multiply the equation (A.1) by u and integrate over the domain Ω. This gives

$$\delta^2||\nabla u||^2 + ||u||^2 = (f, u) \leq \frac{1}{2}||f||^2 + \frac{1}{2}||u||^2,$$

and the first estimate follows. For the second estimate, the equation (A.1) implies -$\delta^2 \triangle u = f - u$ and thus

$$\delta^2||\triangle u|| \leq ||f|| + ||u|| \leq C||f||.$$

The gradient bound in (A.9) is improvable. Indeed, if $f \in H_0^1(\Omega)$ then

$$u \in H^3(\Omega) \bigcap H_0^1(\Omega).$$

Multiply by $-\triangle u$ and integrate. This gives

$$\delta^2||\triangle u||^2 + ||\nabla u||^2 = (\nabla f, \nabla u) \leq \frac{1}{2}||\nabla f||^2 + \frac{1}{2}||\nabla u||^2,$$

and the second estimate (A.10) follows. □

Next we use a reformulation of (A.1) in Tartar [T93]. This reformulation is critical for the next step and the regularity question can be restated for the reformulation. Since $f \in H_0^1(\Omega)$,

$$(u - f) \in H_0^1(\Omega).$$

Subtraction gives the equation

$$-\delta^2\triangle(u - f) + (u - f) = \delta^2\triangle f, \text{ in } \Omega, \tag{A.11}$$

$$(u - f) = 0, \text{ on } \partial\Omega. \tag{A.12}$$

Lemma 69. *There is a C independent of δ such that*

$$||u||_k \leq C||f||_k$$

if and only if

$$||u - f||_k \leq C||f||_k.$$

Proof. This follows from the triangle inequality:

$$||u||_k \leq ||u - f||_k + ||f||_k,$$

$$||u - f||_k \leq ||f||_k + ||u||_k.$$

□

Consider therefore problem (A.11)-(A.12).

Lemma 70. *Under (1.2) we have*

$$\begin{aligned} ||\nabla(u-f)|| &\leq C||\nabla f|| \\ ||u-f|| &\leq C\delta||\nabla f||, \\ ||\nabla(u-f)|| &\leq C\delta||\triangle f||, \\ ||u-f|| &\leq C\delta^2||\triangle f||, \\ ||\triangle(u-f)|| &\leq C||\triangle f||. \end{aligned}$$

Proof. Multiply (A.11) by $u-f$ and integrate. This gives, after the usual manipulations,

$$\delta^2||\nabla(u-f)||^2 + ||u-f||^2 = (\delta^2\triangle f, u) \leq \delta^2||\nabla(u-f)||||\nabla f||.$$

This proves the first two estimates. On the above RHS we can also write $(\delta^2\triangle f, u) \leq \frac{\delta^4}{2}||\triangle f||^2 + \frac{1}{2}||u-f||^2$, which proves the third and fourth estimates. Equation (A.11) now implies

$$\delta^2||\triangle(u-f)|| \leq \delta^2||\triangle f|| + ||u-f|| \leq C\delta^2||\triangle f||.$$

Thus,

$$||\triangle(u-f)|| \leq C||\triangle f||.$$

□

Since $\partial\Omega$ is smooth, $||\triangle u||$ and $|u|_2$ are equivalent. Thus we have the following.

Corollary 71. *For* $k = 0, 1, 2$ *we have*

$$||u||_k \leq C||f||_k.$$

These simple estimates can be continued and give precise information. If

$$f \in H_0^1(\Omega), \text{ then } u \in H^3(\Omega) \bigcap H_0^1(\Omega)$$

and from the equation $-\delta^2\triangle u = f - u \in H_0^1(\Omega)$, so

$$\triangle u \in H_0^1(\Omega).$$

Thus $\triangle u$ satisfies

$$\begin{aligned} -\delta^2\triangle\triangle u + \triangle u &= \triangle f \ \in H_0^1(\Omega), \text{ in } \Omega, \\ \triangle u &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{A.13}$$

Applying the above estimates to $\triangle u$ gives for $k = 0, 1, 2, 3, 4$

$$||u||_k \leq C||f||_k. \tag{A.14}$$

Clearly this can be repeated. Repeating this argument proves Theorem 66.

Theorem 72. *Suppose* $f \in H^{2k}(\Omega) \bigcap H_0^1(\Omega), \triangle^j f \in H_0^1(\Omega), j = 1, \ldots, k-1$. *Then for* $l = 1, \cdots, 2k$

$$||u||_l \leq C||f||_l. \tag{A.15}$$

A.2 The Bootstrap Argument

The usual path to regularity is via a bootstrap argument. The section considers how the classical bootstrap argument applies to the regularity issue. We give a different proof of the basic regularity result of the last section. We return to the case:

$$f \in H^k(\Omega) \bigcap H_0^1(\Omega). \tag{A.16}$$

A.2.1 The Case $k = 3$

The usual procedure, [GT] is first to use a partition of unity. Then, change variables to locally flatten the boundary. The sought estimate is first proven for tangential derivatives through tangential difference quotients. Finally, the last derivative in the normal direction is bounded by tangential derivatives through the equation. This section uses an alternate but related strategy (which was suggested to the author by L. Tartar and used in [T93]) that simplifies the argument considerably. The key is the following observation.

A.2.2 Observation

Let L be a smooth, first order differential operator which acts tangentially to $\partial\Omega$. Then, for any

$$v \in H^2(\Omega) \bigcap H_0^1(\Omega)$$

we have

$$Lv \in H_0^1(\Omega), \text{ and}$$
$$\triangle Lv = L\triangle v + Av,$$

where A is a second order differential operator.

Proposition 73. *Suppose L is a smooth, first order differential operator which acts tangentially to $\partial\Omega$. Then,*

$$||\triangle L(u-f)|| \leq C||f||_3$$

Proof. Applying L to the equation

$$-\delta^2\triangle(u-f)+(u-f)=\delta^2\triangle f, \text{ in } \Omega,$$
$$(u-f)=0, \text{ on } \partial\Omega.$$

gives

$$-\delta^2\triangle L(u-f)+L(u-f)=\delta^2 L\triangle f+\delta^2 A(u-f), \text{ in } \Omega, \tag{A.17}$$
$$L(u-f)=0, \text{ on } \partial\Omega.$$

Apply Lemma (69) to (A.17). This gives

$$||L(u-f)|| \leq C\delta^2\{||L\triangle f||+||A(u-f)||\},$$

and since A is second order, Lemma (68) implies $||A(u-f)|| \leq C||f||_2$. Thus,

$$||L(u-f)|| \leq C\delta^2||f||_3.$$

Next, apply the idea in the proof of Lemma (68) to equation (A.17). This gives

$$\delta^2||\triangle L(u-f)|| \leq \delta^2||f||_3+||L(u-f)|| \leq C\delta^2||f||_3.$$

Thus we have

$$||\triangle L(u-f)|| \leq C||f||_3$$

for any first order differential operator acting tangentially to the boundary. □

There remains only to check the norm of the one third order differential operator acting normal to $\partial\Omega$. This is one additive term in $\nabla\triangle(u-f)$ so that the theorem will hold if $||\triangle(u-f)||_1 \leq C||f||_3$. To verify this estimate, recall that the equation

$$-\delta^2\triangle(u-f)+(u-f)=\delta^2\triangle f, \text{ in } \Omega,$$
$$(u-f)=0, \text{ on } \partial\Omega.$$

implies

$$\delta^2||\triangle(u-f)||_1 \leq \delta^2||\triangle f||_1+C||\nabla(u-f)||. \tag{A.18}$$

At this point, the best estimate of the last term in Lemma (69) is

$$||\nabla(u-f)|| \leq C\delta||\triangle f||.$$

This gives the following.

Corollary 74. *Suppose* $f \in H^3(\Omega)\bigcap H_0^1(\Omega)$. *Then*

$$||u||_3 \leq C(||f||_3 + \delta^{-1}||f||_2)$$

To eliminate the δ^{-1} it suffices that $\triangle f = 0$ on $\partial\Omega$.

Lemma 75. *Suppose* $\triangle f \in H_0^1(\Omega)$, *then*

$$||\nabla(u-f)|| \leq C\delta^2||\triangle f||_1. \tag{A.19}$$

Proof. Begin with the equation

$$\begin{aligned} -\delta^2\triangle(u-f) + (u-f) &= \delta^2\triangle f, \text{ in } \Omega, \\ (u-f) &= 0, \text{ on } \partial\Omega. \end{aligned}$$

Taking the inner product with $-\triangle(u-f)$ gives

$$\delta^2||\triangle(u-f)||^2 + ||\nabla(u-f)|| = \delta^2(\triangle f, -\triangle(u-f)) \leq \delta^2||\triangle f||_1||\triangle(u-f)||_{-1}.$$

The key step depends upon the extra regularity $\triangle f \in H_0^1(\Omega)$. With this we can use the estimate

$$(\triangle f, -\triangle(u-f)) \leq ||\triangle f||_1||\triangle(u-f)||_{-1}.$$

Now $(u-f) \in H_0^1(\Omega)$ so $\triangle(u-f) \in H^{-1}(\Omega)$ and

$$||\triangle(u-f)||_{-1} \leq C||\nabla(u-f)||.$$

Thus,

$$||\nabla(u-f)||^2 \leq C\delta^2||\triangle f||_1||\nabla(u-f)||,$$

completing the proof. □

It is clear that all that is really needed is that the second normal derivative of the RHS be well defined and vanish on the boundary.

Corollary 76. *Suppose* $f \in H^k(\Omega)\bigcap H_0^1(\Omega)$, *and* $\triangle f \in H_0^1(\Omega)$. *Then, for* $k = 0, 1, 2, 3$ *we have*

$$||u||_k \leq C||f||_k.$$

A.3 Examples

Example 77 (When $f \neq 0$ on the boundary). This is an example in which $f \in H^1(\Omega)$ from [L73], page 133. First note that the estimates derived imply that $u \to f$ as $\delta \to 0$ weakly in $L^2(\Omega)$. If the RHS does not vanish on the boundary the gradients cannot converge strongly since $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$.

The following example illustrates this. Consider the $1d$ problem

$$-\delta^2 u'' + u = e^{-x}, \text{ in } (0,\infty),$$
$$u = 0, \text{ at } x = 0.$$

The solution is

$$u(x) = \frac{1}{1-\delta^2} e^{-x} - \frac{1}{1-\delta^2} e^{-\frac{x}{\delta}}.$$

It is easy to verify that on subdomains away from $x = 0$ there is no difficulty: $u(x) \to f(x)$. The derivatives do not converge near $x = 0$ due to the layer at $x = 0$.

Example 78 (Regularity is false in general). The second example is due to P. Rabier and shows that (A.5) cannot hold for all k without extra conditions on the RHS. Consider (A.1) in one dimension, which reduces to

$$-\delta^2 u'' + u = f, \text{ in } (0,1),$$
$$u = 0, \text{ at } x = 0, 1.$$

Pick $f \in C^\infty(0,1)$ with $f''(x) \neq 0$ at all x. In the equation let $x \to 0$. This implies

$$|u''(0)| = \delta^{-2}|f(0)| = 0.$$

Differentiate twice and repeat this argument. We have $-\delta^2 u'''' + u'' = f''$, in $(0,1)$. Thus, at $x = 0$

$$|u''''(0)| = \delta^{-2}|f''(0)| \to \infty, \text{ as } \delta \to 0.$$

By the Sobolev theorem we have

$$|u''''(0)| \leq C||u||_5 \nleq C||f||_k$$

for any k (in particular $k = 5$) since the LHS blows up while the RHS is bounded.

Example 79. The following $1d$ example, due to Xinfu Chen, connects the regularity question to results in asymptotic analysis. Suppose $\Omega = (0,1)$ and $I = [a,b]$ is properly contained in $(0,1)$ so that $0 < a < b < 1$. Let $f(x) \equiv 1$ on I and $\to 0$ smoothly off I. Consider (1.1) in one dimension, which reduces to

$$-\delta^2 u'' + u = f, \text{ in } (0,1), \tag{A.20}$$

$$u = 0, \text{ at } x = 0, 1. \tag{A.21}$$

Differentiating (A.21) and setting $x \in I$, we have

$$u''' = \frac{1}{\delta^2} u',$$
$$u'''' = \delta^{-2} u'' = \delta^{-4} u'$$

and thus

$$u^{(2k)} = \delta^{-2k} u' \text{ on } I.$$

Now, if the Theorem 1.1 holds we must have the apparently impossible relation

$$\delta^{-2k} ||u'||_{L^2(I)} = ||u^{(2k)}||_{L^2(I)} \leq ||u^{(2k)}||_{L^2(0,1)} \leq C(k) ||f||_{2k}. \tag{A.22}$$

From (4.2) it appears that uniform in δ regularity is impossible. However, the solution to (1.1) (and thus (4.1)) is an approximation to $f(x)$ and thus since this particular function satisfies $f'(x) = 0$ on I we should have $u'(x)$ small there as well. Asymptotic analysis of (3.1) indicates that on I, $u'(x)$ is, in fact, exponentially close to $f'(x)$ and thus is exponentially close to zero, e.g., [B75, B79, E79]. Thus, $\sup_{0<\delta\leq 1} \delta^{-2k} ||u'||_{L^2(I)} \leq C(k)$, which is consistent with the possibility of uniform regularity. Indeed, if $f(x)$ is extended to have compact support then the regularity result does hold while it fails for other smooth extensions by Example 2.

Indeed, this last example cannot be a counterexample because the same local argument would apply to the same problem under periodic boundary conditions (where the RHS is extended periodically). In the periodic case we can verify uniform regularity by direct calculation. Indeed, we calculate

$$u(x) = \sum_{j \in \mathbb{Z}} \frac{1}{1 + \delta^2 (\frac{j}{2\pi})^2} f_j e^{ijx/2\pi},$$

where f_j are the Fourier coefficients of $f(x)$. We calculate further that in the periodic case uniform regularity holds with $C(k) = 1$ trivially since:

$$||u^{(k)}||^2 = \sum_{j \in \mathbb{Z}} \left[\frac{1}{1 + \delta^2 (\frac{j}{2\pi})^2} \right]^2 (\frac{j}{2\pi})^{2k} |f_j|^2 \leq$$
$$\leq \sum_{j \in \mathbb{Z}} (\frac{j}{2\pi})^{2k} |f_j|^2 = ||f^{(k)}||^2.$$

A.4 Application to Differential Filters

As noted above, this was motivated by an application to Germano's idea of using differential filters as a basic for large eddy simulation. Given (typically a fluid velocity) $\phi \in H^k(\Omega) \bigcap H_0^1(\Omega)$, its differential filter $\overline{\phi}$ is the unique solution of

$$\begin{aligned} A\overline{\phi} &:= -\delta^2 \triangle \overline{\phi} + \overline{\phi} = \phi, \text{ in } \Omega, \\ \overline{\phi} &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{A.23}$$

The following question occurs in the analysis of the accuracy of approximate deconvolution operators:

For what n and k do we have

$$||A^{-n}\phi||_k \leq C||\phi||_k$$

uniformly in δ?

We trace through now some answers provided by Theorem 66. For $n = 1$ Theorem 66 implies

$$\begin{aligned} ||\overline{\phi}||_k &\leq C||\phi||_k, \text{ for } k = 0, 1, 2 \text{ and that} \\ \triangle\overline{\phi} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Since $\triangle\overline{\phi} = 0$ on $\partial\Omega$, we can apply a higher estimate to the case $n = 2$. Indeed, $A^{-2}\phi = A^{-1}\overline{\phi} = \overline{\overline{\phi}}$ so that

$$\begin{aligned} ||\overline{\overline{\phi}}||_k &\leq C||\overline{\phi}||_k, \text{ for } k = 0, 1, 2, 3, 4 \text{ and that} \\ \triangle\overline{\overline{\phi}} &= \triangle\overline{\phi} = 0 \text{ on } \partial\Omega. \end{aligned}$$

Further, the equation for $\overline{\overline{\phi}}$ is

$$-\delta^2 \triangle \overline{\overline{\phi}} + \overline{\overline{\phi}} = \overline{\phi}, \text{ in } \Omega. \tag{A.24}$$

Taking the Laplacian of this equation gives

$$-\delta^2 \triangle^2 \overline{\overline{\phi}} + \triangle\overline{\overline{\phi}} = \triangle\overline{\phi}, \text{ in } \Omega. \tag{A.25}$$

Now, let $x \to \partial\Omega$ and use $\triangle\overline{\overline{\phi}} = \triangle\overline{\phi} = 0$ on $\partial\Omega$. This implies

$$\triangle^2\overline{\overline{\phi}} = \triangle\overline{\overline{\phi}} = \overline{\overline{\phi}} = 0 \text{ on } \partial\Omega$$

so that even higher uniform regularity can be inferred for $\overline{\overline{\overline{\phi}}}$.

$$||\overline{\overline{\overline{\phi}}}||_k \leq C||\overline{\overline{\phi}}||_k, \text{ for } k = 0, 1, 2, 3, 4, 5, 6.$$

This argument can be continued.

A.5 Remarks

This section is based on [L07] which also treats the nonlinear case. Patrick Rabier, Catalin Trenchea and Luc Tartar gave important help on the estimates in this chapter. The proof of Theorem 66 (for $k > 2$) is due to Patrick Rabier as well as the critical second example of Sect. A.3. The proof of Proposition 73 is due to Luc Tartar and Sect. A.2 is based on a helpful communication of his. Lemmas 68 and 69 are from his paper [T93]. The third example is due to Xinfu Chen and came from a stimulating discussion with him.

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