Chapter 12
Appendix

This appendix is intended to recall some of the basic notions and properties of simplicial complexes and CW-complexes which are used but not explicitly explained in the main text. The reader not familiar with concepts such as simplicial approximations and adjunction spaces, could find this appendix useful. However, more complete expositions on these subjects can be found in Spanier's book [75, Chap. 3] and in Munkres' [61, Chaps. 1 and 2]. Standard references for CW-complexes are also [28, 38, 45].

A.1 Simplicial Complexes

A *simplicial complex* K consists of a set V_K, called the set of vertices, and a set S_K of finite nonempty subsets of V_K, which is called the set of simplices, satisfying that any subset of V of cardinality one is a simplex and any nonempty subset of a simplex is a simplex. By abuse of notation we will write $v \in K$ and $\sigma \in K$ if $v \in V_K$ and $\sigma \in S_K$. Many times, as it is the custom, we will identify a simplicial complex with its set of simplices.

If a simplex σ is contained in another simplex τ, it is called a *face* of τ, and it is a *proper face* if in addition $\sigma \neq \tau$. A simplex with $n+1$ vertices is called an *n-simplex*, and we say that its *dimension* is n. Note that the vertices of K correspond to the 0-simplices. The *dimension* of K is the supremum of the dimensions of its simplices. If K is empty, its dimension is -1 and if K contains simplices of arbitrary large dimension, its dimension is infinite. An *n-complex* is a simplicial complex of dimension n. The maximal simplices (those which are not proper faces of any other simplex) are sometimes called *facets*. A finite dimensional simplicial complex is called *homogeneous* (or *pure*) if all its maximal simplices have the same dimension. A *subcomplex* of a simplicial complex K is a simplicial complex L such that $V_L \subseteq V_K$ and $S_L \subseteq S_K$. A subcomplex $L \subseteq K$ is said to be *full* if any simplex of K with

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all its vertices in L is also a simplex of L. In this case we say that L is the full subcomplex of K spanned by the vertices $v \in V_L$.

Given a simplex $\sigma = \{v_0, v_1, \ldots, v_n\}$ of dimension n, the *closed simplex* $\overline{\sigma}$ is the set of formal convex combinations $\sum\limits_{i=0}^{n} \alpha_i v_i$ with $\alpha_i \geq 0$ for every $0 \leq i \leq n$ and $\sum \alpha_i = 1$. A closed simplex is a metric space with the metric d given by

$$d(\sum_{i=0}^{n} \alpha_i v_i, \sum_{i=0}^{n} \beta_i v_i) = \sqrt{\sum_{i=0}^{n} (\alpha_i - \beta_i)^2}.$$

The *geometric realization* $|K|$ of a simplicial complex K is the set of formal convex combinations $\sum\limits_{v \in K} \alpha_v v$ such that $\{v \mid \alpha_v > 0\}$ is a simplex of K. Therefore, $|K|$ can be regarded as the union of the closed simplices $\overline{\sigma}$ with $\sigma \in K$. The topology of $|K|$ is the final (coherent) topology with respect to the closed simplices. In other words, a set $U \subseteq |K|$ is open (resp. closed) if and only if $U \cap \overline{\sigma}$ is open (resp. closed) in the metric space $\overline{\sigma}$ for every $\sigma \in K$.

The *support* (or *carrier*) of a point $x = \sum\limits_{v \in K} \alpha_v v \in |K|$ is the simplex *support*$(x) = \{v \mid \alpha_v > 0\}$. If σ is a simplex, the *open simplex* $\overset{\circ}{\sigma}$ is the subset of $\overline{\sigma}$ of points whose support is exactly σ. Note that if two points $x, y \in |K|$ lie in the same closed simplex, then the convex combination $tx + (1 - t)y$ is a well defined element in $|K|$. If $L \subseteq K$, $|L|$ is a closed subspace of $|K|$. It is not hard to prove that the topology of the set $\overline{\sigma}$ as a subspace of $|K|$ is the original metric topology on $\overline{\sigma}$. Moreover, if K is a finite simplicial complex (i.e. with a finite number of vertices), the topology of $|K|$ coincides with the metric topology defined as before

$$d(\sum_{v \in K} \alpha_v v, \sum_{v \in K} \beta_v v) = \sqrt{\sum_{v \in K} (\alpha_v - \beta_v)^2}.$$

Moreover, in this case $|K|$ can be imbedded in $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

It is easy to prove that if U is an open subspace of $|K|$, then it has the final topology with respect to the subspaces $U \cap \overline{\sigma} \subseteq \overline{\sigma}$.

A *polyhedron* is the geometric realization of a simplicial complex and a *triangulation* of a polyhedron X is a simplicial complex K whose geometric realization is homeomorphic to X. Any polyhedron is a Hausdorff space.

Since $|K|$ has the final topology with respect to its closed simplices, a map f from $|K|$ to a topological space X is continuous if and only if each of the restrictions $f|_{\overline{\sigma}} : \overline{\sigma} \to X$ is continuous. Moreover, by the exponential law, it can be shown that a map $H : |K| \times I \to X$ is continuous if and only if $H|_{\overline{\sigma} \times I} : \overline{\sigma} \times I \to X$ is continuous for each $\sigma \in K$.

A *simplicial map* $\varphi : K \to L$ between two simplicial complexes K and L is a vertex map $V_K \to V_L$ that sends simplices into simplices. A simplicial map

$\varphi : K \to L$ induces a well defined continuous map $|\varphi| : |K| \to |L|$ between the geometric realizations defined by $|\varphi|(\sum\limits_{v\in K} \alpha_v v) = \sum\limits_{v\in K} \alpha_v \varphi(v)$.

Lemma A.1.1. *Let K be a simplicial complex and let F be a compact subset of $|K|$. Then there exists a finite subcomplex L of K such that $F \subseteq |L|$.*

Proof. Take one point in $F \cap \overset{\circ}{\sigma}$ for every open simplex intersecting F. Denote by D the set of all these points. Let $A \subseteq D$. Since the intersection of A with each closed simplex is finite, it is closed, and then A is closed in $|K|$. Therefore D is discrete and compact, and, in particular, finite. Thus, F intersects only finitely many open simplices. The complex L generated by (i.e. the smallest complex containing) the simplices σ such that $\overset{\circ}{\sigma}$ intersects F is a finite subcomplex of K which satisfies the required property. □

Proposition A.1.2. *Let K and L be two simplicial complexes and let $f, g : |K| \to |L|$ be two continuous maps such that for every $x \in |K|$ there exists $\sigma \in L$ with $f(x), g(x) \in \overline{\sigma}$. Then f and g are homotopic.*

Proof. The map $H : |K| \times I \to |L|$ given by $H(x, t) = tg(x) + (1 - t)f(x)$ is well defined because $g(x)$ and $f(x)$ lie in a same closed simplex. In order to prove that H is continuous it suffices to show that it is continuous in $\overline{\sigma} \times I$ for every $\sigma \in K$. If $\sigma \in K$, $\overline{\sigma}$ is compact and therefore $f(\overline{\sigma})$ and $g(\overline{\sigma})$ are compact. By Lemma A.1.1, $f(\overline{\sigma})$ is contained in the geometric realization of a finite subcomplex L_1 and $g(\overline{\sigma}) \subseteq |L_2|$ for a finite subcomplex $L_2 \subseteq L$. Therefore, $H(\overline{\sigma} \times I)$ is contained in the realization of a finite subcomplex M of L, namely the full subcomplex spanned by the vertices of L_1 and L_2. We have to show then that $H|_{\overline{\sigma}\times I} : \overline{\sigma} \times I \to |M|$ is continuous, where M is a finite simplicial complex. But this is clear since both $\overline{\sigma}$ and $|M|$ have the metric topology.

$$\begin{aligned} d(H(x,t), H(y,s)) &\leq d(tg(x) + (1-t)f(x), sg(x) + (1-s)f(x)) \\ &\quad + d(sg(x) + (1-s)f(x), sg(y) + (1-s)f(y)) \\ &\leq 2|t-s| + d(f(x), f(y)) + d(g(x), g(y)). \end{aligned}$$

Therefore, the continuity of H follows from that of f and g. □

The homotopy H used in the proof of Proposition A.1.2 is called the *linear homotopy* from f to g.

Two simplicial maps $\varphi, \psi : K \to L$ are said to be *contiguous* if for every $\sigma \in K$, $\varphi(\sigma) \cup \psi(\sigma)$ is a simplex of L. In this case, $|\varphi|$ and $|\psi|$ satisfy the hypothesis of Proposition A.1.2, since if $x \in \overline{\sigma}$, both $|\varphi|(x)$ and $|\psi|(x)$ lie in $\overline{\varphi(\sigma) \cup \psi(\sigma)}$. Therefore we deduce the following

Corollary A.1.3. *If φ and ψ are two contiguous maps, $|\varphi|$ and $|\psi|$ are homotopic.*

A *simplicial cone with apex* v is a simplicial complex K with a vertex v satisfying that for every simplex σ of K, $\sigma \cup \{v\}$ is also a simplex.

Corollary A.1.4. *If K is a simplicial cone, $|K|$ is contractible.*

Proof. Let v be the (an) apex of K. The simplicial map that sends every vertex to v is contiguous to the identity by definition of cone. Therefore, by Corollary A.1.3, the identity of $|K|$ is homotopic to a constant. □

Given a simplicial complex K, its *barycentric subdivision* K' is the following simplicial complex. The vertices of K' are the simplices of K, and a simplex of K' is a chain of simplices of K, i.e. a set $\{\sigma_0, \sigma_1, \dots, \sigma_n\}$ of simplices of K such that $\sigma_0 \subsetneq \sigma_1 \subsetneq \dots \subsetneq \sigma_n$. The *barycenter* of a simplex $\sigma \in K$ is the point $b(\sigma) = \sum\limits_{v\in\sigma} \frac{v}{\#\sigma} \in |K|$. The *linear map* $s_K : |K'| \to |K|$ defined by $s_K(\sigma) = b(\sigma)$ is a homeomorphism. By linear we mean a map that preserves convex combinations. The spaces $|K'|$ and $|K|$ are usually identified by means of the map s_K in such a way that s_K becomes the identity map.

A simplicial map $\varphi : K \to L$ is said to be a *simplicial approximation* of a continuous map $f : |K| \to |L|$ if $f(x) \in \overline{\sigma}$ implies $|\varphi|(x) \in \overline{\sigma}$ for every $x \in |K|$. Note that in this situation f and $|\varphi|$ are homotopic by Proposition A.1.2.

Proposition A.1.5. *A vertex map $\varphi : K' \to K$ is a simplicial approximation to the identity $s_K : |K'| \to |K|$ if and only if $\varphi(\sigma) \in \sigma$ for every $\sigma \in K$.*

Proof. Suppose that φ is a simplicial approximation to the identity. If σ is a vertex of K', $s_K(\sigma) = b(\sigma) \in \overline{\sigma}$, then $|\varphi|(\sigma)$ must be contained in $\overline{\sigma}$ as well. Therefore $\varphi(\sigma)$ is a vertex of σ.

Conversely, suppose $\varphi : K' \to K$ is a vertex map in the hypothesis of the proposition. If $\sigma_0 \subsetneq \sigma_1 \subsetneq \dots \subsetneq \sigma_n$ is a chain of simplices of K, then $\varphi(\{\sigma_0, \sigma_1, \dots, \sigma_n\}) \subseteq \sigma_n$. Therefore φ is a simplicial map. Moreover if

$$x = \sum_{i=0}^{n} \alpha_i \sigma_i,$$

with $\alpha_i > 0$ for every i, then

$$s_K(x) = \sum_{i=0}^{n} \alpha_i \sum_{v\in\sigma_i} \frac{v}{\#\sigma_i} \in \overset{\circ}{\sigma}_n.$$

On the other hand, $|\varphi|(x) = \sum\limits_{i=0}^{n} \alpha_i \varphi(\sigma_i) \in \overline{\sigma}_n$. Thus, φ is a simplicial approximation of s_K. □

As an immediate consequence we deduce that there exist simplicial approximations to the identity.

The *n-th barycentric subdivision* of K is defined recursively $K^{(n)} = (K^{(n-1)})'$. A simplicial approximation to the identity $1_{|K|} : |K^{(n)}| \to |K|$ is in this case a simplicial approximation of the map $s_K s_{K'} \dots s_{K^{(n-1)}} : |K^{(n)}| \to |K|$. If $f : |K| \to |L|$ is a continuous map, a simplicial map $\varphi : K^{(n)} \to L$ is called an approximation of f if it is a simplicial approximation of $f s_K s_{K'} \dots s_{K^{(n-1)}}$.

The proof of the following result on simplicial approximations can be found in [75, Corollary 3.4.5, Lemma 3.5.4].

Proposition A.1.6.

1. *The composition of simplicial approximations of two maps is a simplicial approximation of the composition of those maps.*
2. *Two simplicial approximations to the same map are contiguous.*

Two simplicial maps $\varphi, \psi : K \to L$ are said to be in the same *contiguity class* if there is a sequence of simplicial maps $\varphi = \varphi_0, \varphi_1, \dots, \varphi_k = \psi$ from K to L, such that φ_i and φ_{i+1} are contiguous for every $0 \leq i < k$.

The following result is known as the Simplicial Approximation Theorem. Its proof can be found in [75, Theorems 3.4.8 and 3.5.6].

Theorem A.1.7. *Let K be a finite simplicial complex and L a simplicial complex. Given any continuous map $f : |K| \to |L|$ there exist $n \in \mathbb{N}$ and a simplicial approximation $\varphi : K^{(n)} \to L$ to f. Moreover, if $f, g : |K| \to |L|$ are homotopic, there exist $n \in \mathbb{N}$ and simplicial approximations $\varphi, \psi : K^{(n)} \to L$ to f and g in the same contiguity class.*

A.2 CW-Complexes and a Gluing Theorem

If X, Y and Z are three topological spaces, and $f : X \to Y$, $g : X \to Z$ are continuous maps, the *pushout* of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z & \end{tikzcd}$$

is a space P together with maps $\overline{f} : Z \to P$ and $\overline{g} : Y \to P$ such that $\overline{f} g = \overline{g} f$ and with the following universal property: for any space Q and maps $\widetilde{f} : Z \to Q$ and $\widetilde{g} : Y \to Q$ such that $\widetilde{f} g = \widetilde{g} f$, there exists a unique map $h : P \to Q$ such that $h\overline{f} = \widetilde{f}$ and $h\overline{g} = \widetilde{g}$.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "\overline{g}"] \arrow[ddr, bend left, "\widetilde{g}"] & \\
Z \arrow[r, "\overline{f}"] \arrow[drr, bend right, "\widetilde{f}"'] & P \arrow[dr, dashed, "h"] & \\
& & Q
\end{tikzcd}
$$

It is not hard to see that the space P is unique up to homeomorphism and in fact it can be characterized as the space $P = (Z \sqcup Y)/\sim$, where $\sim$ is the relation that identifies $f(x)$ with $g(x)$ for every $x \in X$. The maps $\overline{f}$ and $\overline{g}$ are the canonical inclusions into the disjoint union $Z \sqcup Y$ composed with the quotient map.

For example, if A is a subspace of a space X, the quotient X/A is the pushout of the diagram

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & X \\
*
\end{tikzcd}
$$

If A and B are two closed (or two open) subspaces of a space X and $X = A \cup B$, then X is the pushout of $A \hookleftarrow A \cap B \hookrightarrow B$.

A *topological pair* is an ordered pair of spaces (X, A) with A a subspace of X. In the next definition the inclusions $A \hookrightarrow A \times I$ and $X \hookrightarrow X \times I$ of the spaces A and X in the bases of their cylinders will be denoted by i_0 and j_0 respectively.

Definition A.2.1. A topological pair (X, A) is said to have the *homotopy extension property* if for any space Y and maps $H : A \times I \to Y$, $f : X \to Y$ such that $Hi_0 = f|_A$, there exists a map $\overline{H} : X \times I \to Y$ such that $\overline{H} j_0 = f$ and $\overline{H}|_{A \times I} = H$.

$$
\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, hook] & A \times I \arrow[d, hook] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, hook, "j_0"] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\overline{H}"] & \\
& & Y
\end{tikzcd}
$$

If A is a subspace of a space X, the inclusion $A \hookrightarrow X$ is said to be a *closed cofibration* if A is closed in X and (X, A) has the homotopy extension property.

Definition A.2.2. If $A \subseteq X$, and the inclusion $A \hookrightarrow X$ is a closed cofibration, the pushout Z of a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X & \end{tikzcd}$$

is called the *adjunction space* of X to Y by f. In this case, it can be proved that Y is a closed subspace of Z (see [21]).

When $X = D^n$ is a disk and $A = S^{n-1}$ is its boundary, we say that Z is constructed from Y by *adjoining an n-cell.* More generally, if $X = \bigsqcup_{\alpha \in \Lambda} D^n$ is a union of n-dimensional disks indexed by an arbitrary set Λ and $A = \bigsqcup_{\alpha \in \Lambda} S^{n-1}$, we say that Z is obtained from Y by adjoining n-cells.

Definition A.2.3. A *CW-structure* for a topological space X is a filtration of X by subspaces $X^0 \subseteq X^1 \subseteq \dots$, where X^0 is a discrete space, X^n is constructed from X^{n-1} by adjoining n-cells and X is the union of the spaces X^n, $n \geq 0$, with the final (coherent) topology. The subspace X^n is called the *n-skeleton* of X.

A *CW-complex* is a space X endowed with a CW-structure. Note that, since X^n is obtained from X^{n-1} by adjoining n-cells, there is a pushout

$$\begin{tikzcd} \bigsqcup_{\alpha \in \Lambda_n} S^{n-1} \arrow[r, "\bigsqcup_{\alpha \in \Lambda_n} \varphi_\alpha"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ \bigsqcup_{\alpha \in \Lambda_n} D^n \arrow[r, "\bigsqcup_{\alpha \in \Lambda_n} \psi_\alpha"] & X^n \end{tikzcd}$$

The image of the map $\psi_\alpha : D^n \to X$ is called the *closed cell* $\overline{e^n_\alpha}$. The image of $\varphi_\alpha : S^{n-1} \to X$ is the *boundary* $\dot{e}^n_\alpha$ of the cell and the (*open*) *cell* e^n_α is concretely the subspace $e^n_\alpha = \overline{e^n_\alpha} \smallsetminus \dot{e}^n_\alpha$, which is homeomorphic to the interior of the disk D^n. The maps φ_α and ψ_α are called the *attaching map* and the *characteristic map* of the cell e^n_α, respectively. For us, the attaching and characteristic maps will be part of the structure of the CW-complex. For some authors, though, the CW-structure consists only of the filtration

by skeleta. In that case the characteristic maps are not part of the structure, only their existence is required.

A cell e^n_α is called an *n-cell* or cell of *dimension* n. The *dimension* of a CW-complex X is -1 if it is empty, n if $X^n \neq X^{n-1}$ and $X^m = X^n$ for every $m \geq n$, and infinite if $X^n \neq X$ for every n.

Simplicial complexes are CW-complexes. Their cells are the open simplices. Many properties of polyhedra hold in fact for CW-complexes. For instance, any CW-complex has the final topology with respect to its closed cells and every CW-complex is a Hausdorff space.

A *subcomplex* of a CW-complex X is a closed subspace of X which is a union of cells of X. The following is a basic result about CW-complexes. A proof can be found in [75, 7.6.12] or [28, Corollary 1.4.7].

Theorem A.2.4. *If A is a subcomplex of a CW-complex X, the inclusion $A \hookrightarrow X$ is a closed cofibration.*

The following gluing theorem appears for instance in [21, 7.5.7, Corollary 2].

Theorem A.2.5. *Suppose that the following diagram is a pushout of topological spaces*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r, "\overline{f}"] & Z \end{tikzcd}$$

in which $A \hookrightarrow X$ is a closed cofibration and $f : A \to Y$ is a homotopy equivalence. Then $\overline{f} : X \to Z$ is a homotopy equivalence.

The following are two applications that show how to use the gluing theorem together with Theorem A.2.4 to study the homotopy type of CW-complexes and polyhedra.

Proposition A.2.6. *Let K be a simplicial complex and let v be a vertex of K. If the link $|lk(v)|$ is contractible, $|K|$ and $|K \smallsetminus v|$ are homotopy equivalent.*

Proof. Consider the following diagram

$$\begin{tikzcd} {|lk(v)|} \arrow[r, hook] \arrow[d, hook] & {|st(v)|} \arrow[d, hook] \\ {|K \smallsetminus v|} \arrow[r, hook] & {|K|.} \end{tikzcd}$$

It is a pushout because $|st(v)| \cup |K \smallsetminus v| = |K|$, $|st(v)| \cap |K \smallsetminus v| = |lk(v)|$ and both $|st(v)|$ and $|K \smallsetminus v|$ are closed subspaces of $|K|$. Moreover, $|lk(v)| \hookrightarrow |K \smallsetminus v|$ is a cofibration by Theorem A.2.4 and, since $|lk(v)| \hookrightarrow |st(v)|$ is a homotopy equivalence because $st(v)$ is a cone, $|K \smallsetminus v| \hookrightarrow |K|$ is a homotopy equivalence by Theorem A.2.5. □

Proposition A.2.7. *If Y is a contractible subcomplex of a CW-complex X, the quotient map $X \to X/Y$ is a homotopy equivalence.*

Proof. Since $Y \to *$ is a homotopy equivalence, by the gluing theorem so is $X \to X/Y$. □

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Edited by J.-M. Morel, B. Teissier; P.K. Maini

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