Appendix A

A.1 Convergence of Set Sequences

In this section we collect some facts on the convergence of a sequence of closed sets A_n to A, typically the sets will be spectra of operators and therefore be subsets of $\mathbb{R}_+$ or $\mathbb{C}$. Nevertheless, we need also a weighted distance, so we formulate this chapter in terms of a complete metric space (M, d) and assume $A_n, A \subset M$. We denote the open and closed metric balls by

$$B_\eta(x) := \{\, y \in M \mid d(x, y) < \eta \,\} \qquad \text{and} \qquad \overline{B}_\eta(x) := \{\, y \in M \mid d(x, y) \leq \eta \,\}.$$

Moreover, for a set $B \subset M$ and $a \in M$, we define the distance of a to B by

$$d(a, B) := \inf_{b \in B} d(a, b) \in [0, \infty]. \tag{A.1}$$

Note that if B is compact, then the minimum is achieved, and we can replace inf by min.

Definition A.1.1. We define the myemphmaximal outside distance of A to B by

$$d_+(A, B) := \sup_{a \in A} d(a, B) \in [0, \infty].$$

Moreover, the *maximal inside distance* of A to B is given by

$$d_-(A, B) := d_+(B, A).$$

Finally, the *Hausdorff distance* of A and B is defined by

$$d(A, B) := \max\{d_+(A, B), d_-(A, B)\}.$$

O. Post, *Spectral Analysis on Graph-Like Spaces*, Lecture Notes in Mathematics 2039,
DOI 10.1007/978-3-642-23840-6,

The proof of the following proposition is standard:

Proposition A.1.2. *We have* $d_\pm(A, B) = d_\pm(\overline{A}, \overline{B})$ *and* $d(A, B) = d(\overline{A}, \overline{B})$. *Moreover, the Hausdorff distance is indeed a metric on the space*

$$\mathscr{K}(M) := \{\, K \subset M \mid K \text{ compact}\}.$$

If (M, d) *is complete, so is* $(\mathscr{K}(M), d)$.

Note that the maximal outside and inside distances are *not* symmetric: Assume that A, B are compact. Then $d_+(A, B) = 0$ is equivalent to $A \subset B$, i.e. the maximal outside distance of A to B does not see points in $B \setminus A$, and similarly for d_-.

Lemma A.1.3. *Let* $\eta \geq$. *We have the following equivalent characterisations for compact subsets* $A, B \subset M$*:*

$$d_+(A, B) \leq \eta \qquad \Leftrightarrow \qquad \forall a \in A\ \exists b \in B\colon d(a, b) \leq \eta,$$
$$d_-(A, B) \leq \eta \qquad \Leftrightarrow \qquad \forall b \in B\ \exists a \in A\colon d(a, b) \leq \eta.$$

Note that this characterisation allows to define maps $f_\eta\colon A \longrightarrow B$ and $g_\eta\colon B \longrightarrow A$ such that $d(a, f_\eta(a)) \leq \eta$ and $d(b, g_\eta(b)) \leq \eta$. This point of view is useful when comparing two different metric spaces A, B in the *Gromov-Hausdorff distance* (see e.g. [Ka02, Ka06]).

Definition A.1.4. Let $A_n, A \in \mathscr{K}(M)$ be compact subsets of M.

1. A_n *converges from outside* to A ($A_n \searrow A$) if $d_+(A_n, A) \to 0$.
2. A_n *converges from inside* to A ($A_n \nearrow A$) if $d_-(A_n, A) \to 0$.
3. A_n *converges* to A ($A_n \to A$) if $d(A_n, A) \to 0$.

Remark A.1.5. Again, we warn from taking the terminology too literally: if $A_n \subset A$, then $d_+(A_n, A) = 0$, i.e. a sequence A_n *inside* A converges from outside to A, and similarly for d_-. In particular, if $A_n \searrow A$ converges from outside, then the limit can *suddenly expand*. For example, if $A_n = A_0 \subsetneq A$, then $d_+(A_n, A) = 0$. The convergence from outside does not note what happens *inside* A. Similarly, if $A_n \nearrow A$ converges from inside, then the limit can *suddenly collapse*. For example, if $A_n = A_0 \supsetneq A$, then $d_-(A_n, A) = 0$. The convergence form inside does not note what happens *outside* A. Nevertheless, if $A_n \to A$ (i.e. A_n converges to A in Hausdorff distance), the limit A cannot either expand nor collapse.

Let us give an element-wise characterisation of the convergence from inside and outside. Recall that an assertion (A_n) holds *eventually* if there exists $n_0 \in \mathbb{N}$ such that (A_n) holds for all $n \geq n_0$. Similarly, (A_n) holds *infinitely often* if for all $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that (A_n) holds.

Proposition A.1.6. *Assume that* M *is a compact metric space. Let* $A_n, A \in \mathscr{K}(M)$ *be closed subsets. Then the following conditions are equivalent:*

$$A_n \searrow A \qquad \Leftrightarrow \qquad \forall x \in M \setminus A\ \exists \eta > 0\colon B_\eta(x) \cap A_n = \emptyset \textit{ eventually},$$

$$A_n \nearrow A \qquad \Leftrightarrow \qquad \forall a \in A\ \exists a_n A_n\colon d(a_n, a) \to 0.$$

Proof. Assume that $A_n \searrow A$ then $d(a_n, A) \to 0$. Let $x \notin A$. Since A^c is open, there exists $\eta > 0$ such that $B_{2\eta}(x) \cap A = \emptyset$. Now, if $a_n \in B_\eta(x) \cap A_n$ infinitely often, then

$$d(a, a_n) \geq d(a, x) - d(x, a_n) > 2\eta - \eta = \eta$$

for all $a \in A$, i.e. $d(a_n, A) \geq \eta$, contradicting $d(a_n, A) \to 0$.

For the opposite direction, assume that $A_n \searrow A$ is not true, i.e. there exists $\eta > 0$ and a sequence $a_n \in A_n$ such that $d(a_n, A) \geq \eta$ infinitely often. Since M is compact, we can extract a convergence sub-sequence $a_{n_k} \to x$. In particular, $d(x, A) \geq \eta$, thus $x \in M \setminus A$. From the assumption, we have $B_\eta(x) \cap A_n = \emptyset$ eventually, in contradiction to $a_{n_k} \to x$.

The second equivalence is simpler: The convergence $A_n \nearrow A$ is equivalent to $d(a, A_n) \to 0$ *uniformly* in $a \in A$. In particular, the pointwise convergence $d(a, A_n) \to 0$ for all $a \in A$ follows. For the opposite direction, we use again the compactness of M in order to obtain the *uniform* convergence. □

A.2 Estimates on Abstract Fibred Spaces

In this section, we develop an abstract framework in order to prove Sobolev trace theorems and related estimates on graphs and manifolds. In the abstract setting, we consider (weakly) differentiable functions on an interval with values in a Hilbert space. We allow the Hilbert spaces along the interval to vary, and define abstractly *(warped) products* and spaces close to (warped) products. All relevant examples are covered, e.g., the Sobolev trace estimate for a manifold with boundary or estimates on cone-like manifolds. Our approach here is related (at least formally) to [B89, GG91], as well as to the abstract approach to so-called "Dirac systems" in [BBC08]. There are also relations to "half-line boundary triples" introduced in Sect. 3.5.

A.2.1 Vector-Valued Integrals

Let us start with some facts about vector-valued integrals. More details can be found e.g. in [Y80, GG91]. Let (M, μ) be a measure space and $\mathscr{X}$ a Banach space. A function $u\colon M \longrightarrow \mathscr{X}$ is said to be *weakly measurable* if $\varphi \circ u\colon M \longrightarrow \mathbb{C}$, $s \mapsto \varphi(u(s))$, is measurable for all $\varphi \in \mathscr{X}^*$ in the dual space. A function $u\colon M \longrightarrow \mathscr{X}$ is said to be a *step function* if its range $\operatorname{ran} u = u(M)$ is finite, say, $u(M) = \{x_1, \dots, x_k\}$; and if $\mu(u^{-1}(x_i)) < \infty$ for all $i = 1, \dots, k$, provided $x_i \neq 0$. A function $u\colon M \longrightarrow \mathscr{X}$ is said to be *strongly measurable* if there is a sequence of step functions $u_n\colon M \longrightarrow \mathscr{X}$ (an *approximating sequence*) such that $\|u_n(s) - u(s)\| \to 0$ for μ-almost all $s \in S$. By Pettis' theorem, a weakly

measurable function is strongly measurable provided $\mathscr{X}$ is separable (or at least if the closed subspace generated by $\operatorname{ran} u$ is separable). In our application below, $\mathscr{X}$ will be a separable Hilbert space. Moreover, we use the case $\mathscr{X} = \mathscr{L}(\mathscr{H})$ and $u(s) = -\varphi(s)(H-s)^{-1}$ for a closed operator H (see Theorems 3.3.6 and 3.3.20). Note that in the latter situation, the closed subspace generated by the range of u is an abelian sub-algebra of $\mathscr{L}(\mathscr{H})$ and therefore separable.

For strongly measurable functions, the vector-valued integral is defined in the obvious way. A function $u\colon M \longrightarrow \mathscr{X}$ is said to be *Bochner-integrable*, if u is strongly measurable (with approximating sequence u_n) and if $\int_M \|u(s) - u_n(s)\| \mathrm{d}\mu(s) \to 0$. It can be seen that the definition of the integral in the next theorem is independent of the approximating sequence (two approximating sequences can be combined into a single approximating sequence). By Bochner's theorem, a strongly measurable function is Bochner-integrable iff $s \to \|u(s)\|$ is integrable:

Theorem A.2.1 (Bochner). *Let (M, μ) be a measure space, and $\mathscr{X}$ a Banach space. Assume that $u\colon M \longrightarrow \mathscr{X}$ is strongly measurable. Then u is (Bochner-) integrable iff*

$$\int_M \|u(s)\| \mathrm{d}\mu(s) < \infty. \tag{A.2}$$

In particular, if the integrability condition (A.2) if fulfilled, then the vector-valued integral

$$\int_M u(s) \mathrm{d}\mu(s) := \lim_{n\to\infty} \sum_{x \in \operatorname{ran} u_n \setminus \{0\}} \mu(u_n^{-1}(x))x \in \mathscr{X}$$

exists in the norm-topology on $\mathscr{X}$, where u_n is an approximating sequence. Moreover,

$$\Big\| \int_M u(s) \mathrm{d}\mu(s) \Big\| \le \int_M \|u(s)\| \mathrm{d}\mu(s).$$

Let now $\mathscr{X} = \mathscr{H}_0$ be a (separable) Hilbert space. Then weakly measurable functions are strongly measurable. Moreover, we have a vector-valued generalisation of the Cauchy-Schwarz inequality, namely

$$\Big| \Big\langle \int_M u(s) \mathrm{d}\mu(s), \int_M v(s) \mathrm{d}\mu(s) \Big\rangle_{\mathscr{H}_0} \Big| \overset{\mathrm{CS}}{\le} \int_M \|u(s)\|_{\mathscr{H}_0} \|v(s)\|_{\mathscr{H}_0} \mathrm{d}\mu(s), \tag{A.3}$$

provided the RHS is finite, i.e. $u, v \in \mathsf{L}_2(M, \mathscr{H}_0)$. Here, $\mathsf{L}_2(M, \mathscr{H}_0)$ is the space of (equivalence classes of) measurable functions $u\colon M \longrightarrow \mathscr{H}_0$ such that

$$\|u\|^2_{\mathsf{L}_2(M,\mathscr{H}_0)} := \int_M \|u(s)\|^2_{\mathscr{H}_0} \mathrm{d}\mu(s) < \infty.$$

In particular, $\mathsf{L}_2(M, \mathscr{H}_0)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{\mathsf{L}_2(M,\mathscr{H}_0)} := \int_M \langle u(s), v(s) \rangle_{\mathscr{H}_0} \mathrm{d}\mu(s) = \Big\langle \int_M u(s) \mathrm{d}\mu(s), \int_M u(s) \mathrm{d}\mu(s) \Big\rangle_{\mathscr{H}_0}.$$

Here, the latter equality is true since the Bochner integral commutes with (bounded) linear operators. Note that $\mathsf{L}_2(M, \mathscr{H}_0)$ is also unitarily equivalent with $\mathsf{L}_2(M) \otimes \mathscr{H}_0$ and with the direct integral $\int_M^{\oplus} \mathscr{H}_0 \mathrm{d}\mu(s)$ with constant fibre, see the next subsection.

Moreover, in the vector-valued case, we also have a "scalar-vector" version of the Cauchy-Schwarz inequality, namely

$$\Big\| \int_M f(s)u(s)\mathrm{d}\mu(s)\Big\|^2_{\mathscr{H}_0} \overset{\mathrm{CS}}{\le} \int_M |f(s)|^2 \mathrm{d}\mu(s) \int_M \|u(s)\|^2_{\mathscr{H}_0} \mathrm{d}\mu(s) \tag{A.4}$$

for $f \in \mathsf{L}_2(M, \mathbb{C})$ and $u \in \mathsf{L}_2(M, \mathscr{H}_0)$.

A.2.2 Fibred Spaces Over an Interval

Assume that $I \subset \mathbb{R}$ is an interval and that $\mathscr{H}(s)$ is a family of separable Hilbert spaces based on the same vector space $\mathscr{H}_0$ (i.e. $\mathscr{H}(s) = \mathscr{H}_0$ as vector space). It is unimportant whether we choose I to be open or closed, since individual points are not seen by the Lebesque measure ds. We assume that $\mathscr{H}(s)$ is measurable, i.e. that $s \mapsto \|u_0\|^2_{\mathscr{H}(s)}$ is measurable for all $u_0 \in \mathscr{H}_0$. We set

$$\mathscr{H}(I) = \int_I^{\oplus} \mathscr{H}(s)\mathrm{d}s$$

(for an abstract definition of the direct integral of Hilbert spaces $\int_I^{\oplus} \mathscr{H}(s)\mathrm{d}s$, we refer e.g. to [Di69] or [RS80, Sec. XIII.16]). Equivalently, we can define $\mathscr{H}(I)$ as the completion of the space of continuous functions $\mathsf{C}(I, \mathscr{H}_0)$ under the norm $\|\cdot\|_{\mathscr{H}(I)}$, where

$$\|u\|^2_{\mathscr{H}(I)} := \int_I \|u(s)\|^2_{\mathscr{H}(s)} \mathrm{d}s < \infty.$$

Similarly, we define a fibred Sobolev space $\mathscr{H}^1(I)$ as the completion of the space of functions $\mathsf{C}^1(I, \mathscr{H}_0)$ under the norm $\|\cdot\|_{\mathscr{H}^1(I)}$, where

$$\|u\|^2_{\mathscr{H}^1(I)} := \int_I \big(\|u(s)\|^2_{\mathscr{H}(s)} + \|u'(s)\|^2_{\mathscr{H}(s)}\big)\mathrm{d}s < \infty.$$

We denote by $\mathscr{H}^k(s_0, s_1)$ the corresponding space $\mathscr{H}^k(I)$ with $I = (s_0, s_1)$. Note that

$$I_1 \subset I_2 \qquad \text{implies} \qquad \|u\|_{\mathscr{H}^k(I_1)} \le \|u\|_{\mathscr{H}^k(I_2)}. \tag{A.5}$$

Since the Hilbert spaces $\mathscr{H}(s)$ and $\mathscr{H}_0$ differ only in their inner product, there exists a positive operator-valued, measurable function $s \mapsto A(s)$ such that

$$\|u_0\|^2_{\mathscr{H}(s)} = \langle u_0, u_0\rangle_{\mathscr{H}(s)} = \langle u_0, A(s)u_0\rangle_{\mathscr{H}_0}$$

for all $u_0 \in \mathscr{H}_0$ and almost all $s \in I$ with respect to the Lebesque measure. Moreover, there exist measurable functions $\rho_\pm : I \longrightarrow (0, \infty)$ such that

$$\rho_-(s) \|u_0\|^2_{\mathscr{H}_0} \le \|u_0\|^2_{\mathscr{H}(s)} \le \rho_+(s) \|u_0\|^2_{\mathscr{H}_0}$$

for almost all $s \in I$ and all $u_0 \in \mathscr{H}_0$. For example, we can choose $\rho_-(s) := \inf \sigma(A(s))$ and $\rho_+(s) := \sup \sigma(A(s))$.

We will now make more assumptions on the functions $\rho_\pm$.

Definition A.2.2.

1. We say that the fibred space $\mathscr{H}(I)$ is an *almost warped product* if there is a function ρ called *distortion function* and constants $\kappa_\pm > 0$ such that

   $$\kappa_- \rho(s) \le \rho_-(s) \qquad \text{and} \qquad \rho_+(s) \le \kappa_+ \rho(s)$$

   i.e.

   $$\kappa_- \rho(s) \|u(s)\|^2_{\mathscr{H}_0} \le \|u(s)\|^2_{\mathscr{H}(s)} \le \kappa_+ \rho(s) \|u(s)\|^2_{\mathscr{H}_0} \tag{A.6a}$$

   for almost all $s \in I$ and all $u_0 \in \mathscr{H}_0$. We call $\rho_\infty := \kappa_+/\kappa_-$ the *global relative distortion*.
2. We say that $\mathscr{H}(I)$ is a *warped product* if it is an almost warped product with relative distortion $\rho_\infty = 1$, i.e. if $\rho_-(s) = \rho_+(s) =: \rho(s)$, or equivalently, $A(s) = \rho(s) \operatorname{id}_{\mathscr{H}_0}$, or, what is the same,

   $$\|u_0\|^2_{\mathscr{H}(s)} = \rho(s) \|u_0\|^2_{\mathscr{H}_0} \tag{A.6b}$$

   for all $u_0 \in \mathscr{H}_0$. We call ρ (resp. $\rho_\pm$) the *(upper/lower) distortion function*.

The case when the distortion functions are constant deserves a special name:

Definition A.2.3.

1. We say that $\mathscr{H}(I)$ is an *almost product* if it is an almost warped product with *constant* functions $\rho_\pm$, i.e. $\rho_\pm(s) = \rho_{\pm,0}$ for almost all $s \in I$. In this case, $\rho_\infty = \rho_{+,0}/\rho_{-,0}$.
2. We say that the fibred space $\mathscr{H}(I)$ is a *product* if it is an almost product with relative distortion $\rho_\infty = 1$, i.e. $\|u_0\|^2_{\mathscr{H}(s)} = \|u_0\|^2_{\mathscr{H}_0}$. In this case,

   $$\mathscr{H}(I) = \mathsf{L}_2(I, \rho_0 \mathscr{H}_0) \cong \mathsf{L}_2(I) \otimes \rho_0 \mathscr{H}_0$$

   where ρ_0 denotes the constant value of the distortion function ρ and $\rho_0 \mathscr{H}_0$ is the Hilbert space $\mathscr{H}_0$ with norm defined by $\|u_0\|^2_{\rho_0 \mathscr{H}_0} := \rho_0 \|u_0\|^2_{\mathscr{H}_0}$.

On an (almost) product $\mathscr{H}(I)$, the identification operator from $\mathscr{H}(I)$ onto $\mathsf{L}_2(I) \otimes \rho_0 \mathscr{H}_0$ resp. $\mathsf{L}_2(I) \otimes \rho_{\pm,0} \mathscr{H}_0$ defines a bijective isomorphism resp. an isometry. By suitably inserting the distortion functions $\rho_\pm$ into the inequality (A.4), we can

extend the inequality to (almost) warped products, as we will do in the following lemma:

Lemma A.2.4. *Assume that $\mathscr{H}(I)$ is an almost warped product and that $s_0 < s_1 < s_2$ where $s_i \in \overline{I}$. If $u \in \mathscr{H}^1(I)$ with $u(s_2) = 0$, then*

$$\|u(s_1)\|^2_{\mathscr{H}(s_1)} \le \rho_\infty \eta_1(s_1, s_2)\|u'\|^2_{\mathscr{H}(s_1,s_2)}, \tag{A.7a}$$

$$\|u\|^2_{\mathscr{H}(s_0,s_1)} \le \rho_\infty \eta_2(s_0, s_1, s_2)\|u'\|^2_{\mathscr{H}(s_0,s_2)}, \tag{A.7b}$$

where

$$\eta_1(s_1, s_2) := \rho(s_1)\int_{s_1}^{s_2} \frac{1}{\rho(s)}\mathrm{d}s \qquad \textit{and} \qquad \eta_2(s_0, s_1, s_2) := \int_{s_0}^{s_1} \eta_1(t, s_2, \rho)\mathrm{d}t.$$

Proof. By a density argument, we can assume that u is of class C^1. Then we have

$$u(t) = -\int_t^{s_2} u'(s)\mathrm{d}s.$$

Using the Cauchy-Schwarz inequality for vector-valued integrals (A.4) we obtain

$$\begin{aligned}\|u(t)\|^2_{\mathscr{H}(t)} &\overset{\text{CS}}{\le} \int_t^{s_2} \frac{\kappa_+\rho(t)}{\kappa_-\rho(s)}\mathrm{d}s \int_t^{s_2} \frac{\kappa_-\rho(s)}{\kappa_+\rho(t)}\|u'(s)\|^2_{\mathscr{H}(t)}\mathrm{d}s \\ &\le \rho_\infty \int_t^{s_2} \frac{\rho(t)}{\rho(s)} \int_t^{s_2} \|u'(s)\|^2_{\mathscr{H}(s)}\mathrm{d}s\end{aligned}$$

using (A.6a) and the first estimate follows for $t = s_1$. The existence of the vector-valued integral $u(t)$ is guaranteed once the integrals on the RHS are finite (cf. Theorem A.2.1). The second estimate follows by integrating over t from s_0 to s_1 and using the monotonicity of the norms (A.5). □

Remark A.2.5. Note that the condition $u(s_2) = 0$ is well-defined for functions in $\mathscr{H}^1(I)$ by a similar argument as above.

If u does not vanish at s_2, we can argue as follows.

Proposition A.2.6. *Assume that $\mathscr{H}(I)$ is an almost warped product and that $u \in \mathscr{H}^1(s_0, s_2)$, then*

$$\|u(s_1)\|^2_{\mathscr{H}(s_1)} \le 2\rho_\infty\widetilde{\eta}_1(s_1, s_2, \chi)\|u'\|^2_{\mathscr{H}(s_1,s_2)} + 2\rho_\infty\widetilde{\eta}_1(s_1, s_2, \chi')\|u\|^2_{\mathscr{H}(s_1,s_2)},$$

$$\|u\|^2_{\mathscr{H}(s_0,s_1)} \le 2\rho_\infty\widetilde{\eta}_2(s_0, s_1, s_2, \chi)\|u'\|^2_{\mathscr{H}(s_0,s_2)} + 2\rho_\infty\widetilde{\eta}_2(s_0, s_1, s_2, \chi')\|u\|^2_{\mathscr{H}(s_0,s_2)},$$

where

$$\widetilde{\eta}_1(s_1,s_2,\chi) := \rho(s_1)\int_{s_1}^{s_2} \frac{|\chi(s)|^2}{\rho(s)}\mathrm{d}s, \quad \widetilde{\eta}_2(s_0,s_1,s_2,\chi) := \int_{s_0}^{s_1} \eta_1(t,s_2,\chi,\rho)\mathrm{d}t,$$

and where χ *is a Lipschitz function such that* $\chi(s) = 1$ *for* $s_0 \le s \le s_1$ *and* $\chi(s_2) = 0$.

Proof. Set $\widetilde{u} := \chi u$. Then $\widetilde{u}$ fulfils the assumptions of Lemma A.2.4. Moreover $u(t) = \widetilde{u}(t)$ for $t \in [s_0, s_1]$. We start with

$$\|u(t)\|^2_{\mathscr{H}(t)} \le 2\Big\|\int_t^{s_2} \chi(s)u'(s)\mathrm{d}s\Big\|^2_{\mathscr{H}(t)} + 2\Big\|\int_t^{s_2} \chi'(s)u(s)\mathrm{d}s\Big\|^2_{\mathscr{H}(t)},$$

and argue then as in the proof of Lemma A.2.4. The last estimate follows as before by integration. □

Let us now fix the cut-off function $\chi = \chi_a$: Assume that $0 < a \le s_2 - s_1$. Denote by χ_a the continuous, piecewise affine linear function given by

$$\chi_a(s) = \begin{cases} 1, & \text{for } s_0 \le s \le s_1, \\ 1 - a^{-1}(s - s_1), & \text{for } s_1 \le s \le s_1 + a, \\ 0, & \text{for } s_1 + a \le s \le s_2. \end{cases} \tag{A.8}$$

We obtain the following corollary. For the definitions of the functions η_1 and η_2 see Lemma A.2.4.

Corollary A.2.7. *Assume that* $\mathscr{H}(I)$ *is an almost warped product. If* $u \in \mathscr{H}^1(s_0, s_2)$ *then*

$$\|u(s_1)\|^2_{\mathscr{H}(s_1)} \le 2\rho_\infty \eta_1(s_1, s_1 + a)\Big(\|u'\|^2_{\mathscr{H}(s_1,s_2)} + \frac{1}{a^2}\|u\|^2_{\mathscr{H}(s_1,s_2)}\Big) \tag{A.9a}$$

$$\le \frac{2\rho_\infty \eta_1(s_1, s_1 + a)}{\min\{1, a^2\}}\|u\|^2_{\mathscr{H}^1(s_1,s_2)} \tag{A.9a'}$$

for $0 < a \le s_2 - s_1$ *and* $a \le 1$. *Moreover,*

$$\|u\|^2_{\mathscr{H}(s_0,s_1)} \le 2\rho_\infty \eta_2(s_0, s_1, s_1 + a)\Big(\|u'\|^2_{\mathscr{H}(s_0,s_2)} + \frac{1}{a^2}\|u\|^2_{\mathscr{H}(s_0,s_2)}\Big) \tag{A.9b}$$

$$\le \frac{2\rho_\infty \eta_2(s_0, s_1, s_1 + a)}{\min\{1, a^2\}}\|u\|^2_{\mathscr{H}^1(s_0,s_2)}. \tag{A.9b'}$$

Proof. The result follows from Proposition A.2.6 by simply estimating $|\chi_a(s)| \le 1$ and $|\chi_a'(s)| = 1/a$ for $s_1 \le s \le s_1 + a$. □

If $\mathscr{H}(I)$ is an almost product, we have the following estimate:

Corollary A.2.8. *Assume that $\mathscr{H}(I)$ is an almost product. If $u \in \mathscr{H}^1(s_0, s_2)$ then*

$$\|u(s_1)\|^2_{\mathscr{H}(s_1)} \le \rho_\infty \Big(a\|u'\|^2_{\mathscr{H}(s_1,s_2)} + \frac{2}{a}\|u\|^2_{\mathscr{H}(s_1,s_2)} \Big) \tag{A.10}$$

$$\le \frac{2\rho_\infty}{\min\{1, (s_2 - s_1)\}} \|u\|^2_{\mathscr{H}^1(s_1,s_2)} \tag{A.10'}$$

for $0 < a \le s_2 - s_1$ *and* $a \le 1$.

Proof. Again, the result follows from Proposition A.2.6 by evaluating the integrals $\widetilde{\eta}_i$ for χ_a. □

Remark A.2.9. Note that the estimate (A.7a) is optimal in the following sense: Assume that $\mathscr{H}(I)$ is a warped product and that $u(s) = (\int_{s_1}^{s_2} \rho(s)\mathrm{d}s)^{-1}\varphi_0$ with $\|\varphi_0\|_{\mathscr{H}_0} = 1$, i.e. u is constant and $\|u\|_{\mathscr{H}(s_1,s_2)} = 1$. Moreover, assume that $\chi'(s) < 0$ for all $s_1 \le s \le s_2$. Then we have

$$1 = \big|\chi(s_2) - \chi(s_1)\big|^2 = \Big(\int_{s_1}^{s_2} |\chi'(s)|\mathrm{d}s\Big)^2 \overset{\text{CS}}{\le} \int_{s_1}^{s_2} \rho(s)\mathrm{d}s \int_{s_1}^{s_2} \frac{|\chi'(s)|^2}{\rho(s)}\mathrm{d}s.$$

If the distortion function is $\rho(s) = -\chi'(s) = |\chi'(s)|$, then we have equality in the Cauchy-Schwarz inequality, and in particular,

$$\|u(s_1)\|^2_{\mathscr{H}(s_1)} = \frac{\rho(s_1)}{\int_{s_1}^{s_2} \rho(s)\mathrm{d}s} = \rho(s_1)\int_{s_1}^{s_2} \frac{|\chi'(s)|^2}{\rho(s)}\mathrm{d}s = \widetilde{\eta}_1(s_1, s_2, \chi')\|u\|^2_{\mathscr{H}(s_1,s_2)}$$

by the above equality of Cauchy-Schwarz, the definition of $\widetilde{\eta}_1$ and the fact that u is normalised on $\mathscr{H}(s_1, s_2)$. Therefore, estimate (A.7a) is optimal. A similar remark holds for the other estimates.

Let us now fix a normalised vector $\varphi_0 \in \mathscr{H}_0$. Denote by $P_0 u_0 := \langle \varphi_0, u_0\rangle_{\mathscr{H}_0}\varphi_0$ the corresponding orthogonal projection onto the subspace $\mathbb{C}\varphi_0$. In our applications later on, φ_0 will be an eigenvector associated with the lowest eigenfunction of an operator K on $\mathscr{H}_0$. For a warped product, the projection is independent of s:

Lemma A.2.10. *Let $\mathscr{H}(I)$ be a warped product. Then $\varphi_0 \ne 0$ in $\mathscr{H}(s)$ for almost all $s \in I$, and $\varphi_s := \varphi_0/\|\varphi_0\|_{\mathscr{H}(s)} = \rho(s)^{-1/2}\varphi_0$ is normalised in $\mathscr{H}_0$. Moreover, P_0 is also the orthogonal projection onto φ_s in the Hilbert space $\mathscr{H}(s)$. In particular,*

$$\|P_0 u_0\|^2_{\mathscr{H}(s)} \overset{\text{CS}}{\le} \|u_0\|^2_{\mathscr{H}(s)}$$

for $u_0 \in \mathscr{H}(s) = \mathscr{H}_0$.

Proof. We have $\|\varphi_0\|^2_{\mathscr{H}(s)} = \rho(s)\|\varphi_0\|^2_{\mathscr{H}_0} = \rho(s) > 0$ for almost all $s \in I$; in particular, $\varphi_0 \ne 0 \in \mathscr{H}(s)$. Moreover, the orthogonal projection onto φ_s in $\mathscr{H}(s)$

is given by

$$\langle \varphi_s, u_0 \rangle_{\mathscr{H}(s)} \varphi_s = \rho(s)^{-1} \langle \varphi_0, u_0 \rangle_{\mathscr{H}(s)} \varphi_0 = \langle \varphi_0, u_0 \rangle_{\mathscr{H}_0} \varphi_0 = P_0 u_0.$$

The last estimate follows from Cauchy-Schwarz. □

Proposition A.2.11. *Let $\mathscr{H}(I)$ be a warped product. Then*

$$\begin{aligned}\rho(s_1)\|P_0 u(s_2) - P_0 u(s_1)\|^2_{\mathscr{H}_0} &= \rho(s_1)\big|\langle \varphi_0, u(s_2)\rangle_{\mathscr{H}_0} - \langle \varphi_0, u(s_1)\rangle_{\mathscr{H}_0}\big|^2 \\ &\le \eta_1(s_1, s_2)\|u'\|^2_{\mathscr{H}(s_1,s_2)}\end{aligned}$$

for $u \in \mathscr{H}^1(I)$.

Proof. We have

$$\begin{aligned}\big\|\langle \varphi_0, u(s_2) - u(s_1)\rangle_{\mathscr{H}_0}\varphi_0\big\|^2_{\mathscr{H}_0} &\overset{\text{CS}}{\le} \|u(s_2) - u(s_1)\|^2_{\mathscr{H}_0} \\ &= \Big\|\Big\langle \varphi_0, \int_{s_1}^{s_2} u'(s)\mathrm{d}s\Big\rangle_{\mathscr{H}_0}\varphi_0\Big\|^2_{\mathscr{H}_0} \\ &\overset{\text{CS}}{\le} \int_{s_1}^{s_2} \frac{1}{\rho(s)}\mathrm{d}s \int_{s_1}^{s_2} \|u'(s)\|^2_{\mathscr{H}_0}\rho(s)\mathrm{d}s\end{aligned}$$

using Cauchy-Schwarz for vector-valued integrals (A.4) for the second inequality. The result follows by the definition of η_1 in Lemma A.2.4. □

A.2.3 Examples: Cones and Cylinders

We will give now some special cases coming from a warped product metric on a manifold as applied in Chap. 6. Denote by $X = I \times_s Y$ the warped product of $I = [0, \ell]$ and the compact Riemannian manifold Y with radius function $r\colon I \longrightarrow (0, \infty)$, i.e. X is equipped with the metric $g = \mathrm{d}s^2 + r(s)^2 h$, where h is the metric on Y (see Definition 5.3.2). Then we have

$$\|u\|^2_{\mathsf{L}_2(X,g)} = \int_I \|u(s)\|^2_{\mathsf{L}_2(Y,h)} r(s)^m \mathrm{d}s.$$

In particular, $\mathsf{L}_2(X, g) = \mathscr{H}(I)$ is a warped product with distortion function $\rho(s) = r(s)^m$ and fibre Hilbert space $\mathscr{H}_0 = \mathsf{L}_2(Y, h)$.

For the next corollary, assume that r is constant, say $r = 1$.

Corollary A.2.12 (Product). *Assume that $I = [0, \ell]$ and $X = I \times Y$ with product metric $g = \mathrm{d}s^2 + h$. Then $\mathscr{H}(I)$ is a product (in the sense of Definition A.2.3). Moreover,*

$$\|u(0,\cdot)\|^2_{\mathsf{L}_2(Y)} \le a\|u'\|^2_{\mathsf{L}_2(X)} + \frac{2}{a}\|u\|^2_{\mathsf{L}_2(X)} \le \frac{2}{\min\{1,\ell\}}\|u\|^2_{\mathsf{H}^1(X)} \tag{A.11}$$

for all $0 < a \le \min\{1,\ell\}$ *and* $u \in \mathsf{H}^1(X)$.

Proof. The result follows from Corollary A.2.8. Note that $\mathsf{H}^1(X) \subset \mathscr{H}^1(I)$ since

$$\|u\|^2_{\mathscr{H}^1(I)} = \|u\|^2_{\mathsf{L}_2(X)} + \|u'\|^2_{\mathsf{L}_2(X)} \le \|u\|^2_{\mathsf{L}_2(X)} + \|u'\|^2_{\mathsf{L}_2(X)} + \|\mathrm{d}_Y u\|^2_{\mathsf{L}_2(X)} = \|u\|^2_{\mathsf{H}^1(X)}$$

for $u \in \mathsf{H}^1(X)$. □

We return to the abstract framework, and calculate the functions η_1, η_2 of Lemma A.2.4 for special choices of the distortion function $\rho = r^m$. Let us first define some universal functions occurring in the estimates later on. *Universal* refers here to the fact that the functions depend only on the parameter m.

Definition A.2.13. We set

$$p_m(\tau) := \frac{1}{1-\tau}\int_\tau^1 u^m \mathrm{d}u \qquad \text{and} \qquad \widetilde{p}_m(\tau) := \frac{1}{1-\tau}\int_\tau^1 p_m(t)\mathrm{d}t.$$

We have the following properties of the functions p_m and $\widetilde{p}_m$:

Lemma A.2.14.

1. *We have*

$$p_m(\tau) = \begin{cases} \dfrac{1}{m+1}\displaystyle\sum_{i=0}^m \tau^i, & \text{if } m \ge 0, \\ -\dfrac{\log\tau}{1-\tau}, & \text{if } m = -1, \end{cases}$$

and

$$\widetilde{p}_m(\tau) = \begin{cases} \dfrac{1}{m+1}\displaystyle\sum_{i=0}^m \frac{1}{i+1}\sum_{j=0}^i \tau^j, & \text{if } m \ge 0, \\ \dfrac{\operatorname{dilog}\tau + \tau(1-\log\tau) - 1}{1-\tau} & \text{if } m = -1, \end{cases}$$

where $\operatorname{dilog}\tau := \int_1^\tau \frac{\log t}{1-t}\mathrm{d}t$. *Moreover,* p_m *and* $\widetilde{p}_m$ *are polynomials of degree indicated by the subscript (for non-negative subscripts) with continuous extension* $p_m(1) = 1$ *and* $\widetilde{p}_m(1) = 1$. *In addition,* p_{-1} *and* $\widetilde{p}_{-1}$ *extend to an analytic functions also in* $\tau = 1$ *by setting* $p_{-1}(1) := 1$ *and* $\widetilde{p}_{-1}(1) := 1$.
2. *For* $0 < \tau \le 1$ *and* $m \ge 0$, *we have*

$$\frac{1}{m+1} \le p_m(\tau), \widetilde{p}_m(\tau) \le 1, \qquad -\log\tau \le p_{-1}(\tau) \le 1 - \log\tau.$$

Moreover, we have the asymptotic behaviour

$$p_m(\tau) \sim_0 \frac{1}{m+1}, \qquad p_{-1}(\tau) \sim_0 -\log\tau,$$

$$\widetilde{p}_m(\tau) \sim_0 \frac{1}{m+1}\sum_{i=0}^{m}\frac{1}{i+1}, \qquad \widetilde{p}_{-1}(\tau) \sim_0 \frac{\pi^2}{6}-1,$$

at $\tau = 0$. *Here,* $f(\tau) \sim_a g(\tau)$ *means that* $\lim_{\tau\to a} f(\tau)/g(\tau) = 1$.

3. *For* $1 \le \tau$ *and* $m \ge 0$*, we have*

$$\frac{\tau^m}{m+1} \le p_m(\tau) \le \tau^m, \qquad \frac{\tau^m}{(m+1)^2} \le \widetilde{p}_m(\tau) \le \tau^m.$$

In addition, we have

$$p_m(\tau) \sim_\infty \frac{\tau^m}{m+1}, \qquad p_{-1}(\tau) \sim_\infty \frac{\log\tau}{\tau},$$

$$\widetilde{p}_m(\tau) \sim_\infty \frac{\tau^m}{(m+1)^2} \qquad \widetilde{p}_{-1}(\tau) \sim_\infty \log\tau.$$

Proof. The properties follow by direct calculation, e.g., for $\tau \neq 1$ and $m > -1$, we have

$$p_m(\tau) = \frac{1}{1-\tau}\int_1^{1/\tau}\frac{1}{t^{m+2}}\mathrm{d}t = \frac{1}{(m+1)}\cdot\frac{1-\tau^{m+1}}{1-\tau} = \frac{1}{m+1}\sum_{i=0}^{m}\tau^i$$

substituting $u = 1/t$. For $m = -1$ the result follows similarly. Moreover,

$$\begin{aligned}
\widetilde{p}_m(\tau) = \frac{1}{1-\tau}\int_\tau^1 p_m(\tau)\tau\mathrm{d}\tau &= \frac{1}{m+1}\cdot\frac{1}{1-\tau}\sum_{i=0}^{m}\int_\tau^1\tau^i\mathrm{d}\tau \\
&= \frac{1}{m+1}\sum_{i=0}^{m}\frac{1}{i+1}\cdot\frac{1-\tau^{i+1}}{1-\tau} \\
&= \frac{1}{m+1}\sum_{i=0}^{m}\frac{1}{i+1}\sum_{j=0}^{i}\tau^j
\end{aligned}$$

for $m > -1$ and similarly for $m = -1$. The other assertions can be checked easily. □

In the following lemma, it is convenient to use the differences $a = s_1 - s_0$ and $b = s_2 - s_1$:

Definition A.2.15. We set $s_0 = 0$, $s_1 = a$ and $s_2 = a + b$ for $a, b > 0$ and define

$$\eta_1(b) := \eta_1(a, a+b) \qquad \text{and} \qquad \eta_2(a,b) := \eta_2(0, a, a+b).$$

Moreover, we set

$$\operatorname{vol}(s_0, s_1) := \int_{s_0}^{s_1} \rho(s)\mathrm{d}s, \qquad \operatorname{vol}(a) := \operatorname{vol}(0, a).$$

Let us now fix the radius function r, and therefore the distortion function $\rho = r^m$: Note that if r is constant, then the resulting warped product manifold $X = I \times_r Y$ is a product, or a *cylinder*. If r is affine linear, then $X = I \times_r Y$ is a cone. Recall that we allowed weak regularity on r and therefore on the metric of the manifold (see Sect. 5.1). Therefore, we may consider the following (piecewise) affine linear function r, i.e. we consider two cones (if $r_{i-1} \neq r_i$) or cylinders (if $r_{i-1} = r_i$) attached together:

Lemma A.2.16.

1. ***(Almost) Product:*** *If* $\rho(s) = r_0^m$, *then*

$$\eta_1(b) = b,$$
$$\eta_2(a,b) = \int_{s_0}^{s_1} (s_2 - t)\mathrm{d}t = \frac{1}{2}\big((a+b)^2 - a^2\big) = a(b + a/2) \le a(a+b),$$
$$\operatorname{vol}(a) = a r_0^m.$$

2. ***(Almost) warped product*** *If* $\rho(s) = r(s)^m$, *where* r *is the continuous affine linear function such that* $r(s_i) = r_i$, $i = 0, 1, 2$, *and* $\tau_i := r_{i-1}/r_i$ *are the relative radii, then*

$$\eta_1(b) = b p_{m-2}(\tau_2)\tau_2$$
$$ab p_{m-2}(\tau_2)\tau_2 p_m(\tau_1) \le \eta_2(a,b) \le a\big(a\widetilde{p}_{m-2}(\tau_1) + b p_{m-2}(\tau_2)\tau_2 p_m(\tau_1)\big),$$
$$\operatorname{vol}(a) = a r_0^m p_m(\tau_1^{-1}),$$

Note that η_1 and η_2 depend only on the ratio $\tau_1 = r_0/r_1$ and $\tau_2 = r_1/r_2$ an *not* on the values r_i itself.

Proof. The first result is an easy calculation. For the second, note that

$$r(s) := r_i + (s - s_i)\frac{r_j - r_i}{s_j - s_i}, \qquad 0 \le i < j \le 2, \quad s_i \le s \le s_j.$$

Moreover, for $r_1 \neq r_2$ and $m > 1$, we have

$$\eta_1(s_1, s_2) = \int_{s_1}^{s_2} \Big(\frac{r(s)}{r_1}\Big)^{-m} \mathrm{d}s = \frac{b\tau_2}{1 - \tau_2}\int_1^{1/\tau_2} \tau^{-m}\mathrm{d}\tau = b\tau_2 p_{m-2}(\tau_2)$$

using the substitution $\tau = r(s)/r_1$ and the fact that $\mathrm{d}\tau = (1-\tau_2)(b\tau_2)^{-1}\mathrm{d}s$. If $m = 1$, then

$$\eta_1(s_1, s_2) = -\frac{b\tau_2}{1-\tau_2}\log\tau_2 = bp_{-1}(\tau_2)\tau_2.$$

In addition, if $r_1 = r_2$, i.e. $\tau_2 = 1$, then $\eta_1(b) = b$ by the first part. But $p_m(1) = 1$ for $m = -1, 0, 1, \ldots$, so that the formula for η_1 also holds in this case. In order to estimate η_2, we observe that

$$\eta_1(t, s_2) = \eta_1(t, s_1) + \frac{\rho(t)}{\rho(s_1)}\eta_1(s_1, s_2).$$

The integral over the first term can be estimated as

$$\begin{aligned}\int_{s_0}^{s_1}\eta_1(t, s_1)\mathrm{d}t &= \int_{s_0}^{s_1}(s_1 - t)p_{m-2}\Big(\frac{r(t)}{r_1}\Big)\frac{r(t)}{r_1}\mathrm{d}t \\ &\le a\int_{s_0}^{s_1}p_{m-2}\Big(\frac{r(t)}{r_1}\Big)\frac{r(t)}{r_1}\mathrm{d}t = \frac{a^2}{1-\tau_1}\int_{\tau_1}^{1}p_{m-2}(\tau)\tau\mathrm{d}\tau\end{aligned}$$

using the substitution $\tau = r(t)/r_1$ and $\mathrm{d}\tau = (1-\tau_1)a^{-1}\mathrm{d}t$.

The second term can be integrated by

$$\int_{s_0}^{s_1}\frac{\rho(t)}{\rho(s_1)}\mathrm{d}t = \int_{s_0}^{s_1}\Big(\frac{r(t)}{r_1}\Big)^m\mathrm{d}t = \frac{a}{1-\tau_1}\int_{\tau_1}^{1}\tau^m\mathrm{d}\tau = ap_m(\tau_1).$$

The calculation for $\mathrm{vol}(a)$ follows similarly. □

We will use the following special case in Sects. 6.3 and 6.8 on warped products:

Proposition A.2.17. *Assume that $I = [0, a+b]$, where $0 < a \le b$. Let ρ be the continuous, piecewise affine linear function with $\rho(0) = r_0$, $\rho(a) = r_1$ and $\rho(a+b) = r_1$, where $r_0 \le r_1$. Then $\mathscr{H}(I)$ is a warped product and,*

$$\|u\|^2_{\mathscr{H}(0,a)} \le 4ab\Big(\|u'\|^2_{\mathscr{H}(0,a+b)} + \frac{1}{a^2}\|u\|^2_{\mathscr{H}(0,a+b)}\Big) \quad \textit{and} \tag{A.12a}$$

$$\frac{a}{m+1}r_1^m \le \mathrm{vol}(a) \le ar_1^m \tag{A.12b}$$

for $u \in \mathscr{H}^1(I)$. Moreover, if P_0 is a projection in $\mathscr{H}_0$ with one-dimensional range, then

$$r_0^m\|P_0u(0)\|^2_{\mathscr{H}_0} \le 2ap_{m-2}\Big(\frac{r_0}{r_1}\Big)\frac{r_0}{r_1} + 2r_0^m\|P_0u(a)\|^2_{\mathscr{H}_0} \tag{A.12c}$$

for $u \in \mathscr{H}^1(I)$.

Proof. The first estimate follows from Corollary A.2.7 and the estimate

$$\eta_2(a,b) \le a\big(a\widetilde{p}_{m-2}(\tau_1) + bp_m(\tau_1)\big) \le a^2 + ab \le 2ab.$$

Here, we used Lemma A.2.16 with the setting $\tau_1 = r_0/r_1 \le 1$, $\tau_2 = 1$, $\widetilde{p}_{m-2}(\tau_1) \le 1$, $p_m(\tau_1) \le 1$ and $a \le b$. The second estimate estimate is a consequence of Lemma A.2.16 and

$$\frac{1}{(m+1)}\tau_1^{-1} \le p_m(\tau_1^{-1}) \le \tau_1^{-m}$$

using Lemma A.2.14. Finally, the last estimate can be seen from Proposition A.2.11 and $\eta_1(a) = ap_{m-2}(\tau_1)\tau_1$. □

Let us finish this section with some more examples illustrating the abstract setting. The possibly simplest example is given by $\mathscr{H}(s) = \mathbb{C}$. The estimate follows from Corollary A.2.8:

Corollary A.2.18 (Interval). *Let $I = [0,\ell]$. Assume that $\mathscr{H}(s) = \mathbb{C}$ and that the distortion function is constant, say $\rho(s) = 1$. Then $\mathscr{H}(I) = \mathsf{L}_2(I)$ is a product and*

$$|f(0)|^2 \le a\|f'\|^2_{\mathsf{L}_2(I)} + \frac{2}{a}\|f\|^2_{\mathsf{L}_2(I)} \le \frac{2}{\ell_-}\|f\|^2_{\mathsf{H}^1(I)}$$

for all $0 < a \le \ell_- = \min\{1,\ell\}$ and $f \in \mathsf{H}^1(I) = \mathscr{H}^1(I)$.

The following result is only needed in the special case $\mathscr{H}(s) = \mathbb{C}$:

Proposition A.2.19. *Assume that $I = [0,\ell]$, then*

$$\|f'\|^2_{\mathsf{L}_2(I)} \le \frac{1025}{\ell_-^2}\|f\|^2_{\mathsf{L}_2(I)} + 2\|f''\|^2_{\mathsf{L}_2(I)} \le \frac{1025}{\ell_-^2}\Big(\|f\|^2_{\mathsf{L}_2(I)} + \|f''\|^2_{\mathsf{L}_2(I)}\Big)$$

for $f \in \mathsf{H}^2(I)$.

Proof. Partial integration and Cauchy-Young's inequality yield

$$\|f'\|^2 \overset{\text{CY}}{\le} \frac{1}{2}\|f\|^2 + \frac{1}{2}\|f''\|^2 + \big|f(0)f'(0)\big| + \big|f(\ell)f'(\ell)\big|.$$

The boundary term at $s = 0$ can be estimated by

$$\begin{aligned}\big|f(0)f'(0)\big| &\overset{\text{CY}}{\le} \frac{\eta}{2}\big|f'(0)\big|^2 + \frac{1}{2\eta}\big|f(0)\big|^2\\ &\le \frac{\eta}{2}\Big(b'\|f''\|^2 + \frac{2}{b'}\|f'\|^2\Big) + \frac{1}{2\eta}\Big(b\|f'\|^2 + \frac{2}{b}\|f\|^2\Big)\\ &= \frac{1}{\eta b}\|f\|^2 + \frac{\eta b'}{2}\|f''\|^2 + \frac{1}{2}\Big(\frac{2\eta}{b'} + \frac{b}{\eta}\Big)\|f'\|^2\end{aligned}$$

for $\eta >$ and $b, b' \in (0, \ell]$, applying Corollary A.2.18 to f and f'. A similar result holds for the boundary term at $s = \ell$, so that we end up with the inequality

$$\|f'\|^2 \le \Big(\frac{1}{2} + \frac{2}{\eta b}\Big)\|f\|^2 + \Big(\frac{1}{2} + \eta b'\Big)\|f''\|^2 + \Big(\frac{2\eta}{b'} + \frac{b}{\eta}\Big)\|f'\|^2.$$

If we set $\eta := a/8$, $b := a/32$ and $b' := a$ for $0 < a \le \ell$, then the coefficient of $\|f'\|^2$ on the RHS equals $1/2$. Bringing this term on the LHS and multiplying by 2 yields the desired estimate with $a = \ell_-$. Note that $1 + 4/(\eta b) = 1 + 1{,}024/a^2 \le 1{,}025/a^2$ and $12\eta b' = 1 + a^2/4 \le 2$ since $a \le 1$. $\square$

More generally than in Corollary A.2.18, we may assume that the measure on I is weighted, i.e. we replace the Lebesque measure ds by $w(s)$ds. Then from Corollary A.2.7 we conclude (with the estimate $\eta_1(a) \le \omega a$):

Corollary A.2.20 (Interval with weights). *Assume that* $I = [0, \ell]$, *that* $w\colon (0, \ell) \longrightarrow (0, \infty)$ *is a measurable function, and that* $\mathscr{H}(s) = \mathbb{C}w(s)$. *Then* $\mathscr{H}(I) = \mathsf{L}_2(I, w\mathrm{d}s)$ *is a warped product with distortion function* $\rho = w$. *If in addition*

$$\omega := \frac{w(0)}{\inf w([0, \min\{1, \ell\}])} < \infty,$$

then

$$|f(0)|^2 \le 2a\omega\|f'\|^2_{\mathsf{L}_2(I,w(s)\mathrm{d}s)} + \frac{2\omega}{a}\|f\|^2_{\mathsf{L}_2(I,w(s)\mathrm{d}s)} \le \frac{2\omega}{\min\{1,a\}}\|f\|^2_{\mathsf{H}^1(I,w(s)\mathrm{d}s)}$$

for $0 < a \le \min\{1, \ell\}$ *and* $f \in \mathsf{H}^1(I, w(s)\mathrm{d}s) = \mathscr{H}^1(I)$.

Let us finally give another example, which is needed for topological perturbations like removing balls from a manifold or adding handles to it. We will present such constructions in a subsequent work.

Example A.2.21 (Polar coordinates). Assume that (X, g) is a Riemannian manifold. Here, we apply the warped product structure of a metric g on a manifold X given in polar coordinates.

Denote by $\kappa_x(r)$ the maximal absolute value of the sectional curvature on $\overline{B}_x(r)$. Let $r_\kappa(x)$ be the maximal radius $r > 0$ such that $r \le \pi/(2\sqrt{\kappa_x(r)})$ with the convention $1/0 = \infty$. We set

$$r_0(x) := \min\{\operatorname{inj\,rad} x, r_\kappa(x)\}. \tag{A.13}$$

For $x \in X$, we can parametrise $M_x := B(x, r_0(x))$ with *polar coordinates* $(s, y) \in I_x \times Y$, $I_x := (0, r_0(x))$, $Y = \mathbb{S}^m$. The metric in these coordinates is given by

$$g = \mathrm{d}s^2 + h_s,$$

where h_s is an s-depend metric on Y. We denote the standard metric on $\mathbb{S}^{d-1}$ by h. Recall that the warped product metric $g_0 = \mathrm{d}s^2 + s^2 h$ is the *flat* metric around x, i.e. $\mathbb{R}_+ \times_r \mathbb{S}^m \cong \mathbb{R}^{m+1}$, where $r(s) = s$ is the radius function (see Definition 5.3.2). From [Au82, Thm. 1.53] and the definition of $r_0(x)$ in (A.13), it follows that we have the estimate

$$\mathrm{d}s^2 + s^2\Big(\frac{2}{\pi}\Big)^2 h \leq g = \mathrm{d}s^2 + h_s \leq \mathrm{d}s^2 + s^2\Big(\sinh\Big(\frac{\pi}{2}\Big)\frac{2}{\pi}\Big)^2 h.$$

In particular, $\mathsf{L}_2(B_x)$ is an almost warped product with distortion function $\rho(s) = s^m$ and relative distortion $\rho_\infty = (\sinh(\pi/2))^m$.

References

[AC71] J. Aguilar and J. M. Combes, *A class of analytic perturbations for one-body Schrödinger Hamiltonians*, Comm. Math. Phys. **22** (1971), 269–279.

[AS00] M. Aizenman and J. H. Schenker, *The creation of spectral gaps by graph decoration*, Lett. Math. Phys. **53** (2000), 253–262.

[AADH94] S. Alama, M. Avellaneda, P. Deift, and R. Hempel, *On the existence of eigenvalues of a divergence-form operator $A + \lambda B$ in a gap of $\sigma(A)$*, Asymptotic Anal. **8** (1994), 311–344.

[ADH89] S. Alama, P. A. Deift, and R. Hempel, *Eigenvalue branches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$*, Commun. Math. Phys. **121** (1989), 291–321.

[ACF07] S. Albeverio, C. Cacciapuoti, and D. Finco, *Coupling in the singular limit of thin quantum waveguides*, J. Math. Phys. **48** (2007), 032103.

[AGH$^+$05] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, second ed., AMS Chelsea Publishing, Providence, RI, 2005, With an appendix by Pavel Exner.

[AK00] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Mathematical Society Lecture Note Series, vol. 271, Cambridge University Press, Cambridge, 2000, Solvable Schrödinger type operators.

[Al83] S. Alexander, *Superconductivity of networks. A percolation approach to the effects of disorder*, Phys. Rev. B (3) **27** (1983), 1541–1557.

[ALM04] C. Amovilli, F. E. Leys, and N. H. March, *Topology, connectivity, and electronic structure of C and B cages and the corresponding nanotubes*, J. Chem. Inf. Comput. Sci. **44** (2004), 122–135.

[ACh92] M. T. Anderson and J. Cheeger, *C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below*, J. Differential Geom. **35** (1992), 265–281.

[ACP09] C. Anné, G. Carron, and O. Post, *Gaps in the differential forms spectrum on cyclic coverings*, Math. Z. **262** (2009), 57–90.

[AC93] C. Anné and B. Colbois, *Opérateur de Hodge-Laplace sur des variétés compactes privées d'un nombre fini de boules*, J. Funct. Anal. **115** (1993), 190–211.

[AC95] ———, *Spectre du Laplacien agissant sur les p-formes différentielles et écrasement d'anses*, Math. Ann. **303** (1995), 545–573.

[A87] C. Anné, *Spectre du Laplacien et écrasement d'anses*, Ann. Sci. Éc. Norm. Super., IV. Sér. **20** (1987), 271–280.

[A90] ———, *Fonctions propres sur des variétés avec des anses fines, application à la multiplicité*, Comm. Partial Differential Equations **15** (1990), 1617–1630.

[A94] ———, *Laplaciens en interaction*, Manuscr. Math. **83** (1994), 59–74.

[AtE10] W. Arendt and A. F. M. ter Elst, *The Dirichlet-to-Neumann operator on rough domains*, Preprint `arXiv:1005.0875` (2010).

O. Post, *Spectral Analysis on Graph-Like Spaces*, Lecture Notes in Mathematics 2039, DOI 10.1007/978-3-642-23840-6, © Springer-Verlag Berlin Heidelberg 2012

[Ar00] Y. Arlinskii, *Abstract boundary conditions for maximal sectorial extensions of sectorial operators*, Math. Nachr. **209** (2000), 5–36.

[Au82] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften, vol. 252, Springer, New York, 1982.

[AHL+01] P. Auscher, S. Hofmann, M. Lacey, J. Lewis, A. McIntosh, and P. Tchamitchian, *The solution of Kato's conjectures*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), 601–606.

[ABGM91] Y. Avishai, D. Bessis, B. G. Giraud, and G. Mantica, *Quantum bound states in open geometries*, Phys. Rev. B **44** (1991), 8028–8034.

[AEY94] J. E. Avron, P. Exner, and Y. Last, *Periodic Schrödinger operators with large gaps and Wannier-Stark ladders* , Phys. Rev. Lett. **72** (1994), 896–899.

[BBK07] A. Badanin, J. Brüning, and E. Korotyaev, *Schrödinger operators on armchair nanotubes. II*, Preprint `arXiv:0707.3900` (2007).

[BF06] M. Baker and X. Faber, *Metrized graphs, Laplacian operators, and electrical networks*, Quantum graphs and their applications, Contemp. Math., vol. 415, Amer. Math. Soc., Providence, RI, 2006, pp. 15–33.

[BR07] M. Baker and R. Rumely, *Harmonic analysis on metrized graphs*, Canad. J. Math. **59** (2007), 225–275.

[BBC08] W. Ballmann, J. Brüning, and G. Carron, *Regularity and index theory for Dirac-Schrödinger systems with Lipschitz coefficients*, J. Math. Pures Appl. (9) **89** (2008), 429–476.

[BC71] E. Balslev and J. M. Combes, *Spectral properties of many-body Schrödinger operators with dilatation-analytic interactions*, Comm. Math. Phys. **22** (1971), 280–294.

[BSS06] R. Band, T. Shapira, and U. Smilansky, *Nodal domains on isospectral quantum graphs: the resolution of isospectrality*, J. Phys. A **39** (2006), 13999–14014.

[BBK06] M. T. Barlow, R. F. Bass, and T. Kumagai, *Stability of parabolic Harnack inequalities on metric measure spaces*, J. Math. Soc. Japan **58** (2006), 485–519.

[BCh05] R. A. Bartnik and P. T. Chruściel, *Boundary value problems for Dirac-type equations*, J. Reine Angew. Math. **579** (2005), 13–73.

[Ba10] H. Baumgärtel, *Resonances of quantum mechanical scattering systems and Lax-Phillips scattering theory*, J. Math. Phys. **51** (2010), 113508, 20.

[Be73] J. T. Beale, *Scattering frequencies of resonators*, Commun. pure appl. Math. **26** (1973), 549–563.

[BL07] J. Behrndt and M. Langer, *Boundary value problems for elliptic partial differential operators on bounded domains*, J. Funct. Anal. **243** (2007), 536–565.

[BL10] ——, *Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples*, Preprint (2010).

[BBG94] P. Bérard, G. Besson, and S. Gallot, *Embedding Riemannian manifolds by their heat kernel*, Geom. Funct. Anal. **4** (1994), 373–398.

[Bi53] M. Š. Birman, *On the theory of self-adjoint extensions of positive definite operators*, Doklady Akad. Nauk SSSR (N.S.) **91** (1953), 189–191.

[BCD89] P. Briet, J.-M. Combes, and P. Duclos, *Spectral stability under tunneling*, Comm. Math. Phys. **126** (1989), 133–156.

[BGW09] B. M. Brown, G. Grubb, and I. G. Wood, *M-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems*, Math. Nachr. **282** (2009), 314–347.

[BHM+09] B. M. Brown, J. Hinchcliffe, M. Marletta, S. Naboko, and I. Wood, *The abstract Titchmarsh-Weyl M-function for adjoint operator pairs and its relation to the spectrum*, Integral Equations Operator Theory **63** (2009), 297–320.

[BMNW08] B. M. Brown, M. Marletta, S. Naboko, and I. Wood, *Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices*, J. Lond. Math. Soc. (2) **77** (2008), 700–718.

[BHM94] R. M. Brown, P. D. Hislop, and A. Martinez, *Lower bounds on the interaction between cavities connected by a thin tube*, Duke Math. J. **73** (1994), 163–176.

[B89] J. Brüning, *On Schrödinger operators with discrete spectrum*, J. Funct. Anal. **85** (1989), 117–150.

[BEG03] J. Brüning, P. Exner, and V. Geyler, *Large gaps in point-coupled periodic systems of manifolds*, J. Phys. A **36** (2003), 4875–4890.

[BG03] J. Brüning and V. Geyler, *Scattering on compact manifolds with infinitely thin horns*, J. Math. Phys. **44** (2003), 371–405.

[BG05] ———, *Geometric scattering on compact Riemannian manifolds and spectral theory of automorphic functions*, Preprint (2005).

[BGL05] J. Brüning, V. Geyler, and I. Lobanov, *Spectral properties of Schrödinger operators on decorated graphs*, Mat. Zametki **77** (2005), 152–156.

[BGP07] J. Brüning, V. Geyler, and K. Pankrashkin, *Cantor and band spectra for periodic quantum graphs with magnetic fields*, Comm. Math. Phys. **269** (2007), 87–105.

[BGP08] ———, *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*, Rev. Math. Phys. **20** (2008), 1–70.

[BL92] J. Brüning and M. Lesch, *Hilbert complexes*, J. Funct. Anal. **108** (1992), 88–132.

[CE07] C. Cacciapuoti and P. Exner, *Nontrivial edge coupling from a Dirichlet network squeezing: the case of a bent waveguide*, J. Phys. A **40** (2007), L511–L523.

[Ca00] R. Carlson, *Nonclassical Sturm-Liouville problems and Schrödinger operators on radial trees*, Electron. J. Differential Equations (2000), No. 71, 24 pp. (electronic).

[CW07] D. I. Cartwright and W. Woess, *The spectrum of the averaging operator on a network (metric graph)*, Illinois J. Math. **51** (2007), 805–830.

[Ct97] C. Cattaneo, *The spectrum of the continuous Laplacian on a graph*, Monatsh. Math. **124** (1997), 215–235.

[Ch84] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, Orlando, 1984.

[CF78] I. Chavel and E. A. Feldman, *Spectra of domains in compact manifolds*, J. Funct. Anal. **30** (1978), 198–222.

[CF81] ———, *Spectra of manifolds with small handles*, Comment. Math. Helv. **56** (1981), 83–102.

[CLPZ02] G. A. Chechkin, D. Lukkassen, A. L. Piatnitski, and V. V. Zhikov, *On homogenization of networks and junctions*, Asymptot. Anal. **30** (2002), 61–80.

[Che76] S. Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), 43–55.

[CE04] T. Cheon and P. Exner, *An approximation to δ' couplings on graphs*, J. Phys. A **37** (2004), L329–L335.

[CET10] T. Cheon, P. Exner, and O. Turek, *Approximation of a general singular vertex coupling in quantum graphs*, Ann. Physics **325** (2010), 548–578.

[Chu97] F. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, vol. 92, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997.

[CGY96] F. Chung, A. Grigor′yan, and S.-T. Yau, *Upper bounds for eigenvalues of the discrete and continuous Laplace operators*, Adv. Math. **117** (1996), 165–178.

[CGY00] ———, *Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs*, Comm. Anal. Geom. **8** (2000), 969–1026.

[CC91] B. Colbois and G. Courtois, *Convergence de variétés et convergence du spectre du Laplacien*, Ann. Sci. École Norm. Sup. (4) **24** (1991), 507–518.

[CdV86] Y. Colin de Verdière, *Sur la multiplicité de la première valeur propre non nulle du laplacien*, Comment. Math. Helv. **61** (1986), 254–270.

[CdV87] ———, *Construction de laplaciens dont une partie finie du spectre est donnée*, Ann. Sci. École Norm. Sup. (4) **20** (1987), 599–615.

[CdV98] ———, *Spectres de graphes*, Cours Spécialisés [Specialized Courses], vol. 4, Société Mathématique de France, Paris, 1998.

[Co69] J.-M. Combes, *Relatively compact interactions in many particle systems*, Commun. Math. Phys. **12** (1969), 283–295.

[CDKS87] J.-M. Combes, P. Duclos, M. Klein, and R. Seiler, *The shape resonance*, Comm. Math. Phys. **110** (1987), 215–236.

[CT73] J.-M. Combes and L. Thomas, *Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators*, Comm. Math. Phys. **34** (1973), 251–270.

[CMM94] E. Curtis, E. Mooers, and J. Morrow, *Finding the conductors in circular networks from boundary measurements*, RAIRO Modél. Math. Anal. Numér. **28** (1994), 781–814.

[CIM98] E. B. Curtis, D. Ingerman, and J. A. Morrow, *Circular planar graphs and resistor networks*, Linear Algebra Appl. **283** (1998), 115–150.

[CM91] E. B. Curtis and J. A. Morrow, *The Dirichlet to Neumann map for a resistor network*, SIAM J. Appl. Math. **51** (1991), 1011–1029.

[CM00] ———, *Inverse problems for electrical networks*, World Scientific, Singapore, 2000.

[DT82] B. E. J. Dahlberg and E. Trubowitz, *A remark on two-dimensional periodic potentials*, Comment. Math. Helv. **57** (1982), 130–134.

[Da08] D. Daners, *Domain perturbation for linear and semi-linear boundary value problems*, Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 1–81.

[D95] E. B. Davies, *Spectral theory and differential operators*, Cambridge University Press, Cambridge, 1995.

[D97] ———, *Spectral enclosures and complex resonances for general self-adjoint operators*, LMS J. Comput. Math. **1** (1998), 42–74 (electronic).

[DS92] E. B. Davies and B. Simon, *Spectral properties of Neumann Laplacian of horns*, Geom. Funct. Anal. **2** (1992), 105–117.

[DEL10] E. B. Davies, P. Exner, and J. Lipovský, *Non-Weyl asymptotics for quantum graphs with general coupling conditions*, J. Phys. A **43** (2010), 474013, 16.

[DN00] B. Dekoninck and S. Nicaise, *The eigenvalue problem for networks of beams*, Linear Algebra Appl. **314** (2000), 165–189.

[DT04] G. F. Dell'Antonio and L. Tenuta, *Semiclassical analysis of constrained quantum systems*, J. Phys. A **37** (2004), 5605–5624.

[DT06] ———, *Quantum graphs as holonomic constraints*, J. Math. Phys. **47** (2006), 072102, 21.

[DeA07] G. Dell'Antonio, *Dynamics on quantum grpahs as constrained systems*, Rep. Math. Phys. **59** (2007), 267–279.

[DHMS00] V. Derkach, S. Hassi, M. Malamud, and H. de Snoo, *Generalized resolvents of symmetric operators and admissibility*, Methods Funct. Anal. Topology **6** (2000), 24–55.

[DHMS06] ———, *Boundary relations and their Weyl families*, Trans. Amer. Math. Soc. **358** (2006), 5351–5400 (electronic).

[DM91] V. Derkach and M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), 141–242.

[DM95] ———, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. (New York) **73** (1995), 141–242.

[Di69] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann)*, Gauthier-Villars Éditeur, Paris, 1969.

[Do84] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984), 787–794.

[EH89] W. D. Evans and D. J. Harris, *On the approximation numbers of Sobolev embeddings for irregular domains*, Quart. J. Math. Oxford Ser. (2) **40** (1989), 13–42.

[Ex95] P. Exner, *Lattice Kronig-Penney models*, Phys. Rev. Lett. **74** (1995), 3503–3506.

[Ex97] ———, *A duality between Schrödinger operators on graphs and certain Jacobi matrices*, Ann. Inst. H. Poincaré Phys. Théor. **66** (1997), 359–371.

[EG96] P. Exner and R. Gawlista, *Band spectra of rectangular graph superlattices*, Phys. Rev. B **53** (1996), 7275–7286.

[EKK+08] P. Exner, J. P. Keating, P. Kuchment, T. Sunada, and A. Teplyaev (eds.), *Analysis on graphs and its applications*, Proc. Symp. Pure Math., vol. 77, Providence, R.I., Amer. Math. Soc., 2008.

[EP05] P. Exner and O. Post, *Convergence of spectra of graph-like thin manifolds*, Journal of Geometry and Physics **54** (2005), 77–115.

[EP07] ———, *Convergence of resonances on thin branched quantum wave guides*, J. Math. Phys. **48** (2007), 092104 (43pp).

[EP09] ———, *Approximation of quantum graph vertex couplings by scaled Schrödinger operators on thin branched manifolds*, J. Phys. A **42** (2009), 415305 (22pp).

[EŠ89] P. Exner and P. Šeba, *Electrons in semiconductor microstructures: a challenge to operator theorists*, Schrödinger operators, standard and nonstandard (Dubna, 1988), World Sci. Publishing, Teaneck, NJ, 1989, pp. 78–100.

[EKW10] P. Exner, P. Kuchment, and B. Winn, *On the location of spectral edges in $\mathbb{Z}$-periodic media*, J. Phys. A **43** (2010), 474022, 8.

[FH05] K. Fissmer and U. Hamenstädt, *Spectral convergence of manifold pairs*, Comment. Math. Helv. **80** (2005), 725–754.

[Fre96] M. Freidlin, *Markov processes and differential equations: asymptotic problems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1996.

[FW93] M. I. Freidlin and A. D. Wentzell, *Diffusion processes on graphs and the averaging principle*, Ann. Probab. **21** (1993), 2215–2245.

[FT04a] J. Friedman and J.-P. Tillich, *Calculus on graphs*, Preprint `arXiv:cs.DM/0408028` (2004).

[FT04b] ———, *Wave equations for graphs and the edge-based Laplacian*, Pacific J. Math. **216** (2004), 229–266.

[Fri34] K. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren*, Math. Ann. **109** (1934), 465–487.

[Fu87] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), 517–547.

[FKW07] S. Fulling, P. Kuchment, and J. H. Wilson, *Index theorems for quantum graphs*, J. Phys. A **40** (2007), 14165–14180.

[GO91] B. Gaveau and M. Okada, *Differential forms and heat diffusion on one-dimensional singular varieties*, Bull. Sci. Math. **115** (1991), 61–79.

[GnS06] S. Gnutzmann and U. Smilansky, *Quantum graphs: Applications to quantum chaos and universal spectral statistics*, Adv. in Phys. **55** (2006), 527–625.

[GG91] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for operator differential equations*, Mathematics and its Applications (Soviet Series), vol. 48, Kluwer Academic Publishers Group, Dordrecht, 1991.

[GY83] S. Graffi and K. Yajima, *Exterior complex scaling and the AC-Stark effect in a Coulomb field*, Comm. Math. Phys. **89** (1983), 277–301.

[G08a] D. Grieser, *Spectra of graph neighborhoods and scattering*, Proc. Lond. Math. Soc. (3) **97** (2008), 718–752.

[G08b] ———, *Thin tubes in mathematical physics, global analysis and spectral geometry*, in [EKK+08] (2008), 565–593.

[GJ09] D. Grieser and D. Jerison, *Asymptotics of eigenfunctions on plane domains*, Pacific J. Math. **240** (2009), 109–133.

[Gr03] A. Grigor′yan, *Heat kernels and function theory on metric measure spaces*, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, pp. 143–172.

[Gr10] ———, *Heat kernels on metric measure spaces with regular volume growth*, Handbook of geometric analysis, No. 2, Adv. Lect. Math. (ALM), vol. 13, Int. Press, Somerville, MA, 2010, pp. 1–60.

[GSC99] A. Grigor′yan and L. Saloff-Coste, *Heat kernel on connected sums of Riemannian manifolds*, Math. Res. Lett. **6** (1999), 307–321.

[Grv85] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[Gro81] M. Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques, vol. 1, CEDIC, Paris, 1981, Edited by J. Lafontaine and P. Pansu.

[Gro99] ———, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999.

[Gru68] G. Grubb, *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 425–513.

[Gru08] ———, *Krein resolvent formulas for elliptic boundary problems in nonsmooth domains*, Rend. Semin. Mat. Univ. Politec. Torino **66** (2008), 271–297.

[Gru10a] ———, *Extension theory for elliptic partial differential operators with pseudodifferential methods*, Preprint `arXiv:1008.1081` (2010).

[Gru10b] ———, *Krein-like extensions and the lower boundedness problem for elliptic operators on exterior domains*, Preprint `arXiv:1002.4549` (2010).

[GSm01] B. Gutkin and U. Smilansky, *Can one hear the shape of a graph?*, J. Phys. A **34** (2001), 6061–6068.

[Ha00] M. Harmer, *Hermitian symplectic geometry and extension theory*, J. Phys. A **33** (2000), 9193–9203.

[HKSW07] J. M. Harrison, P. Kuchment, A. Sobolev, and B. Winn, *On occurrence of spectral edges for periodic operators inside the Brillouin zone*, J. Phys. A **40** (2007), 7597–7618.

[He96] E. Hebey, *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Mathematics. Vol. 1635, Springer, Berlin, 1996.

[HM87] B. Helffer and A. Martinez, *Comparaison entre les diverses notions de résonances*, Helv. Phys. Acta **60** (1987), 992–1003.

[HSS91] R. Hempel, L. A. Seco, and B. Simon, *The essential spectrum of Neumann Laplacians on some bounded singular domains*, J. Funct. Anal. **102** (1991), 448–483.

[H06] R. Hempel, *On the lowest eigenvalue of the Laplacian with Neumann boundary condition at a small obstacle*, J. Comput. Appl. Math. **194** (2006), 54–74.

[HP03] R. Hempel and O. Post, *Spectral gaps for periodic elliptic operators with high contrast: an overview*, Progress in analysis, Vol. I, II (Berlin, 2001), World Sci. Publ., River Edge, NJ, 2003, pp. 577–587.

[HS99] Y. Higuchi and T. Shirai, *A remark on the spectrum of magnetic Laplacianon a graph*, Yokohama Math. J. **47** (1999), 129–141.

[HS04] ———, *Some spectral and geometric properties for infinite graphs*, Discrete geometric analysis, Contemp. Math., vol. 347, Amer. Math. Soc., Providence, RI, 2004, pp. 29–56.

[Hi92] P. D. Hislop, *Singular perturbations of Dirichlet and Neumann domains and resonances for obstacle scattering*, Astérisque (1992), 8, 197–216, Méthodes semi-classiques, Vol. 2 (Nantes, 1991).

[HiL01] P. D. Hislop and C. V. Lutzer, *Spectral asymptotics of the Dirichlet-to-Neumann map on multiply connected domains in* $\mathbb{R}^d$, Inverse Problems **17** (2001), 1717–1741.

[HiM91] P. D. Hislop and A. Martinez, *Scattering resonances of a Helmholtz resonator*, Indiana Univ. Math. J. **40** (1991), 767–788.

[HiS89] P. D. Hislop and I. M. Sigal, *Semiclassical theory of shape resonances in quantum mechanics*, Mem. Amer. Math. Soc. **78** (1989), 123.

[HiP09] P. D. Hislop and O. Post, *Anderson localization for radial tree-like quantum graphs*, Waves Random Complex Media **19** (2009), 216–261.

[Hu86] W. Hunziker, *Distortion analyticity and molecular resonance curves*, Ann. Inst. H. Poincaré Phys. Théor. **45** (1986), 339–358.

[JS10] P. Joly and A. Semin, *Study of propagation of acoustic waves in junction of thin slots*, Preprint (2010).

[Ju01] C. M. Judge, *Tracking eigenvalues to the frontier of moduli space. I. Convergence and spectral accumulation*, J. Funct. Anal. **184** (2001), 273–290.

[Ka66] M. Kac, *Can one hear the shape of a drum?* Am. Math. Mon. **73** (1966), 1–23.

[KKVW09] U. Kant, T. Klauss, J. Voigt, and M. Weber, *Dirichlet forms for singular one-dimensional operators and on graphs*, J. Evol. Equ. **9** (2009), 637–659.

[KP88] L. Karp and M. Pinsky, *First-order asymptotics of the principal eigenvalue of tubular neighborhoods*, Geometry of random motion (Ithaca, N.Y., 1987), Contemp. Math., vol. 73, Amer. Math. Soc., Providence, RI, 1988, pp. 105–119.

[Ka02] A. Kasue, *Convergence of Riemannian manifolds and Laplace operators. I*, Ann. Inst. Fourier (Grenoble) **52** (2002), 1219–1257.

[Ka06] ———, *Convergence of Riemannian manifolds and Laplace operators. II*, Potential Anal. **24** (2006), 137–194.

[KK94] A. Kasue and H. Kumura, *Spectral convergence of Riemannian manifolds*, Tohoku Math. J. (2) **46** (1994), 147–179.

[KK96] ———, *Spectral convergence of Riemannian manifolds. II*, Tohoku Math. J. (2) **48** (1996), 71–120.

[K66] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.

[K67] ———, *Scattering theory with two Hilbert spaces*, J. Functional Analysis **1** (1967), 342–369.

[KL07] E. Korotyaev and I. Lobanov, *Schrödinger operators on zigzag nanotubes*, Ann. Henri Poincaré **8** (2007), 1151–1176.

[KPS07] V. Kostrykin, J. Potthoff, and R. Schrader, *Heat kernels on metric graphs and a trace formula*, Adventures in Mathematical Physics, Contemp. Math., vol. 447, Amer. Math. Soc., Providence, RI, 2007, pp. 175–198.

[KS99] V. Kostrykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A **32** (1999), 595–630.

[KS03] ———, *Quantum wires with magnetic fluxes*, Comm. Math. Phys. **237** (2003), 161–179, Dedicated to Rudolf Haag.

[KS06] ———, *Laplacians on metric graphs: eigenvalues, resolvents and semigroups*, Quantum graphs and their applications, Contemp. Math., vol. 415, Amer. Math. Soc., Providence, RI, 2006, pp. 201–225.

[Ko00] S. Kosugi, *A semilinear elliptic equation in a thin network-shaped domain*, J. Math. Soc. Japan **52** (2000), 673–697.

[Ko02] ———, *Semilinear elliptic equations on thin network-shaped domains with variable thickness*, J. Differential Equations **183** (2002), 165–188.

[KSm97] T. Kottos and U. Smilansky, *Quantum chaos on graphs*, Phys. Rev. Lett. **79** (1997), 4794–4797.

[KSm99] ———, *Periodic orbit theory and spectral statistics for quantum graphs*, Ann. Physics **274** (1999), 76–124.

[KSm03] ———, *Quantum graphs: a simple model for chaotic scattering*, J. Phys. A **36** (2003), 3501–3524, Random matrix theory.

[KOZ94] S. M. Kozlov, O. A. Oleĭnik, and V. V. Zhikov, *Homogenization of differential operators and integral functionals*, Springer, Berlin, 1994.

[Kr47] M. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I*, Rec. Math. [Mat. Sbornik] N.S. **20(62)** (1947), 431–495.

[Ku01] P. Kuchment, *The mathematics of photonic crystals*, Mathematical modeling in optical science, SIAM, Philadelphia, PA, 2001, pp. 207–272.

[Ku04] ———, *Quantum graphs: I. Some basic structures*, Waves Random Media **14** (2004), S107–S128.

[Ku05] ———, *Quantum graphs: II. Some spectral properties of quantum and combinatorial graphs*, J. Phys. A **38** (2005), 4887–4900.

[Ku08] ———, *Quantum graphs: an introduction and a brief survey*, in [EKK+08] (2008), 291–312.

[KuP07] P. Kuchment and O. Post, *On the spectra of carbon nano-structures*, Comm. Math. Phys. **275** (2007), 805–826.

[KuZ01] P. Kuchment and H. Zeng, *Convergence of spectra of mesoscopic systems collapsing onto a graph*, J. Math. Anal. Appl. **258** (2001), 671–700.

[KuZ03] ———, *Asymptotics of spectra of Neumann Laplacians in thin domains*, Advances in differential equations and mathematical physics (Birmingham, AL, 2002), Contemp. Math., vol. 327, Amer. Math. Soc., Providence, RI, 2003, pp. 199–213.

[Ks08] P. Kurasov, *Graph Laplacians and topology*, Ark. Mat. **46** (2008), 95–111.

[KN05] P. Kurasov and M. Nowaczyk, *Inverse spectral problem for quantum graphs*, J. Phys. A **38** (2005), 4901–4915.

[KN06] ———, *Corrigendum to [KN05]*, J. Phys. A **39** (2006), 993.

[KZh98] S. Kusuoka and X. Y. Zhou, *Waves on fractal-like manifolds and effective energy propagation*, Probab. Theory Related Fields **110** (1998), 473–495.

[KSh03] K. Kuwae and T. Shioya, *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry*, Comm. Anal. Geom. **11** (2003), 599–673.

[LU01] M. Lassas and G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. École Norm. Sup. (4) **34** (2001), 771–787.

[LPPV08] D. Lenz, N. Peyerimhoff, O. Post, and I. Veselić, *Continuity properties of the integrated density of states on manifolds*, Jpn. J. Math. **3** (2008), 121–161.

[LY86] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[LM68] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 1*, Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968.

[LP07] F. Lledó and O. Post, *Generating spectral gaps by geometry*, Prospects in Mathematical Physics, Young Researchers Symposium of the 14th International Congress on Mathematical Physics, Lisbon, July 2003, Contemporary Mathematics, vol. 437, 2007, pp. 159–169.

[LP08a] ———, *Eigenvalue bracketing for discrete and metric graphs*, J. Math. Anal. Appl. **348** (2008), 806–833.

[LP08b] ———, *Existence of spectral gaps, covering manifolds and residually finite groups*, Rev. Math. Phys. **20** (2008), 199–231.

[Lo01] J. Lott, *On the spectrum of a finite-volume negatively-curved manifold*, Amer. J. Math. **123** (2001), 185–205.

[Lü02] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete., vol. 44, Springer, Berlin, 2002.

[Ma10] M. M. Malamud, *Spectral theory of elliptic operators in exterior domains*, Russ. J. Math. Phys. **17** (2010), 96–125.

[McI72] A. McIntosh, *On the comparability of $A^{1/2}$ and $A^{*1/2}$*, Proc. Amer. Math. Soc. **32** (1972), 430–434.

[Me01] T. A. Mel′nyk, *Hausdorff convergence and asymptotic estimates of the spectrum of a perturbed operator*, Z. Anal. Anwendungen **20** (2001), 941–957.

[Me03] ———, *A scheme for studying the spectrum of a family of perturbed operators and its application to spectral problems in thick junctions*, Nelīnīĭnī Koliv. **6** (2003), 233–251.

[Me08] ———, *Homogenization of a boundary-value problem with a nonlinear boundary condition in a thick junction of type* $3 : 2 : 1$, Math. Methods Appl. Sci. **31** (2008), 1005–1027.

[MW89] B. Mohar and W. Woess, *A survey on spectra of infinite graphs*, Bull. London Math. Soc. **21** (1989), 209–234.

[MV06] S. Molchanov and B. Vainberg, *Transition from a network of thin fibers to the quantum graph: an explicitly solvable model*, Quantum graphs and their applications, Contemp. Math., vol. 415, Amer. Math. Soc., Providence, RI, 2006, pp. 227–239.

[MV07] ———, *Scattering solutions in networks of thin fibers: small diameter asymptotics*, Comm. Math. Phys. **273** (2007), 533–559.

[MV08] ———, *Laplace operator in networks of thin fibers: spectrum near the threshold*, Stochastic analysis in mathematical physics, World Sci. Publ., Hackensack, NJ, 2008, pp. 69–93.

[Mo94] U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal. **123** (1994), 368–421.

[MNP10] D. Mugnolo, R. Nittka, and O. Post, *Convergence of sectorial operators on varying Hilbert spaces*, Preprint `arXiv:1007.3932` (2010).

[MM06] J. Müller and W. Müller, *Regularized determinants of Laplace-type operators, analytic surgery, and relative determinants*, Duke Math. J. **133** (2006), 259–312.

[Na88] A. I. Nachman, *Reconstructions from boundary measurements*, Ann. of Math. (2) **128** (1988), 531–576.

[Ni87] S. Nicaise, *Approche spectrale des problèmes de diffusion sur les réseaux*, Séminaire de Théorie du Potentiel, Paris, No. 8, Lecture Notes in Math., vol. 1235, Springer, Berlin, 1987, pp. 120–140.

[No07] M. Nowaczyk, *Inverse spectral problem for quantum graphs with rationally dependent edges*, Operator theory, analysis and mathematical physics, Oper. Theory Adv. Appl., vol. 174, Birkhäuser, Basel, 2007, pp. 105–116.

[OSY92] O. A. Oleĭnik, A. S. Shamaev, and G. A. Yosifian, *Mathematical problems in elasticity and homogenization*, Studies in Mathematics and its Applications, vol. 26, North-Holland Publishing Co., Amsterdam, 1992.

[Oz81] S. Ozawa, *Singular variation of domains and eigenvalues of the Laplacian*, Duke Math. J. **48** (1981), 767–778.

[Oz82] ———, *Spectra of domains with small spherical Neumann boundary*, Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), 190–192.

[Pan06] K. Pankrashkin, *Spectra of Schrödinger operators on equilateral quantum graphs*, Lett. Math. Phys. **77** (2006), 139–154.

[Par08] L. Parnovski, *Bethe-Sommerfeld conjecture*, Ann. Henri Poincaré **9** (2008), 457–508.

[Pas04] S. E. Pastukhova, *On the convergence of hyperbolic semigroups in a variable Hilbert space*, Tr. Semin. im. I. G. Petrovskogo (2004), 215–249, 343.

[Pav02] B. Pavlov, *Resonance quantum switch: matching domains*, Surveys in analysis and operator theory (Canberra, 2001), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 40, Austral. Nat. Univ., Canberra, 2002, pp. 127–156.

[Pav07] ———, *A star-graph model via operator extension*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 365–384.

[PKWA10] R. C. Penner, M. Knudsen, C. Wiuf, and J. E. Andersen, *Fatgraph models of proteins*, Comm. Pure Appl. Math. **63** (2010), 1249–1297.

[PWZ08] Y. Pinchover, G. Wolansky, and D. Zelig, *Spectral properties of Schrödinger operators on radial N-dimensional infinite trees*, Israel J. Math. **165** (2008), 281–328.

[Pc08] A. Posilicano, *Self-adjoint extensions of restrictions*, Oper. Matrices **2** (2008), 483–506.

[PcR09] A. Posilicano and L. Raimondi, *Krein's resolvent formula for self-adjoint extensions of symmetric second-order elliptic differential operators*, J. Phys. A **42** (2009), 015204 (11pp).

[P03a] O. Post, *Eigenvalues in spectral gaps of a perturbed periodic manifold*, Mathematische Nachrichten **261–262** (2003), 141–162.

[P03b] ———, *Periodic manifolds with spectral gaps*, J. Diff. Equations **187** (2003), 23–45.

[P05] ———, *Branched quantum wave guides with Dirichlet boundary conditions: the decoupling case*, Journal of Physics A: Mathematical and General **38** (2005), 4917–4931.

[P06] ———, *Spectral convergence of quasi-one-dimensional spaces*, Ann. Henri Poincaré **7** (2006), 933–973.

[P07] ———, *First order operators and boundary triples*, Russ. J. Math. Phys. **14** (2007), 482–492.

[P08] ———, *Equilateral quantum graphs and boundary triples*, in [EKK+08] (2008), 469–490.

[P09a] ———, *First order approach and index theorems for discrete and metric graphs*, Ann. Henri Poincaré **10** (2009), 823–866.

[P09b] ———, *Spectral analysis of metric graphs and related spaces*, in "Limits of graphs in group theory", eds. G. Arzhantseva and A. Valette, Presses Polytechniques et Universitaires Romandes, 109–140 (2009), 109–140.

[RT75] J. Rauch and M. Taylor, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal. **18** (1975), 27–59.

[RS80] M. Reed and B. Simon, *Methods of modern mathematical physics I–IV*, Academic Press, New York, 1980.

[Ros97] S. Rosenberg, *The Laplacian on a Riemannian manifold*, London Mathematical Society Student Texts. 31, Cambridge University Press, Cambridge, 1997.

[Rot84] J.-P. Roth, *Le spectre du laplacien sur un graphe*, Théorie du potentiel (Orsay, 1983), Lecture Notes in Math., vol. 1096, Springer, Berlin, 1984, pp. 521–539.

[RuS01a] J. Rubinstein and M. Schatzman, *Variational problems on multiply connected thin strips. I. Basic estimates and convergence of the Laplacian spectrum*, Arch. Ration. Mech. Anal. **160** (2001), 271–308.

[RuS01b] ———, *Variational problems on multiply connected thin strips. II. Convergence of the Ginzburg-Landau functional*, Arch. Ration. Mech. Anal. **160** (2001), 309–324.

[RSc53] K. Ruedenberg and C. W. Scherr, *Free–electron network model for conjugated systems, I. Theory*, J. Chem. Phys. **21** (1953), 1565–1581.

[Ry07] V. Ryzhov, *A general boundary value problem and its Weyl function*, Opuscula Math. **27** (2007), 305–331.

[Sa00] Y. Saito, *The limiting equation for Neumann Laplacians on shrinking domains.*, Electron. J. Differ. Equ. **31** (2000), 25 p.

[SP80] E. Sánchez-Palencia, *Nonhomogeneous media and vibration theory*, Lecture Notes in Physics, vol. 127, Springer, Berlin, 1980.

[SRW89] R. L. Schult, D. G. Ravenhall, and H. W. Wyld, *Quantum bound states in a classically unbound system of crossed wires*, Phys. Rev. B **39** (1989), 5476–5479.

[Sh00] T. Shirai, *The spectrum of infinite regular line graphs*, Trans. Amer. Math. Soc. **352** (2000), 115–132.

[Si72] B. Simon, *Quadratic form techniques and the Balslev-Combes theorem*, Comm. Math. Phys. **27** (1972), 1–9.

[Si79] ———, *The definition of molecular resonance curves by the method of exterior complex scaling*, Phys. Lett. A **71** (1979), 211–214.

[Si96] ———, *Operators with singular continuous spectrum. VI. Graph Laplacians and Laplace-Beltrami operators*, Proc. Amer. Math. Soc. **124** (1996), 1177–1182.

[Sk79] M. M. Skriganov, *Proof of the Bethe-Sommerfeld conjecture in dimension* 2, Dokl. Akad. Nauk SSSR **248** (1979), 39–42.

[Sk85] ———, *The spectrum band structure of the three-dimensional Schrödinger operator with periodic potential*, Invent. Math. **80** (1985), 107–121.

[Sm07] U. Smilansky, *Quantum chaos on discrete graphs*, J. Phys. A **40** (2007), F621–F630.

[SmS06] U. Smilansky and M. Solomyak, *The quantum graph as a limit of a network of physical wires*, Quantum graphs and their applications, Contemp. Math., vol. 415, Amer. Math. Soc., Providence, RI, 2006, pp. 283–291.

[So04] M. Solomyak, *On the spectrum of the Laplacian on regular metric trees*, Waves Random Media **14** (2004), S155–S171, Special section on quantum graphs.

[Su08] T. Sunada, *Discrete geometric analysis*, Analysison Graphs and its Applications (Providence, R.I.) (P. Exner, J. P. Keating, P. Kuchment, T. Sunada, and A. Teplayaev, eds.), Proc. Symp. Pure Math., vol. 77, Amer. Math. Soc., 2008, pp. 51–83.

[Ta96] M. E. Taylor, *Partial differential equations. Basic theory*, Springer-Verlag, New York, 1996.

[Tel83] N. Teleman, *The index of signature operators on Lipschitz manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), 39–78 (1984).

[Tep98] A. Teplyaev, *Spectral analysis on infinite Sierpiński gaskets*, J. Funct. Anal. **159** (1998), 537–567.

[TW09] S. Teufel and J. Wachsmuth, *Effective hamiltonians for constrained quantum systems*, Preprint `arXiv:0907.0351` (2009).

[Vi52] M. I. Vishik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obšč. **1** (1952), 187–246 (English translation in Am. Math. Soc., Transl. (2) 24 (1963), 107–172).

[vB85] J. von Below, *A characteristic equation associated to an eigenvalue problem on C^2-networks*, Linear Algebra Appl. **71** (1985), 309–325.

[vN30] J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102** (1930), 49–131.

[W84] J. Weidmann, *Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet*, Math. Scand. **54** (1984), 51–69.

[Y80] K. Yosida, *Functional analysis*, sixth ed., Grundlehren der Mathematischen Wissenschaften, vol. 123, Springer-Verlag, Berlin, 1980.

[Ze05] D. Zelig, *Properties of solutions of partial differential equations on human lung-shaped domains*, Ph.D. thesis, Ph-D thesis Technion, Haifa, 2005.

[Zh02] V. V. Zhikov, *Homogenization of elasticity problems on singular structures*, Izv. Ross. Akad. Nauk Ser. Mat. **66** (2002), 81–148.

[ZP07] V. V. Zhikov and S. E. Pastukhova, *On the Trotter-Kato theorem in a variable space*, Funktsional. Anal. i Prilozhen. **41** (2007), 22–29, 96.

[Zw99] M. Zworski, *Resonances in physics and geometry*, Notices Amer. Math. Soc. **46** (1999), 319–328.

Notation

We give an overview of some general notation used in this work. Other commonly used symbols are listed in the index.

General notation

- $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$,
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$,
- $\mathbb{R}_+ = [0, \infty)$,$\partial_+ I := \overline{I} \cap \overline{\mathbb{R}_+ \setminus I}$ denotes the topological boundary in the *relative* topology $\mathbb{R}_+$ ($I \subset \mathbb{R}_+$), e.g., $\partial_+[0, \lambda] = \{\lambda\}$.
- We use the short hand notation

$$\partial_s f(s) = \frac{\partial f(s)}{\partial s}$$

Moreover, $\mathrm{d}u$ denotes the exterior derivative of $u \in \mathsf{C}^\infty(M)$.

Sets and topology

- $|A|$ denotes the number of elements in the set A
- $C = A \mathbin{\dot\cup} B$ means that $C = A \cup B$ and that A, B are disjoint ($A \cap B = \emptyset$)

Let A, B, C be subsets of a topological space X.

- $\overline{A}$ denotes the *closure* of A, $\mathring{A}$ denotes the *interior* of A.

O. Post, *Spectral Analysis on Graph-Like Spaces*, Lecture Notes in Mathematics 2039,
DOI 10.1007/978-3-642-23840-6, © Springer-Verlag Berlin Heidelberg 2012

Measure spaces

Let X be a measure space.

- $X = A \mathbin{\overline{\cup}} B$ (“X is the disjoint union of A and B up to measure 0”) means that $X = A \cup B$ and that $A \cap B$ has measure 0.
- $X = \overline{\bigcup}_i A_i$ (“X is the disjoint union of all A_i up to measure 0”) means that $X = \bigcup_i A_i$ and that $A_i \cap A_j$ has measure 0 for all $i \neq j$.
- $\operatorname{len}(I)$ denotes the Lebesque measure of I (length of the interval $I \subset \mathbb{R}$).

Hilbert spaces, operators and function spaces

All our Hilbert spaces are assumed to be separable, i.e. having a *countable* orthonormal basis. The inner product and all sesquilinear forms are anti-linear in their *first* argument.

- $a \overset{\text{CS}}{\leq} b$ means that we applied the Cauchy-Schwarz inequality.
- The Cauchy-Young reads as

$$2\big|\langle f, g\rangle\big| \overset{\text{CY}}{\leq} \eta\|f\|^2 + \frac{1}{\eta}\|g\|^2 \tag{1}$$

  for any $\eta > 0$. In particular, if $\eta = 1$,

$$2\big|\langle f, g\rangle\big| \leq \|f\|^2 + \|g\|^2.$$

- $\mathscr{H}_1 \oplus \mathscr{H}_2$ denotes the *orthogonal sum* of the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$; elements of $\mathscr{H}_1 \oplus \mathscr{H}_2$ are written as $f = (f_1, f_2)$ or $f = f_1 \oplus f_2$. Similarly, elements of $\bigoplus_e \mathscr{H}_e$ written as $f = \{f_e\}_e$ or sometimes as $\bigoplus_e f_e$.
- $\mathscr{H}_1 \dotplus \mathscr{H}_2$ denotes the *topological sum* of the spaces $\mathscr{H}_i \subset \mathscr{H}$, i.e., the direct (but not necessarily orthogonal) sum, and $\mathscr{H}_i$ is assumed to be closed in $\mathscr{H}$.
- If a linear operator A is not defined for all elements of $\mathscr{H}$, we write $\operatorname{dom} A$ for the *domain of* A. Similarly, for a quadratic form $\mathfrak{h}$, we write $\operatorname{dom}\mathfrak{h}$ for its domain. The corresponding sesquilinear form is then definded on $\mathfrak{h}\colon \operatorname{dom}\mathfrak{h} \times \operatorname{dom}\mathfrak{h} \longrightarrow \mathbb{C}$.
- For a linear operator $A\colon \mathscr{H} \longrightarrow \mathscr{G}$ we denote the *range* of A by $\operatorname{ran} A := \{\, g \in \mathscr{G} \mid \exists h \in \operatorname{dom} A\colon Ah = g \,\}$ and the *kernel* of A by $\ker A := \{\, h \in \operatorname{dom} A \mid Ah = 0 \,\}$.
- $\mathsf{C}_{\mathrm{c}}^{\infty}(I)$ denotes the space of smooth functions with compact support in $\mathring{I}$.
- Sobolev spaces $\mathsf{H}^k(I) := \{\, f \in \mathsf{L}_2(I) \mid f^{(i)} \in \mathsf{L}_2(I) \quad \forall\, i = 1, \dots, k \,\}$ with norm $\|f\|^2_{\mathsf{H}^k(I)} := \sum_{i=0}^{k} \|f^{(i)}\|^2_{\mathsf{L}_2(I)}$, where $f^{(i)}$ is the i-th (weak) derivative, $\mathring{\mathsf{H}}^k(I)$ is the closure of $\mathsf{C}_{\mathrm{c}}^{\infty}(I)$ w.r.t. the norm $\|\cdot\|_{\mathsf{H}^k(I)}$.

Index

Symbols

O. Post, *Spectral Analysis on Graph-Like Spaces*, Lecture Notes in Mathematics 2039, DOI 10.1007/978-3-642-23840-6,

E

F

I

J

K

L

M

N

O

P

Q

R

S

X

Y

Z

Editorial Policy (for the publication of monographs)

1. Lecture Notes aim to report new developments in all areas of mathematics and their applications - quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome.
   Monograph manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They may be based on specialised lecture courses. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for doctoral theses to be accepted for the Lecture Notes series, though habilitation theses may be appropriate.

2. Manuscripts should be submitted either online at www.editorialmanager.com/lnm to Springer's mathematics editorial in Heidelberg, or to one of the series editors. In general, manuscripts will be sent out to 2 external referees for evaluation. If a decision cannot yet be reached on the basis of the first 2 reports, further referees may be contacted: The author will be informed of this. A final decision to publish can be made only on the basis of the complete manuscript, however a refereeing process leading to a preliminary decision can be based on a pre-final or incomplete manuscript. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter, a bibliography and several sample chapters.
   Authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer refereeing times and nevertheless unclear referees' recommendations, making further refereeing of a final draft necessary.
   Authors should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

3. Manuscripts should in general be submitted in English. Final manuscripts should contain at least 100 pages of mathematical text and should always include

   – a table of contents;
   – an informative introduction, with adequate motivation and perhaps some historical remarks: it should be accessible to a reader not intimately familiar with the topic treated;
   – a subject index: as a rule this is genuinely helpful for the reader.

   For evaluation purposes, manuscripts may be submitted in print or electronic form (print form is still preferred by most referees), in the latter case preferably as pdf- or zipped psfiles. Lecture Notes volumes are, as a rule, printed digitally from the authors' files. To ensure best results, authors are asked to use the LaTeX2e style files available from Springer's web-server at:

   ftp://ftp.springer.de/pub/tex/latex/svmonot1/ (for monographs) and
   ftp://ftp.springer.de/pub/tex/latex/svmultt1/ (for summer schools/tutorials).

Additional technical instructions, if necessary, are available on request from lnm@springer.com.

4. Careful preparation of the manuscripts will help keep production time short besides ensuring satisfactory appearance of the finished book in print and online. After acceptance of the manuscript authors will be asked to prepare the final LaTeX source files and also the corresponding dvi-, pdf- or zipped ps-file. The LaTeX source files are essential for producing the full-text online version of the book (see http://www.springerlink.com/openurl.asp?genre=journal&issn=0075-8434 for the existing online volumes of LNM). The actual production of a Lecture Notes volume takes approximately 12 weeks.

5. Authors receive a total of 50 free copies of their volume, but no royalties. They are entitled to a discount of 33.3 % on the price of Springer books purchased for their personal use, if ordering directly from Springer.

6. Commitment to publish is made by letter of intent rather than by signing a formal contract. Springer-Verlag secures the copyright for each volume. Authors are free to reuse material contained in their LNM volumes in later publications: a brief written (or e-mail) request for formal permission is sufficient.

Addresses:

Professor J.-M. Morel, CMLA,
École Normale Supérieure de Cachan,
61 Avenue du Président Wilson, 94235 Cachan Cedex, France
E-mail: morel@cmla.ens-cachan.fr

Professor B. Teissier, Institut Mathématique de Jussieu,
UMR 7586 du CNRS, Équipe “Géométrie et Dynamique”,
175 rue du Chevaleret
75013 Paris, France
E-mail: teissier@math.jussieu.fr

For the “Mathematical Biosciences Subseries” of LNM:

Professor P. K. Maini, Center for Mathematical Biology,
Mathematical Institute, 24-29 St Giles,
Oxford OX1 3LP, UK
E-mail : maini@maths.ox.ac.uk

Springer, Mathematics Editorial, Tiergartenstr. 17,
69121 Heidelberg, Germany,
Tel.: +49 (6221) 4876-8259

Fax: +49 (6221) 4876-8259
E-mail: lnm@springer.com