Appendix A

Linear Elliptic Partial Differential Equations

A.1 Sobolev Spaces

We are going to use the integration theory of Lebesgue. Therefore, we shall always identify functions which differ only on a set of measure zero. Thus, when we speak about a *function*, we actually always mean an equivalence class of functions under the above identification. In particular, a statement like "the function f is continuous" is to be interpreted as "f differs from a continuous function at most on a set of measure zero" or equivalently "the equivalence class of f contains a continuous function".

Replacing functions by their equivalence classes is necessary in order to make the L^p- and Sobolev spaces Banach spaces.

Definition A.1.1. $\Omega \subset \mathbb{R}^d$ open, $p \in \mathbb{R}$, $p \geq 1$,

$$\begin{aligned} L^p(\Omega) := \Big\{ & f : \Omega \to \mathbb{R} \cup \{\pm\infty\} \text{ measurable} \\ & \text{and } \|f\|_{L^p(\Omega)} := \Big(\int_\Omega |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty \Big\}, \\ L^\infty(\Omega) := \Big\{ & f : \Omega \to \mathbb{R} \cup \{\pm\infty\} \text{ measurable} \\ & \text{and } \|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty \Big\}, \quad \text{with} \end{aligned}$$

$$\operatorname*{ess\,sup}_{x \in \Omega} f(x) := \inf\big\{ a \in \mathbb{R} \cup \{\infty\} : f(x) \leq a \text{ for almost all } x \in \Omega \big\}.$$

J. Jost, *Riemannian Geometry and Geometric Analysis*, Universitext,
DOI 10.1007/978-3-642-21298-7,

Theorem A.1.1. *With norm* $\|\cdot\|_{L^p(\Omega)}$, $L^p(\Omega)$ *is a Banach space for* $1 \leq p \leq \infty$.

Theorem A.1.2 (Hölder's Inequality). *Let* $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *(*$q = \infty$ *for* $p = 1$ *and vice versa),* $f \in L^p(\Omega), g \in L^q(\Omega)$. *Then* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |f(x)g(x)|\,dx \leq \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q dx\right)^{\frac{1}{q}}.$$

More generally, for $p_1, \ldots, p_m \geq 1$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, $f_i \in L^{p_i}(\Omega)$, $i = 1, \ldots, m$,

$$\int \Big|\prod_{i=1}^m f_i(x)\Big| dx \leq \prod_{i=1}^m \left(\int |f_i(x)|^{p_i}\right)^{\frac{1}{p_i}}.$$

Theorem A.1.3. *If* $(f_n)_{n\in\mathbb{N}}$ *converges to* f *in* $L^p(\Omega)$, *then a subsequence converges pointwise almost everywhere to* f.

Theorem A.1.4. $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$ *for* $1 \leq p < \infty$ *(but not for* $p = \infty$*).*

Theorem A.1.5. *If* $f \in L^2(\Omega)$ *and*

$$\int_\Omega f(x)\varphi(x)\,dx = 0, \quad \textit{for every } \varphi \in C_0^\infty(\Omega),$$

then

$$f = 0.$$

We let

$$L^p_{\text{loc}}(\Omega) := \big\{f : \Omega \to \mathbb{R} \cup \{\pm\infty\} : f \in L^p(\Omega') \text{ for } \forall\ \Omega' \Subset \Omega\big\}.$$

Definition A.1.2. Let $f \in L^1_{\text{loc}}(\Omega)$. We call $v \in L^1_{\text{loc}}(\Omega)$ the *weak derivative* of f in the direction of x^i, $v = D_i f$, if

$$\int_\Omega v(x)\varphi(x)\,dx = -\int_\Omega f(x)\frac{\partial\varphi(x)}{\partial x^i}\,dx,$$

for all $\varphi \in C_0^1(\Omega)$. Here $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.

Weak derivatives of higher order are similarly defined (notation $D_{\boldsymbol{\alpha}} f$ for a multiindex $\boldsymbol{\alpha}$).

Definition A.1.3. Let $k \in \mathbb{N}$, $1 \le p \le \infty$. We define the *Sobolev spaces* and *Sobolev norms* as follows:

$$
\begin{aligned}
W^{k,p}(\Omega) &:= \{f \in L^p(\Omega) : \forall \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| \le k : D_{\boldsymbol{\alpha}} f \in L^p(\Omega)\}, \\
\|f\|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\boldsymbol{\alpha}| \le k} \int_\Omega |D_{\boldsymbol{\alpha}} f|^p \right)^{\frac{1}{p}} \text{ for } 1 \le p < \infty, \\
\|f\|_{W^{k,\infty}(\Omega)} &:= \sum_{|\boldsymbol{\alpha}| \le k} \operatorname{ess\,sup}_{\boldsymbol{\alpha} \in \Omega} |D_{\boldsymbol{\alpha}} f(x)|, \\
H_0^{k,p}(\Omega) &:= \text{closure of } C_0^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{W^{k,p}(\Omega)}, \\
H^{k,p}(\Omega) &:= \text{closure of } C^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{W^{k,p}(\Omega)}.
\end{aligned}
$$

Theorem A.1.6. *$W^{k,p}(\Omega) = H^{k,p}(\Omega)$ for $1 \le p < \infty, k \in \mathbb{N}$. $W^{k,p}(\Omega)$ is a Banach space for $1 \le p \le \infty$, $k \in \mathbb{N}$.*

Some local properties of Sobolev functions:

Lemma A.1.1. *$\Omega \subset \mathbb{R}^d$ open, $f \in H^{1,1}(\Omega)$, $i \in \{1, \ldots, d\}$. Then for almost all $\lambda \in \mathbb{R}, f|_{\{x^i = \lambda\}}$ is absolutely continuous.*

Let $f \in L^1(\Omega)$, Ω open in $\mathbb{R}^d$. Then for almost all $x_0 \in \Omega$,

$$\lim_{r \to 0} \frac{1}{|B(x_0, r)|} \int |f(x) - f(x_0)|\, dx = 0$$

($|B(x_0, r)| = \omega_d r^d$ denotes the Lebesgue measure of the ball $B(x_0, r)$).

An x_0 satisfying this property is called a Lebesgue point. If x_0 is a Lebesgue point, then f is approximately continuous at x_0; this means the following:
For $\varepsilon > 0$, let

$$S_\varepsilon := \{y \in \Omega : |f(y) - f(x_0)| < \varepsilon\}.$$

Then

$$\lim_{r \to 0} \frac{|S_\varepsilon \cap B(x_o, r)|}{|B(x_0, r)|} = 1 \quad \text{for all } \varepsilon > 0.$$

Similarly, $f \in H^{1,1}(\Omega)$ is called approximately differentiable at $x_0 \in \Omega$, with approximate derivative $\nabla f(x_0)$, if for

$$S_\varepsilon^1 := \{y \in \Omega : |f(y) - f(x_0)(y - x_0) - \nabla f(x_0)| \le \varepsilon |y - x_0|\},$$

$$\lim_{r \to 0} \frac{|S_\varepsilon^1 \cap B(x_0, r)|}{|B(x_0, r)|} = 1 \quad \text{for all } \varepsilon > 0.$$

We then have

Lemma A.1.2. *A function $f \in H^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, is approximately differentiable almost everywhere, and the weak derivative coincides with the approximate derivative almost everywhere.*

Lemma A.1.3. *$\Omega \subset \mathbb{R}^d$ open, $\ell : \mathbb{R} \to \mathbb{R}$ Lipschitz, $f \in H^{1,p}(\Omega)$. If $\ell \circ f \in L^p(\Omega)$, then $\ell \circ f \in H^{1,p}(\Omega)$ and for almost all $x \in \Omega$,*

$$D_i(\ell \circ f)(x) = \ell'(f(x))D_i f(x), \quad i = 1, \dots, d.$$

Theorem A.1.7 (Sobolev Embedding Theorem). *$\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H_0^{1,p}(\Omega)$. Then*

$$\begin{aligned} &f \in L^{\frac{np}{n-p}} && \text{for } p < n, \\ &f \in C^0(\overline{\Omega}) && \text{for } p > n. \end{aligned}$$

More precisely, there exist constants $c = c(n,p)$:

$$\begin{aligned} \|f\|_{L^{\frac{np}{n-p}}(\Omega)} &\leq c\|Df\|_{L^p(\Omega)} && \text{for } p < n, \\ \sup_{x\in\Omega} |f(x)| &\leq c\text{Vol}\,(\Omega)^{\frac{1}{n}-\frac{1}{p}}\|Df\|_{L^p(\Omega)} && \text{for } p > n. \end{aligned}$$

For $n = p$, $f \in L^q(\Omega)$ for all $q < \infty$.

Remark. $H^{1,n}(\Omega)$ is not contained in $C^0(\overline{\Omega})$ or $L^\infty(\Omega)$.

Let us consider the following example:
$d \geq 2$, $\Omega = \overset{\circ}{B}(0, \frac{1}{e}) \subset \mathbb{R}^d$, $f(x) := \log\log\frac{1}{|x|}$ is in $H_0^{1,d}(\Omega)$, but has a singularity at $x = 0$ and is unbounded there. Using this example, we may even produce functions in $H^{1,d}$ with a dense set of singular points. For example, take $\Omega = \overset{\circ}{B}(0, \frac{1}{2e}) \subset \mathbb{R}^d$, let $(p_\nu)_{\nu\in\mathbb{N}}$ be a dense sequence of points in Ω and consider

$$g(x) := \sum_\nu 2^{-\nu} f(x - p_\nu).$$

Corollary A.1.1 (Poincaré Inequality). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then for all $f \in H_0^{1,2}(\Omega)$*

$$\|f\|_{L^2(\Omega)} \leq \text{const Vol}\,(\Omega)^{\frac{1}{n}}\|Df\|_{L^2(\Omega)}. \tag{A.1.1}$$

When in place of a domain Ω in Euclidean space, we have a compact Riemannian manifold M, this becomes

Corollary A.1.2 (Poincaré Inequality). *Let M be a compact Riemannian manifold. If $f \in H^{1,2}(M)$ satisfies $\int_M f = 0$, then*

$$\|f\|_{L^2(M)} \leq \text{const Vol}\,(M)^{\frac{1}{n}}\|Df\|_{L^2(M)}. \tag{A.1.2}$$

We recall here that we assume all our manifolds to be connected. Otherwise, we would have to impose the condition that the integral of f be 0 on every component of M.

Corollary A.1.3. *$\Omega \subset \mathbb{R}^n$ open, bounded, then,*

$$H_0^{k,p}(\Omega) \subset \begin{cases} L^{\frac{np}{n-kp}}(\Omega) & \text{for } kp < n, \\ C^m(\overline{\Omega}) & \text{for } 0 \le m < k - \frac{n}{p}. \end{cases}$$

In particular, if $f \in H_0^{k,p}(\Omega)$ for all $k \in \mathbb{N}$ and some fixed p, then $f \in C^\infty(\overline{\Omega})$.

Theorem A.1.8 (Rellich-Kondrachov Compactness Theorem). *$\Omega \subset \mathbb{R}^n$ open, bounded. Suppose $1 \le q < \frac{np}{n-p}$ if $p < d$, and $1 \le q < \infty$ if $p \ge d$. Then $H_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, i.e. if $(f_n)_{n\in\mathbb{N}} \subset H_0^{1,p}(\Omega)$ satisfies*

$$\|f_n\|_{W^{1,p}(\Omega)} \le \text{const},$$

then a subsequence converges in $L^q(\Omega)$.

Corollary A.1.4. *Ω as before. Then $H_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Similarly, on a compact Riemannian manifold M, $H^{1,2}(M)$ is compactly embedded in $L^2(M)$.*

$H^{k,2}(\Omega)$ is a Hilbert space, the scalar product is

$$(f,g)_{H^{k,2}(\Omega)} := \sum_{|\boldsymbol{\alpha}|\le k} \int_\Omega D_{\boldsymbol{\alpha}} f(x) D_{\boldsymbol{\alpha}} g(x)\, dx.$$

Finally, we recall the concept of weak convergence:

Let H be a Hilbert space with norm $\|\cdot\|$ and a product $\langle\cdot,\cdot\rangle$. Then $(v_n)_{n\in\mathbb{N}} \subset H$ is called *weakly convergent* to $v \in H$,

$$v_n \rightharpoonup v,$$

iff

$$\langle v_n, w\rangle \to \langle v, w\rangle \quad \text{for all } w \in H.$$

Theorem A.1.9. *Every bounded sequence $(v_n)_{n\in\mathbb{N}}$ in H contains a weakly convergent subsequence, and if the limit is v,*

$$\|v\| \le \liminf_{n\to\infty} \|v_n\|$$

(where (v_n) now is the weakly convergent subsequence).

Example. Let (e_n) be an orthonormal sequence in an infinite dimensional Hilbert space. Then $e_n \rightharpoonup 0$. In particular, the inequality in Theorem A.1.9 may be strict.

A.2 Existence and Regularity Theory for Solutions of Linear Elliptic Equations

Ω will always be an open subset of $\mathbb{R}^m$.

For technical purposes, one often has to approximate weak derivatives if they are not yet known to exist by difference quotients which are supposed to exist. Thus, let

$$\begin{aligned}&f \in L^2(\Omega, \mathbb{R}),\\&(e_1, \dots, e_m) \text{ an orthonormal basis of } \mathbb{R}^m,\\&h \in \mathbb{R}, \quad h \neq 0.\end{aligned}$$

We put

$$\Delta_i^h f(x) := \frac{f(x + he_i) - f(x)}{h} \quad (\text{if dist}\,(x, \partial\Omega) > |h|).$$

If $\varphi \in L^2(\Omega)$, $\operatorname{supp}\varphi \Subset \Omega$, $|h| < \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$, we have

$$\int_\Omega (\Delta_i^h f(x))\varphi(x)\,dx = -\int_\Omega f(x)\Delta_i^{-h}\varphi(x)\,dx. \tag{A.2.1}$$

Lemma A.2.1. *If $f \in H^{1,2}(\Omega)$, $\Omega' \Subset \Omega$, $|h| < \operatorname{dist}(\Omega', \partial\Omega)$, then $\Delta_i^h f \in L^2(\Omega')$ and*

$$\|\Delta_i^h f\|_{L^2(\Omega')} \le \|D_i f\|_{L^2(\Omega)} \quad \text{for } i = 1, \dots, m.$$

Conversely,

Lemma A.2.2. *If $f \in L^2(\Omega)$ and if for some $K < \infty$*

$$\|\Delta_i^{h_n} f\|_{L^2(\Omega')} \le K$$

for some sequence $h_n \to 0$ and all $\Omega' \Subset \Omega$ with $h_n < \operatorname{dist}(\Omega', \partial\Omega)$, then the weak derivative $D_i f$ exists and

$$\|D_i f\|_{L^2(\Omega)} \le K.$$

The fundamental elliptic regularity theorems for Sobolev norms may be proved by approximating weak derivatives by difference quotients.

We now formulate the general regularity theorem.

We consider an operator

$$Lf(x) := \frac{\partial}{\partial x^\alpha}\Big(a^{\alpha\beta}(x)\frac{\partial}{\partial x^\beta} f(x)\Big) \tag{A.2.2}$$

for $x \in \Omega$, $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^m$.

We assume that there exist constants $0 < \lambda \le \mu$ with

$$\lambda|\xi|^2 \le a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \le \mu|\xi|^2 \tag{A.2.3}$$

for all $x \in \Omega, \xi \in \mathbb{R}^m$. We say that L *is uniformly elliptic*. Let $k \in L^2(\Omega)$. Then $f \in H^{1,2}(\Omega)$ is called a *weak solution* of

$$Lf = k$$

if

$$\int_\Omega a^{\alpha\beta}(x) D_\beta f(x) D_\alpha \varphi(x)\, dx = -\int_\Omega k(x)\varphi(x)\, dx \tag{A.2.4}$$

for all $\varphi \in H_0^{1,2}(\Omega)$.

Theorem A.2.1. *Let $f \in H^{1,2}(\Omega)$ be a weak solution of (A.2.4). Suppose $k \in H^{\nu,2}(\Omega)$, $a^{\alpha\beta} \in C^{\nu+1}(\Omega)$ ($\nu \in \mathbb{N}$).*

Then

$$f \in H^{\nu+2,2}(\Omega')$$

for every $\Omega' \Subset \Omega$.

If

$$\|a^{\alpha\beta}\|_{C^{\nu+1}(\Omega)} \le K_\nu,$$

then

$$\|f\|_{H^{\nu+2,2}(\Omega')} \le c(\|f\|_{L^2(\Omega)} + \|k\|_{H^{\nu,2}(\Omega)}), \tag{A.2.5}$$

where c depends on m, λ, ν, K_ν and dist $(\Omega', \partial\Omega)$.

Iterating this result with respect to the order ν of regularity, we obtain

Corollary A.2.1. *Let $f \in H^{1,2}(\Omega)$ be a weak solution of (A.2.4). Suppose $k, a^{\alpha\beta} \in C^\infty(\Omega)$.*

Then

$$f \in C^\infty(\Omega')$$

for every $\Omega' \Subset \Omega$.

The Harnack inequalities of Moser are of fundamental importance for the theory of elliptic partial differential equations:

Theorem A.2.2. *Let L be a uniformly elliptic operator as in (A.2.2), (A.2.3).*

(i) Let u be a weak subsolution, i.e.

$$Lu \ge 0 \quad \text{in a ball } B(x_0, 4R) \subset \mathbb{R}^m$$

($\int a^{\alpha\beta} D_p u\, D_\beta \varphi \le 0$ *for all $\varphi \in H_0^{1,2}(B(x_0, 4R))$). For $p > 1$ then*

$$\sup_{B(x_0,R)} u \le c_1 \Big(\frac{p}{p-1}\Big)^{\frac{2}{p}} \Big(\frac{1}{\omega_m (2R)^m} \int_{B(x_0,2R)} \max(u(x), 0)^p\, dx\Big)^{\frac{1}{p}},$$

where c_1 depends only on m and $\frac{\mu}{\lambda}$ in (A.2.3).

(ii) Let u be a positive supersolution, i.e.

$$Lu \le 0 \quad \text{in a ball } B(x_0, 4R) \subset \mathbb{R}^m.$$

For $m \ge 3$ *and* $0 < p < \frac{m}{m-2}$ *then*

$$\left(\frac{1}{\omega_m (2R)^m} \int_{B(x_0,2R)} u^p \right)^{\frac{1}{p}} \le \frac{c_2}{(\frac{m}{m-2} - p)^2} \inf_{B(x_0,R)} u,$$

c_2 *again depending only on* m *and* $\frac{\mu}{\lambda}$*. For* $m = 2$ *and* $0 < p < \infty$*, the same estimate holds when* $\frac{c_2}{(\frac{m}{m-2}-p)^2}$ *is replaced by a constant* c_3 *depending on* p *and* $\frac{\mu}{\lambda}$*.*

The Harnack inequality also translates into estimates for the fundamental solutions of the Laplace–Beltrami operator, and their generalizations, the Green functions. The Green function $G(x_0, x)$ of a ball $B \subset M$ (or another sufficiently regular domain), for x_0 in the interior of B, is symmetric in x and x_0, smooth for $x \ne x_0$, becomes singular like $\frac{1}{(d-2)\omega_d} d(x, x_0)^{2-d}$ in case $d = \dim M \ge 3$ ($\omega_d = \operatorname{Vol} S^{d-1}$) (and like $\frac{1}{\omega_2} \log d(x_0, x)$ for $d = 2$), vanishes for $x \in \partial B$, and satisfies

$$h(x_0) = \int_B \Delta h(x)\, G(x_0, x)\, d\mathrm{Vol}\,(x) \quad \text{for all}\, h \in C_0^2(B).$$

A geometric approximation of the Green function (that is exact in the Euclidean case) has been investigated in §5.7. An analytic alternative that allows to avoid the singularity is the use of the mollified Green function. For simplicity, and because that typically suffices for applications, we only consider the case of a ball. The mollified Green function $G^R(x_0, x)$ on the ball $B(x_0, R)$ relative to the ball $B(x_0, 2R)$ of double radius, $G^R(x_0, \cdot) \in H^{1,2} \cap C_0^0(B(x_0, 2R))$, satisfies

$$\begin{aligned}\int_{B(x_0,2R)} \Delta\varphi(x)\, G^R(x_0, x)\, d\mathrm{Vol}\,(x) &= \int_{B(x_0,2R)} \langle d\varphi(x), dG^R(x_0, x)\rangle\, d\mathrm{Vol}\,(x) \\ &= \fint_{B(x_0,R)} \varphi(x)\, d\mathrm{Vol}\,(x),\end{aligned}$$

for all $\varphi \in H^{1,2}$ with $\operatorname{supp} \varphi \Subset B(x_0, 2R)$.

For purposes of normalization, it is convenient to consider

$$w^R(x) := \frac{|B(x_0, 2R)|}{R^2} G^R(x_0, x)$$

with $|B| := \operatorname{Vol} B$.

We then have

$$\int_{B(x_0,2R)} \langle d\varphi(x), dw^R(x)\rangle = \frac{1}{R^2} \int_{B(x_0,R)} \varphi(x),$$

for all $\varphi \in H^{1,2}$ with $\operatorname{supp} \varphi \Subset B(x_0, 2R)$.

We then have the estimates

Corollary A.2.2.

$$0 \le w^R \le \gamma_1 \quad \text{in } B(x_0, 2R),$$
$$w^R \ge \gamma_2 > 0 \quad \text{in } B(x_0, R),$$

for constants γ_1, γ_2 *that do not depend on* R.

The estimates of J. Schauder are also very important:

Theorem A.2.3. *Let* L *be as in (A.2.2), (A.2.3), and suppose that the coefficients* $a^{\alpha\beta}(x)$ *are Hölder continuous in* Ω*, i.e. contained in* $C^\sigma(\Omega)$ *for some* $0 < \sigma < 1$*.*

(i) If u *is a weak solution of*

$$Lu = k$$

and if k *is in* $L^\infty(\Omega)$*, then* u *is in* $C^{1,\sigma}(\Omega)$*, and on every* $\Omega_0 \Subset \Omega$*, its* $C^{1,\sigma}$*-norm can be estimated in terms of its* L^2*-norm and the* L^∞*-norm of* k*, with a structural constant depending on* Ω*,* Ω_0*,* m*,* σ*,* λ*,* μ *and the* C^σ*-norm of the* $a^{\alpha\beta}(x)$*.*

(ii) If u *is a weak solution of*

$$Lu = k$$

for some $k \in C^{\nu,\sigma}(\Omega)$*,* $\nu = 0, 1, 2, \dots$*,* $0 < \sigma < 1$*, and if the coefficients* $a^{\alpha\beta}$ *are also in* $C^{\nu,\sigma}(\Omega)$*, then* u *is in* $C^{\nu+2,\sigma}(\Omega)$*, and a similar estimate as in (i) holds, this time involving the* $C^{\nu,\sigma}$*-norm of* k *and the* $a^{\alpha\beta}$*.*

Finally, we quote the maximum principle.

Theorem A.2.4. *Let* $\Omega \subset \mathbb{R}^m$ *(or, more generally,* $\Omega \subset M$*,* M *a Riemannian manifold) be open and bounded,* $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *with*

$$Lf \ge 0 \quad \text{in } \Omega,$$

L *as in (A.2.2), (A.2.3). Then* f *assumes its maximum on the boundary* $\partial\Omega$*.*

All the preceding results naturally apply to the Laplace–Beltrami operator on a ball $B(x_0, r)$ in a Riemannian manifold M, putting

$$L = -\Delta = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma} \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right),$$

$(\gamma_{\alpha\beta})_{\alpha,\beta=1,\dots,m}$ the metric tensor of M in local coordinates, $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$, $\gamma = \det(\gamma_{\alpha\beta})$.

References for the material in this appendix are: Gilbarg and Trudinger[113], Jost[166] and, with a more elementary presentation, Jost[167]. The results of Corollary A.2.2 about Green functions are systematically derived in [134], and in a more general context in [28]. Some further points about Sobolev spaces can be found in Ziemer[315].

A.3 Existence and Regularity Theory for Solutions of Linear Parabolic Equations

In this section, we consider differential equations on $\Omega \times [0, \infty)$ where Ω is an open subset of $\mathbb{R}^m$ as in §A.2, and we continue to use the notations introduced there.

In particular, as before, let the operator L be a *uniformly elliptic* operator of the form

$$Lf(x) := \frac{\partial}{\partial x^\alpha}\big(a^{\alpha\beta}(x)\frac{\partial}{\partial x^\beta} f(x)\big) \tag{A.3.1}$$

with constants $0 < \lambda \le \mu$ satisfying

$$\lambda|\xi|^2 \le a^{\alpha\beta}(x,t)\xi_\alpha\xi_\beta \le \mu|\xi|^2 \tag{A.3.2}$$

for all $x \in \Omega, 0 \le t, \xi \in \mathbb{R}^m$. The equation we wish to study then is

$$\frac{\partial}{\partial t} f(x,t) - Lf(x,t) = k(x,t) \text{ for } x \in \Omega,\ t \ge 0 \tag{A.3.3}$$

$$f(x,0) = \phi(x) \tag{A.3.4}$$

for some continuous function $\phi(x)$ and some bounded function $k(x,t)$ (and suitable boundary conditions, but since in the text, we are interested in compact manifolds M in place of the open domain Ω, these will not play an essential role and consequently are not emphasized here). (A.3.3) is a linear parabolic partial differential equation.

We first state the parabolic maximum principle.

Theorem A.3.1. *Let $\Omega \subset \mathbb{R}^m$ (or, more generally, $\Omega \subset M$, M a Riemannian manifold) be open and bounded, $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with respect to x and in $C^1((0,T)) \cap C^0([0,T])$ with respect to t, with*

$$\frac{\partial}{\partial t} f - Lf \le 0 \quad \text{in } \Omega \times [0,T]. \tag{A.3.5}$$

Then f assumes its maximum for (x,t) with $x \in \partial\Omega$ or for $t = 0$, that is, either on the spatial boundary or at the initial time. In particular, when M is a compact manifold (without boundary), the supremum of $f(\cdot,t)$ is a decreasing function of t.

We have the following existence and regularity theorem for solutions of (A.3.3), with Schauder type estimates:

Theorem A.3.2. *Let L be as in (A.3.1), (A.3.2), and suppose that the coefficients $a^{\alpha\beta}(x,t)$ are Hölder continuous in $\Omega \times [0,\infty)$, i.e. contained in $C^\sigma(\Omega \times [0,\infty))$ for some $0 < \sigma < 1$. If we prescribe some boundary values, say $f(y,t) = g(y)$ for all $y \in \partial\Omega$, for some given, e.g. continuous, function g, the solution of (A.3.3) then exists for all $t \ge 0$.*

Furthermore, we have the following estimates:

(i) If u is a weak solution of

$$Lu = k \tag{A.3.6}$$

and if k is in $L^\infty(\Omega \times [0, \infty))$, then as a function of x, u is in $C^{1,\sigma}(\Omega)$, and for every $\Omega_0 \Subset \Omega$ and $t_0 > 0$, its (spatial) $C^{1,\sigma}(\Omega)$-norm on $\Omega_0 \times [t_0, \infty)$ can be estimated in terms of its L^∞-norm and the L^∞-norm of k, with a structural constant depending on Ω, Ω_0, t_0, m, σ, λ, μ and the C^σ-norm of the $a^{\alpha\beta}(x)$.

(ii) If u is a weak solution of

$$Lu = k$$

for some $k \in C^{\nu,\sigma}(\Omega \times [0, \infty))$, $\nu = 0, 1, 2, \ldots$, $0 < \sigma < 1$, and if the coefficients $a^{\alpha\beta}$ are also in $C^{\nu,\sigma}(\Omega \times [0, \infty))$, then u is in $C^{\nu+2,\sigma}(\Omega)$ with respect to x and of class $C^{\nu+1,\sigma}$ with respect to t, and the corresponding norms can be estimated analogously to (i), this time involving the $C^{\nu,\sigma}$-norm of k and the $a^{\alpha\beta}$.

The restriction to $t \geq t_0 > 0$ can be avoided if the initial values f_0 satisfy appropriate regularity results. The estimates on $[0, \infty)$ will then naturally also involve the corresponding norms of f_0.

Theorem A.3.2 concerns a linear parabolic equation. In the text, we shall encounter nonlinear parabolic equations and systems. For those, the global existence and regularity cannot be deduced from a general result, but rather needs to invoke the detailed structure of the system. What one can deduce from Theorem A.3.2, however, is the short time existence of solutions when the linearization of the differential operator satisfies the assumptions of that theorem. This follows by linearization and the implicit function theorem. That means that for such nonlinear systems, we can obtain the existence of a solution on some interval $[0, T)$ whose length depends on the regularity properties of the initial values. This also implies that the maximal interval of existence for nonlinear parabolic systems is open. For the closedness of the interval of existence, and consequently the existence of a solution for all "time" $t \geq 0$, one then needs to derive specific a-priori estimates that prevent solutions from becoming singular in finite time.

A reference for parabolic differential equations and systems is [196]. For a textbook treatment, we refer to [166].

Appendix B

Fundamental Groups and Covering Spaces

In this appendix, we briefly list some topological results. We assume that M is a connected manifold, although the results hold for more general spaces.

A *path* or *curve* in M is a continuous map

$$c : [0, a] \to M \quad (a \geq 0).$$

A *loop* is a path with $c(0) = c(a)$, and that point then is called the *base point* of the loop. The *inverse* of a path c is

$$c^{-1} : [0, a] \to M,$$
$$c^{-1}(t) := c(a - t).$$

If $c_i : [0, a_i] \to M$ are paths ($i = 1, 2$) with $c_2(0) = c_1(a_1)$, we can define the product $c_1 \cdot c_2$ as the path $c : [0, a_1 + a_2] \to M$,

$$c(t) = \begin{cases} c_1(t) & \text{for } 0 \leq t \leq a_1, \\ c_2(t - a_1) & \text{for } a_1 \leq t \leq a_1 + a_2. \end{cases}$$

Two paths $c_i : [0, a_i]$ with $c_1(0) = c_2(0)$ and $c_1(a_1) = c_2(a_2)$ are called *equivalent* or *homotopic* if there exists a continuous function

$$H : [0, 1] \times [0, 1] \to M$$

with

$$\begin{aligned} H(t,0) &= c_1\left(\frac{t}{a_1}\right), \\ H(t,1) &= c_2\left(\frac{t}{a_2}\right), \qquad \text{for all } t, \\ H(0,s) = c_1(0) &= c_2(0), \\ H(1,s) = c_1(a_1) &= c_2(a_2), \qquad \text{for all } s. \end{aligned}$$

In particular, $c : [0,a] \to M$ is equivalent to $\tilde{c} : [0,1] \to M$ with $\tilde{c}(t) = c(\frac{t}{a})$, and so we may assume that all paths are parametrized on the unit interval.

We obtain an equivalence relation on the space of all paths. The equivalence class of c is denoted $[c]$, and it is not hard to verify that $[c_1c_2]$ and $[c^{-1}]$ are independent of the choice of representations. Thus, we may define

$$\begin{aligned} [c_1 \cdot c_2] &=: [c_1] \cdot [c_2], \\ [c^{-1}] &=: [c]^{-1}. \end{aligned}$$

In particular, the equivalence or homotopy classes of loops with fixed base point $p \in M$ form a group $\pi_1(M,p)$, the *fundamental group* of M with *base point* p.

If p and q are in M and $\gamma : [0,1] \to M$ satisfies $\gamma(0) = p$, $\gamma(1) = q$, then for every loop c with base point q, $\gamma^{-1}c\gamma$ is a loop with base point p, and this induces an isomorphism between $\pi_1(M,q)$ and $\pi_1(M,p)$. We may thus speak of the fundamental group $\pi_1(M)$ of M without reference to a base point. M is called *simply connected* if $\pi_1(M) = \{1\}$. A continuous map $f : M \to N$ induces a homomorphism $f_\# : \pi_1(M,p) \to \pi_1(N,f(p))$ of fundamental groups.

Let X be another connected manifold. A continuous map

$$\pi : X \to M$$

is called a *covering map* if each $p \in M$ has a neighborhood U with the property that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U. If $p \in M$ and H is a subgroup of $\pi_1(M,p)$, there exists a covering $\pi : X \to M$ with the property that for any $x \in X$ with $\pi(x) = p$, we have $\pi_*\ (\pi_1(X,x)) = H$.

If we choose $H = \{1\}$, we obtain a simply connected manifold $\tilde{M}$ and a covering

$$\pi : \tilde{M} \to M.$$

$\tilde{M}$ is called the *universal covering* of M.

If $\pi : X \to M$ is a covering, $c : [0,1] \to M$ a path, $x_0 \in \pi^{-1}(c(0))$, then there exists a unique path

$$\tilde{c} : [0,1] \to X$$

with $\tilde{c}(0) = x_0$ and $c(t) = \pi(\tilde{c}(t))$. $\tilde{c}$ is called the lift of c through x_0.

More generally, if M' is another manifold, $f : M' \to M$ is continuous, $p_0 \in M$, $y_0 \in f^{-1}(p_0)$, $x_0 \in \pi^{-1}(p_0)$, there exists a continuous

$$\tilde{f} : M' \to X$$

with $\tilde{f}(y_0) = x_0$ and $f = \pi \circ \tilde{f}$ if and only if $f_\#(\pi_1(M', y_0)) \subset \pi_\#(\pi_1(X, x_0))$. $\tilde{f}$ is unique if it exists.

Let $\pi : \tilde{M} \to M$ be the universal covering of M. A *deck transformation* is a homeomorphism $\varphi : \tilde{M} \to \tilde{M}$ with

$$\pi = \pi \circ \varphi.$$

Let $\pi(x_0) = p_0$. $\pi_1(M, p_0)$ then bijectively corresponds to $\pi^{-1}(p_0)$. More precisely, $x_1 \in \pi^{-1}(p_0)$ corresponds to the homotopy class of $\pi(\gamma_{x_1})$, where $\gamma_{x_1} : [0,1] \to \tilde{M}$ is any path with $\gamma_{x_1}(0) = x_0$, $\gamma_{x_1}(1) = x_1$. The deck transformations form a group that acts simply transitively on $\pi^{-1}(p_0)$, and associating to a deck transformation $\varphi(x_0) \in \pi^{-1}(p_0)$ then yields an isomorphism between the group of deck transformations and $\pi_1(M, p_0)$.

If M and N are manifolds with universal coverings $\tilde{M}$ and $\tilde{N}$, resp., and if

$$f : M \to N$$

is a continuous map, we consider the induced homomorphism

$$\rho := f_\sharp : \pi_1(M, p) \to \pi_1(N, f(p))$$

of fundamental groups. If $\pi : \tilde{M} \to M$ is the universal covering, we can lift $f \circ \pi : \tilde{M} \to N$ to a map

$$\tilde{f} : \tilde{M} \to \tilde{N},$$

because the above lifting condition is trivially satisfied as $\pi_1(\tilde{M}) = \{1\}$. $\tilde{f}$ is equivariant w.r.t. the above homomorphism ρ in the sense that for every $\lambda \in \pi_1(M, p)$, acting as a deck transformation on $\tilde{M}$, we have

$$\tilde{f}(\lambda x) = \rho(\lambda)\tilde{f}(x) \quad \text{for every } x \in \tilde{M}, \tag{B.1}$$

where $\rho(\lambda)$ acts as a deck transformation on $\tilde{N}$. We say that $\tilde{f}$ is a ρ-equivariant map between the universal covers $\tilde{M}$ and $\tilde{N}$.

Conversely, given any homomorphism

$$\rho : \pi_1(M, p) \to \pi_1(N, q)$$

and any ρ-equivariant map

$$g : \tilde{M} \to \tilde{N} \quad (\text{with } g(p) = q),$$

not necessarily continuous, then g induces a map

$$g' : M \to N$$

whose lift to universal covers is g. g' is continuous if g is.

Finally, if $\tilde{M}$ is the universal cover of a compact Riemannian manifold M, a so-called fundamental domain $F(M)$ for M in $\tilde{M}$ can be constructed as follows:

For simplicity of notation, we denote the group $\pi_1(M, x_0)$ operating by deck transformations on $\tilde{M}$ by Γ, and its trivial element by e.

Let $d(.,.)$ be the Riemannian distance function on $\tilde{M}$. We select any $z_0 \in \tilde{M}$. We then put

$$F(M) := \{z \in \tilde{M} : d(z, z_0) < d(\gamma z, z_0) \quad \text{for all } \gamma \in \Gamma, \gamma \neq e\}.$$

$F(M)$ is open. Since Γ operates by isometries, i.e.

$$d(\lambda z_1, \lambda z_2) = d(z_1, z_2) \quad \text{for all } \lambda \in \Gamma, z_1, z_2 \in \tilde{M},$$

we may also write

$$F(M) = \{z \in \tilde{M} : d(z, z_0) < d(z, \lambda z_0) \quad \text{for all } \lambda \in \Gamma, \lambda \neq e\}.$$

By its definition, $F(M)$ cannot contain any two points that are equivalent under the operation of Γ. On the other hand, for any $z \in \tilde{M}$, we may find some $\mu \in \Gamma$ such that

$$\mu z \in \overline{F(M)}.$$

Thus, the closure of $F(M)$ contains at least one point from every orbit of Γ in $\tilde{M}$.

If $f : M \to \mathbb{R}$ is an integrable function, and if $\tilde{f} : \tilde{M} \to \mathbb{R}$ is its lift to the universal cover of M, then

$$\int_M f(x)\, d\mathrm{Vol}\,(x) = \int_{F(M)} \tilde{f}(y)\, d\mathrm{Vol}\,(y).$$

Examples of fundamental groups.

1. $\pi_1(\mathbb{R}^n) = \{1\}$ for all n.
2. $\pi_1(S^1) = \mathbb{Z}$.

   A generator is given by

$$c : [0,1] \to S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$
$$c(t) = (\cos 2\pi t, \sin 2\pi t).$$

   The universal covering of S^1 is $\mathbb{R}^1$, and the covering map is likewise given by

$$\pi(t) = (\cos 2\pi t, \sin 2\pi t).$$

3. $\pi_1(S^n) = \{1\}$ for $n \geq 2$.
4. $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$.

The preceding results can be found in any reasonable textbook on Algebraic Topology, for example in [119] or [274].

Bibliography

[1] A. Abbondandolo and P. Majer. Morse homology on Hilbert spaces. *Comm.Pure Appl.Math.*, 54:689–760, 2001.

[2] U. Abresch and W. Meyer. Pinching below $\frac{1}{4}$, injectivity radius, and conjugate radius. *J.Diff.Geom.*, 40:643–691, 1994.

[3] U. Abresch and W. Meyer. A sphere theorem with pinching constant below $\frac{1}{4}$. *J.Diff.Geom.*, 44:214–261, 1996.

[4] S.I. Al'ber. On n-dimensional problems in the calculus of variations in the large. *Sov. Math. Dokl.*, 5:700–704, 1964.

[5] S.I. Al'ber. Spaces of mappings into a manifold with negative curvature. *Sov. Math. Dokl.*, 9:6–9, 1967.

[6] F. Almgren. Questions and answers about area-minimizing surfaces and geometric measure theory. *Proc. Symp. Pure Math.*, 54(I):29–53, 1993.

[7] S. Aloff and N. Wallach. An infinite family of 7-manifolds admitting positively curved Riemannian structures. *Bull.AMS*, 81:93–97, 1975.

[8] M. Atiyah, R. Bott, and A. Shapiro. Clifford moduls. *Topology*, 3(Suppl. I):3–38, 1964.

[9] M. Atiyah and I. Singer. The index of elliptic operators: III. *Ann.Math.*, 87:546–604, 1968.

[10] T. Aubin. *Nonlinear analysis on manifolds. Monge–Ampère equations.* Springer-Verlag, Berlin, 1982.

[11] W. Ballmann. Der Satz von Ljusternik und Schnirelman. *Bonner Math. Schriften*, 102:1–25, 1978.

[12] W. Ballmann. *Lectures on spaces of nonpositive curvature.* DMV Seminar Vol.25, Birkhäuser, 1995.

[13] W. Ballmann, M. Gromov, and V. Schroeder. *Manifolds of nonpositive curvature.* Birkhäuser, 1985.

[14] V. Bangert. On the existence of closed geodesics on two–spheres. *Internat. J. Math*, 4:1–10, 1993.

[15] H. Bass and J. Morgan (eds.). *The Smith conjecture.* Academic Press, 1984.

[16] Y. Bazaikin. On a family of 13-dimensional closed Riemannian manifolds of positive curvature. *Sib.Math.J.*, 37:1068–1085, 1996.

[17] Y. Bazaikin. A manifold with positive sectional curvature and fundamental group $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. *Sib.Math.J.*, 40:834–836, 1999.

[18] V.N. Berestovskij and I.G. Nikolaev. Multidimensional generalized Riemannian spaces. In Y.G. Reshetnyak, editor, *Geometry IV*, pages 165–243. Encyclopedia Math. Sciences Vol 70, Springer, 1993.

[19] M. Berger. Les variétés Riemanniennes $(\frac{1}{4})$–pincées. *Ann. Scuola Norm. Sup. Pisa*, III(14):161–170, 1960.

[20] M. Berger. *A panoramic view of Riemannian geometry.* Springer, 2003.

[21] M. Berger, P. Gauduchon, and E. Mazet. *Le spectre d'une varieté riemannienne.* Springer, Lecture Notes in Mathematics 194, 1974.

[22] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators.* Springer, 1992.

[23] A. Besse. *Einstein manifolds.* Springer, 1987.

[24] L. Bessières, G. Besson, M. Boileau, S. Maillot, and J. Porti. *Geometrisation of 3-manifolds.* Europ.Math.Soc., 2010.

[25] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *GAFA*, 5:731–799, 1995.

[26] G. Besson, G. Courtois, and S. Gallot. Minimal entropy and Mostow's rigidity theorems. *Ergod. Th. & Dynam. Sys*, 16:623–649, 1996.

[27] M. Betz and R.Cohen. Graph moduli spaces and cohomology operations. *Turkish J.Math.*, 18:23–41, 1994.

[28] M. Biroli and U. Mosco. A Saint-Venant principle for Dirichlet forms on discontinuous media. *Annali Mat.Pura Appl.(IV)*, 169:125–181, 1995.

[29] S. Bochner. Harmonic surfaces in Riemannian metric. *Trans.AMS*, 47:146–154, 1940.

[30] S. Bochner and K. Yano. *Curvature and Betti numbers.* Princeton Univ. Press, 1953.

[31] C. Boehm and B. Wilking. Manifolds with positive curvature operator are space forms. *Annals Math.*, 167:1079–1097, 2008.

[32] J. Borchers and W. Garber. Analyticity of solutions of the O(N) nonlinear σ–model. *Comm. Math. Phys*, 71:299–309, 1980.

[33] J.P. Bourguignon. *Eugenio Calabi and Kähler metrics*, pages 61–85. In: P. de Bartolomeis (ed.), Manifolds and geometry, Symp. Math. 36, Cambridge Univ. Press, 1996.

[34] J.P. Bourguignon and H. Karcher. Curvature operators: Pinching estimates and geometric examples. *Ann. scient. Éc. Norm. Sup*, 11:71–92, 1978.

[35] S. Bradlow. Vortices in holomorphic line bundles over closed Kähler manifolds. *Comm. Math. Phys*, 135:1–17, 1990.

[36] S. Bradlow. Special metrics and stability for holomorphic bundles with global sections. *J. Diff. Geom*, 33:169–214, 1991.

[37] S. Brendle. A general convergence result for the Ricci flow. *Duke Math. J.*, 145:585–601, 2008.

[38] S. Brendle. *Ricci flow and the sphere theorem*, volume 111 of *Grad. Studies Math.* AMS, 2010.

[39] S. Brendle and R. Schoen. Classification of manifolds with weakly 1/4-pinched curvature. *Acta Math.*, 200:1–13, 2008.

[40] S. Brendle and R. Schoen. Manifolds with 1/4-pinched curvature are space forms. *J.Amer.Math.Soc.*, 22:287–307, 2009.

[41] H. Brézis and J. Coron. Large solutions for harmonic maps in two dimensions. *Comm. Math. Phys*, 92:203–215, 1983.

[42] C. Le Brun. Einstein metrics and Mostow rigidity. *Math. Res. Lett.*, 2:1–8, 1995.

[43] Yu. Burago, M. Gromov, and G. Perel'man. A. D. Alexandrov's spaces with curvatures bounded from below. *Russ. Math. Surveys*, 42:1–58, 1992.

[44] P. Buser and H. Karcher. *Gromov's almost flat manifolds.* Astérisque, 1981.

[45] L. Caffarelli and Y.S. Yang. Vortex condensation in the Chern–Simons–Higgs Model. An existence theorem. *Comm. Math. Phys*, 168:321–336, 1995.

[46] E. Calabi and E. Vesentini. On compact, locally symmetric Kähler manifolds. *Ann. Math*, 71:472–507, 1960.

[47] H.D. Cao and X.P. Zhu. A complete proof of the Poincaré and geometrization conjectures - application of the Hamilton–Perelman theory of the Ricci flow. *Asian J. Math.*, 10(2):195–492, 2006.

[48] H.D. Cao and X.P. Zhu. Hamilton–Perelman's proof of the Poincaré conjecture and the geometrization conjecture. arXiv:math.DG/0612069, 2006.

[49] J. Cao and F. Xavier. Kähler parabolicity and the Euler number of compact Kähler manifolds of non–positive sectional curvature. *Math. Ann*, 319:483–491, 2001.

[50] K.C. Chang. Heat flow and boundary value problem for harmonic maps. *Anal. Nonlinéaire*, 6:363–396, 1989.

[51] K.C. Chang. *Infinite dimensional Morse theory and multiple solution problems.* Birkhäuser, 1993.

[52] I. Chavel. *Eigenvalues in Riemannian geometry.* Academic Press, 1984.

[53] I. Chavel. *Riemannian geometry - A modern introduction.* Cambridge University Press, 1993.

[54] J. Cheeger. Finiteness theorems for Riemannian manifolds. *Amer. J. Math.*, 92:61–74, 1970.

[55] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in Analysis*, pages 195–199. Princeton Univ. Press, 1970.

[56] J. Cheeger and D. Ebin. *Comparison theorems in Riemannian geometry.* North Holland, 1975.

[57] J. Cheeger and D. Gromoll. The splitting theorems for manifolds of nonnegative Ricci curvature. *J. Diff. Geom*, 6:119–129, 1971.

[58] J. Cheeger and M. Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. I. *J. Diff. Geom.*, 23:309–346, 1983.

[59] J. Cheeger and M. Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. II, *J. Diff. Geom.*, 32:269–298, 1990.

[60] Q. Chen, J. Jost, J.Y. Li, and G.F. Wang. Regularity theorems and energy identities for Dirac-harmonic maps. *Math.Zeitschr.*, 251:61–84, 2005.

[61] Q. Chen, J. Jost, J.Y. Li, and G.F. Wang. Dirac-harmonic maps. *Math.Zeitschr.*, 254:409–432, 2006.

[62] Q. Chen, J. Jost, G.F. Wang, and M.M. Zhu. The boundary value problem for Dirac-harmonic maps. 2011.

[63] W.Y. Chen and J. Jost. A Riemannian version of Korn's inequality. *Calc. Var.*, 14:517–530, 2001.

[64] S.Y. Cheng. Eigenvalue comparison theorems and its geometric applications. *Math.Z.*, 143:289–297, 1975.

[65] B. Chow and D. Knopf. *The Ricci flow: An introduction.* Amer. Math. Soc., 2004.

[66] B. Chow, P. Lu, and L. Ni. *Hamilton's Ricci flow.* Amer. Math. Soc. and Intern. Press, 2007.

[67] S.N. Chow and J. Hale. *Methods of bifurcation theory.* Springer, 1982.

[68] C. Conley. *Isolated invariant sets and the Morse index.* CBMS Reg. Conf. Ser. Math. 38, AMS, Providence, RI, 1978.

[69] C. Conley and E. Zehnder. Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure Appl. Math.*, 37:207–253, 1984.

[70] K. Corlette. Archimedean superrigidity and hyperbolic geometry. *Ann. Math.*, 135:165–182, 1992.

[71] R. Courant and D. Hilbert. *Methoden der Mathematischen Physik I.* Springer, 1924, 1968.

[72] G. dal Maso. *An introduction to* Γ*–convergence.* Birkhäuser, 1993.

[73] D. de Turck and J. Kazdan. Some regularity theorems in Riemannian geometry. *Ann. Sci. École Norm. Sup.*, 4(14):249–260, 1981.

[74] U. Dierkes, S. Hildebrandt, and F. Sauvigny. *Minimal surfaces.* Springer, 2010.

[75] U. Dierkes, S. Hildebrandt, and A. Tromba. *Global analysis of minimal surfaces.* Springer, 2010.

[76] U. Dierkes, S. Hildebrandt, and A. Tromba. *Regularity of minimal surfaces.* Springer, 2010.

[77] W.Y. Ding. Lusternik–Schnirelman theory for harmonic maps. *Acta Math. Sinica*, 2:105–122, 1986.

[78] W.Y. Ding, J. Jost, J.Y. Li, X.W. Peng, and G.F. Wang. Self duality equations for Ginzburg–Landau and Seiberg–Witten type functionals with 6^{th} order potentials. *Comm. Math. Phys.*, 217(2):383–407, 2001.

[79] W.Y. Ding, J. Jost, J.Y. Li, and G.F. Wang. An analysis of the two–vortex case in the Chern–Simons–Higgs model. *Calc. Var.*, 7:87–97, 1998.

[80] W.Y. Ding, J. Jost, J.Y. Li, and G.F. Wang. Multiplicity results for the two vortex Chern–Simons–Higgs model on the two-sphere. *Commentarii Math. Helv.*, 74:118–142, 1999.

[81] M. do Carmo. *Riemannian geometry.* Birkhäuser, 1992.

[82] S. Donaldson and P. Kronheimer. *The geometry of four-manifolds.* Oxford Univ. Press, 1990.

[83] H. Donnelly and F. Xavier. On the differential form spectrum of negatively curved Riemannian manifolds. *Amer. J. Math.*, 106:169–185, 1984.

[84] P. Eberlein. *Rigidity problems for manifolds of nonpositive curvature.* Springer LNM 1156, 1984.

[85] P. Eberlein. *Geometry of nonpositively curved manifolds.* Univ. Chicago Press, 1996.

[86] P. Eberlein, U. Hamenstädt, and V. Schroeder. Manifolds of nonpositive curvature. *Proc. Symp. Pure Math*, 54:Part 3, 179–227, 1993.

[87] P. Eberlein and B. O'Neill. Visibility manifolds. *Pacific J. Math.*, 46:45–109, 1973.

[88] J. Eells and L. Lemaire. A report on harmonic maps. *Bull. London Math. Soc.*, 10:1–68, 1978.

[89] J. Eells and L. Lemaire. Selected topics in harmonic maps. *CBMS Reg. Conf. Ser.*, 50, 1983.

[90] J. Eells and L. Lemaire. Another report on harmonic maps. *Bull. London Math. Soc.*, 20:385–524, 1988.

[91] J. Eells and J. Sampson. Harmonic mappings of Riemannian manifolds. *Am. J. Math.*, 85:109–160, 1964.

[92] J. Eschenburg. New examples of manifolds with strictly positive curvature. *Inv.math.*, 66:469–480, 1982.

[93] J. Eschenburg and J. Jost. *Differentialgeometrie und Minimalflächen.* Springer, 2007.

[94] P. Deligne et al. *Quantum fields and strings: a course for mathematicians, Vol. I.* Amer.Math.Soc. and Inst. Adv. Study, Princeton, NJ, 1999.

[95] P. Deligne et al. *Quantum fields and strings: a course for mathematicians, Vol. II.* Amer.Math.Soc. and Inst. Adv. Study, Princeton, NJ, 1999.

[96] F. Fang and X. Rong. Positive pinching, volume and second Betti number. *GAFA*, 9:641–674, 1999.

[97] H. Federer. *Geometric measure theory.* Springer, 1979.

[98] A. Floer. Witten's complex and infinite dimensional Morse theory. *J. Diff. Geom.*, 30:207–221, 1989.

[99] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Math. Z.*, 212:13–38, 1993.

[100] J. Franks. Morse–Smale flows and homotopy theory. *Topology*, 18:199–215, 1979.

[101] J. Franks. Geodesics on S^2 and periodic points of annulus diffeomorphisms. *Inv. math.*, 108:403–418, 1992.

[102] D. Freed. Special Kähler manifolds. *Comm.Math.Phys.*, 203:31–52, 1999.

[103] D. Freed and K. Uhlenbeck. *Instantons and 4-manifolds.* Springer, 1984.

[104] Th. Friedrich. *Neue Invarianten der 4–dimensionalen Mannigfaltigkeiten.* SFB 288, Berlin, 1995.

[105] Th. Friedrich. *Dirac-Operatoren in der Riemannschen Geometrie.* Vieweg, 1997.

[106] K. Fukaya. A boundary of the set of Riemannian manifolds with bounded curvatures and diameters. *J. Diff. Geom.*, 28:1–21, 1988.

[107] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry.* Springer, 1987.

[108] O. García-Prada. Invariant connections and vortices. *Comm. Math. Phys.*, 156:527–546, 1993.

[109] O. García-Prada. Dimensional reduction of stable bundles, vortices and stable pairs. *Intern. J. Math*, 5:1–52, 1994.

[110] O. García-Prada. Proof for the vortex equations over a compact Riemann surface. *Bull. London Math. Soc.*, 26:88–96, 1994.

[111] M. Giaquinta and E. Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[112] M. Giaquinta and E. Giusti. The singular set of the minima of certain quadratic functionals. *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.*, (4) 11:45–55, 1984.

[113] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order.* Springer, 1977.

[114] C. Gordon. *Handbook of Differential Geometry*, volume 1, chapter Survey of isospectral manifolds, pages 747–778. Elsevier, 2000.

[115] C. Gordon. Isospectral deformations of metrics on spheres. *Inv.math.*, 145:317–331, 2001.

[116] C. Gordon, D. Webb, and S. Wolpert. Isospectral plane domains and surfaces via Riemannian orbifolds. *Inv.math.*, 110:1–22, 1992.

[117] M. Goresky and R. MacPherson. *Stratified Morse theory.* Ergebnisse 14, Springer, 1988.

[118] M. Grayson. Shortening embedded curves. *Ann. Math.*, 120:71–112, 1989.

[119] M. Greenberg. *Lectures on algebraic topology.* Benjamin, Reading, MA., 1967.

[120] R. Greene and H. Wu. Lipschitz convergence of Riemannian manifolds. *Pacific J. Math.*, 131:119–141, 1988.

[121] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley–Interscience, 1978.

[122] D. Gromoll. Spaces of nonnegative curvature. *Proc. Sym. Pure Math.*, 54(Part 3):337–356, 1993.

[123] D. Gromoll, W. Klingenberg, and W. Meyer. *Riemannsche Geometrie im Großen.* Springer LNM 55, 2, 1975.

[124] D. Gromoll and J. Wolf. Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature. *Bull. AMS*, 77:545–552, 1971.

[125] M. Gromov. *Structures métriques pour les variétés riemanniennes.* Rédigé par J. Lafontaine and P. Pansu. Cedic-Nathan, Paris, 1980.

[126] M. Gromov. Pseudoholomorphic curves in symplectic geometry. *Inv. math.*, 82:307–347, 1985.

[127] M. Gromov. Kähler hyperbolicity and L^2-Hodge theory. *J. Diff. Geom.*, 33:263–292, 1991.

[128] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser, 1999.

[129] M. Gromov and B. Lawson. The classification of simply connected manifolds of positive scalar curvature. *Ann. Math.*, 111:423–434, 1980.

[130] M. Gromov and B. Lawson. Spin and scalar curvature in the presence of a fundamental group. *Ann. Math.*, 111:209–230, 1980.

[131] M. Gromov and R. Schoen. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Publ. Math.IHES*, 76:165–246, 1992.

[132] K. Grove and W. Ziller. Cohomogeneity one manifolds with positive Ricci curvature. *Inv.Math.*, 149:619–646, 2002.

[133] M. Grüter. Regularity of weak H-surfaces. *J. reine angew. Math.*, 329:1–15, 1981.

[134] M. Grüter and K.-O. Widman. The Green function for uniformly elliptic equations. *Man.Math.*, 37:303–342, 1982.

[135] M. Günther. Zum Einbettungssatz von J. Nash. *Math. Nachr.*, 144:165–187, 1989.

[136] R. Hamilton. *Harmonic maps of manifolds with boundary.* Springer LNM 471, 1975.

[137] R. Hamilton. Three–manifolds with positive Ricci curvature. *J. Diff. Geom.*, 17:255–306, 1982.

[138] R. Hamilton. The Harnack estimate for the Ricci flow. *J. Diff. Geom.*, 37:225–243, 1993.

[139] R. Hardt and L. Simon. Boundary regularity and embedded solutions for the oriented Plateau problem. *Ann. Math.*, 110:439–486, 1979.

[140] P. Hartman. On homotopic harmonic maps. *Can. J. Math.*, 19:673–687, 1967.

[141] S. Hawking and Ellis. *The large scale structure of space–time*. Cambridge University Press, 1973.

[142] F. Hélein. Régularité des applications faiblement harmoniques entre une surface et une varieté riemannienne. *C.R. Acad. Sci. Paris*, 312:591–596, 1991.

[143] F. Hélein. *Harmonic maps, conservation laws and moving frames*. Cambridge Univ. Press, 2002.

[144] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. Academic Press, 1978.

[145] S. Hildebrandt, H. Kaul, and K.O. Widman. Harmonic mappings into Riemannian manifolds with non–positive sectional curvature. *Math. Scand.*, 37:257–263, 1975.

[146] S. Hildebrandt, H. Kaul, and K.O. Widman. An existence theorem for harmonic mappings of Riemannian manifolds. *Acta Math.*, 138:1–16, 1977.

[147] N. Hingston. On the growth of the number of closed geodesics on the two–sphere. *Intern. Math. Res.*, 9:253–262, 1993.

[148] N. Hitchin. The self–duality equations on a Riemann surface. *Proc. London Math. Soc.*, 55:19–126, 1987.

[149] M.C. Hong, J. Jost, and M. Struwe. Asymptotic limits of a Ginzburg–Landau type functional. In J. Jost, editor, *Geometric Analysis and the Calculus of Variations for Stefan Hildebrandt*, pages 99–123. International Press, Boston, 1996.

[150] D. Husemoller. *Fibre bundles*. Springer, GTM 20, 1975.

[151] T. Ishihara. A mapping of Riemannian manifolds which preserves harmonic functions. *J. Math. Kyoto Univ.*, 19:215–229, 1979.

[152] W. Jäger and H. Kaul. Uniqueness and stability of harmonic maps and their Jacobi fields. *Man. math.*, 28:269–291, 1979.

[153] J. Jost. *Harmonic mappings between Riemannian manifolds*. ANU–Press, Canberra, 1984.

[154] J. Jost. The Dirichlet problem for harmonic maps from a surface with boundary onto a 2–sphere with non–constant boundary values. *J. Diff. Geom.*, 19:393–401, 1984.

[155] J. Jost. The geometric calculus of variations: A short survey and a list of open problems. *Expo.Math.*, 6:111–143, 1988.

[156] J. Jost. A nonparametric proof of the theorem of Ljusternik and Schnirelman. *Arch. Math.*, 53:497–509, 1989. Correction in *Arch. Math.* 56 (1991), 624.

[157] J. Jost. *Two-dimensional geometric variational problems.* Wiley–Interscience, 1991.

[158] J. Jost. Unstable solutions of two–dimensional geometric variational problems. *Proc. Symp. Pure Math*, 54(I):205–244, 1993.

[159] J. Jost. Equilibrium maps between metric spaces. *Calc. Var.*, 2:173–204, 1994.

[160] J. Jost. Convex functionals and generalized harmonic maps into spaces of nonpositive curvature. *Comment. Math. Helv.*, 70:659–673, 1995.

[161] J. Jost. Generalized harmonic maps between metric spaces. In J. Jost, editor, *Geometric analysis and the calculus of variations for Stefan Hildebrandt*, pages 143–174. Intern. Press, Boston, 1996.

[162] J. Jost. Generalized Dirichlet forms and harmonic maps. *Calc.Var.*, 5:1–19, 1997.

[163] J. Jost. *Nonpositive curvature: Geometric and analytic aspects.* Birkhäuser, 1997.

[164] J. Jost. *Geometry and physics.* Springer, 2009.

[165] J. Jost. *Nonlinear methods in Riemannian and Kählerian geometry.* Birkhäuser, 1991.

[166] J. Jost. *Partial differential equations.* Springer, 2007.

[167] J. Jost. *Postmodern analysis.* Springer, 2005.

[168] J. Jost. *Compact Riemann surfaces.* Springer, 2006.

[169] J. Jost. *Riemannian geometry and geometric analysis.* Springer, 2011.

[170] J. Jost and H. Karcher. Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen. *Man. math.*, 40:27–77, 1982.

[171] J. Jost and X. Li-Jost. *Calculus of variations.* Cambridge Univ. Press, 1998.

[172] J. Jost, X.W. Peng, and G.F. Wang. Variational aspects of the Seiberg–Witten functional. *Calc. Var.*, 4:205–218, 1996.

[173] J. Jost and M. Struwe. Morse–Conley theory for minimal surfaces of varying topological type. *Inv. math.*, 102:465–499, 1990.

[174] J. Jost and L. Todjihounde. Harmonic nets in metric spaces. *Pacific J. Math.*, 231:437–444, 2007.

[175] J. Jost and Y. L. Xin. Vanishing theorems for L^2-cohomology groups. *J. reine angew. Math.*, 525:95–112, 2000.

[176] J. Jost and S.T. Yau. Harmonic mappings and Kähler manifolds. *Math. Ann.*, 262:145–166, 1983.

[177] J. Jost and S.T. Yau. A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry. *Acta Math.*, 170:221–254, 1993. Corr. in *Acta Math.* 173 (1994), 307.

[178] J. Jost and S.T. Yau. Harmonic maps and superrigidity. *Proc. Symp. Pure Math.*, 54(I):245–280, 1993.

[179] J. Jost and K. Zuo. Harmonic maps of infinite energy and rigidity results for Archimedean and non-Archimedean representations of fundamental groups of quasiprojective varieties. *J.Diff.Geom.*, 47:469–503, 1997.

[180] J. Jost and K. Zuo. Vanishing theorems for L^2-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry. *Comm. Anal. Geom.*, 8:1–30, 2000.

[181] E. Kähler. Über eine bemerkenswerte Hermitesche Metrik. *Abh. Math. Sem. Univ. Hamburg*, 9:173–186, 1933.

[182] E. Kähler. Der innere Differentialkalkül. *Rend.Mat.Appl.(5)*, 21:425–523, 1962.

[183] E. Kähler. *Mathematische Werke – Mathematical Works.* Edited by R. Berndt and O. Riemenschneider; de Gruyter, 2003.

[184] H. Karcher. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math.*, 30:509–541, 1977.

[185] W. Kendall. Convexity and the hemisphere. *J. London Math. Soc.*, 43(2):567–576, 1991.

[186] Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv.*, 34:257–270, 1960.

[187] B. Kleiner and J. Lott. Notes on Perelman's papers. *arXiv*, math.DG/0605667.p:47–54, 2006.

[188] W. Klingenberg. Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung. *Comm. Math. Helv.*, 35:47–54, 1961.

[189] W. Klingenberg. *Lectures on closed geodesics.* Springer, 1978.

[190] W. Klingenberg. *Riemannian geometry.* de Gruyter, 1995.

[191] S. Kobayashi and K. Nomizu. *Foundations of differential geometry, I.* Wiley–Interscience, 1963.

[192] S. Kobayashi and K. Nomizu. *Foundations of differential geometry, II.* Wiley–Interscience, 1969.

[193] N. Korevaar and R. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1:561–659, 1993.

[194] N. Korevaar and R. Schoen. Global existence theorems for harmonic maps to non–locally compact spaces. *Comm. Anal. Geom.*, 5:213–266, 1997.

[195] P. Kronheimer and T. Mrowka. The genus of embedded surfaces in the projective plane. *Math. Res. Letters*, 1:797–808, 1994.

[196] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian, Amer. Math. Soc., Providence, RI, 1968.

[197] O.A. Ladyzhenskaya and N.N. Ural'ceva. *Linear and quasilinear elliptic equations.* Translated from the Russian, Academic Press, 1968.

[198] H.B. Lawson and M.L. Michelsohn. *Spin geometry.* Princeton University Press, Princeton, 1989.

[199] H.B. Lawson and S.T. Yau. Compact manifolds of nonpositive curvature. *J. Diff. Geom.*, 7:211–228, 1972.

[200] L. Lemaire. Applications harmoniques de surfaces riemanniennes. *J. Diff. Geom.*, 13:51–78, 1978.

[201] P. Li. On the Sobolev constant and the p-spectrum of a compact Riemannian manifold. *Ann.Sci.Ec.Norm.Sup., Paris*, 13:419–435, 1980.

[202] P. Li and S.T. Yau. Estimates of eigenvalues of a compact Riemannian manifold. *AMS Proc.Symp.Pure Math.*, 36:205–240, 1980.

[203] P. Li and S.T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156:153–201, 1986.

[204] F.H. Lin. Analysis on singular spaces. In Ta-Tsien Li, editor, *Geometry, Analysis and Mathematical Physics, in honor of Prof. Chaohao Gu*, pages 114–126. World Scientific, 1997.

[205] J. Lohkamp. Negatively Ricci curved manifolds. *Bull. AMS*, 27:288–292, 1992.

[206] J. Lohkamp. Metrics of negative Ricci curvature. *Ann. Math.*, 140:655–683, 1994.

[207] U. Ludwig. Stratified Morse theory with tangential conditions. *arXiv: math/0310019v1.*

[208] K. Marathe. *Topics in physical mathematics.* Springer, 2010.

[209] G.A. Margulis. Discrete groups of motion of manifolds of nonpositive curvature. *AMS Transl.*, 190:33–45, 1977.

[210] G.A. Margulis. *Discrete subgroups of semisimple Lie groups*. Springer, 1991.

[211] Y. Matsushima. On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces. *Ann. Math.*, 75:312–330, 1962.

[212] W. Meeks and S.T. Yau. The classical Plateau problem and the topology of three-dimensional manifolds. *Top.*, 21:409–440, 1982.

[213] M. Micallef and J.D. Moore. Minimal two–spheres and the topology of manifolds with positive curvature on totally isotropic two–planes. *Ann. Math.*, 127:199–227, 1988.

[214] A. Milgram and P. Rosenbloom. Harmonic forms and heat conduction, I: Closed Riemannian manifolds. *Proc.Nat.Acad.Sci.*, 37:180–184, 1951.

[215] J. Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. Math.*, 64:399–405, 1956.

[216] J. Milnor. Differentiable structures on spheres. *Am. J. Math.*, 81:962–972, 1959.

[217] J. Milnor. *Morse theory, Ann. Math. Studies 51*. Princeton Univ. Press, 1963.

[218] J. Milnor. Eigenvalues of the Laplace operator on certain manifolds. *Proc.Nat.Ac.Sc.*, 51:542, 1964.

[219] J. Milnor. *Lectures on the h-cobordism theorem*. Princeton Univ. Press, 1965.

[220] N. Mok. Geometric Archimedean superrigidity in the Hermitian case. Unpublished.

[221] N. Mok. Aspects of Kähler geometry on arithmetic varieties. *Proc.Symp. Pure Math.*, 52:335–396, 1991.

[222] N. Mok, Y.T. Siu, and S.K. Yeung. Geometric superrigidity. *Inv. math.*, 113:57–84, 1993.

[223] J. Morgan. *The Seiberg–Witten equations and applications to the topology of smooth four manifolds*. Princeton University Press, 1996.

[224] J. Morgan, Z. Szabó, and C. Taubes. A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture. *J. Diff. Geom.*, 44:706–788, 1996.

[225] J. Morgan and G. Tian. *Ricci Flow and the Poincare Conjecture*. AMS, 2007.

[226] C. Morrey. The problem of Plateau on a Riemannian manifold. *Ann.Math.*, 49:807–851, 1948.

[227] G. Mostow. *Strong rigidity of locally symmetric spaces*. Ann. Math. Studies 78, Princeton Univ. Press, 1973.

[228] A. Nadel. Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature. *Ann.Math.*, 132:549–596, 1990.

[229] I.G. Nikolaev. Smoothness of the metric of spaces with curvature that is bilaterally bounded in the sense of A.D. Aleksandrov. *Sib. Math. J.*, 24:247–263, 1983.

[230] I.G. Nikolaev. Bounded curvature closure of the set of compact Riemannian manifolds. *Bull. AMS*, 24:171–177, 1991.

[231] I.G. Nikolaev. *Synthetic methods in Riemannian geometry.* Lecture Notes. Univ. Illinois at Urbana–Champaign, 1992.

[232] J. Nitsche. *Lectures on minimal surfaces, Vol. I.* Cambridge Univ. Press, 1989.

[233] M. Obata. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J.Math.Soc.Japan*, 14:333–340, 1962.

[234] G. Perel'man. The entropy formula for the Ricci flow and its geometric application. arxiv:math.DG/0211159, 2002.

[235] G. Perel'man. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arxiv:math.DG/0307245, 2003.

[236] G. Perel'man. Ricci flow with surgery on three-manifolds. arxiv:math.DG/0303109, 2003.

[237] S. Peters. Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds. *J. reine angew. Math.*, 394:77–82, 1984.

[238] S. Peters. Convergence of Riemannian manifolds. *Compos. Math.*, 62:3–16, 1987.

[239] P. Petersen. *Riemannian geometry.* Springer, 2006.

[240] P. Petersen and T. Tao. Classification of almost quarter-pinched manifolds. *Proc.AMS*, 137:2437–2440, 2009.

[241] P. Petersen and F. Wilhelm. An exotic sphere with positive sectional curvature. arxiv:0805.0812, 2008.

[242] A. Petrunin, X. Rong, and W. Tuschmann. Collapsing vs. positive pinching. *GAFA*, 9:699–735, 1999.

[243] A. Petrunin and W. Tuschmann. Diffeomorphism finiteness, positive pinching, and second homotopy. *GAFA*, 9:736–774, 1999.

[244] D. Quillen. Determinants of Cauchy–Riemann operators over a Riemann surface. *Funct. Anal. Appl.*, 19:31–34, 1985.

[245] Y.G. Reshetnyak. Inextensible mappings in a space of curvature no greater than K. *Sib. Math. J.*, 9:683–689, 1968.

[246] E. Ruh. Almost flat manifolds. *J. Diff. Geom.*, 17:1–14, 1982.

[247] I.K. Sabitov and S.Z. Shefel'. Connections between the order of smoothness of a surface and that of its metric. *Sib. Math. J.*, 17:687–694, 1976.

[248] R. Sachs and H. Wu. *General relativity for mathematicians.* Springer GTM, 1977.

[249] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Ann. Math.*, 113:1–24, 1981.

[250] T. Sakai. *Riemannian geometry.* Amer. Math. Soc., 1995.

[251] D. Salamon. *Spin geometry and Seiberg–Witten invariants.* 1995.

[252] J. Sampson. Applications of harmonic maps to Kähler geometry. *Contemp. Math.*, 49:125–134, 1986.

[253] M. Schlicht. Another proof of Bianchi's identity in arbitrary bundles. *Ann. Global Anal. Geom.*, 13:19–22, 1995.

[254] R. Schoen and K. Uhlenbeck. A regularity theory for harmonic maps. *J. Diff. Geom.*, 17:307–335, 1982.

[255] R. Schoen and K. Uhlenbeck. Boundary regularity and miscellaneous results on harmonic maps. *J. Diff. Geom.*, 18:253–268, 1983. Correction in *J. Diff. Geom.* 18 (1983), 329.

[256] R. Schoen and S.T. Yau. Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with nonnegative scalar curvature. *Ann. Math.*, 110:127–142, 1979.

[257] R. Schoen and S.T. Yau. The structure of manifolds with positive scalar curvature. *Man. math.*, 28:159–183, 1979.

[258] R. Schoen and S.T. Yau. *Lectures on differential geometry.* Internat. Press, 1994.

[259] J.A. Schouten. *Der Riccikalkül.* Springer, 1924.

[260] J.A. Schouten. *Ricci-calculus.* Springer, 1954.

[261] D. Schüth. Isospectral manifolds with different local geometries. *J.reine angew.Math.*, 534:41–94, 2001.

[262] D. Schüth. Isospectral metrics on five-dimensional spheres. *J.Diff.Geom.*, 58:87–111, 2001.

[263] M. Schwarz. *Morse homology.* Birkhäuser, 1993.

[264] M. Schwarz. Equivalences for Morse homology. *Contemp. Math.*, 246:197–216, 1999.

[265] N. Seiberg and E. Witten. Electromagnetic duality, monopol condensation, and confinement in $N = 2$ supersymmetric Yang–Mills theory. *Nucl. Phys.*, 431(B):581–640, 1994.

[266] N. Seiberg and E. Witten. Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD. *Nucl. Phys. B*, 431:581–640, 1994.

[267] K. Shankar. On the fundamental group of positively curved manifolds. *J.Diff.Geom.*, 49:179–182, 1998.

[268] Y.T. Siu. The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. *Ann. Math.*, 112:73–111, 1980.

[269] Y.T. Siu. The existence of Kähler–Einstein metrics on manifolds with positive anticanonical line bundle and suitable finite symmetry group. *Ann.Math.*, 127:585–627, 1988.

[270] Y.T. Siu and S.T. Yau. Compact Kähler manifolds of positive bisectional curvature. *Inv. math.*, 59:189–204, 1980.

[271] S. Smale. On gradient dynamical systems. *Ann. Math.*, 74:199–206, 1961.

[272] E. Spanier. *Algebraic topology*. McGraw Hill, 1966.

[273] K. Steffen. *An introduction to harmonic mappings*. SFB 256, Vorlesungsreihe, Vol. 18, Bonn, 1991.

[274] R. Stoecker and H. Zieschang. *Algebraische Topologie*. Teubner, 1994.

[275] S. Stolz. Simply connected manifolds of positive scalar curvature. *Ann. Math.*, 136:511–540, 1992.

[276] M. Struwe. On the evolution of harmonic mappings. *Comm. Math. Helv.*, 60:558–581, 1985.

[277] M. Struwe. *Variational methods*. Springer, 2008.

[278] T. Sunada. Riemannian coverings and isospectral manifolds. *Ann. Math.*, 121:169–186, 1985.

[279] Z. Szabó. Isospectral pairs of metrics on balls, spheres, and other manifolds with different local geometries. *Ann. Math.*, 154:437–475, 2001.

[280] G. Tarantello. Multiple condensate solutions for the Chern–Simons–Higgs theory. *J. Math. Phys.*, 37:3769–3796, 1996.

[281] C. Taubes. Arbitrary n–vortex solutions to the first order Ginzburg–Landau equations. *Comm. Math. Phys.*, 72:227–292, 1980.

[282] C. Taubes. The Seiberg–Witten invariant and symplectic forms. *Math. Res. Letters*, 1:809–822, 1994.

[283] C. Taubes. More constraints on symplectic manifolds from Seiberg–Witten invariants. *Math. Res. Letters*, 2:9–14, 1995.

[284] C. Taubes. The Seiberg–Witten and the Gromov invariants. *Math. Res. Letters*, 9:809–822, 1995.

[285] C. Taubes. Counting pseudoholomorphic submanifolds in dimension 4. *J. Diff. Geom.*, 44:818–893, 1996.

[286] R. Thom. Sur une partition cellules associés à une fonction sur une variété. *C. R. Acad. Sci. Paris*, 228:973–975, 1949.

[287] W. Thurston. Hyperbolic structures on 3-manifolds. Preprint, 1980.

[288] W. Thurston. *Three-dimensional geometry and topology, Vol. 1.* Princeton University Press, 1997.

[289] G. Tian. On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. *Inv. math.*, 89:225–246, 1987.

[290] G. Tian. Kähler–Einstein metrics with positive scalar curvature. *Invent.Math.*, 130:1–39, 1997.

[291] G. Tian and S.T. Yau. Kähler–Einstein metrics on complex surfaces with $c_1(M) > 0$. *Comm. Math. Phys.*, 112:175–203, 1987.

[292] W. Tuschmann. Collapsing, solvmanifolds, and infrahomogenous spaces. *Diff. Geom. Appl.*, 7:251–264, 1997.

[293] W. Tuschmann. Endlichkeitssätze und positive Krümmung. Habilitation thesis, Leipzig, 2000.

[294] M.-F. Vignéras. Variétés riemanniannes isospectrales et non isométriques. *Ann.Math.*, 112:21–32, 1980.

[295] J. Weber. The Morse–Witten complex via dynamical systems. arXiv:math.GT/0411465, 2004.

[296] R. Wells. *Differential analysis on complex manifolds.* Springer, 1980.

[297] H. Weyl. *Die Idee der Riemannschen Fläche.* Teubner, Leipzig, Berlin, 1913.

[298] H. Weyl. *Space, time, matter.* Dover, 1952. Translated from the German.

[299] H. Weyl. *Das Kontinuum und andere Monographien.* Chelsea, New York, 1973. Reprint.

[300] B. Wilking. Manifolds with positive sectional curvature almost everywhere. *Invent.Math.*, 148:117–141, 2002.

[301] B. Wilking. Nonegatively and positively curved manifolds. *Surveys Diff.Geom.*, 2007.

[302] E. Witten. Supersymmetry and Morse theory. *J. Diff. Geom*, 17:661–692, 1982.

[303] E. Witten. Monopoles and 4-manifolds. *Math. Res. Letters*, 1:764–796, 1994.

[304] J. Wolf. *Spaces of constant curvature.* Publish or Perish, Boston, 1974.

[305] H. Wu. The Bochner technique in differential geometry. *Math. Rep.*, 3(2):289–538, 1988.

[306] Y.L. Xin. *Geometry of harmonic maps.* Birkhäuser, 1996.

[307] Y.L. Xin. *Minimal submanifolds and related topics.* World Scientific, 2004.

[308] D. Yang. Lower bound estimates of the first eigenvalue for compact manifolds with positive Ricci curvature. *Pacif.J.Math.*, 190:383–399, 1999.

[309] S.T. Yau. On the fundamental group of compact manifolds of non-positive curvature. *Ann. Math.*, 93:579–585, 1971.

[310] S.T. Yau. Isoperimetric constants and the first eigenvalue of a compact manifold. *Ann.Sci.Ec.Norm.Sup.Paris*, 8:487–507, 1975.

[311] S.T. Yau. Calabi's conjecture and some new results in algebraic geometry. *Proc.Nat.Acad.Sc.*, 74:1798–1801, 1977.

[312] S.T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Comm. Pure Appl. Math.*, 31:339–411, 1978.

[313] S.T. Yau. Open problems in geometry. *Proc. Symp. Pure Math.*, 54(I):1–28, 1993.

[314] J.Q. Zhong and H.C. Yang. On the estimate of the first eigenvalue of a compact Riemannian manifold. *Scientia Sinica*, 27(12):1265–1273, 1984.

[315] W. Ziemer. *Weakly differentiable functions.* Springer, 1989.

[316] W. Ziller. Examples of Riemannian manifolds with non-negative sectional curvature. *Surveys Diff.Geom.*, 2007.

[317] R. Zimmer. *Ergodic theory and semisimple groups.* Birkhäuser, 1984.

Index