Appendix A
Notation

We summarize here the notation that is used for describing the approximations of the quasi-stationary distribution q and of the stationary distributions $p^{(1)}$ and $p^{(0)}$ of the two auxiliary processes of the stochastic SIS model. All quantities included are expressed in terms of the two essential parameters N and R_0.

The first quantities to be defined have the subscript 1. They are important for describing results when R_0 is distinctly above one. The quantities $\bar{\mu}_1$ and $\bar{\sigma}_1$ then serve as mean and standard deviation, respectively, of the approximation of the body and the near-end tails of the quasi-stationary distribution. Furthermore, the expected time to extinction from quasi-stationarity then grows exponentially with N. Its logarithm is asymptotically proportional to $\gamma_1 N$. The quantities γ_1 and β_1 are related since $\beta_1^2 = 2\gamma_1 N$. This implies that $\varphi(\beta_1) = \exp(-\gamma_1 N)/\sqrt{2\pi}$. Note also that $\gamma_1 \geq 0$ for $R_0 > 0$. We have

$$\bar{\mu}_1 = \frac{R_0 - 1}{R_0} N, \tag{A.1}$$

$$\bar{\sigma}_1 = \sqrt{\frac{N}{R_0}}, \tag{A.2}$$

$$y_1(n) = \frac{n - \bar{\mu}_1}{\bar{\sigma}_1}, \tag{A.3}$$

$$z_1(n) = \frac{n - \bar{\mu}_1}{\bar{\sigma}_1^2}, \tag{A.4}$$

$$\beta_1 = \operatorname{sign}(R_0 - 1)\sqrt{2N\left(\log R_0 - \frac{R_0 - 1}{R_0}\right)}, \tag{A.5}$$

$$\gamma_1 = \log R_0 - \frac{R_0 - 1}{R_0}. \tag{A.6}$$

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The next quantities have the subscript 2. They appear in approximations of the left tail of the quasi-stationary distribution when R_0 is distinctly above the value one. They are

$$\bar{\mu}_2 = N \log R_0, \tag{A.7}$$

$$\bar{\sigma}_2 = \sqrt{N}, \tag{A.8}$$

$$y_2(n) = \frac{n - \bar{\mu}_2}{\bar{\sigma}_2}, \tag{A.9}$$

$$z_2(n) = \frac{n - \bar{\mu}_2}{\bar{\sigma}_2^2}, \tag{A.10}$$

$$\beta_2 = \sqrt{N} \log R_0. \tag{A.11}$$

The study of the transition region requires a rescaling of R_0 with the aid of a quantity ρ defined as follows:

$$\rho = (R_0 - 1)\sqrt{N}. \tag{A.12}$$

The transition region is characterized by the requirement that $\rho = O(1)$ as $N \to \infty$.

The next quantities have the subscript 3. They are useful in describing the quasi-stationary distribution in the transition region:

$$\bar{\mu}_3 = \rho\sqrt{N}, \tag{A.13}$$

$$\bar{\sigma}_3 = \bar{\sigma}_2 = \sqrt{N}, \tag{A.14}$$

$$y_3(n) = \frac{n - \bar{\mu}_3}{\bar{\sigma}_3}, \tag{A.15}$$

$$z_3(n) = \frac{n - \bar{\mu}_3}{\bar{\sigma}_3^2}, \tag{A.16}$$

$$\beta_3 = \frac{R_0 - 1}{R_0}\sqrt{N}. \tag{A.17}$$

It is useful to note that $\rho = 0$ when $R_0 = 1$, and that ρ has the same sign as $R_0 - 1$. The quantities β_1, β_2, β_3 have similar behaviors. They deviate slightly from each other in the transition region, as shown by the following asymptotic approximations:

$$\beta_1 \sim \rho - \frac{2}{3}\frac{\rho^2}{\sqrt{N}}, \quad \rho = O(1), \quad N \to \infty, \tag{A.18}$$

$$\beta_2 \sim \rho - \frac{1}{2}\frac{\rho^2}{\sqrt{N}}, \quad \rho = O(1), \quad N \to \infty, \tag{A.19}$$

$$\beta_3 \sim \rho - \frac{\rho^2}{\sqrt{N}}, \quad \rho = O(1), \quad N \to \infty. \tag{A.20}$$

The description of the uniform results in Chap. 12 uses the following notation:

$$\bar{\mu}_{12} = \begin{cases} \bar{\mu}_1, & R_0 \geq 1, \\ \bar{\mu}_2, & R_0 \leq 1, \end{cases} \tag{A.21}$$

$$\bar{\sigma}_{12} = \begin{cases} \bar{\sigma}_1, & R_0 \geq 1, \\ \bar{\sigma}_2, & R_0 \leq 1, \end{cases} \tag{A.22}$$

$$y_{12}(n) = \begin{cases} y_1(n), & R_0 \geq 1, \\ y_2(n), & R_0 \leq 1, \end{cases} \tag{A.23}$$

$$r_0 = \begin{cases} \sqrt{R_0}, & R_0 \geq 1, \\ R_0, & R_0 \leq 1, \end{cases} \tag{A.24}$$

$$\beta_0 = \rho, \tag{A.25}$$

and

$$\beta_{ij} = \begin{cases} \beta_i, & R_0 \geq 1, \quad i = 0,1,2,3, \\ \beta_j, & R_0 \leq 1, \quad j = 0,1,2,3. \end{cases} \tag{A.26}$$

We note that all of the quantities $\beta_0 = \rho$, β_1, β_2, and β_3 are asymptotic to $\rho = (R_0 - 1)\sqrt{N}$ in the transition region. Also, all of them are equal to zero when $\rho = 0$. Thus, all of the quantities β_{ij} are continuous functions of ρ. They are equal to zero when $\rho = 0$, and they all have the same sign as ρ. This means that they are positive for $R_0 > 1$ and negative for $R_0 < 1$.

The functions G_1, G_2, and G are defined as follows:

$$G_1(z) = \begin{cases} 1 - \dfrac{1}{z} + \dfrac{1}{\exp(z) - 1}, & z \neq 0, \\ \dfrac{1}{2}, & z = 0, \end{cases} \tag{A.27}$$

$$G_2(x,z) = \begin{cases} \dfrac{1 - \exp(-xz)}{z}, & z \neq 0, \\ x, & z = 0. \end{cases} \tag{A.28}$$

$$G(z) = \begin{cases} \dfrac{1}{z} \log \dfrac{\exp(z) - 1}{z}, & z \neq 0, \\ \dfrac{1}{2}, & z = 0. \end{cases} \tag{A.29}$$

The functions H_1, H_1^*, H_0, and H are defined as follows:

$$H_1(y) = \frac{\Phi(y)}{\varphi(y)}, \tag{A.30}$$

$$H_1^*(y) = \frac{\Phi^*(y)}{\varphi^*(y)}, \tag{A.31}$$

$$H_0(y) = \begin{cases} H_a(y), & y \le \rho_b, \\ H_a(\rho_b) + \displaystyle\int_{\rho_b}^{y} H_1(t)\,dt, & \rho > \rho_b, \end{cases} \tag{A.32}$$

where the auxiliary function H_a is defined by

$$H_a(y) = -\log|y| + \sum_{k=1}^{m_d} (-1)^k \frac{a_k}{2k} \frac{1}{y^{2k}}, \quad y < 0, \tag{A.33}$$

and m_d is a positive integer depending on ρ_b. The Maple procedure in Appendix B uses $\rho_b = -3$ and $m_d = 4$. The coefficients a_k are defined by

$$a_k = \frac{(2k)!}{k!2^k}, \quad k = 0, 1, 2, \ldots. \tag{A.34}$$

Finally, the functions H, $\tilde{\rho}$, and R are defined as follows:

$$H(y) = \frac{1}{y + 1/H(y)} \int_{-1/H(y)}^{y} H_1(t)\,dt, \tag{A.35}$$

$$\tilde{\rho}(y) = y + \frac{1}{H(y)}, \tag{A.36}$$

and

$$R(N, y) = 1 + \frac{\tilde{\rho}(y)}{\sqrt{N}} = 1 + \frac{y + 1/H(y)}{\sqrt{N}}. \tag{A.37}$$

Appendix B
A Maple Module for Numerical Evaluations

The numerical evaluations that are reported in this monograph have all been done with Maple. We have used the Maple procedures that are included in the Maple module named SIS given below. The module contains 21 procedures. We give some brief comments on them.

We begin with six scalar functions. The normal distribution function Φ and the normal density function φ are evaluated with the Maple functions **PHI** and **phi**, respectively. Similarly, the three functions H, H_0, and H_1 are evaluated using the Maple functions **H**, **H0**, and **H1**, respectively. In addition, the Maple procedure **h** is used to determine **H**. It calls **H1**.

The remaining 15 procedures all deal with probability vectors. These probability vectors are treated in the form of one-dimensional Maple Arrays. With the exception of the last two procedures (**ratio** and **short**), the lengths of the vectors are N, the population size for the SIS model. The first four of these procedures are **PSI**, **PSImn**, **rf**, and **cdf**. The procedure **PSI** is used to determine the image under the map Ψ of some vector p. A related procedure is the one called **PSImn**. It is used to determine the image of the vector $p^{(1)}$ or $p^{(0)}$ after application of the map Ψ n times. The ratio r_k defined in (3.35) can be studied with the procedure **rf**. It is called by the procedure **PSI**. Another useful procedure is called **cdf**. It converts a probability vector p into its distribution function. It is called by both **rf** and by **PSI**.

There are three procedures associated with each of the two stationary distributions $p^{(0)}$ and $p^{(1)}$, and three more associated with the quasi-stationary distribution q. First of all, each of these distributions can be evaluated numerically with the aid of the procedures called **p0f**, **p1f**, and **qsf**, respectively. Furthermore, we note that uniform approximations of body and of left tail of these distributions are given in Chap. 13. Procedures that evaluate these approximations numerically are contained in **p0bodyappr**, **p0tailappr**, **p1bodyappr**, **p1tailappr**, **qsbodyappr**, and **qstailappr**, respectively.

In addition, the module contains two procedures that can be useful in certain circumstances. They are called **ratio** and **short**. The first of these takes a vector of length M with components $p_1, p_2, \ldots, p_M$, and produces a vector of length $M-1$

whose component k equals the ratio p_{k+1}/p_k. The last procedure shortens a Maple Array to its first M elements. It can be useful in plotting distributions.

We conclude this appendix by a listing of the Maple module named SIS:

```
SIS:=module()
  export PHI,phi,H,H0,H1,h,PSI,PSImn,rf,cdf,
    p0f,p1f,qsf,p0bodyappr,p0tailappr,p1bodyappr,
    p1tailappr,qsbodyappr,qstailappr,ratio,short:
  option package;
  description "The qsd of the SIS model, and related
    quantities";

  PHI:=y->(1+erf(y/sqrt(2)))/2;

  phi:=y->exp(-y^2/2)/sqrt(2*Pi):

  H:=x->1/fsolve(h(x,v)=1,v):

  H0:=proc(x)
    local y;
    if evalf(x)<= -3 then
      evalf(-log(abs(x))-1/2/x^2+3/4/x^4-15/6/x^6
           +105/8/x^8);
    else
      evalf(-log(3)-835/17496 + Int(H1(y),y=-3..x));
    fi;
  end proc:

  H1:=x->PHI(x)/phi(x):

  h:=proc(x,v)
    local y;
    if v=-x then
      -x*H1(x);
    else
      (v/(x+v))*evalf(Int(H1(y),y=-v..x));
    fi;
  end proc:

  PSI:=proc(N,R0,p)
    local p0,pin,r,S,pp,ppsum:
    description ``Determine the image under the map PSI
      of the vector p'':
    p0:=p0f(N,R0):
```

```
  pin:=p0/p0[1]:
  r:=rf(N,R0,p):
  S:=cdf(r):
  pp:=pin*S:
  ppsum:=add(pp[k],k=1..N):
  pp:=pp/ppsum:
end proc:

PSImn:=proc(N,R0,m,n)
  local p0,p1,pin,p,k:
  description ``Determines PSI^n(p^(m)), m=0 or 1'':
  p0:=p0f(N,R0):
  p1:=p1f(N,R0):
  pin:=p0/p0[1]:
  if m=0 then
    p:=p0:
  else
    p:=p1:
  fi:
  if n>0 then
    for k from 1 to n do
      p:=PSI(N,R0,p):
    od:
  fi:
  p:
end proc:

rf:=proc(N,R0,p)
  local p1,rho,pcum,rr,k:
  description ``The ratio r with vector p in
    the numerator'':
  p1:=p1f(N,R0):
  rho:=p1/p1[1]:
  pcum:=cdf(p):
  rr:=Array(1..N,1):
  for k from 2 to N do
    rr[k]:=(1-pcum[k-1])/rho[k]:
  od:
  rr:
end proc:

cdf:=proc(p)
  local F,N,k;
  description "The sum of p[j] from 1 to k":
```

```
  N:=op(2,ArrayDims(p));
  F:=Array(1..N);
  F[1]:=p[1];
  for k from 2 to N do
    F[k]:=F[k-1]+p[k];
  od;
  F;
end proc;

p0f:=proc(N,R0)
  local p0,k0,k,sump0;
  description "The stationary distribution p0":
  p0:=Array(1..N,1.);
  k0:=max(1,floor((R0-1)*N/R0));
  for k from k0+1 to N do
    p0[k]:=p0[k-1]*(1-1/k)*(1-(k-1)/N)*R0;
  od;
  for k from k0-1 by -1 to 1 do
    p0[k]:=p0[k+1]*(1+1/k)/R0/(1-k/N);
  od;
  sump0:=add(p0[k],k=1..N):
  p0:=p0/sump0;
  p0:
end proc:

p1f:=proc(N,R0)
  local p1,k0,k,sump1;
  description "The stationary distribution p1":
  p1:=Array(1..N,1.):
  k0:=max(1,floor((R0-1)*N/R0));
  for k from k0+1 to N do
    p1[k]:=p1[k-1]*(1-(k-1)/N)*R0;
  od;
  for k from k0-1 by -1 to 1 do
    p1[k]:=p1[k+1]/R0/(1-k/N);
  od;
  sump1:=add(p1[k],k=1..N):
  p1:=p1/sump1:
end proc:

qsf:=proc(N,R0)
  local qs1,kvot,qs2,kvotmax,kvotmin,kv:
  description "The quasi-stationary distribution":
  if R0>1 then
```

```
    qs1:=p0f(N,R0):
  else
    qs1:=p1f(N,R0):
  fi:
  kv:=2:
  while kv > 1+1e-6 do
  qs2:=PSI(N,R0,qs1):
  kvot:=qs1/qs2:
  kvotmax:=max(kvot):
  kvotmin:=min(kvot):
  kv:=max(kvotmax,1/kvotmin):
  qs1:=qs2:
  od:
  qs2:
  end proc:

p0bodyappr:=proc(N,R0)
  local rho,beta2,beta3,mu1,sigma1,mu2,sigma2,a,
    beta30,beta32,mu12,sigma12,r0,aa,f,p0:
  description ``Approximation of the body of p0'':
  rho:=(R0-1)*sqrt(N):
  beta2:=sqrt(N)*log(R0):
  beta3:=(R0-1)*sqrt(N)/R0:
  mu1:=(R0-1)*N/R0: sigma1:=sqrt(N/R0):
  mu2:=N*log(R0): sigma2:=sqrt(N):
  if R0>1 then
    r0:=1: beta30:=beta3: beta32:=beta3: mu12:=mu1:
      sigma12:=sigma1:
  else
    r0:=R0: beta30:=rho: beta32:=beta2: mu12:=mu2:
      sigma12:=sigma2:
  fi:
  a:=sqrt(R0/r0):
  aa:=evalf(a/(0.5*log(N)+H0(beta30))/phi(beta32)):
  f:=n->aa*evalf(phi((n-mu12)/sigma12)/n):
  p0:=Array(1..N,f):
end proc:

p0tailappr:=proc(N,R0)
  local rho,beta1,beta2,beta3,mu2,sigma2,a,beta10,
    aa,f,p0:
  description ``Approximation of the left tail of
    p0'':
  rho:=(R0-1)*sqrt(N):
```

```
  beta1:=sqrt(2*N*(log(R0)-(R0-1)/R0)):
  beta2:=sqrt(N)*log(R0):
  beta3:=(R0-1)*sqrt(N)/R0:
  mu2:=N*log(R0): sigma2:=sqrt(N):
  if R0>1 then
    a:=beta3/beta1: beta10:=beta1:
  else
    a:=1: beta10:=rho:
  fi:
  aa:=evalf(a/(0.5*log(N)+H0(beta10))/phi(beta2)):
  f:=n->aa*evalf(phi((n-mu2)/sigma2)/n):
  p0:=Array(1..N,f):
end proc:

p1bodyappr:=proc(N,R0)
  local beta1,beta2,beta3,mu1,sigma1,mu2,sigma2,
    beta12,beta13,mu12,sigma12,a,f,p1:
  description ``Approximation of the body of p1'':
  beta1:=sqrt(2*N*(log(R0)-(R0-1)/R0)):
  beta2:=sqrt(N)*log(R0):
  beta3:=(R0-1)*sqrt(N)/R0:
  mu1:=(R0-1)*N/R0: sigma1:=sqrt(N/R0):
  mu2:=N*log(R0): sigma2:=sqrt(N):
  if R0>1 then
    beta12:=beta1: beta13:=beta1: mu12:=mu1:
      sigma12:=sigma1:
  else
    beta12:=beta2: beta13:=beta3: mu12:=mu2:
      sigma12:=sigma2:
  fi:
  a:=evalf(1/H1(beta13)/sigma12/phi(beta12)):
  f:=n->a*evalf(phi((n-mu12)/sigma12)):
  p1:=Array(1..N,f):
  end proc:

p1tailappr:=proc(N,R0)
  local beta1,beta2,beta3,mu2,sigma2,beta13,a,f,p1:
  description ``Approximation of the left tail of
    p1'':
  beta1:=sqrt(2*N*(log(R0)-(R0-1)/R0)):
  beta2:=sqrt(N)*log(R0):
  beta3:=(R0-1)*sqrt(N)/R0:
  mu2:=N*log(R0): sigma2:=sqrt(N):
  if R0>1 then
```

```
    beta13:=beta1:
  else
    beta13:=beta3:
  fi:
  a:=evalf(1/H1(beta13)/sigma2/phi(beta2)):
  f:=n->a*evalf(phi((n-mu2)/sigma2)):
  p1:=Array(1..N,f):
end proc:

qsbodyappr:=proc(N,R0)
  local rho,R,beta2,mu1,sigma1,mu2,sigma2,beta02,r0,
    mu12,sigma12,a,aa,f,qs:
  description ``Approximation of the body of qs'':
  rho:=(R0-1)*sqrt(N):
  R:=evalf(1+(rho+1/H(rho))/sqrt(N)):
  beta2:=sqrt(N)*log(R0):
  mu1:= (R0-1)*N/R0: sigma1:=sqrt(N/R0):
  mu2:=N*log(R0): sigma2:=sqrt(N):
  if R0>1 then
    r0:=1: beta02:=rho: mu12:=mu1: sigma12:=sigma1:
  else
    r0:=R0: beta02:=beta2: mu12:=mu2: sigma12:=sigma2:
  fi:
  a:=1/sqrt(R0*r0):
  aa:=evalf(a/(1+rho*H(rho))/phi(beta02)):
  f:=n->aa*evalf(phi((n-mu12)/sigma12)*(1-1/R^n)/n):
  qs:=Array(1..N,f):
  end proc:

qstailappr:=proc(N,R0)
  local rho,R,beta1,beta2,beta10,mu2,sigma2,
    a,aa,f,qs:
  description ``Approximation of the left tail of
    qs'':
  rho:=(R0-1)*sqrt(N):
  R:=evalf(1+(rho+1/H(rho))/sqrt(N)):
  beta1:=sqrt(2*N*(log(R0)-(R0-1)/R0)):
  beta2:=sqrt(N)*log(R0):
  mu2:=N*log(R0): sigma2:=sqrt(N):
  if R0>1 then
    a:=rho/R0/beta1: beta10:=beta1:
  else
    a:=1/R0: beta10:=rho:
  fi:
```

```
    aa:=evalf(a/(1+beta10*H(*beta10))/phi(beta2)):
    f:=n->aa*evalf(phi((n-mu2)/sigma2)*(1-1/R^n)/n):
    qs:=Array(1..N,f):
  end proc:

  ratio:=proc(p)
    local rat,k,M;
    description "The ratio p[k+1]/p[k]":
    M:=op(2,ArrayDims(p));
    rat:=Array(1..M-1):
    for k from 1 to M-1 do
      rat[k]:=p[k+1]/p[k];
    od;
    rat:
  end proc;

  short:=proc(p,M)
    local sh,k:
    description "Shortens the Array p to its first M
      elements":
    sh:=Array(1..M):
    for k from 1 to M do
      sh[k]:=p[k]:
    od:
    sh:
  end proc:

end module:
```

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Index

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Edited by J.-M. Morel, B. Teissier, P.K. Maini

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