Appendix A
Discrete Renewal Theory: Basic (and a Few Less Basic) Facts and Estimates

A.1 A Crash Course on Renewal Theory

A.1.1 Renewal and Markov Chains

We start by working in a general (discrete) framework, that is we choose a discrete probability density $K(\cdot)$ on $\mathbb{N} \cup \{\infty\}$ ($K(\infty) < 1$ to avoid trivialities) and we introduce $\tau := \{\tau_j\}_{j=0,1,\dots}$ as the sequence of partial sums of an IID sequence of $K(\cdot)$ distributed variables, that we call *inter-arrival variables*. We call τ $K(\cdot)$*-renewal* and we stress that $\tau_0 = 0$ unless explicitly stated. We also freely switch from looking at τ as a sequence of random variables and as a (random) subset of $\mathbb{N} \cup \{0\}$ (*point process*): note that we do not include infinity in this case, because it is always the case that either τ is a finite set (when $\tau_n = \infty$ for some n) or τ contains infinitely many points (but not ∞). The point process notational convention is rather practical and compact: for example $\{$there exists j such that $\tau_j = n\}$ shrinks down to $\{n \in \tau\}$. We say that τ is *persistent* when $|\tau| = \infty$ ($|\tau|$ is the number of points in τ); otherwise we say that it is *terminating*. Of course τ is persistent if and only if $K(\infty) = 0$.

Renewal processes enjoy the renewal property, i.e. if $A \subset \mathscr{P}(\{0,1,\dots,n\})$ and $B \subset \mathscr{P}(\{n+1, n+2\dots,\})$ we have

$$\mathbf{P}(\tau \cap [0,n] \in A, n \in \tau, \tau \cap [n+1,\infty) \in B) = \mathbf{P}(\tau \cap [0,n] \in A, n \in \tau)\,\mathbf{P}(\tau + n \in B). \quad \text{(A.1)}$$

There is natural link between renewal processes and Markov chains (see [1, Chap. I] for a quick self-contained review on Markov chains and for all basic notions). In fact, by the strong Markov property the sequence of successive returns of a Markov chain to a fixed (recurrent or transient) state is a renewal process. But this works also in the opposite direction: any renewal process is the return time sequence of a suitable Markov chain. In fact if we define $\mathrm{A}_n := \mathrm{A}_n(\tau) := n - \sup\{\tau_k : \tau_k \le n\}$, then the sequence $\mathrm{A} := \{\mathrm{A}_n\}_{n=0,1,\dots}$ is a Markov chain called

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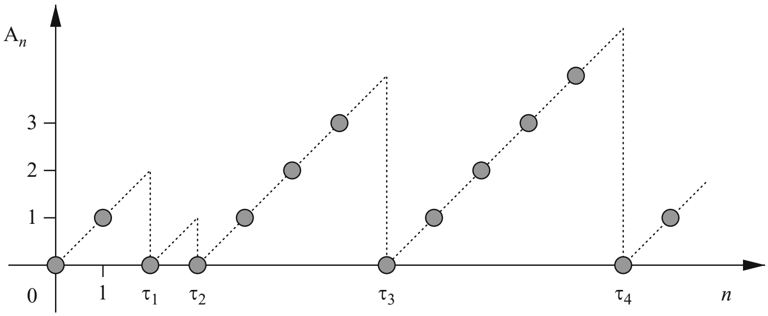

Fig. A.1 The A process (large gray dots) associated to the renewal τ. In this case $\tau_1 = 2$, $\tau_2 = 3$, $\tau_3 = 7$, $\tau_4 = 12$ and $\tau_5 > 13$

backward recurrence time. Note that $\mathrm{A}_n \in \mathbb{N} \cup \{0\}$ is the time elapsed since the last renewal when looking from n, see Fig. A.1. The probability transition from $\mathrm{A}_n = i$ to $\mathrm{A}_{n+1} = j$ is non-zero only if $j = i+1$ or $j = 0$ and the probability that the process moves up is $(\overline{K}(i+1) + K(\infty))/(\overline{K}(i) + K(\infty))$, independently of $\mathrm{A}_0, \mathrm{A}_1, \dots, \mathrm{A}_{n-1}$ (we have used $\overline{K}(i) := \sum_{n>i} K(n)$ and in this sum $K(\infty)$ is not included).

We say that $K(\cdot)$ has period $p \in \mathbb{N}$ if $\{n : K(n) > 0\}$ is contained in $\{pn : n \in \mathbb{N}\}$ and if p is the largest number with this property. If $p = 1$ we say that $K(\cdot)$ is aperiodic. The aperiodicity of $K(\cdot)$ implies the aperiodicity of the Markov chain A. Note that when $K(\infty) = 0$, the state space of A is $\{0, 1, \dots, \sup\{n : K(n) > 0\}\} \setminus \{\infty\}$ and that A is irreducible (that is all states communicate in a finite number of steps, with positive probability).

For ease of notation set $m_K := \sum_{n \in \mathbb{N} \cup \{\infty\}} nK(n) \in [1, \infty]$. Of course $m_K = \infty$ may arise also when $K(\infty) = 0$: in this case A is a recurrent Markov chain, but it is immediate to see that it is a null recurrent chain, since τ_1 coincides with $\inf\{n > 0 : \mathrm{A}_n = 0\}$. On the other hand, A is clearly positive recurrent if $m_K < \infty$. We will therefore say that τ is positive (respectively null) persistent if A is.

A.1.2 The Renewal Theorem

We state now the Theorem.

Theorem A.1. *If* $K(\cdot)$ *is aperiodic then* $\lim_{n\to\infty} \mathbf{P}(n \in \tau) = 1/m_K$*, with* $1/\infty = 0$.

We direct the reader to [1, Chap. I, Theorem 2.2] for a proof, which is based on the fundamental formula (direct consequence of the renewal property), often called *renewal equation* (and $n \mapsto \mathbf{P}(n \in \tau)$ is the *renewal function*), that says

$$\mathbf{P}(n \in \tau) = \mathbf{1}_{n=0} + \sum_{k=0}^{n} \mathbf{P}(k \in \tau) K(n-k). \tag{A.2}$$

One way to extract information from (A.2) is to pass to Laplace transform: for $s > 0$

$$\sum_{n=0}^{\infty} \exp(-sn)\mathbf{P}(n \in \tau) = \left(1 - \sum_{n=1}^{\infty} \exp(-sn)K(n)\right)^{-1}. \tag{A.3}$$

It helps the intuition to note that Theorem A.1 can be proven also by looking at the ergodic properties of the backward recurrence time process A [1, Sect. VII.2]. In particular one sees that the condition $m_K < \infty$ is precisely the condition for A to be positive recurrent. This implies directly the existence of a unique invariant probability measure which we can write explicitly:

$$p_{\mathrm{A}}(n) = \frac{1}{m_K} \sum_{j \geq n+1} K(j), \quad n = 0, 1, \ldots. \tag{A.4}$$

By the Ergodic Theorem for irreducible aperiodic Markov chains one also has that

$$\lim_{N \to \infty} \mathbf{P}(\mathrm{A}_N = n) = p_{\mathrm{A}}(n). \tag{A.5}$$

If $\mathrm{A} = \{\mathrm{A}_n\}_n$ is redefined so that A_0 is distributed according to $p_{\mathrm{A}}(\cdot)$, so that A is stationary, then $\{n : \mathrm{A}_n = 0\}$ is a renewal process translated by a random quantity (independent of the renewal). In other terms, τ_0 is no longer degenerate, but it has its own distribution (on $\mathbb{N} \cup \{0\}$). We call such a process a *delayed renewal* and when τ_0 has distribution $p_{\mathrm{A}}(\cdot)$ [cf. (A.4)] we call *stationary renewal* such a delayed renewal (note that also a delayed renewal enjoys the renewal property).

We will often need information on the number of renewal points up to a certain time n, that is on $|\tau \cap (0, n]|$. The fact that

$$\lim_{n \to \infty} \frac{1}{n} |\tau \cap (0, n]| = \frac{1}{m_K}, \tag{A.6}$$

both in the almost sure and in the L^1 sense (as a matter of fact, in L^p for any $p \geq 1$), is a direct consequence of Kolmogorov law of large numbers.

A.1.3 Beyond the Renewal Theorem

Theorem A.1 does apply when $m_K = \infty$, but in this case it calls for refinements. We state here results in this direction in the framework in which we typically work: that is, we assume (2.30).

First of all, the transient case is covered by the following result.

Theorem A.2. *If $K(\infty) > 0$ then*

$$\mathbf{P}(N \in \tau) \overset{N \to \infty}{\sim} \frac{K(N)}{(K(\infty))^2}. \tag{A.7}$$

A proof can be found in [7, A.5.2].

For the null recurrent case we have of course $\alpha \in (0,1]$. The following results can be found in [2, Theorem 8.7.3 and Theorem 8.7.5].

Theorem A.3. *For $\alpha \in (0,1)$ and $K(\infty) = 0$ we have that*

$$\mathbf{E}\left[|\tau \cap (0,N]|\right] = \sum_{n=1}^{N} \mathbf{P}(n \in \tau) \overset{N\to\infty}{\sim} \frac{\sin(\pi\alpha)}{\mathtt{c}_K \pi} N^{\alpha}. \tag{A.8}$$

For $\alpha = 1$ and $K(\infty) = 0$ we have that $\sum_{n=1}^{N} \mathbf{P}(n \in \tau) \sim N/(\mathtt{c}_K \log N)$.

Theorem A.3 gives *integral* bounds and it can be obtained by standard Tauberian arguments [2] applied to (A.3). Getting *local* estimates is substantially harder: sharp local estimates however have been obtained and this is the content of the next theorem.

Theorem A.4. *For $\alpha \in (0,1)$ and $K(\infty) = 0$ we have*

$$\mathbf{P}(N \in \tau) \overset{N\to\infty}{\sim} \frac{\alpha \sin(\pi\alpha)}{\mathtt{c}_K \pi} \frac{1}{N^{1-\alpha}}. \tag{A.9}$$

For $\alpha = 1$ instead

$$\mathbf{P}(N \in \tau) \overset{N\to\infty}{\sim} \frac{1}{\mathtt{c}_K \log N}. \tag{A.10}$$

Formula (A.9) is due to Doney [4, Theorem B], that completed a partial result of [6]. The second result instead, that is (A.10), can be found in [2, Theorem 8.7.5].

Of course, from (A.9) one directly extracts the existence for $\alpha \in (0,1)$ of $C = C(K(\cdot)) > 0$ such that for every $N \in \mathbb{N}$

$$\frac{1}{C} \le N^{1-\alpha} \mathbf{P}(N \in \tau) \le C. \tag{A.11}$$

A.1.4 Convergence of Renewal and Point Processes

A renewal or, more generally, a discrete point process can be seen as a random variable $\tau : \Omega \to \mathscr{P}(\mathbb{N} \cup \{0\}) = 2^{\mathbb{N}\cup\{0\}}$, with $(\Omega, \mathscr{F}, \mathbf{P})$ a generic probability space and $\mathscr{P}(\mathbb{N} \cup \{0\})$ is equipped with the σ-algebra $\mathscr{G} := \sigma(\cup_{n\in\mathbb{N}} \mathscr{G}_n))$, where $\{\mathscr{G}_n\}_{n=0,1,\ldots}$ is the natural filtration of the process $\mathrm{A} = \mathrm{A}(\tau)$ defined in Sect. A.1. It is convenient to introduce a metric space for which $\mathscr{G}$ is the Borel σ-algebra. For this we introduce the semi-metric d_n on $\mathscr{P}(\mathbb{N} \cup \{0\})$ by setting $d_n(B_1, B_2) = \mathbf{1}_{B_1 \cap [0,n] \neq B_2 \cap [0,n]}$. Then one directly verifies that $\mathscr{G}$ is the Borel σ-algebra for the metric $d(B_1, B_2) := \sum_n 2^{-n} d_n(B_1, B_2)$. We introduce the notation

$$\mathscr{P}_0 := (\mathscr{P}(\mathbb{N} \cup \{0\}), d), \tag{A.12}$$

for the metric space. In general we do not deal only with renewal processes: the typical case we consider is the one of a sequence of measures $\{\mathbf{P}_N\}_N$ on $\mathscr{P}_0$ converging to a limit (that happens to be a renewal process). In any case the convergence in law of $\{\mathbf{P}_N\}_N$ to a limit $\mathbf{P}_\infty$ means simply if $B \in \mathscr{G}_n$ for some n then $\lim_N \mathbf{P}_N(B) = \mathbf{P}_\infty(B)$.

A.2 Some Pinning Oriented Renewal Issues

A.2.1 On Boundary Effects

Next is a bound on boundary effects: we just state the inequality that we use, but also the reciprocal bound can be proven (with a different constant) along the same line.

Lemma A.5. *Assume* (2.30)*. For every $K(\cdot)$-renewal τ there exists $C_{bc} > 0$ such that*

$$\mathbf{E}\left[F_n(\tau)\,\middle|\,2n \in \tau\right] \le C_{bc}\,\mathbf{E}\left[F_n(\tau)\right], \tag{A.13}$$

for every $n \in \mathbb{N}$ and every $F_n : \mathscr{P}(\mathbb{N}) \longrightarrow [0,\infty)$ ($\mathscr{P}(\mathbb{N})$ is the set of subsets of $\mathbb{N}$) which is measurable with respect to the σ-algebra generated by $\{\{A \in \mathscr{P}(\mathbb{N}) : j \in A\} : j = 1,2,\ldots,n\}$ (that is $F_n(A) = F_n(A \cap (0,n])$ for every $A \subset \mathbb{N}$).

Proof. The various constants c_1, c_2,... appearing below depend only on $K(\cdot)$. Set $X_n = X_n(\tau) := \max\{j = 0,1,\ldots,n : j \in \tau\}$. By the measurability properties of $F_n(\cdot)$ we see that $F_n(\tau) = F_n(\tau \cap (0,j])$ if $X_n(\tau) = j$ (read $(0,0]$ as $\emptyset$) and therefore

$$\mathbf{E}\left[F_n(\tau)\,\middle|\,2n \in \tau\right] = \sum_{j=0}^{n} \mathbf{E}\left[F_n(\tau \cap (0,j])\,\middle|\,X_n = j\right] \mathbf{P}\left(X_n = j\,\middle|\,2n \in \tau\right), \tag{A.14}$$

and therefore it is sufficient to show that

$$\mathbf{P}\left(X_n = j\,\middle|\,2n \in \tau\right) \le C_{bc}\,\mathbf{P}\left(X_n = j\right). \tag{A.15}$$

For this we write

$$\begin{aligned}\mathbf{P}(X_n = j, 2n \in \tau) &= \mathbf{P}(j \in \tau) \sum_{m=n+1}^{2n} K(m-j)\mathbf{P}(2n - m \in \tau) \\ &= \mathbf{P}(j \in \tau)\left(\sum_{m=n+1}^{\lfloor 3n/2 \rfloor} \ldots + \sum_{m=\lfloor 3n/2 \rfloor + 1}^{2n} \ldots\right) =: T_1 + T_2 .\end{aligned} \tag{A.16}$$

For what concerns T_1 we use the fact that $2n-m \geq n/2$ so that $\mathbf{P}(2n-m \in \tau) \leq c_1 \mathbf{P}(2n \in \tau)$ and

$$T_1 \leq c_1 \mathbf{P}(2n \in \tau)\mathbf{P}(j \in \tau) \sum_{m=n+1}^{\lfloor 3n/2 \rfloor} K(m-j)\,. \tag{A.17}$$

For T_2 we use that $\mathbf{P}(j \in \tau) \leq c_2 (j+1)^{\alpha-1}$ for every j and that $m-j \geq \lfloor n/2 \rfloor$, which implies $K(m-j) \leq c_3/n^{1+\alpha}$, so that

$$T_2 \leq c_2 c_3 \mathbf{P}(j \in \tau) n^{-1-\alpha} \sum_{m=\lfloor 3n/2 \rfloor+1}^{2n} (2n-m+1)^{\alpha-1} \leq c_4 \mathbf{P}(j \in \tau) n^{-1}\,, \tag{A.18}$$

for every $n \in \mathbb{N}$. Since we have $\mathbf{P}(2n \in \tau) \geq c_5 n^{\alpha-1}$ and $\sum_{m=\lfloor 3n/2 \rfloor+1}^{\infty} K(m-j) \geq c_6 n^{-\alpha}$ (for $j \leq n$), from (A.18) we get that

$$T_2 \leq \frac{c_4}{c_5 c_6} \mathbf{P}(2n \in \tau)\mathbf{P}(j \in \tau) \sum_{m=\lfloor 3n/2 \rfloor+1}^{\infty} K(m-j)\,. \tag{A.19}$$

By putting (A.17) and (A.19) together we have that

$$\begin{aligned} &\mathbf{P}\left(X_n = j, 2n \in \tau\right) \\ &\qquad \leq c_7 \mathbf{P}(2n \in \tau)\mathbf{P}(j \in \tau) \sum_{m=n+1}^{\infty} K(m-j) = c_7 \mathbf{P}(2n \in \tau)\mathbf{P}(X_n = j)\,, \end{aligned} \tag{A.20}$$

which completes the proof with $C_{bc} = c_7$. □

A.2.2 Two Scaling Results for Renewal Processes

The first result is applied in Chap. 6 when $\alpha \in (1/2, 1)$, but it plays a central role also for the case $\alpha = 1/2$, since it is used in Proposition A.7 below, that, in turn, is used in Chap. 6.

Proposition A.6. *For every $K(\cdot)$-renewal τ with $K(\cdot)$ as in (2.30) and $\alpha \in (0,1)$ we have*

$$\mathscr{L}\text{-}\lim_{n \to \infty} \frac{1}{n^\alpha} \left|\tau \cap (0,n]\right| = \frac{Y_\alpha}{\mathtt{c}_K}\,, \tag{A.21}$$

where Y_α is a random variable that depends only on α with the property $\mathbf{P}(Y_\alpha > 0) = 1$. In particular $Y_{1/2} = |Z|/\sqrt{2\pi}$, with $Z \sim \mathscr{N}(0,1)$.

Proof. This is treated for example in [5]. The point is simply that $|\tau \cap (0,n]| < m$ is equivalent to $\tau_m > n$ and τ_m is a sum of m IID variables and the question is therefore

an issue of domain of stability of stable laws. In [5, XI.5, p. 373] it is proven that for every $x > 0$

$$\lim_{n\to\infty} \mathbf{P}\left(\overline{K}(n)\,|\tau\cap(0,n]| \geq \frac{2-\alpha}{\alpha}\frac{1}{x^\alpha}\right) = G_\alpha(x)\,, \tag{A.22}$$

where $G_\alpha(\cdot)$ is the distribution function of the one sided stable distribution satisfying $\lim_{x\to\infty} x^\alpha(1-G_\alpha(x)) = (2-\alpha)/\alpha$, characterized by the Laplace transform

$$\int_0^\infty \exp(-\lambda x)\mathrm{d}G_\alpha(x) = \exp\left(-c_\alpha\lambda^\alpha\right) \quad \text{with } c_\alpha := \frac{(2-\alpha)\Gamma(1-\alpha)}{\alpha}\,, \tag{A.23}$$

for $\lambda > 0$. Such stable laws are treated in detail for example in [5, XIII.6, Theorem 1], where (A.23) is proven along with the fact that $\lim_{x\searrow 0} G_\alpha(x) = 0$, so that $(0,\infty)$ is of full measure under this distribution (for completeness: in this limit $G_\alpha(x) = o(\exp(-cx^{-\alpha}))$, with $c = \alpha/((2-\alpha)\Gamma(1-\alpha)))$. By a change of variable and by using $\overline{K}(n) \sim (\mathtt{c}_K/\alpha)n^{-\alpha}$ in (A.22) we see that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{|\tau\cap(0,n]|}{n^\alpha} \geq y\right) = G_\alpha\left(\left(\frac{2-\alpha}{\mathtt{c}_K}\right)^{\frac{1}{\alpha}} y^{-\frac{1}{\alpha}}\right) =: 1-F_\alpha(y)\,, \tag{A.24}$$

and the asymptotic properties of $G_\alpha(x)$ mentioned just above directly yield that $F_\alpha(y)$ tends to 0 as $y\searrow 0$ (more precisely: $F_\alpha(y) \sim \mathtt{c}_K y/\alpha$) and that $\lim_{y\to\infty} F_\alpha(y) = 1$ (more precisely, $1-F_\alpha(y) = o(\exp(-cy))$ for a $c>0$). These facts suffice to conclude the converge in law that we claim in the statement and that the limit variable Y_α is a.s. positive. A number of further properties of Y_α can be derived by exploiting the properties of the stable distribution $G_\alpha(\cdot)$, for example that $G_\alpha(t) = \int_0^t g_\alpha(s)\mathrm{d}s$ for a suitable probability density $g_\alpha(\cdot)$ (this follows immediately from the fact that the characteristic function $\psi_\alpha(t) = \int_0^\infty \exp(itx)\mathrm{d}G_\alpha(x) = \exp(-c_\alpha t^\alpha(\cos(\pi\alpha/2)+i\sin(\pi\alpha/2))$ for $t\geq 0$, so that $|\psi_\alpha(t)| = \exp(-c_\alpha|t|^\alpha(\cos(\pi\alpha/2))$ and therefore $\int_{\mathbb{R}}|\psi(t)|\mathrm{d}t < \infty$). However, *it seems impossible to express stable densities in a closed form* [5, p. 581], with the notable exception of $\alpha = 1/2$ for which we can use that if the random variable X has density $f_X(x) = x^{-3/2}(2\sqrt{\pi})^{-1}\exp(-1/(4x))\mathbf{1}_{x>0}$ we have

$$\mathbf{E}\left[\exp(-\lambda X)\right] = \exp\left(-\sqrt{\lambda}\right)\,, \quad \text{for } \lambda > 0. \tag{A.25}$$

A straightforward, but rather painful, constant tracking exercise [via (A.23) and (A.24)] leads to $Y_{1/2} = |Z|/\sqrt{2\pi}$. $\square$

Proposition A.7. *For every $K(\cdot)$-renewal τ with $K(\cdot)$ as in (2.30) and $\alpha = 1/2$ we have*

$$\mathscr{L}\text{-}\lim_{n\to\infty} \frac{1}{\sqrt{n}\log n}\sum_{1\leq i<j\leq n}\frac{\delta_i\delta_j}{\sqrt{j-i}} = \frac{|Z|}{(2\pi)^{3/2}\,\mathtt{c}_K^2}\,, \tag{A.26}$$

where $Z \sim \mathcal{N}(0,1)$.

Proof. We introduce the notation

$$Y_n^{(i)} := \sum_{j=i+1}^{n} \frac{\delta_j}{\sqrt{j-i}}, \tag{A.27}$$

that allows writing

$$\frac{1}{\sqrt{n}\log n} \sum_{1 \le i < j \le n} \frac{\delta_i \delta_j}{\sqrt{j-i}} = \frac{1}{\sqrt{n}\log n} \sum_{i=1}^{n-1} \delta_i Y_n^{(i)} =: X_n . \tag{A.28}$$

Note that, by the renewal property of τ, $Y_n^{(i)}$ (under $\mathbf{P}(\cdot \mid \delta_i = 1)$) is distributed like $Y_{n-i} := Y_{n-i}^{(0)}$ (under $\mathbf{P}$). The first step in the proof is observing that, by (A.11), we have

$$\mathbf{E}\left[\frac{1}{\sqrt{n}\log n} \sum_{i=(1-\varepsilon)n}^{n-1} \delta_i Y_n^{(i)}\right] = \frac{1}{\log n \sqrt{n}} \sum_{i=(1-\varepsilon)n}^{n-1} \sum_{j=i+1}^{n} \frac{\mathbf{P}(i \in \tau)\mathbf{P}(j-i \in \tau)}{\sqrt{j-i}} = O(\varepsilon), \tag{A.29}$$

uniformly in n: we have introduced the short-cut convention (that we will keep throughout this proof) that summing from $(1-\varepsilon)n$ means summing from $\lfloor (1-\varepsilon)n \rfloor + 1$ and, just below, summing up to $(1-\varepsilon)n$ means up to $\lfloor (1-\varepsilon)n \rfloor$. What (A.29) is telling us is that we can focus on studying $X_{n,\varepsilon}$, defined as X_n, but stopping the sum over i at $(1-\varepsilon)n$:

$$X_{n,\varepsilon} := \frac{1}{\sqrt{n}\log n} \sum_{i=1}^{(1-\varepsilon)n} \delta_i Y_n^{(i)} . \tag{A.30}$$

At this point we use that

$$\lim_{n\to\infty} \frac{Y_n}{\log n} = \frac{1}{2\pi \mathrm{c}_K} =: \widehat{c}_K, \tag{A.31}$$

in $L^2(\mathbf{P})$ (and hence in $L^1(\mathbf{P})$). We postpone the proof of (A.31) and observe that, since the normalization is the logarithm of n, it implies that for every $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{q\in[\varepsilon,1]} \mathbf{E}\left[\left|\frac{1}{\log n} \sum_{j=1}^{\lfloor qn \rfloor} \frac{\delta_j}{\sqrt{j}} - \widehat{c}_K\right|\right] = 0. \tag{A.32}$$

Let us write

$$R_n := X_{n,\varepsilon} - \frac{\widehat{c}_K}{\sqrt{n}} \sum_{i=1}^{(1-\varepsilon)n} \delta_i \tag{A.33}$$

and note that $n^{-1/2}\sum_{i=1}^{(1-\varepsilon)n}\delta_i$ converges in law toward $\sqrt{(1-\varepsilon)/(2\pi c_K^2)}\,|Z|$ by Proposition A.6. It suffices therefore to show that for every $\varepsilon \in (0,1)$ we have $\lim_{n\to\infty}\mathbf{E}[|R_n|]=0$. And in fact

$$\begin{aligned}\mathbf{E}\left[|R_n|\right] &\le \frac{1}{\sqrt{n}}\sum_{i=1}^{(1-\varepsilon)n}\mathbf{E}[\delta_i]\mathbf{E}\left[\left|\frac{Y_n^{(i)}}{\log n}-\widehat{c}_K\right|\,\middle|\,\delta_i=1\right]\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{(1-\varepsilon)n}\mathbf{E}[\delta_i]\mathbf{E}\left[\left|\frac{Y_{n-i}}{\log n}-\widehat{c}_K\right|\right]\stackrel{n\to\infty}{\longrightarrow}0,\end{aligned}\qquad \text{(A.34)}$$

where in the last step we have used (A.32) and (A.11).

We are therefore left with proving (A.31). This result has been established in [3, Theorem 6] in the case in which τ is given by the successive returns to zero of a centered, aperiodic and irreducible random walk on $\mathbb{Z}$ with bounded increment variance. Note that, by well established local limit theorems, for such a class of random walks we have (A.9). In [3] it is proven more, namely that (A.31) holds also almost surely and this is extracted from the estimate $\mathrm{var}_{\mathbf{P}}(Y_n)=O(\log n)$. What we are going to do is simply to re-obtain such a bound, by repeating the steps in [3] and using (A.9)–(A.11), for the general renewal processes that we consider (and one can verify that almost sure convergence comes as a bonus, but we will not use it).

The proof goes as follows: by (A.9) one directly sees that $\lim_{n\to\infty}\mathbf{E}[Y_n/\log n]=\widehat{c}_K$, therefore we are done if we show that $\lim_{n\to\infty}\mathrm{var}_{\mathbf{P}}(Y_n/\log n)=0$. So we start by observing that

$$\mathrm{var}_{\mathbf{P}}(Y_n)=\sum_{i,j}\frac{\mathbf{E}[\delta_i\delta_j]-\mathbf{E}[\delta_i]\mathbf{E}[\delta_j]}{\sqrt{ij}}=2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\mathbf{E}[\delta_i\delta_j]-\mathbf{E}[\delta_i]\mathbf{E}[\delta_j]}{\sqrt{ij}}+O(1),\qquad \text{(A.35)}$$

by (A.11). Now we compute

$$\begin{aligned}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\mathbf{E}[\delta_i\delta_j]-\mathbf{E}[\delta_i]\mathbf{E}[\delta_j]}{\sqrt{ij}} &= \sum_{i=1}^{n-1}\frac{\mathbf{E}[\delta_i]}{\sqrt{i}}\left[\sum_{j=1}^{n-i}\frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}}-\sum_{j=i+1}^{n}\frac{\mathbf{E}[\delta_j]}{\sqrt{j}}\right]\\ &\le \sum_{i=1}^{n-1}\frac{\mathbf{E}[\delta_i]}{\sqrt{i}}\left[\sum_{j=1}^{n-i}\frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}}-\sum_{j=i+1}^{n}\frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}}\right]\\ &\le \sum_{i=1}^{n-1}\frac{\mathbf{E}[\delta_i]}{\sqrt{i}}\sum_{j=1}^{i}\frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}}\le\sum_{i=1}^{n-1}\frac{\mathbf{E}[\delta_i]}{i}\sum_{j=1}^{i}\mathbf{E}[\delta_j]\\ &\le C^2\sum_{i=1}^{n-1}\frac{1}{i^{3/2}}\sum_{j=1}^{i}\frac{1}{j^{1/2}}=O(\log n),\end{aligned}\qquad \text{(A.36)}$$

where, in the last line, we have used (A.11). In view of (A.35), we have obtained $\mathrm{var}_{\mathbf{P}}(Y_n) = O(\log n)$ so that the proof (A.31) is complete and, with it, the proof of Lemma A.7. □

A.2.3 On the Derivatives of the Free Energy Near Criticality

We have seen that, assuming (2.30) and $\sum_n K(n) = 1$, for $\alpha \in (0,1)$ we have

$$\textsc{f}(h) \overset{h\searrow 0}{\sim} c\, h^{1/\alpha} =: \textsc{f}_{\mathrm{cr}}(h)\,, \tag{A.37}$$

where $c = (\alpha/(\mathtt{c}_K \Gamma(1-\alpha)))^{1/\alpha} > 0$ (cf. Theorem 2.10). The subscript *cr* is used to indicate that the function captures the leading critical behavior. Recall that $\textsc{f}(\cdot)$ is real analytic except at the origin. Here we prove that:

Proposition A.8. *For $\alpha \in (0,1)$ and $1/\alpha \notin \mathbb{N}$ we have that for every $j \in \mathbb{N}$*

$$\left(\frac{\mathrm{d}}{\mathrm{d}h}\right)^j \textsc{f}(h) \overset{h\searrow 0}{\sim} \left(\frac{\mathrm{d}}{\mathrm{d}h}\right)^j \textsc{f}_{cr}(h) = c h^{-j+1/\alpha} \prod_{i=1}^{j} \left(\frac{1}{\alpha} - i + 1\right). \tag{A.38}$$

If $1/\alpha \in \mathbb{N}$ then (A.38) holds for $j \le 1/\alpha$.

This result largely suffices for our purposes, but let us point out that generalizing the statement to $j > 1/\alpha$ when $1/\alpha \in \mathbb{N}$ requires more on $K(\cdot)$ than (2.30).

Proof. Let us start by setting up some notation:

$$\Psi(x) \overset{x\ge 0}{=} 1 - \sum_{n=1}^{\infty} K(n) \exp(-nx) \overset{x\searrow 0}{\sim} \mathtt{c}_K \frac{\Gamma(1-\alpha)}{\alpha} x^{\alpha} =: \Psi_{\mathrm{cr}}(x). \tag{A.39}$$

Let us recall that the relation defining $\textsc{f}(h)$ for $h > 0$ is

$$\Psi(\textsc{f}(h)) = 1 - \exp(-h) \overset{h\searrow 0}{\sim} h\,. \tag{A.40}$$

This formula has the important companion:

$$\Psi_{\mathrm{cr}}(\textsc{f}_{\mathrm{cr}}(h)) = h\,. \tag{A.41}$$

In the sequel we use the notation $f^{(j)}(h) := (\mathrm{d}/\mathrm{d}h)^j f(h)$ and we point out that a standard Riemann sum approximation yields for $j \in \mathbb{N}$:

$$\begin{aligned}\Psi^{(j)}(x) &= (-1)^{j+1} \sum_n n^j K(n) \exp(-nx) \overset{x\searrow 0}{\sim} (-1)^{j+1} \Gamma(j-\alpha) \mathtt{c}_K x^{\alpha-j} \\ &= \left(\prod_{i=1}^{j-1} (\alpha - i)\right) \Gamma(1-\alpha) \mathtt{c}_K x^{\alpha-j} = \Psi_{\mathrm{cr}}^{(j)}(x)\,, \end{aligned} \tag{A.42}$$

with the convention $\prod_{i=1}^{0}(\alpha - i) = 1$.

Let us now compute the asymptotic behavior of $\mathrm{F}'(h)$ and of $\mathrm{F}''(h)$: this will serve the double purpose of getting acquainted with the general case and of serving to verify the first step in the induction argument for the general case. First of all from the relation (A.40) defining $\mathrm{F}(h)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}h}\Psi(\mathrm{F}(h)) = \Psi'(\mathrm{F}(h))\mathrm{F}'(h) = \exp(-h) \overset{h\searrow 0}{\sim} 1\,, \tag{A.43}$$

so that

$$\mathrm{F}'(h) \overset{h\searrow 0}{\sim} \frac{1}{\Psi'(\mathrm{F}(h))} \sim \frac{1}{\Psi'_{\mathrm{cr}}(\mathrm{F}_{\mathrm{cr}}(h))} = \mathrm{F}'_{\mathrm{cr}}(h)\,, \tag{A.44}$$

where the last equality follows by using (A.37) and (A.42) or (more easily!) by taking the derivative of (A.41). For what concerns $\mathrm{F}''(h)$ we compute and use once again the relation (A.40) to get

$$\left(\frac{\mathrm{d}}{\mathrm{d}h}\right)^2\Psi(\mathrm{F}(h)) = \Psi''(\mathrm{F}(h))\left(\mathrm{F}'(h)\right)^2 + \Psi'(\mathrm{F}(h))\mathrm{F}''(h) \overset{h\searrow 0}{\sim} -1. \tag{A.45}$$

By (A.42) and (A.44) we see that

$$\Psi''(\mathrm{F}(h))\left(\mathrm{F}'(h)\right)^2 \overset{h\searrow 0}{\sim} \frac{c}{h}\,, \tag{A.46}$$

with $c \neq 0$, so that

$$\Psi''(\mathrm{F}(h))\left(\mathrm{F}'(h)\right)^2 \overset{h\searrow 0}{\sim} -\Psi'(\mathrm{F}(h))\,\mathrm{F}''(h)\,, \tag{A.47}$$

from which we extract the asymptotic relation of $\mathrm{F}''(h) \sim \mathrm{F}''_{\mathrm{cr}}(h)$ by using again (A.42) and (A.44), that is by using the asymptotic behavior of F, F', Ψ' and Ψ''. Note however that, once again, there is a much cheaper way to go from (A.47) to $\mathrm{F}''(h) \sim \mathrm{F}''_{\mathrm{cr}}(h)$: by taking two derivatives of (A.41) one obtains (A.47) with the subscripts *cr* added (six subscripts in total) and with $\sim$ replaced by $=$, but we already know that we can add the subscripts in (A.47) without altering the validity of the statement to all the functions except a priori F'', and this implies $\mathrm{F}''(h) \sim \mathrm{F}''_{\mathrm{cr}}(h)$.

We have therefore established the claim for $j = 1$ and 2, but explicit expressions for the jth derivative of the composition of two functions are rather involved for arbitrary j. To get the result we want we will go around this point (much in the spirit of the alternative approach used twice above) by observing that

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}h}\right)^j\Psi(\mathrm{F}(h)) = {}& \Psi^{(1)}(\mathrm{F}(h))\mathrm{F}^{(j)}(h) \\ & + P_j\left(\mathrm{F}^{(1)}(h),\ldots,\mathrm{F}^{(j-1)}(h),\Psi^{(1)}(\mathrm{F}(h)),\ldots,\Psi^{(j)}(\mathrm{F}(h))\right)\,,\end{aligned} \tag{A.48}$$

where P_j is a polynomial: actually $P_1(x)$ is just zero [cf. (A.43)] and $P_2(x,y,z) = zx^2$ [cf. (A.45)], and that (A.48) has the companion

$$\left(\frac{\mathrm{d}}{\mathrm{d}h}\right)^j \Psi_{\mathrm{cr}}(\mathrm{F}_{\mathrm{cr}}(h)) = \Psi_{\mathrm{cr}}^{(1)}(\mathrm{F}_{\mathrm{cr}}(h))\mathrm{F}_{\mathrm{cr}}^{(j)}(h) \\ +P_j\left(\mathrm{F}_{\mathrm{cr}}^{(1)}(h),\ldots,\mathrm{F}_{\mathrm{cr}}^{(j-1)}(h),\Psi_{\mathrm{cr}}^{(1)}(\mathrm{F}_{\mathrm{cr}}(h)),\ldots,\Psi_{\mathrm{cr}}^{(j)}(\mathrm{F}(h))\right). \tag{A.49}$$

But the expression in (A.49) is equal to one if $j=1$ and it is equal to zero if $j \geq 2$, while the analogous expression without the *cr* subscripts takes the value one, or minus one, as $h \searrow 0$. Therefore in this limit

$$\Psi^{(1)}(\mathrm{F}(h)) - \Psi_{\mathrm{cr}}^{(1)}(\mathrm{F}_{\mathrm{cr}}(h)) = O(1). \tag{A.50}$$

For simplicity below we write $P_j(\mathrm{F}^{(1)},\ldots)$ and $P_j(\mathrm{F}_{\mathrm{cr}}^{(1)},\ldots)$ for the more cumbersome complete expressions.

We start now the induction procedure that consists in obtaining that $\mathrm{F}^{(j)}(h) \sim \mathrm{F}_{\mathrm{cr}}^{(j)}(h)$ for all $j < \widehat{n}$, with $\widehat{n} = 1 + 1/\alpha$ if $1/\alpha \in \mathbb{N}$ and $\widehat{n} = \infty$ otherwise, knowing that $\mathrm{F}^{(k)}(h) \sim \mathrm{F}_{\mathrm{cr}}^{(k)}(h)$ for $k = 0,1,\ldots,j-1$. Note that this follows if we can show that

$$P_j(\mathrm{F}^{(1)}(h),\ldots) \overset{h\searrow 0}{\sim} P_j(\mathrm{F}_{\mathrm{cr}}^{(1)}(h),\ldots), \tag{A.51}$$

and that these expressions diverge in this limit. This is because by using (A.50) and the fact that the asymptotic behaviors of the functions in the right-hand sides of (A.48) and (A.49) are all known, except for $\mathrm{F}^{(j)}(h)$ that is then necessarily the same as the one of $\mathrm{F}_{\mathrm{cr}}^{(j)}(h)$ and we are done.

Let us therefore establish (A.51) and the fact that the quantities in it diverge. That they diverge can be established by observing that for $2 \leq j < \widehat{n}$

$$P_j(\mathrm{F}_{\mathrm{cr}}^{(1)}(h),\ldots) = -\Psi_{\mathrm{cr}}^{(1)}(\mathrm{F}_{\mathrm{cr}}(h))\mathrm{F}_{\mathrm{cr}}^{(j)}(h) = c\,h^{1-j}, \tag{A.52}$$

where $c \neq 0$ is an explicit constant (note that if $1/\alpha \in \mathbb{N}$ then $\mathrm{F}_{\mathrm{cr}}^{(j)}(h) = 0$ for $j > 1/\alpha$). For what concerns (A.51) we start by observing that the leading behavior of each of the term constituting $P_j(\mathrm{F}^{(1)},\ldots)$ (P_j is of course a sum of monomials) is the same of the corresponding term in $P_j(\mathrm{F}_{\mathrm{cr}}^{(1)}(h),\ldots)$. All these monomial terms are of the same order (in a strong sense: any ratio converge to a non-zero value): this can be either verified on the expression with or without the subscript *cr* (in a rather illogical way we verify it for the quantities without subscript: the prof is slightly more involved, since we need to use the induction assumption, but

formulas are lighter without subscripts). In fact in taking the derivative that builds $P_j(\dots)$ we repeat two types of operations:

1. Taking a derivative of $\Psi^k(\mathrm{F}(h))$ for $k \le j$ and for this we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}h}\Psi^k(\mathrm{F}(h)) = \Psi^{k+1}(\mathrm{F}(h))\mathrm{F}'(h) &\overset{h\searrow 0}{\sim} -(j+1-\alpha)\Psi^k(\mathrm{F}(h))\frac{\mathrm{F}'(h)}{\mathrm{F}(h)} \\
&\overset{h\searrow 0}{\sim} -\left(\frac{j-1+\alpha}{\alpha}\right)\frac{\Psi^k(\mathrm{F}(h))}{h},
\end{aligned}
\tag{A.53}
$$

   that is such a derivative makes the term more singular (by a factor $1/h$).
2. Taking derivatives of $\mathrm{F}^{(k)}(h)$ for $k = 1,\dots,j-2$ (by definition of P_j, the derivative of $\mathrm{F}^{(j-1)}(h)$ does not enter P_j): but the asymptotic behaviors of $\mathrm{F}^{(k)}(h)$ and $\mathrm{F}^{(k+1)}(h)$ are in the induction assumption and one directly verifies that $h\mathrm{F}^{(k+1)}(h)/\mathrm{F}^{(k)}(h)$ tends to a non-zero constant (recall that $k < j < \widehat{n}$). Once again, such a derivative asymptotically just introduces a multiplicative $1/h$ factor (times a non-zero constant).

If we now recall that in the starting step ($j = 2$) of the induction we had just two terms, cf. (A.45), and each of order $1/h$ [cf. (A.45)] we see that $P_j(\mathrm{F}^{(1)}(h),\dots)$ (and $P_j(\mathrm{F}^{(1)}_{\mathrm{cr}}(h),\dots)$) is a sum of terms of order h^{1-j}, so that (A.52) is telling us that the asymptotic behavior of $P_j(\mathrm{F}^{(1)}_{\mathrm{cr}}(h),\dots)$ is of the same order of each of the monomial terms that constitute it (and it is not the result of the cancellation between the leading orders of larger terms). Therefore (A.51) holds for $j < \widehat{n}$ and the proof is complete. □

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