Appendix A
Gâteaux and Fréchet Differentiability

Following [LP03] there are two basic notions of differentiability for functions $f : X \to Y$ between Banach spaces X and Y.

Definition A.1. A function f is said to be *Gâteaux differentiable* at x if there exists a bounded linear[1] operator $T_x \in \mathcal{B}(X, Y)$ such that $\forall\, v \in X$,

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = T_x v.$$

The operator T_x is called the *Gâteaux derivative* of f at x.

If for some fixed v the limits

$$\delta_v f(x) := \frac{d}{dt}\Big|_{t=0} f(x + tv) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

exists, we say f has a directional derivative at x in the direction v. Hence f is Gâteaux differentiable at x if and only if all the directional derivatives $\delta_v f(x)$ exist and form a bounded linear operator $Df(x) : v \mapsto \delta_v f(x)$.

If the limit (in the sense of the Gâteaux derivative) exists *uniformly* in v on the unit sphere of X, we say f is *Fréchet differentiable* at x and T_x is the *Fréchet derivative* of f at x. Equivalently, if we set $y = tv$ then $t \to 0$ if and only if $y \to 0$. Thus f is Fréchet differentiable at x if for all y,

$$f(x + y) - f(x) - T_x(y) = o(\|y\|)$$

and we call $T_x = Df(x)$ the derivative of f at x.

Note that the distinction between the two notion of differentiability is made by how the limit is taken. The importance being that the limit in the Fréchet case only depends on the norm of y.[2]

[1] Some authors drop the requirement for linearity here.

[2] In terms of ε-δ notation the differences can expressed as follows. Gâteaux: $\forall \varepsilon > 0$ and $\forall v \neq 0$, $\exists\, \delta = \delta(\varepsilon, v) > 0$ such that, $\|f(x + tv) - f(x) - tTv\| \leq \varepsilon |t|$ whenever $|t| < \delta$. Fréchet: $\forall \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ such that $\|f(x + v) - f(x) - Tv\| \leq \varepsilon \|v\|$ whenever $\|v\| < \delta$.

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A.1 Properties of the Gâteaux Derivative

If the Gâteaux derivative exists it unique, since the limit in the definition is unique if it exists.

A function which is Fréchet differentiable at a point is continuous there, but this is not the case for Gâteaux differentiable functions (even in the finite dimensional case). For example, the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0,0) = 0$ and $f(x,y) = x^4 y/(x^6 + y^3)$ for $x^2 + y^2 > 0$ has 0 as its Gâteaux derivative at the origin, but fails to be continuous there. This also provides an example of a function which is Gâteaux differentiable but not Fréchet differentiable. Another example is the following: If X is a Banach space, and $\varphi \in X'$ a discontinuous linear functional, then the function $f(x) = \|x\|\varphi(x)$ is Gâteaux differentiable at $x = 0$ with derivative 0, but $f(x)$ is not Fréchet differentiable since φ does not have limit zero at $x = 0$.

Proposition A.2 (Mean Value Formula). *If f is Gâteaux differentiable then*

$$\|f(y) - f(x)\| \leq \|x - y\| \sup_{0 \leq \theta \leq 1} \|Df(\theta x + (1-\theta)y)\|.$$

Proof. Choose $u^* \in X$ such that $\|u^*\| = 1$ and $\|f(y) - f(x)\| = \langle u^*, f(y) - f(x)\rangle$. By applying the mean value theorem to $h(t) = \langle u^*, f(x + t(y-x))\rangle$ we find that $|\langle u^*, f(y)\rangle - \langle u^*, f(x)\rangle| = \|h(1) - h(0)\| \leq \sup_{0 \leq t \leq 1} \|h'(t)\|$ and

$$\begin{aligned} h'(t) &= \left\langle u^*, \frac{d}{dt} f(x + t(y-x)) \right\rangle \\ &= \left\langle u^*, \lim_{s \to 0} \frac{f(x + (t+s)(y-x)) - f(x + t(y-x))}{s} \right\rangle \\ &= \langle u^*, Df(x + t(y-x))\,(y-x)\rangle\,. \end{aligned}$$

So $|h'(t)| \leq \|Df(x + t(y-x))\,(y-x)\| \leq \|Df(x + t(y-x))\|\,\|y - x\|$. □

If the Gateaux derivative exist and is continuous in the following sense, then the two notions coincide.

Proposition A.3. *If f is Gâteaux differentiable on an open neighbourhood U of x and $Df(x)$ is continuous,*[3] *then f is also Fréchet differentiable at x.*

[3] In the sense that $Df : U \to \mathcal{B}(X,Y)$ is continuous at x so that $\lim_{\tilde{x} \to x} \|Df(x) - Df(\tilde{x})\| = 0$. In words, the derivative depends continuous on the point x.

Proof. Fix v and let $g(t) = f(x+tv) - f(x) - t\,Df(x)v$, so $g(0) = 0$. By continuity of the Gâteaux derivative with the mean value theorem we find that

$$\begin{aligned}\|f(x+tv) - f(x) - t\,Df(x)v\| &= \|g(1)\| \\ &\leq \|v\| \sup_{0\leq t\leq 1} \|Df(x+tv) - Df(x)\| \\ &= o(\|v\|)\end{aligned}$$

□

The notion of Gâteaux differentiability and Fréchet differentiability also coincide if f is Lipschitz and $\dim(X) < \infty$, that is:

Proposition A.4. *Suppose $f : X \to Y$ is a Lipschitz function from a finite-dimensional Banach space X to a (possibly infinite-dimensional) Banach space Y. If f is Gâteaux differentiable at some point x, then it is also Fréchet differentiable at that point.*

Proof. As the unit sphere S_X of X is compact, it is totally bounded. So given $\varepsilon > 0$ there exists a finite set $F = F(\varepsilon) \subset X$ such that $S_X = \bigcup_{u_j\in F} B_\varepsilon(u_j)$. Thus for all $u \in S_X$ there is an index j such that $\|u - u_j\| < \varepsilon$.

By hypothesis choose $\delta > 0$ such that

$$\|f(x+tu_j) - f(x) - t\,Df(x)u_j\| < \varepsilon\,|t|$$

for $|t| < \delta$ and any index j. It follows that for any $u \in S_X$,

$$\begin{aligned}\|f(x+tu) - f(x) - tDf(x)u\| &\leq \|f(x+tu) - f(x+tu_j)\| \\ &\quad + \|f(x+tu_j) - f(x) - t\,Df(x)u_j\| \\ &\quad + \|t\,Df(x)(u_j - u)\| \\ &\leq (C + \|Df(x)\| + 1)\varepsilon\,|t|\end{aligned}$$

for $|t| < \delta$, where C is the Lipschitz constant of f. Hence δ is independent of u and so f is also Fréchet differentiable at x. □

In the infinite dimensional case the story is very different. Broadly speaking in such a situation there are reasonably satisfactory results on the existence of Gâteaux derivatives of Lipschitz functions, while results on existence of Fréchet derivatives are rare and usually very hard to prove. On the other hand, in many applications it is important to have Fréchet derivatives of f, since they provide genuine local linear approximation to f, unlike the much weaker Gâteaux derivatives.

Appendix B
Cones, Convex Sets and Support Functions

The geometric concept of tangency is one of the most important tools in analysis. Tangent lines to curves and tangent planes to surfaces are defined classically in terms of differentiation. In convex analysis, the opposite approach is exploited. A generalised tangency is defined geometrically in terms of separation; it is expressed by supporting hyperplanes and half-spaces. Here we look at convex sets (particularly when they are defined by a set of linear inequalities) and the characterisation of their tangent and normal cones. This will be needed in the proof of the maximum principle for vector bundles discussed in Sect. 7.4.

B.1 Convex Sets

Let E be a (finite-dimensional) inner product space, and E^* its dual space. A subset $A \subset E$ is *convex set* if for every $v, w \in A$, $\theta v + (1-\theta)w \in A$ for all $\theta \in [0,1]$. A set $\Gamma \subset E$ is a *cone* with vertex $u \in E$ if for every $v \in \Gamma$ we have $u + \theta(v-u) \in \Gamma$ for all $\theta \geq 0$. A *half-space* is a set of the form $\{x \in E : \ \ell(x) \leq c\}$ where ℓ is a non-trivial linear function on E, i.e. $\ell \in E^* \setminus \{0\}$. In such a case we normalise so that ℓ is an element of $S^* = \{\omega \in E^* : \ \|\omega\| = 1\}$.

A *supporting half-space* to a closed convex set A is a half-space which contains A and has points of A arbitrarily close to its boundary. A *supporting hyperplane* to A is a hyperplane which is the boundary of a supporting half-space to A. That is, supporting hyperplanes to A take the form $\{x : \ \ell(x) = c\}$ where $\ell \in E^* \setminus \{0\}$ and $c = \sup\{\ell(v) : \ v \in A\}$.

B.2 Support Functions

If A is a closed convex set in E, the *support function* of A is a function $s = s_A : \ E^* \to \mathbb{R} \cup \{\infty\}$ defined by

$$s(\ell) = \sup\{\ell(x) : \ x \in A\}$$

for each $\ell \in E^* \setminus \{0\}$. Here s is a homogeneous degree one convex function on E^*. For each ℓ with $s(\ell) < \infty$, the half-space $\{x : \ \ell(x) \leq s(\ell)\}$ is the unique supporting half-space of A which is parallel to $\{x : \ell(x) \leq 0\}$.

Theorem B.1. *The convex set A is the intersection of its supporting half-spaces:*

$$A = \bigcap_{\ell \in S^*} \{x \in E : \ \ell(x) \leq s(\ell)\}.$$

Proof. Firstly, the set A is contained in this intersection since it is contained in each of the half-spaces. To prove the reverse inclusion it suffices to show for any $y \notin A$ there exists $\ell \in S^*$ such that $\ell(y) > s(\ell)$.

Let x be the closest point to y in A, and define $\ell \in E^*$ by $\ell(z) = \langle z, y - x \rangle$. Suppose $\ell(w) > \ell(x)$ for some $w \in A$. Then $x + t(w - x) \in A$ for $0 \leq t \leq 1$, and

$$\frac{d}{dt} \|y - (x + t(w - x))\|^2 \bigg|_{t=0} = -2\langle y - x, w - x \rangle = -2(\ell(w) - \ell(x)) < 0,$$

contradicting the fact that x is the closed point to y in A. Therefore $\ell(z) \leq \ell(x)$ for all $z \in A$, so $s(\ell) = \sup_A \ell = \ell(x) < \ell(y)$. The same holds for $\tilde{\ell} = \ell / \|\ell\| \in S^*$. □

B.3 The Distance From a Convex Set

For a closed convex set A in E, the function $d_A : \ E \to \mathbb{R}$ given by

$$d_A(x) = \inf\{\|x - y\| : \ y \in A\}$$

is Lipschitz continuous, with Lipschitz constant 1, and is strictly positive on $E \setminus A$. We call this the distance to A. It has the following characterisation in terms of the support function of A.

Theorem B.2. *For any $y \notin A$,*

$$d_A(y) = \sup\{\ell(y) - s(\ell) : \ \ell \in S^*\}.$$

Proof. Let x be the closest point to y in A. So for any $\ell \in S^*$ we have

$$\begin{aligned} \ell(y) - s(\ell) = \ell(y) - \sup_A \ell \leq \ell(y) - \ell(x) = \ell(y - x) \\ \leq \|\ell\| \, \|y - x\| = \|y - x\| = d_A(y), \end{aligned}$$

while the particular choice of $\ell(\,\cdot\,) = \langle y - x, \cdot \rangle / \|y - x\|$ gives equality throughout. □

B.4 Tangent and Normal Cones

A convex set may have non-smooth boundary, so there will not in general be a well-defined normal vector or tangent plane. Nevertheless we can make sense of a *set* of normal vectors, as follows:

Definition B.3. Let A be a closed bounded convex set in E, and let $x \in \partial A$. The *normal cone* to A at x is defined by

$$\mathcal{N}_x A = \{\ell \in E^* : \ \ell(x) = s(\ell)\}.$$

In other words, $\mathcal{N}_x A$ is the set of linear functions which achieve their maximum over A at the point x (so that the corresponding supporting half-spaces have x in their boundary). The set $\mathcal{N}_x A$ is a convex cone in E^* with vertex at the origin.

Complementary to this is the following definition:

Definition B.4. The *tangent cone* $\mathcal{T}_x A$ to A at x is the set

$$\mathcal{T}_x A = \bigcap_{\ell \in \mathcal{N}_x A} \{z \in E : \ \ell(z) \le 0\}.$$

That is, $x + \mathcal{T}_x A$ is the intersection of the supporting half-spaces of A with x on their boundary. It follows that $A - x \subset \mathcal{T}_x A$. Indeed $\mathcal{T}_x A$ may alternatively be characterised as the closure of $\bigcup \{\frac{1}{h}(A - x) : h > 0\}$. The tangent cone $\mathcal{T}_x A$ is a closed convex cone in E with vertex at the origin (in fact it is the smallest such cone containing $A - x$).

B.5 Convex Sets Defined by Inequalities

In many cases the convex set A of interest is explicitly presented as an intersection of half-spaces, in the form

$$A = \bigcap_{\ell \in B} \{x \in E : \ \ell(x) \le \phi(\ell)\} \tag{B.1}$$

where B is a given closed subset of $E^* \setminus \{0\}$ and $\phi : B \to \mathbb{R}$ is given. If B does not intersect every ray from the origin, this definition will involve only a subset of the supporting half-spaces of A. In this situation we have the following characterisation of the support function of A:

Theorem B.5. *Let E be of dimension n, and suppose A is defined by* (B.1)*. For any $\ell \in E^*$ with $s(\ell) < \infty$ there exist $\ell_1, \ldots, \ell_{n+1} \in B$ and $\lambda_1, \ldots, \lambda_{n+1} \geq 0$ such that*

$$\ell = \sum_{i=1}^{n+1} \lambda_i \ell_i \quad \textit{and} \quad s(\ell) = \sum_{i=1}^{n+1} \lambda_i \phi(\ell_i).$$

It follows that the support function s of A on all of E^* can be recovered from the given function ϕ on B.

Proof. Firstly, for $\ell \in B$ note that if $\ell(x) = \phi(\ell)$ for some $x \in A$, then $\ell(x) = \sup\{\ell(y) : y \in A\} = s(\ell)$. Now define

$$\widetilde{B} = \mathbb{R}^+\{\ell \in B : \exists\, x \in A \text{ with } \ell(x) = \phi(\ell)\}.$$

That is, $\widetilde{B}$ consists of positive scalar multiples of those ℓ in B for which equality holds in equation (B.1). Note that $\widetilde{B}$ is closed. Also let

$$\widetilde{\phi}(\vartheta) = \begin{cases} c\phi(\ell) & \text{if } \vartheta = c\ell \text{ where } c \geq 0, \ell \in \widetilde{B} \\ +\infty & \text{otherwise} \end{cases}$$

Thus we have that $\widetilde{\phi}(\vartheta) = s(\vartheta)$ for $\vartheta \in \widetilde{B}$. From (B.1) and by the construction of $\widetilde{\phi}$ we have

$$A = \bigcap_{\vartheta \in E^*} \{x \in E : \vartheta(x) \leq \widetilde{\phi}(\vartheta)\}.$$

In which case we see that

$$\begin{aligned} s(\ell) &= \sup\{\ell(x) : x \in A\} \\ &= \sup\{\ell(x) : x \in E, \vartheta(x) \leq \widetilde{\phi}(\vartheta), \forall \vartheta \in E^*\} \\ &= \sup\{\ell^*(\ell) : \ell^* \leq \widetilde{\phi}, \ell^* \in (E^*)^*\} \end{aligned}$$

since $(E^*)^* = E$. That is, the epigraph of s is the convex hull of the epigraph of $\widetilde{\phi}$ (cf. [Roc70, Corollary 12.1.1]). Now we observe by the Caratheodory theorem [Roc70, Corollary 17.1.3] that

$$s(\ell) = \inf\left\{\sum_{i=1}^{n+1} \lambda_i \widetilde{\phi}(\ell_i) : \ \ell_i \in \widetilde{B},\ \lambda_i \geq 0,\ \sum_{i=1}^{n+1} \lambda_i \ell_i = \ell\right\}.$$

The infimum is attained since $\widetilde{B}$ is closed. The result follows since each $\ell_i \in \widetilde{B}$ is a non-negative multiple of some element $\overline{\ell}_i$ of B with $\phi(\overline{\ell}_i) = s(\overline{\ell}_i)$. $\square$

From this theorem we obtain a useful result for the normal cone:

Theorem B.6. *Let E be of dimension n, and suppose A is defined by* (B.1)*. Then for any $x \in \partial A$, $\mathcal{N}_x A$ is the convex cone generated by $B \cap \mathcal{N}_x A$. That is, for any $\ell \in \mathcal{N}_x A$ there exist $k \leq n+1$ and $\ell_1, \ldots, \ell_k \in B \cap \mathcal{N}_x A$ and $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\ell = \sum_{i=1}^{k} \lambda_i \ell_i$.*

Proof. Let $\ell \in \mathcal{N}_x A$. By Theorem B.5 there exist $\ell_1, \ldots, \ell_{n+1}$ and $\lambda_i \geq 0$ such that $s(\ell) = \sum_{i=1}^{n+1} \lambda_i \phi(\ell_i)$ and $\ell = \sum_{i=1}^{n+1} \lambda_i \ell_i$. Since $\ell \in \mathcal{N}_x A$ we have

$$\ell(x) = s(\ell) = \sum_{i=1}^{n+1} \lambda_i s(\ell_i) \geq \sum_{i=1}^{n+1} \lambda_i \ell_i(x) = \ell(x),$$

so that equality holds throughout, and $s(\ell_i) = \ell_i(x)$ (hence $\ell_i \in \mathcal{N}_x A$) for each i with $\lambda_i > 0$. $\square$

This in turn gives a useful characterisation of the tangent cone:

Theorem B.7. *Let E be of dimension n, and suppose A is defined by* (B.1)*. Then for any $x \in \partial A$,*

$$\mathcal{T}_x A = \bigcap_{\ell \in B:\, \ell(x) = \phi(\ell)} \{z \in E : \ \ell(z) \leq 0\}$$

and the interior of $\mathcal{T}_x A$ is given by the intersection of the corresponding open half-spaces.

Proof. Any point z in $\mathcal{T}_x A$ satisfies $\ell(z) \leq 0$ for every $\ell \in E \setminus \{0\}$ with $\ell(x) = s(\ell)$. In particular, if $\ell \in B$ and $\ell(x) = \phi(x)$, then $\phi(x) = s(\ell)$ and $\ell(z) \leq 0$. Conversely, if $\ell(z) \leq 0$ for all $\ell \in B$ with $\ell(x) = \phi(\ell)$ (equivalently, for all $\ell \in B \cap \mathcal{N}_x A$) and ϑ is any element of $\mathcal{N}_x A$, then by Theorem B.6 there exist $\ell_i \in B \cap \mathcal{N}_x A$ and $\lambda_i > 0$ for $i = 1, \ldots, k$ such that $\vartheta = \sum_{i=1}^{k} \lambda_i \ell_i$, and so $\vartheta(z) = \sum_{i=1}^{k} \lambda_i \ell_i(z) \leq 0$. Since this is true for all $\vartheta \in \mathcal{N}_x A$, z is in $\mathcal{T}_x A$. $\square$

Appendix C
Canonically Identifying Tensor Spaces with Lie Algebras

In studying the algebraic decomposition of the curvature tensor, one needs to make several natural identification between tensor spaces and Lie algebras. By doing so, one is able to use the Lie algebra structure in conjunction with the tensor space construction to elucidate the structure of the quadratic terms in the curvature evolution equation.

C.1 Lie Algebras

A *Lie algebra* consists of a finite-dimensional vector space V over a field $\mathbb{F}$ with a bilinear *Lie bracket* $[,] : (X, Y) \mapsto [X, Y]$ that satisfies the properties:

1. $[X, X] = 0$
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

for all vectors X, Y and Z.

Any *algebra* $\mathscr{A}$ over a field $\mathbb{F}$ can be made into a Lie algebra by defining the bracket

$$[X, Y] := X \cdot Y - Y \cdot X.$$

A special case of this arises when $\mathscr{A} = \mathrm{End}(V)$ is the algebra of operator endomorphisms of a vector space V. In which case the corresponding Lie algebra is called the *general Lie algebra* $\mathfrak{gl}(V)$. Concretely, setting $V = \mathbb{R}^n$ gives the *general linear Lie algebra* $\mathfrak{gl}(n, \mathbb{R})$ of all $n \times n$ real matrices with bracket $[X, Y] := XY - YX$. Furthermore, the *special linear Lie algebra* $\mathfrak{sl}(n, \mathbb{R})$ is the set of real matrices of trace 0; it is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. The *special orthogonal Lie algebra* $\mathfrak{so}(n, \mathbb{R}) = \{X \in \mathfrak{sl}(n, \mathbb{R}) : X^T = -X\}$ is the set of skew-symmetric matrices.

C.2 Tensor Spaces as Lie Algebras

Suppose $U = (U, \langle\cdot,\cdot\rangle)$ is a real N-dimensional inner product space with orthonormal basis $(e_\alpha)_{\alpha=1}^N$. Let $E_{\alpha\beta}$ be the matrix of zero's with a 1 in the (α,β)-th entry. The matrix product then satisfies $E_{\alpha\beta}E_{\lambda\eta} = \delta_{\beta\lambda}E_{\alpha\eta}$.

The tensor space $U \otimes U$ is equipped with an inner product

$$\langle x \otimes y, u \otimes v\rangle = \langle x, u\rangle \langle y, v\rangle .$$

The set $(e_\alpha \otimes e_\beta)_{\alpha,\beta=1}^N$ forms an orthonormal basis. We identify $U \otimes U \simeq \mathfrak{gl}(N,\mathbb{R})$ by defining the linear transformation

$$x \otimes y : z \mapsto \langle y, z\rangle x \tag{C.1}$$

for any $x \otimes y \in U \otimes U$. The map simply identifies y with its dual. Under this identification, the inner product on $\mathfrak{gl}(N,\mathbb{R})$ is given by the trace norm:

$$\langle A, B\rangle = \operatorname{tr} A^T B,$$

for any $A, B \in \mathfrak{gl}(N,\mathbb{R})$. To see why, observe that $e_\alpha \otimes e_\beta \simeq E_{\alpha\beta}$ and so

$$\operatorname{tr} E_{\alpha\beta}^T E_{\lambda\eta} = \operatorname{tr} E_{\beta\alpha}E_{\lambda\eta} = \operatorname{tr} \delta_{\alpha\lambda}E_{\beta\eta} = \delta_{\alpha\lambda}\delta_{\beta\eta} = \langle e_\alpha \otimes e_\beta, e_\lambda \otimes e_\eta\rangle .$$

C.3 The Space of Second Exterior Powers as a Lie Algebra

Consider the n-dimensional real inner product space $V = (V, \langle\cdot,\cdot\rangle)$ with orthonormal basis $(e_i)_{i=1}^n$. As usual, let (e^i) be the corresponding dual basis for V^*. Define $\bigwedge^2 V = V \otimes V/\mathcal{I}$ to be the quotient algebra of the tensor space $V \otimes V$ by the ideal $\mathcal{I}$ generated from $x \otimes x$ for $x \in V$. In which case

$$x \wedge y = x \otimes y \pmod{\mathcal{I}},$$

for any $x, y \in V$. The space $\bigwedge^2 V$ is called the *second exterior power* of V and elements $x \wedge y$ are referred to as *bivectors*.[1] The canonical inner product

[1] The geometric interpretation of $x \wedge y$ is that of an oriented area element in the plane spanned by x and y. The object $x \wedge y$ is referred to as a bivector as it is a two-dimensional analog to a one-dimensional vector. Whereas a vector is often utilised to represent a one-dimensional directed quantity (often visualised geometrically as a directed line-segment), a bivector is used to represent a two-dimensional directed quantity (often visualised as an oriented plane-segment).

on $\bigwedge^2 V$ is given by

$$\langle x \wedge y, u \wedge v \rangle = \langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle . \tag{C.2}$$

With respect to this, the set $(e_i \wedge e_j)_{i<j}$ forms an orthonormal basis for the $n(n-1)/2$-dimensional vector space $\bigwedge^2 V$. We identify $\bigwedge^2 V \simeq \mathfrak{so}(n)$ by mapping $e_i \wedge e_j$ to the linear map $L(e_i \wedge e_j)$ of rank 2 which is a rotation with angle $\pi/2$ in the (i,j)-th plane. This is equivalent defining the linear transformation

$$x \wedge y : z \mapsto \langle y, z \rangle x - \langle x, z \rangle y. \tag{C.3}$$

Under this identification, the inner product on $\mathfrak{so}(n)$ is given by the trace norm

$$\langle A, B \rangle = \frac{1}{2} \mathrm{tr}\, A^T B = -\frac{1}{2} \mathrm{tr}\, AB$$

where $A, B \in \mathfrak{so}(n)$. To see this, note that

$$\begin{aligned}(e_i \wedge e_j)^T \cdot (e_k \wedge e_\ell) &= (E_{ji} - E_{ij})(E_{k\ell} - E_{\ell k}) \\ &= \delta_{ik} E_{j\ell} - \delta_{i\ell} E_{jk} + \delta_{j\ell} E_{ik} - \delta_{jk} E_{i\ell}\end{aligned}$$

and so $\mathrm{tr}\,(e_i \wedge e_j)^T \cdot (e_k \wedge e_\ell) = 2(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) = 2 \langle e_i \wedge e_j, e_k \wedge e_\ell \rangle$.

Example C.1. When $n = 3$ and $V = \mathbb{R}^3$ we observe that

$$e_2 \wedge e_3 \longmapsto R_x = E_{23} - E_{32} = \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & 1 \\ & -1 & 0 \end{array}\right)$$

$$e_1 \wedge e_3 \longmapsto R_y = E_{13} - E_{31} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$e_1 \wedge e_2 \longmapsto R_z = E_{12} - E_{21} = \left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & 0 \end{array}\right)$$

where R_x, R_y, R_z are $\pi/2$-rotations about the x, y and z axis. Whence any $X \in \mathfrak{so}(3)$ can be written as

$$X = \begin{pmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{pmatrix} = aR_x + bR_y + cR_z,$$

since $X^T = -X$ and $\mathrm{tr} X = 0$ by definition. Furthermore, if $Y = uR_x + vR_y + wR_z$ then the inner product $\langle X, Y \rangle = au + bv + cw = (a, b, c) \cdot (u, v, w)$ is the usual Euclidean inner product.

C.3.1 The space $\bigwedge^2 V^*$ as a Lie Algebra

As done in the above passage, $\bigwedge^2 V^* = V^* \otimes V^* / \mathcal{I}$ is the quotient algebra of $V^* \otimes V^*$ by the ideal $\mathcal{I} = \langle x \otimes x \,|\, x \in V^* \rangle$. The canonical inner product given by (C.2), except now applied to dual vectors. The wedge $\wedge$ is an anti-symmetric bilinear product with the additional property that

$$(e^i \wedge e^j)(e_k, e_\ell) = \det \begin{pmatrix} e^i(e_k) & e^i(e_\ell) \\ e^j(e_k) & e^j(e_\ell) \end{pmatrix} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}.$$

Any $\varphi \in \bigwedge^2 V^*$ may be written as

$$\varphi = \frac{1}{2} \sum_{i,j} \varphi_{ij} e^i \wedge e^j = \sum_{i<j} \varphi_{ij} e^i \wedge e^j \tag{C.4}$$

where $\varphi_{ij} := \varphi(e_i, e_j)$. Moreover, the pairing of bivectors with its dual is given by $(e^i \wedge e^j)(e_k \wedge e_\ell) = (e^i \wedge e^j)(e_k, e_\ell)$ in order to preserve orthonormality.

Remark C.2. A quick consistency check confirms the summation convention used in (C.4) allows the coefficients φ_{ij} that appear in the sum to agree with the component $\varphi(e_i, e_j)$. Indeed, we observe that

$$\left(\frac{1}{2} \sum_{i,j} \varphi_{ij} e^i \wedge e^j \right) (e_k, e_\ell) = \frac{1}{2} \varphi_{ij} (\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) = \frac{1}{2}(\varphi_{k\ell} - \varphi_{\ell k}) = \varphi_{k\ell}$$

which is equal to $\varphi(e_k, e_\ell)$ by definition. Furthermore one also find that $\left\langle \varphi, e^k \wedge e^\ell \right\rangle = \frac{1}{2} \sum_{i,j} \varphi_{ij} \left\langle e^i \wedge e^j, e^k \wedge e^\ell \right\rangle = \varphi_{k\ell}$. Thus the convention is consistent.

We identify $\bigwedge^2 V^*$ with the Lie algebra $\mathfrak{so}(n)$ by sending $e^i \wedge e^j \mapsto E_{ij} - E_{ji}$ as before. This equips $\bigwedge^2 V^*$ with a Lie algebra structure. In particular the bracket

$$\begin{aligned}
[e^i \wedge e^j, e^k \wedge e^\ell] &= (e^i \wedge e^j) \cdot (e^k \wedge e^\ell) - (e^k \wedge e^\ell) \cdot (e^i \wedge e^j) \\
&= (E_{ij} - E_{ji})(E_{k\ell} - E_{\ell k}) - (E_{k\ell} - E_{\ell k})(E_{ij} - E_{ji}) \\
&= E_{ij}E_{k\ell} - E_{ij}E_{\ell k} - E_{ji}E_{k\ell} + E_{ji}E_{\ell k} \\
&\quad - E_{k\ell}E_{ij} + E_{k\ell}E_{ji} + E_{\ell k}E_{ij} - E_{\ell k}E_{ji} \\
&= \delta_{i\ell} e^j \wedge e^k + \delta_{jk} e^i \wedge e^\ell - \delta_{ik} e^j \wedge e^\ell - \delta_{j\ell} e^i \wedge e^k
\end{aligned}$$

In which case, given any $\phi, \psi \in \bigwedge^2 V^*$ one computes

$$\begin{aligned}
[\phi, \psi] &= \frac{1}{4}\phi_{ij}\psi_{k\ell}[e^i \wedge e^j, e^k \wedge e^\ell] \\
&= \frac{1}{4}\phi_{ij}\psi_{k\ell}(\delta_{i\ell}e^j \wedge e^k + \delta_{jk}e^i \wedge e^\ell - \delta_{ik}e^j \wedge e^\ell - \delta_{j\ell}e^i \wedge e^k) \\
&= \frac{1}{4}\big(\phi_{pj}\psi_{kp}e^j \wedge e^k + \phi_{ip}\psi_{p\ell}e^i \wedge e^\ell - \phi_{pj}\psi_{p\ell}e^j \wedge e^\ell - \phi_{ip}\psi_{kp}e^i \wedge e^k\big) \\
&= \frac{1}{2}\sum_{i,j}(\phi_{ip}\psi_{pj} - \psi_{ip}\phi_{pj})e^i \wedge e^j
\end{aligned}$$

Therefore we (naturally) define the components of the bracket, with respect to the basis $(e_i \wedge e_j)_{i<j}$, by

$$[\phi, \psi]_{ij} := \phi_{ip}\psi_{pj} - \psi_{ip}\phi_{pj} \tag{C.5}$$

for any $\phi, \psi \in \bigwedge^2 V^*$.

C.3.1.1 Structure Constants

Now suppose (φ^α) is an orthonormal basis for $\bigwedge^2 V^*$. The *structure constants* $c_\gamma^{\alpha\beta}$ for the bracket (C.5), with respect to the basis (φ^α), are defined by

$$[\varphi^\alpha, \varphi^\beta] = c_\gamma^{\alpha\beta}\varphi^\gamma.$$

As (φ^α) are orthonormal, the structure constants can be directly computed from

$$c_\gamma^{\alpha\beta} = \big\langle [\varphi^\alpha, \varphi^\beta], \varphi^\gamma \big\rangle .$$

It is easy to check that the tri-linear form $\big\langle [\varphi^\alpha, \varphi^\beta], \varphi^\gamma \big\rangle$ is fully antisymmetric, thus the structure constants $c_\gamma^{\alpha\beta}$ are anti-symmetric in all three components. Moreover, if (σ_α) orthonormal basis for $\bigwedge^2 V$ dual to (φ^α), then the corresponding structure constants $c_{\alpha\beta}^\gamma$ are given by

$$[\sigma_\alpha, \sigma_\beta] = c_{\alpha\beta}^\gamma \sigma_\gamma.$$

From the identification of $\bigwedge^2 V$ with $\bigwedge^2 V^*$ we also have $c_{\alpha\beta}^\gamma = c_\gamma^{\alpha\beta}$.

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