Appendix

Appendix A
List of Symbols

In this appendix, we collect a list of all the symbols that have been used in this book. If necessary, cross-references are also given.

Functions

$\|\cdot\|_\infty$ Chebyshev norm; $\|f\|_\infty = \sup_{a\le x\le b}|f(x)|$

$\|\cdot\|_p$ L_p norm ($1 \le p < \infty$); $\|f\|_p = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}$

$\lfloor\cdot\rfloor$ Floor function, $\lfloor x\rfloor = \max\{z \in \mathbb{Z} : z \le x\}$

$\lceil\cdot\rceil$ Ceiling function, $\lceil x\rceil = \min\{z \in \mathbb{Z} : z \ge x\}$

$\binom{n}{k}$ Binomial coefficient, $\binom{n}{k} = n(n-1)(n-2)\cdots(n-k+1)/k!$ for $n \in \mathbb{R}$ and $k \in \mathbb{N}_0$

$B_N[f]$ Nth Bernstein polynomial for the function f, $B_N[f](t) = \sum_{k=0}^N \binom{N}{k} t^k (1-t)^{N-k} f(k/N)$ (see Appendix D.5)

B Euler's Beta function, $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ (cf. Appendix D.1)

Γ Euler's Gamma function, $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$ (cf. Definition 1.2)

ψ Digamma function, $\psi(x) = \Gamma'(x)/\Gamma(x)$

E_n Mittag-Leffler function of order n, $E_n(x) = \sum_{j=0}^\infty x^j/\Gamma(jn+1)$ (cf. Definition 4.1)

E_{n_1,n_2} two-parameter Mittag-Leffler function, $E_{n_1,n_2}(x) = \sum_{j=0}^\infty x^j/\Gamma(jn_1+n_2)$ (cf. Definition 4.2)

${}_1F_1$ Kummer's confluent hypergeometric function [2, Chapter 13],

$$ {}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^\infty \frac{\Gamma(a+k)}{\Gamma(b+k)k!} z^k \qquad (a \in \mathbb{R},\ -b \notin \mathbb{N}_0) $$

$_2F_1$ Gauss' hypergeometric function [2, Chapter 15],
$_2F_1(a,b;c;z) =$
$\dfrac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \dfrac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k \quad (a,b \in \mathbb{R},\ -c \notin \mathbb{N}_0)$

o, O Landau symbols

$T_j[f;a]$ Taylor polynomial of degree j for the function f centered at the point a

Sets

A^n, $A^n[a,b]$ Set of functions with absolutely continuous derivative of order $n-1$ (cf. Definition 1.5)
C, $C[a,b]$ Set of continuous functions (cf. Definition 1.3)
C^k, $C^k[a,b]$ Set of functions with continuous kth derivative (cf. Definition 1.3)
H^*, $H^*[a,b]$ Cf. Definition 1.4
H_μ, $H_\mu[a,b]$ Hölder space (cf. Definition 1.3)
L_p, $L_p[a,b]$ Lebesgue space (cf. Definition 1.3)
$\mathbb{N}$ $= \{1,2,3,\ldots\}$, the set of natural numbers
$\mathbb{N}_0$ $= \mathbb{N} \cup \{0\}$
$\mathbb{R}$ The set of real numbers
$\mathbb{R}_+$ $= \{x \in \mathbb{R} : x > 0\}$, the set of strictly positive real numbers
$\mathbb{Z}$ $= \{0, \pm 1, \pm 2, \pm 3 \ldots\}$, the set of integer numbers

Operators

Δ_h^n Finite difference of order n; cf. (2.11)
D Differential operator, $Df(x) = f'(x)$ (cf. Definition 1.1)
D^n $n \in \mathbb{N}$: n-fold iterate of the differential operator D (cf. Definition 1.1)
D_a^n $n \in \mathbb{R}_+$: Riemann–Liouville fractional differential operator (cf. Definition 2.2)
$\widetilde{D}_a^n$ $n \in \mathbb{R}_+$: Grünwald–Letnikov fractional differential operator (cf. Definition 2.3)
$\widehat{D}_a^n$ Cf. Definition 3.1
D_{*a}^n $n \in \mathbb{R}_+$: Caputo fractional differential operator (cf. Definition 3.2)
$\mathscr{D}^n$ $n \in \mathbb{R}_+$: Gel'fond–Leont'ev operator (cf. Definition 3.3)
I Identity operator
J_a Integral operator, $J_a f(x) = \int_a^x f(t)\,\mathrm{d}t$ (cf. Definition 1.1)

J_a^n $n \in \mathbb{N}$: n-fold iterate of the integral operator J_a (cf. Definition 1.1)
$n \in \mathbb{R}_+ \setminus \mathbb{N}$: Riemann–Liouville fractional integral operator (cf. Definition 2.1)
$n = 0$: identity operator (cf. Definition 2.1)
$\widetilde{J}_a^n$ $n \in \mathbb{R}_+$: Grünwald–Letnikov fractional integral operator (cf. Definition 2.4)
$\mathscr{L}$ Laplace transform operator (cf. Appendix D.3)
ω Modulus of continuity of the function $g : [a,b] \to \mathbb{R}$,
$\omega(g;h) := \sup\{|g(y_1) - g(y_2)| : y_1, y_2 \in [a,b], |y_1 - y_2| \le h\}$

Other Symbols

$\sim$ $a_j \sim b_j \Leftrightarrow \exists A, B > 0\, \exists j_0 \in \mathbb{N}\, \forall j \ge j_0 : A \le |a_j/b_j| \le B$

Remarks

1. The power series for both types of Mittag-Leffler functions converge in the entire complex plane (cf. Theorem 4.1).
2. The power series for Kummer's confluent hypergeometric function converges in the entire complex plane.
3. For the Gauss hypergeometric function, the power series converges for all complex z with $|z| < 1$ and may be extended analytically into the entire complex plane with a branch cut along the positive real axis from $+1$ to $+\infty$. (In the formulas in Appendix B, we need to evaluate this function for $z < 0$, so the branch cut for $z \ge 1$ gives no problems.)

Appendix B
A Table of Caputo Derivatives

For the convenience of the reader, we provide this appendix where we give some Caputo-type derivatives of certain important functions. We do not strive for completeness in any sense, but we do want to give at least the derivatives of the classical examples.

Throughout this appendix, n will always denote the order of the Caputo-type differential operator under consideration. We shall only consider the case $n > 0$ and $n \notin \mathbb{N}$, and we use the notation $m := \lceil n \rceil$ to denote the smallest integer greater than (or equal to) n. Recall that for $n \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order, and for $n < 0$, the Caputo differential operator of negative order can be interpreted as the Riemann–Liouville differential operator of the same order. Tables of the latter are given in various places in the literature (cf., e.g., Podlubny [153] or Samko et al. [167]); we are not going to repeat those results here.

Various special functions will arise in this connection; for the precise definitions we refer to Appendix A. By $\mathrm{i} = \sqrt{-1}$ we denote the imaginary unit.

1. Let $f(x) = x^j$. Here we have to distinguish some cases:

$$(D^n_{*0} f)(x) = \begin{cases} 0 & \text{if } j \in \mathbb{N}_0 \text{ and } j < m, \\ \dfrac{\Gamma(j+1)}{\Gamma(j+1-n)} x^{j-n} & \text{if } j \in \mathbb{N}_0 \text{ and } j \geq m \\ & \text{or } j \notin \mathbb{N} \text{ and } j > m-1. \end{cases}$$

2. Let $f(x) = (x+c)^j$ for arbitrary $c > 0$ and $j \in \mathbb{R}$. Then

$$(D^n_{*0} f)(x) = \frac{\Gamma(j+1)}{\Gamma(j+1-m)} \frac{c^{j-m-1} x^{m-n}}{\Gamma(m-n+1)} {}_2F_1(1, m-j; m-n+1; -x/c).$$

3. Let $f(x) = \exp(jx)$ for some $j \in \mathbb{R}$. Then

$$(D^n_{*0} f)(x) = j^m x^{m-n} E_{1,m-n+1}(jx).$$

4. Let $f(x) = x^j \ln x$ for some $j > m-1$. Then

$$(D^n_{*0} f)(x) = x^{j-n} \sum_{k=0}^{m-1} (-1)^{m-k+1} \binom{j}{k} \frac{m!}{m-k} \frac{\Gamma(j-m+1)}{\Gamma(j-n+1)} + \frac{\Gamma(j+1)}{\Gamma(j-n+1)} x^{j-n} (\psi(j-m+1) - \psi(j-n+1) + \ln x).$$

5. Let $f(x) = \sin jx$ for some $j \in \mathbb{R}$. Here again we have two cases:

$$(D^n_{*0} f)(x) = \begin{cases} \dfrac{j^m \mathrm{i} (-1)^{m/2} x^{m-n}}{2\Gamma(m-n+1)} \left[-{}_1F_1(1; m-n+1; \mathrm{i}jx) \right. \\ \qquad \left. + {}_1F_1(1; m-n+1; -\mathrm{i}jx) \right] & (m \text{ even}), \\ \dfrac{j^m (-1)^{(m-1)/2} x^{m-n}}{2\Gamma(m-n+1)} \left[{}_1F_1(1; m-n+1; \mathrm{i}jx) \right. \\ \qquad \left. + {}_1F_1(1; m-n+1; -\mathrm{i}jx) \right] & (m \text{ odd}). \end{cases}$$

6. Finally we consider $f(x) = \cos jx$ with some $j \in \mathbb{R}$. As in the previous example, we obtain two cases:

$$(D^n_{*0} f)(x) = \begin{cases} \dfrac{j^m (-1)^{m/2} x^{m-n}}{2\Gamma(m-n+1)} \left[{}_1F_1(1; m-n+1; \mathrm{i}jx) \right. \\ \qquad \left. + {}_1F_1(1; m-n+1; -\mathrm{i}jx) \right] & (m \text{ even}), \\ \dfrac{j^m \mathrm{i} (-1)^{(m-1)/2} x^{m-n}}{2\Gamma(m-n+1)} \left[{}_1F_1(1; m-n+1; \mathrm{i}jx) \right. \\ \qquad \left. - {}_1F_1(1; m-n+1; -\mathrm{i}jx) \right] & (m \text{ odd}). \end{cases}$$

Appendix C
Numerical Solution of Fractional Differential Equations

For most fractional differential equations we cannot provide methods to compute the exact solutions analytically. Therefore it is necessary to revert to numerical methods. In order to give the reader a tool that can be applied to a very wide class of equations, we now present a method that is well understood and that has proven to be efficient in many practical applications [50, 51, 65, 182, 183]. We begin by discussing this problem for single-term equations and later extend our idea to multi-term problems.

C.1 An Algorithm for Single-Term Equations

The method can be called *indirect* because, rather than discretizing the differential equation

$$D^n_{*0}y(x) = f(x, y(x))$$

with appropriate initial conditions

$$D^k y(0) = y_0^{(k)}, \qquad k = 0, 1, \ldots, \lceil n \rceil - 1,$$

directly, it requires some preliminary analytical manipulation, namely an application of Lemma 6.2 in order to convert the initial value problem for the differential equation into an equivalent Volterra integral equation,

$$y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} D^k y(0) + \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t, y(t))\, \mathrm{d}t \tag{C.1}$$

where $m = \lceil n \rceil$. We shall therefore now look at a method for the numerical solution of (C.1).

The algorithm that we shall consider can be interpreted as a fractional variant of the classical second-order Adams–Bashforth–Moulton method. It has been introduced and briefly discussed in [50]; more information is given in [51]. Some additional results for a specific initial value problem are contained in [44], a detailed mathematical analysis is provided in [49], and additional practical remarks can be

found in [48]. Numerical experiments and comparisons with other methods are reported in [52, 65, 182, 183]. Here we shall give an even more detailed analysis under quite general assumptions.

Remark C.1. Before starting the investigations, we need to give a note of caution. It is common to construct methods for fractional differential equations by taking methods for classical (typically first-order) equations and then generalizing the concepts in an appropriate way. The resulting formulas are then usually given the same name as the underlying classical algorithm, possibly extended by the adjective "fractional". However, many classical numerical schemes can be extended in more than one way. This may lead to the problem that, in two different items of literature, two different algorithms are denoted in identical ways. Of course, this is a potential source for confusion, and the reader must be very careful in this respect. For example, the fractional Adams–Moulton rules of Galeone and Garrappa [70] do not coincide with the methods of the same name that we shall develop below.

Classical Formulation

In order to motivate the construction of the method, we shall first briefly recall the idea behind the classical Adams–Bashforth–Moulton algorithm for first-order equations. So, for a start, we focus our attention on the well-known initial-value problem for the first-order differential equation

$$Dy(x) = f(x, y(x)), \tag{C.2a}$$

$$y(0) = y_0. \tag{C.2b}$$

We assume the function f to be such that a unique solution exists on some interval $[0, T]$, say. Following [88, §III.1], we suggest to use the predictor-corrector technique of Adams where, for the sake of simplicity, we assume that we are working on a uniform grid $\{t_j = jh : j = 0, 1, \ldots, N\}$ with some integer N and $h = T/N$. In some applications it may be more efficient to use a non-uniform grid, and we will develop the numerical approximation formulas in this generalized sense. However, for the subsequent analysis of the properties of the scheme we will then restrict ourselves to the equispaced case.

The basic idea is, assuming that we have already calculated approximations $y_j \approx y(t_j)$ $(j = 1, 2, \ldots, k)$, that we try to obtain the approximation y_{k+1} by means of the equation

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(z, y(z))\,\mathrm{d}z. \tag{C.3}$$

This equation follows upon integration of (C.2a) on the interval $[t_k, t_{k+1}]$. Of course, we know neither of the expressions on the right-hand side of (C.3) exactly, but we do have an approximation for $y(t_k)$, namely y_k, that we can use instead. The integral is then replaced by the two-point trapezoidal quadrature formula

$$\int_a^b g(z)\,\mathrm{d}z \approx \frac{b-a}{2}\,(g(a)+g(b)), \tag{C.4}$$

thus giving an equation for the unknown approximation y_{k+1}, it being

$$y_{k+1} = y_k + \frac{t_{k+1}-t_k}{2}\,(f(t_k,y(t_k)) + f(t_{k+1},y(t_{k+1}))), \tag{C.5}$$

where again we have to replace $y(t_k)$ and $y(t_{k+1})$ by their approximations y_k and y_{k+1}, respectively. This yields the equation for the implicit one-step *Adams–Moulton method*, which is

$$y_{k+1} = y_k + \frac{t_{k+1}-t_k}{2}\,(f(t_k,y_k) + f(t_{k+1},y_{k+1})). \tag{C.6}$$

The problem with this equation is that the unknown quantity y_{k+1} appears on both sides, and due to the nonlinear nature of the function f, we cannot solve for y_{k+1} directly in general. Therefore, we may use (C.6) in an iterative process, inserting a preliminary approximation for y_{k+1} in the right-hand side in order to determine a better approximation that we can then use.

The preliminary approximation y_{k+1}^{P}, the so-called predictor, is obtained in a very similar way, only replacing the trapezoidal quadrature formula by the rectangle rule

$$\int_a^b g(z)\,dz \approx (b-a)g(a), \tag{C.7}$$

giving the explicit (forward Euler or one-step *Adams–Bashforth*) method

$$y_{k+1}^{\mathrm{P}} = y_k + hf(t_k,y_k). \tag{C.8}$$

It is well known [88, p. 372] that the process defined by (C.8) and

$$y_{k+1} = y_k + \frac{h}{2}\left(f(t_k,y_k) + f(t_{k+1},y_{k+1}^{\mathrm{P}})\right), \tag{C.9}$$

known as the one-step *Adams–Bashforth–Moulton* technique, is convergent of order 2, i.e.

$$\max_{j=1,2,\ldots,N} |y(t_j) - y_j| = O(h^2). \tag{C.10}$$

Moreover, this method behaves satisfactorily from the point of view of its numerical stability [89, Chap. IV]. It is said to be of the PECE (Predict, Evaluate, Correct, Evaluate) type because, in a concrete implementation, we would start by calculating the predictor in (C.8), then we evaluate $f(t_{k+1},y_{k+1}^{\mathrm{P}})$, use this to calculate the corrector in (C.9), and finally evaluate $f(t_{k+1},y_{k+1})$. This result is stored for future use in the next integration step.

Fractional Formulation

Having introduced this concept, we now try to carry over the essential ideas to the fractional-order problem with some unavoidable modifications. The key is to derive an equation similar to (C.3). Fortunately, such an equation is available, namely (C.1). This equation looks somewhat different from (C.3), because the range of integration now starts at 0 instead of t_k. This is a consequence of the non-local structure of the fractional-order differential operators. This however does not cause major problems in our attempts to generalize the Adams method. What we do is simply use the product trapezoidal quadrature formula to replace the integral, i.e. we use the nodes t_j ($j = 0, 1, \ldots, k+1$) and interpret the function $(t_{k+1} - \cdot)^{n-1}$ as a weight function for the integral. In other words, we apply the approximation

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} g(z)\,\mathrm{d}z \approx \int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \tilde{g}_{k+1}(z)\,\mathrm{d}z, \tag{C.11}$$

where $\tilde{g}_{k+1}$ is the piecewise linear interpolant for g with nodes and knots chosen at the t_j, $j = 0, 1, 2, \ldots, k+1$.

It is clear by construction that the required weighted trapezoidal quadrature formula can be represented as a weighted sum of function values of the integrand g, taken at the points t_j. Specifically, we find that we can write the integral on the right-hand side of (C.11) as

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \tilde{g}_{k+1}(z)\,\mathrm{d}z = \sum_{j=0}^{k+1} a_{j,k+1} g(t_j) \tag{C.12a}$$

where

$$a_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \phi_{j,k+1}(z)\,\mathrm{d}z \tag{C.12b}$$

and

$$\phi_{j,k+1}(z) = \begin{cases} (z - t_{j-1})/(t_j - t_{j-1}) & \text{if } t_{j-1} < z \le t_j, \\ (t_{j+1} - z)/(t_{j+1} - t_j) & \text{if } t_j < z < t_{j+1}, \\ 0 & \text{else.} \end{cases} \tag{C.12c}$$

This is clear because the functions $\phi_{j,k+1}$ satisfy

$$\phi_{j,k+1}(t_\mu) = \begin{cases} 0 & \text{if } j \neq \mu, \\ 1 & \text{if } j = \mu, \end{cases}$$

and that they are continuous and piecewise linear with breakpoints at the nodes t_μ, so that they must be integrated exactly by our formula.

An easy explicit calculation yields that, for an arbitrary choice of the t_j, (C.12b) and (C.12c) produce

$$a_{0,k+1} = \frac{(t_{k+1} - t_1)^{n+1} + t_{k+1}^n [n t_1 + t_1 - t_{k+1}]}{t_1 n (n+1)}, \tag{C.13a}$$

$$a_{j,k+1} = \frac{(t_{k+1}-t_{j-1})^{n+1} + (t_{k+1}-t_j)^n[n(t_{j-1}-t_j)+t_{j-1}-t_{k+1}]}{(t_j-t_{j-1})n(n+1)} + \frac{(t_{k+1}-t_{j+1})^{n+1} - (t_{k+1}-t_j)^n[n(t_j-t_{j+1})-t_{j+1}+t_{k+1}]}{(t_{j+1}-t_j)n(n+1)}, \quad \text{(C.13b)}$$

if $1 \le j \le k$, and

$$a_{k+1,k+1} = \frac{(t_{k+1}-t_k)^n}{n(n+1)}. \quad \text{(C.13c)}$$

In the case of equispaced nodes ($t_j = jh$ with some fixed h), these relations reduce to

$$a_{j,k+1} = \begin{cases} \dfrac{h^n}{n(n+1)}\left(k^{n+1} - (k-n)(k+1)^n\right) & \text{if } j = 0, \\ \dfrac{h^n}{n(n+1)}\left((k-j+2)^{n+1} + (k-j)^{n+1} - 2(k-j+1)^{n+1}\right) & \text{if } 1 \le j \le k, \\ \dfrac{h^n}{n(n+1)} & \text{if } j = k+1. \end{cases} \quad \text{(C.14)}$$

This then gives us our corrector formula (i.e. the fractional variant of the one-step *Adams–Moulton method*), which is

$$y_{k+1} = \sum_{j=0}^{m-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(n)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^{\mathrm{P}}) \right). \quad \text{(C.15)}$$

The remaining problem is the determination of the predictor formula that we require to calculate the value y_{k+1}^{P}. The idea we use to generalize the one-step Adams–Bashforth method is the same as the one described above for the Adams–Moulton technique: We replace the integral on the right-hand side of (C.1) by the product rectangle rule

$$\int_0^{t_{k+1}} (t_{k+1}-z)^{n-1} g(z)\,\mathrm{d}z \approx \sum_{j=0}^{k} b_{j,k+1} g(t_j), \quad \text{(C.16)}$$

where now

$$b_{j,k+1} = \int_{t_j}^{t_{j+1}} (t_{k+1}-z)^{n-1}\,\mathrm{d}z = \frac{(t_{k+1}-t_j)^n - (t_{k+1}-t_{j+1})^n}{n}. \quad \text{(C.17)}$$

This expression for weights can be derived in a way similar to the method used in the derivation of (C.13). However, here we are dealing with a piecewise constant approximation and not a piecewise linear one, and hence we have to replace the "hat-shaped" functions ϕ_{kj} by functions being of constant value 1 on $[t_j, t_{j+1}]$ and 0

on the remaining parts of the interval $[0, t_{k+1}]$. Again, in the equispaced case, we have the simpler expression

$$b_{j,k+1} = \frac{h^n}{n}\left((k+1-j)^n - (k-j)^n\right). \tag{C.18}$$

Thus, the predictor y^{P}_{k+1} is determined by the fractional *Adams–Bashforth method*

$$y^{\mathrm{P}}_{k+1} = \sum_{j=0}^{m-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j). \tag{C.19}$$

Our basic algorithm, the fractional *Adams–Bashforth–Moulton method*, is therefore completely described now by (C.19) and (C.15) with the weights $a_{j,k+1}$ and $b_{j,k+1}$ being defined according to (C.13) and (C.17), respectively.

Error Analysis

For the error analysis of this algorithm, we restrict our attention to the case of an equispaced grid, i.e. from now on we assume that $t_j = jh = jT/N$ with some $N \in \mathbb{N}$. Essentially we follow the structure of [49] and begin by stating some auxiliary results.

What we need for our purposes is some information on the errors of the quadrature formulas that we have used in the derivation of the predictor and the corrector, respectively. We first give a statement on the product rectangle rule that we have used for the predictor.

Theorem C.1. *(a) Let $z \in C^1[0,T]$. Then,*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \le \frac{1}{n} \|z'\|_\infty t_{k+1}^n h.$$

(b) Let $z(t) = t^p$ for some $p \in (0,1)$. Then,

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \le C^{\mathrm{Re}}_{n,p} t_{k+1}^{n+p-1} h$$

where $C^{\mathrm{Re}}_{n,p}$ is a constant that depends only on n and p.

Proof. By construction of the product rectangle formula, we find in both cases that the quadrature error has the representation

$$\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j)$$
$$= \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1} (z(t)-z(t_j))\,\mathrm{d}t. \tag{C.20}$$

To prove statement (a), we apply the Mean Value Theorem of Differential Calculus to the second factor of the integrand on the right-hand side of (C.20) and derive

$$\begin{aligned}
&\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \\
&\quad \le \|z'\|_\infty \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1} (t-jh)\,\mathrm{d}t \\
&\quad = \|z'\|_\infty \frac{h^{1+n}}{n} \sum_{j=0}^{k} \left(\frac{1}{1+n} [(k+1-j)^{1+n} - (k-j)^{1+n}] - (k-j)^n \right) \\
&\quad = \|z'\|_\infty \frac{h^{1+n}}{n} \left(\frac{(k+1)^{1+n}}{1+n} - \sum_{j=0}^{k} j^n \right) \\
&\quad = \|z'\|_\infty \frac{h^{1+n}}{n} \left(\int_0^{k+1} t^n\,\mathrm{d}t - \sum_{j=0}^{k} j^n \right).
\end{aligned}$$

Here the term in parentheses is simply the remainder of the standard rectangle quadrature formula, applied to the function t^n, and taken over the interval $[0, k+1]$. Since the integrand is monotonic, we may apply some standard results from quadrature theory [19, Thm. 97] to find that this term is bounded by the total variation of the integrand, viz. the quantity $(k+1)^n$. Thus,

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \le \|z'\|_\infty \frac{h^{1+n}}{n} (k+1)^n.$$

Similarly, to prove (b), we use the monotonicity of z in (C.20) and derive

$$\begin{aligned}
&\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \\
&\quad \le \sum_{j=0}^{k} |z(t_{j+1}) - z(t_j)| \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1}\,\mathrm{d}t \\
&\quad = \frac{h^{n+p}}{n} \sum_{j=0}^{k} ((j+1)^p - j^p)((k+1-j)^n - (k-j)^n)
\end{aligned}$$

$$\leq \frac{h^{n+p}}{n}\left((k+1)^n - k^n + (k+1)^p - k^p + pn\sum_{j=1}^{k-1} j^{p-1}(k-j+q)^{n-1}\right)$$
$$\leq \frac{h^{n+p}}{n}\left(n(k+q)^{n-1} + pk^{p-1} + pn\sum_{j=1}^{k-1} j^{p-1}(k-j+q)^{n-1}\right)$$

by additional applications of the Mean Value Theorem. Here $q = 0$ if $n \leq 1$, and $q = 1$ otherwise. In either case a brief asymptotic analysis using the Euler–MacLaurin formula [188, Thm. 3.7] yields that the term in parentheses is bounded from above by $C^{\mathrm{Re}}_{n,p}(k+1)^{p+n-1}$ where $C^{\mathrm{Re}}_{n,p}$ is a constant depending on n and p but not on k. □

Next we come to a corresponding result for the product trapezoidal formula that we have used for the corrector. The proof of this theorem is very similar to the proof of Theorem C.1; we therefore omit the details.

Theorem C.2. *(a) If $z \in C^2[0,T]$ then there is a constant C^{Tr}_n depending only on n such that*

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j)\right| \leq C^{\mathrm{Tr}}_n \|z''\|_\infty t^n_{k+1} h^2.$$

(b) Let $z \in C^1[0,T]$ and assume that z' fulfils a Lipschitz condition of order μ for some $\mu \in (0,1)$. Then, there exist positive constants $B^{\mathrm{Tr}}_{n,\mu}$ (depending only on n and μ) and $M(z,\mu)$ (depending only on z and μ) such that

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j)\right| \leq B^{\mathrm{Tr}}_{n,\mu} M(z,\mu) t^n_{k+1} h^{1+\mu}.$$

(c) Let $z(t) = t^p$ for some $p \in (0,2)$ and $\rho := \min(2, p+1)$. Then,

$$\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\,\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j)\right| \leq C^{\mathrm{Tr}}_{n,p} t^{n+p-\rho}_{k+1} h^\rho$$

where $C^{\mathrm{Tr}}_{n,p}$ is a constant that depends only on n and p.

Remark C.2. Notice that in part (c) of Theorem C.2 it may happen that $n < 1$ and $p < 1$. This implies $\rho = p+1$. Thus, the exponent of t_{k+1} on the right-hand side of the inequality is equal to $n-1$ which is negative. At first sight this may seem counter-intuitive because it means that the overall integration error becomes larger if the size of the interval of integration becomes smaller. The explanation for this phenomenon is that by making t_{k+1} smaller we do not only shorten the length of the integration interval (which should lead to a smaller error) but we also change the weight function in a way that makes the integral more difficult, and this second feature leads to an increase in the error.

A similar observation can be made in Theorem C.1 (b).

We now present the main results concerning the error of our Adams scheme. It is useful to distinguish a number of cases. Specifically, we shall see that the precise behaviour of the error differs depending on whether $n < 1$ or $n > 1$. Moreover, the smoothness properties of the given function f and the unknown solution y play an important role. In view of the results of Sect. 6.4, we find that smoothness of one of these functions will imply non-smoothness of the other unless some special conditions are fulfilled. Therefore we shall also investigate the error under those two different smoothness assumptions.

Based on the error estimates above we shall first present a general convergence result for the Adams–Bashforth–Moulton method. In the theorems below we shall specialize this result to particularly important special cases.

Lemma C.3. *Assume that the solution y of the initial value problem is such that*

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_{*0}^n y(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} D_{*0}^n y(t_j) \right| \le C_1 t_{k+1}^{\gamma_1} h^{\delta_1}$$

and

$$\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_{*0}^n y(t)\,\mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} D_{*0}^n y(t_j) \right| \le C_2 t_{k+1}^{\gamma_2} h^{\delta_2}$$

with some $\gamma_1, \gamma_2 \ge 0$ and $\delta_1, \delta_2 > 0$. Then, for some suitably chosen $T > 0$, we have

$$\max_{0 \le j \le N} |y(t_j) - y_j| = O(h^q)$$

where $q = \min\{\delta_1 + n, \delta_2\}$ and $N = \lfloor T/h \rfloor$.

Proof. We will show that, for sufficiently small h,

$$|y(t_j) - y_j| \le Ch^q \tag{C.21}$$

for all $j \in \{0, 1, \ldots, N\}$, where C is a suitable constant. The proof will be based on mathematical induction. In view of the given initial condition, the induction basis ($j = 0$) is presupposed. Now assume that (C.21) is true for $j = 0, 1, \ldots, k$ for some $k \le N-1$. We must then prove that the inequality also holds for $j = k+1$. To do this, we first look at the error of the predictor y_{k+1}^{P}. By construction of the predictor we find that

$$\begin{aligned} |y(t_{k+1}) - y_{k+1}^{\mathrm{P}}| &= \frac{1}{\Gamma(n)} \left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} f(t, y(t))\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j) \right| \\ &\le \frac{1}{\Gamma(n)} \left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_{*0}^n y(t)\,\mathrm{d}t - \sum_{j=0}^{k} b_{j,k+1} D_{*0}^n y(t_j) \right| \\ &\quad + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} b_{j,k+1} |f(t_j, y(t_j)) - f(t_j, y_j)| \end{aligned}$$

$$\leq \frac{C_1 t_{k+1}^{\gamma_1}}{\Gamma(n)} h^{\delta_1} + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} b_{j,k+1} L C h^q$$
$$\leq \frac{C_1 T^{\gamma_1}}{\Gamma(n)} h^{\delta_1} + \frac{CLT^n}{\Gamma(n+1)} h^q. \tag{C.22}$$

Here we have used the Lipschitz property of f, the assumption on the error of the rectangle formula, and the facts that, by construction of the quadrature formula underlying the predictor, $b_{j,k+1} > 0$ for all j and k and

$$\sum_{j=0}^{k} b_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} \, \mathrm{d}t = \frac{1}{n} t_{k+1}^n \leq \frac{1}{n} T^n.$$

On the basis of the bound (C.22) for the predictor error we begin the analysis of the corrector error. We recall the relation (C.14) which we shall use in particular for $j = k+1$ and find, arguing in a similar way to above, that

$$\begin{aligned}
&|y(t_{k+1}) - y_{k+1}| \\
&\quad = \frac{1}{\Gamma(n)} \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} f(t, y(t)) \, \mathrm{d}t \right. \\
&\qquad\qquad \left. - \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) - a_{k+1,k+1} f(t_{k+1}, y_{k+1}^{\mathrm{P}}) \right| \\
&\quad \leq \frac{1}{\Gamma(n)} \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} D_{*0}^n y(t) \, \mathrm{d}t - \sum_{j=0}^{k+1} a_{j,k+1} D_{*0}^n y(t_j) \right| \\
&\qquad + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} a_{j,k+1} |f(t_j, y(t_j)) - f(t_j, y_j)| \\
&\qquad + \frac{1}{\Gamma(n)} a_{k+1,k+1} |f(t_{k+1}, y(t_{k+1})) - f(t_{k+1}, y_{k+1}^{\mathrm{P}})| \\
&\quad \leq \frac{C_2 t_{k+1}^{\gamma_2}}{\Gamma(n)} h^{\delta_2} + \frac{CL}{\Gamma(n)} h^q \sum_{j=0}^{k} a_{j,k+1} + a_{k+1,k+1} \frac{L}{\Gamma(n)} \left(\frac{C_1 T^{\gamma_1}}{\Gamma(n)} h^{\delta_1} + \frac{CLT^n}{\Gamma(n+1)} h^q \right) \\
&\quad \leq \left(\frac{C_2 T^{\gamma_2}}{\Gamma(n)} + \frac{CLT^n}{\Gamma(n+1)} + \frac{C_1 L T^{\gamma_1}}{\Gamma(n)\Gamma(n+2)} + \frac{CL^2 T^n}{\Gamma(n+1)\Gamma(n+2)} h^n \right) h^q
\end{aligned}$$

in view of the nonnegativity of γ_1 and γ_2 and the relations $\delta_2 \leq q$ and $\delta_1 + n \leq q$. By choosing T sufficiently small, we can make sure that the second summand in the parentheses is bounded by $C/2$. Having fixed this value for T, we can then make the sum of the remaining expressions in the parentheses smaller than $C/2$ too (for sufficiently small h) simply by choosing C sufficiently large. It is then obvious that the entire upper bound does not exceed Ch^q. □

As a first application of this Lemma we assume that the given data is such that the solution y itself is sufficiently differentiable. As mentioned above, the result depends on whether $n > 1$ or $n < 1$.

Theorem C.4. *Let $0 < n$ and assume $D^n_{*0}y \in C^2[0,T]$ for some suitable T. Then,*

$$\max_{0\le j\le N} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if } n \ge 1, \\ O(h^{1+n}) & \text{if } n < 1. \end{cases}$$

Before we come to the proof, we note one particular point: The order of convergence depends on n, and it is a non-decreasing function of n. This is due to the fact that we discretize the integral operator in (C.1) which behaves more smoothly (and hence can be approximated with a higher accuracy) as n increases. In contrast, so-called *direct methods* like the backward differentiation method of [34] use a different approach; as the name suggests they directly discretize the differential operator in the given initial value problem. The smoothness properties of such operators (and thus the ease with which they may be approximated) deteriorate as n increases, and so we find that the convergence order of the method from [34] is a non-increasing function of n; in particular no convergence is achieved there for $n \ge 2$. It is a distinctive advantage of the Adams scheme presented here that it converges for all $n > 0$.

Remark C.3. We formally recover the error bound (C.10) if we set $n = 1$.

Proof (of Theorem C.4). In view of Theorems C.1 and C.2, we may apply Lemma C.3 with $\gamma_1 = \gamma_2 = n > 0$, $\delta_1 = 1$ and $\delta_2 = 2$. Thus, defining

$$q = \min\{1+n, 2\} = \begin{cases} 2 & \text{if } n \ge 1, \\ 1+n & \text{if } n < 1, \end{cases}$$

we find an $O(h^q)$ error bound. □

Note that in a certain sense the theorem above deals with the "optimal" situation: The function that we approximate in our process is $f(\cdot, y(\cdot)) = D^n_{*0}y$. In order to obtain very good error bounds, we need to make sure that the quadrature errors for this function are (asymptotically) as small as possible. A sufficient condition for this to hold is, as is well known from quadrature theory [19], that this function is in C^2 on the interval of integration. This is precisely the setting discussed in Theorem C.4. So this theorem shows us what kind of performance the Adams method can give under optimal circumstances, and it also states sufficient conditions for such results to hold.

An objection against the use of this Adams–Bashforth–Moulton scheme may be the very slow rate of convergence if n is close to 0. However, a careful inspection of the proof of the error bound reveals a fact that is well known in the error analysis for methods of this structure for first-order equations (cf., e.g., [179]): The application of the corrector formula improves the accuracy of its input (the predictor) by a factor of h^n until an order of $O(h^2)$ (i.e. a saturation) is reached. Thus we may replace the plain PECE structure by a P(EC)$^\mu$E method, i.e. by introducing additional corrector iterations.

Remark C.4. An interesting observation here is that by choosing a larger number of corrector iterations, we essentially leave the computational complexity unchanged: A corrector iteration is of the form

$$y_{j+1}^{[\ell]} = \sum_{r=0}^{\lceil n\rceil-1} \frac{t_{j+1}^r}{r!} y_0^{(r)} + \frac{h^n}{\Gamma(n+2)} f(t_{j+1}, y_{j+1}^{[\ell-1]}) + \frac{h^n}{\Gamma(n+2)} \sum_{r=0}^{j} a_{r,j+1} f(t_r, y_r),$$

cf. (C.15). Here $y_{j+1}^{[\ell]}$ denotes the approximation after ℓ corrector steps, $y_{j+1}^{[0]} = y_{j+1}^{\mathrm{P}}$ is the predictor, and $y_{j+1} := y_{j+1}^{[\mu]}$ is the final approximation after μ corrector steps that we actually use. We can rewrite this as

$$y_{j+1}^{[\ell]} = \beta_{j+1} + \frac{h^n}{\Gamma(n+2)} f(t_{j+1}, y_{j+1}^{[\ell-1]})$$

where

$$\beta_{j+1} = \sum_{r=0}^{\lceil n\rceil-1} \frac{t_{j+1}^r}{r!} y_0^{(r)} + \frac{h^n}{\Gamma(n+2)} \sum_{r=0}^{j} a_{r,j+1} f(t_r, y_r)$$

is independent of ℓ. Thus the total arithmetic complexity of the corrector part of the $(j+1)$st step (taking us from t_j to t_{j+1}) is $O(j)$ for the calculation of β_{j+1} plus $O(\mu)$ for the μ corrector steps, which (since μ is constant) is asymptotically the same as the complexity in the case $\mu = 1$.

For the error of the scheme outlined in Remark C.4 we find, as indicated above, by a repeated application of the considerations of the proof of Theorem C.4 (see [36] for details):

Theorem C.5. *Under the assumptions of Theorem C.4, the approximation obtained by the $P(EC)^\mu E$ method described above satisfies*

$$\max_{0\le j\le N} |y(t_j) - y_j| = O(h^q)$$

where $q = \min\{2, 1+\mu n\}$.

For the moment we leave the topic of general P(EC)$^\mu$E methods in favour of a more detailed investigation of its special case $\mu = 1$, i.e. the original Adams–Bashforth–Moulton method introduced in (C.19) and (C.15). In this context we note an apparent disadvantage in the formulation of the hypotheses of the theorems above: They are stated in terms of the solution y (or, more precisely, its Caputo derivative of order n), which is unknown in general. Even though it is sometimes possible to determine the smoothness properties of $D_{*0}^n y$ from the given data, there still is some need for a corresponding error theory for the Adams method under assumptions formulated directly in terms of the given data, i.e. in terms of the function f. Such results will be the derived later in this section.

Before we come to those results however, we want to give some more information under assumptions similar to those of Theorem C.4. Specifically we want to state the conjecture that the error of our scheme, taken at a fixed abscissa, possesses an asymptotic expansion in powers of the step size h under additional smoothness conditions on $D^n_{*0}y$. If this were true (as, e.g., numerical results in Example C.1 below indicate), we could construct a Richardson extrapolation algorithm based on the Adams method in a way similar to the construction in [53] which is based on the scheme of [34]. The use of this extrapolation procedure then would permit us to obtain more accurate numerical approximations for the desired solution.

Conjecture C.1. *Let $n > 0$ and assume that $D^n_{*0}y \in C^k[0,T]$ for some $k \geq 3$ and some suitable T. Then,*

$$y(T) - y_{T/h} = \sum_{j=1}^{k_1} c_j h^{2j} + \sum_{j=1}^{k_2} d_j h^{j+n} + O(h^{k_3})$$

where k_1, k_2 and k_3 are certain constants depending only on k and satisfying $k_3 > \max(2k_1, k_2+n)$.

Notice that the asymptotic expansion begins with an h^2 term and continues with h^{1+n} for $1 < n < 3$, whereas it begins with h^{1+n}, followed by h^2, for $0 < n < 1$.

Example C.1. Consider the equation

$$D^n_{*0}y(x) = \frac{40320}{\Gamma(9-n)}x^{8-n} - 3\frac{\Gamma(5+n/2)}{\Gamma(5-n/2)}x^{4-n/2} + \frac{9}{4}\Gamma(n+1) + \left(\frac{3}{2}x^{n/2} - x^4\right)^3 - [y(x)]^{3/2}$$

for $x \in [0,1]$ with homogeneous initial conditions ($y(0) = 0$, $y'(0) = 0$; the latter only in the case $n > 1$).

The exact solution of this initial value problem is

$$y(x) = x^8 - 3x^{4+n/2} + \frac{9}{4}x^n,$$

and hence

$$D^n_{*0}y(x) = \frac{40320}{\Gamma(9-n)}x^{8-n} - 3\frac{\Gamma(5+n/2)}{\Gamma(5-n/2)}x^{4-n/2} + \frac{9}{4}\Gamma(n+1),$$

i.e. $D^n_{*0}y \in C^2[0,1]$ if $n \leq 4$, and thus the conditions of Theorem C.4 are fulfilled. Moreover, assuming that Conjecture C.1 holds, the application of Richardson extrapolation is also justified. We display some of the results in Tables C.1 and C.2 where, e.g., the notation $-5.53(-3)$ stands for $-5.53 \cdot 10^{-3}$. In each case, the leftmost column shows the step size used, the following column gives the error of our

Table C.1 Errors for Example C.1 with $n = 1.25$, taken at $x = 1$

Step size	Error of Adams scheme	Extrapolated values			
1/10	−5.53(−3)				
1/20	−1.59(−3)	−2.80(−4)			
1/40	−4.33(−4)	−4.60(−5)	1.63(−5)		
1/80	−1.14(−4)	−8.17(−6)	1.90(−6)	2.13(−7)	
1/160	−2.97(−5)	−1.54(−6)	2.24(−7)	2.71(−8)	1.47(−8)
1/320	−7.66(−6)	−3.04(−7)	2.56(−8)	2.28(−9)	6.24(−10)
1/640	−1.96(−6)	−6.16(−8)	2.85(−9)	1.73(−10)	3.25(−11)
EOC	1.97	2.30	3.17	3.72	4.26

Table C.2 Errors for Example C.1 with $n = 0.25$, taken at $x = 1$

Step size	Error of Adams scheme	Extrapolated values			
1/10	2.50(−1)				
1/20	1.81(−2)	−1.50(−1)			
1/40	3.61(−3)	−6.91(−3)	4.09(−2)		
1/80	1.45(−3)	−1.10(−4)	2.16(−3)	−8.15(−3)	
1/160	6.58(−4)	8.19(−5)	1.46(−4)	−3.89(−4)	1.28(−4)
1/320	2.97(−4)	3.49(−5)	1.92(−5)	−1.45(−5)	1.05(−5)
1/640	1.31(−4)	1.12(−5)	3.37(−6)	−8.50(−7)	6.01(−8)
EOC	1.18	1.63	2.51	4.09	7.44

scheme at $x = 1$, and the columns after that give the extrapolated values. The bottom line (marked "EOC") states the experimentally determined order of convergence for each of the columns on the right of the table. According to our theoretical considerations, these values should be $1+n$, 2, $2+n$, $3+n$, 4, $4+n$, ... in the case $0 < n < 1$ and 2, $1+n$, $2+n$, 4, $3+n$, $4+n$, ... for $1 < n < 2$. The numerical data in the following tables show that these values are reproduced approximately at least for $n > 1$ (see Table C.1). In the case $0 < n < 1$, displayed in Table C.2, the situation seems to be less obvious. Apparently, we need to use much smaller values for h than in the case $n > 1$ before we can see that the asymptotic behaviour really sets in. This would normally correspond to the situation that the coefficients of the leading terms are small in magnitude compared to the coefficients of the higher-order terms.

Our belief in the truth of Conjecture C.1 is not only supported by the numerical results but also by the results of de Hoog and Weiss [32, §5] who show that asymptotic expansions of this form hold if we use the fractional Adams–Moulton method (i.e. if we solve the corrector equation exactly) and that a similar expansion can be derived for the fractional Adams–Bashforth method (using the predictor as the final approximation rather than correcting once with the Adams–Moulton formula). For the moment however, we leave the question of the influence of the corrector step (that combines the two approaches) on this expansion open.

Rather, we turn our attention to another related problem. In the previous theorems we had formulated our hypotheses in the form of smoothness assumptions on $D^n_{*0}y$. Now we want to replace this by similar assumptions on y itself. In view of Theorem 3.15 we must be aware of the fact that smoothness of y in general implies non-smoothness of $D^n_{*0}y$ (the function that we have to approximate), so some difficulties are likely. Fortunately Theorem 3.15 also informs us about the precise nature of the singularities in the derivatives of $D^n_{*0}y$. We can exploit this information to obtain the following results.

Theorem C.6. *Let $n > 1$ and assume that $y \in C^{1+\lceil n\rceil}[0,T]$ for some suitable T. Then,*

$$\max_{0\le j\le N} |y(t_j) - y_j| = O(h^{1+\lceil n\rceil - n}).$$

Proof. By Theorem 3.15 we find that $D^n_{*0}y(x) = cx^{\lceil n\rceil - n} + g(x)$ where $g \in C^1[0,T]$ and g' fulfils a Lipschitz condition of order $\lceil n\rceil - n$. Thus, according to Theorems C.1 and C.2 we can apply Lemma C.3 with $\gamma_1 = 0$, $\gamma_2 = n - 1 > 0$, $\delta_1 = 1$ and $\delta_2 = 1 + \lceil n\rceil - n$. Because of $n > 1$ we then find that $\delta_1 + n = 1 + n > 2 > \delta_2$, and hence $\min\{\delta_1 + n, \delta_2\} = \delta_2$. So the overall error bound is $O(h^{\delta_2})$. □

Notice that a reformulation of Theorem C.6 yields that, if $1 < n = k_1 + k_2$ with $k_1 \in \mathbb{N}$ and $0 < k_2 < 1$, then the error is $O(h^{2-k_2})$. Thus the fractional part of n plays the decisive role for the order of the error. In particular, we find slow convergence if the fractional part of n is large. Consequently, under these assumptions we cannot expect the convergence order to be a monotone function of n any more. Nevertheless we can prove that the method converges for all $n > 0$:

Theorem C.7. *Let $0 < n < 1$ and assume that $y \in C^2[0,T]$ for some suitable T. Then, for $1 \le j \le N$ we have*

$$|y(t_j) - y_j| \le Ct_j^{n-1} \times \begin{cases} h^{1+n} & \text{if } 0 < n < 1/2, \\ h^{2-n} & \text{if } 1/2 \le n < 1, \end{cases} \tag{C.23}$$

where C is a constant independent of j and h.

We obtain two immediate consequences.

Corollary C.8. *Under the assumptions of Theorem C.7, we have*

$$\max_{0\le j\le N} |y(t_j) - y_j| = \begin{cases} O(h^{2n}) & \text{if } 0 < n < 1/2, \\ O(h) & \text{if } 1/2 \le n < 1. \end{cases}$$

Moreover, for every $\varepsilon \in (0,T)$ we have

$$\max_{t_j\in[\varepsilon,T]} |y(t_j) - y_j| = \begin{cases} O(h^{1+n}) & \text{if } 0 < n < 1/2, \\ O(h^{2-n}) & \text{if } 1/2 \le n < 1. \end{cases}$$

Proof (of Theorem C.7). The first steps of the proof are as in the proof of Theorem C.6. The key difference is that now $\gamma_2 < 0$ (note that we still have $\gamma_2 = n-1$, but now $n < 1$). Thus we cannot apply Lemma C.3. Instead we modify its proof so that it fits to our requirements: We keep the inductive structure and remember that our claim is now (C.23) rather than (C.21). With this change in the induction hypothesis we proceed much as in the proof of Lemma C.3. However, because of this new hypothesis, we now have to estimate terms of the form $\sum_{j=1}^{k-1} b_{j,k+1} t_j^{\gamma_2}$ and $\sum_{j=1}^{k-1} a_{j,k+1} t_j^{\gamma_2}$. By the Mean Value Theorem we have $0 \le b_{j,k+1} \le h^n (k-j)^{n-1}$ and $0 \le a_{j,k+1} \le c h^n (k-j)^{n-1}$ for $1 \le j \le k-1$ (where the constant c is independent of j and k), respectively, so that the problem reduces to finding a bound for $S_k := \sum_{j=1}^{k-1} j^{\gamma_2} (k-j)^{n-1}$. Under our assumptions, both the exponents γ_2 and $n-1$ are in the interval $(-1,0)$, and then it is easily seen that $S_k = O(k^{\gamma_2+n})$. Using this relation we can complete the proof of Theorem C.7 by following along the lines of the rest of the proof of Lemma C.3. □

We conclude the discussion of error bounds with a result where we formulate the hypotheses in terms of the given data and not in terms of the unknown solution. We give a result in the case $n > 1$ and later discuss properties of the numerical scheme when $n < 1$.

Theorem C.9. *Let $n > 1$. Then, if $f \in C^2(G)$,*

$$\max_{0 \le j \le N} |y(t_j) - y_j| = O(h^2).$$

Proof (of Theorem C.9). We begin by discussing the case $n \ge 2$. Then, according to Theorem 6.25, we find that $y \in C^2[0,T]$. Thus, in view of the smoothness assumption on f and the chain rule, $D_{*0}^n y := f(\cdot, y(\cdot)) \in C^2[0,T]$ too, and the claim follows by virtue of Theorem C.4.

For the case $1 < n < 2$, we want to apply Lemma C.3 and hence we have to determine the constants $\gamma_1, \gamma_2, \delta_1$ and δ_2 in its hypotheses. In order to do so we need more precise information about the behaviour of y. This information can be found in Theorem 6.38 which asserts that $y(x) = cx^n + \psi(x)$ with some $c \in \mathbb{R}$ and some $\psi \in C^2[0,T]$. This implies, in particular, that $y \in C^1[0,T]$. As in the case $n > 2$ above we can then deduce $D_{*0}^n y \in C^1[0,T]$ too, and by Theorem C.1(a), we find that we may choose $\gamma_1 = n$ and $\delta_1 = 1$. Moreover, using again the fact that $y(x) = cx^n + \psi(x)$ with some $c \in \mathbb{R}$ and some $\psi \in C^2[0,T]$ and applying Theorem C.2(a) and (c), we determine the correct values for the remaining quantities as $\gamma_2 = \min\{n, 2n-2\} = 2n-2 \ge 0$ and $\delta_2 = 2$. The claim then follows from Lemma C.3. □

In the case $n < 1$ the situation seems to be less clear. According to the theorems presented at the end of Sect. 6.4, smoothness conditions on f imply that the exact solution is of the form

$$y(t) = \psi(t) + \sum_{\nu=1}^{\hat{\nu}} c_\nu t^{\nu n} + \sum_{\nu=1}^{\tilde{\nu}} d_\nu t^{1+\nu n}$$

where ψ is twice differentiable. The first sum consists of terms which are not differentiable, and the second sum is of terms that are differentiable once but not twice. As remarked by Lubich [120] it seems unlikely that numerical schemes will be rapidly convergent over any interval that contains the origin. Indeed we can prove that the error $y(t_1) - y_1$ of the approximation after just one step behaves as $O(h^{2n})$ if $f \in C^2(G)$. Simple numerical experiments indicate that this result cannot be improved. However this error introduced in the initial phase is transient and from numerical results reported in [49, Table 4.5] we believe the following conjecture to be true.

Conjecture C.2. *Let* $0 < n < 1$. *Then, if* $f \in C^3(G)$, *for every* $\varepsilon > 0$ *we have*

$$\max_{t_j \in [\varepsilon, T]} |y(t_j) - y_j| = O(h^{1+n}).$$

C.2 Numerical Schemes for Multi-Term Equations

We now come to the extension of the numerical methods discussed in the previous section to multi-term equations. The most important theoretical properties of these multi-term equations have been discussed in Chap. 8. As in that chapter we restrict our attention to equations of the form

$$D_{*0}^{n_k} y(x) = f(x, y(x), D_{*0}^{n_1} y(x), D_{*0}^{n_2} y(x), \ldots, D_{*0}^{n_{k-1}} y(x)) \tag{C.24a}$$

(where $0 < n_1 < n_2 < \ldots < n_k$) with a suitable function f and initial conditions

$$y^{(j)}(0) = y_0^{(j)}, \qquad j = 0, 1, \ldots, \lceil n_k \rceil - 1. \tag{C.24b}$$

For initial value problems of this type we shall discuss the various approaches introduced in Chap. 8 and find out their respective advantages and disadvantages.

Conversion to Single-Order Systems

Our first attempt consists in a direct application of the result of Theorem 8.1 or 8.2 (depending on whether $n_k > 1$ or not) to the given initial value problem. In this way we transform the given initial value problem into a system of equations of the form

$$\begin{aligned} D^{\gamma}_{*0} y_0(x) &= y_1(x), \\ D^{\gamma}_{*0} y_1(x) &= y_2(x), \\ &\vdots \\ D^{\gamma}_{*0} y_{N-2}(x) &= y_{N-1}(x), \\ D^{\gamma}_{*0} y_{N-1}(x) &= f(x, y_0(x), y_{n_1/\gamma}(x), \ldots, y_{n_{k-1}/\gamma}(x)), \end{aligned} \tag{C.25a}$$

together with the initial conditions

$$y_j(0) = \begin{cases} y_0^{(j\gamma)} & \text{if } j\gamma \in \mathbb{N}_0, \\ 0 & \text{else,} \end{cases} \tag{C.25b}$$

with the precise choice of the new parameters γ and N being according to Theorems 8.1 or 8.2, as appropriate. We have thus formally obtained an equation of the type

$$D^{\gamma}_{*0} Y(x) = F(x, Y(x)), \qquad Y(0) = Y_0, \tag{C.26}$$

with certain vector-valued functions F (known) and Y (unknown) and an initial condition vector Y_0, i.e. a single-term equation of order γ with vector-valued data. Thus we may apply any numerical method for such single-term equations and calculate an approximate solution for this system; for the sake of simplicity we shall restrict ourselves to the Adams–Bashforth–Moulton scheme developed above. The first component of the (numerical) solution vector (with index 0) is then the required approximate solution for the original equation. We illustrate the procedure by a simple example taken from [44].

Example C.2. Solve the Bagley–Torvik equation

$$Ay''(x) + BD^{3/2}_{*0} y(x) + Cy(x) = C(x+1)$$

(where $A \neq 0$ and $B, C \in \mathbb{R}$) with initial conditions

$$y(0) = y'(0) = 1$$

with the approach described above.

It is easily verified that the exact solution of this problem is

$$y(x) = x + 1$$

independent of the choice of the coefficients A, B, and C. Thus, the resulting system is

$$D^{1/2}_{*0} \begin{pmatrix} y_0(x) \\ y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \frac{1}{A} \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ -By_3(x) - Cy_0(x) + C(x+1) \end{pmatrix},$$

$$\begin{pmatrix} y_0(0) \\ y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We have solved this problem on the interval $[0,5]$ with the Adams–Bashforth–Moulton scheme. The numerical results at $x = 5$ were as shown in Table C.3.

These data indicate that the convergence behaviour is $O(h^{3/2})$. To understand how this relates to the error estimates of Sect. C.1, we must recall that we now construct an approximate solution for the entire system and not just for its first component. Thus we see that, as a by-product of the method, we do not only obtain information on y but also on its fractional derivatives of order $\gamma, 2\gamma, \ldots, (N-1)\gamma$. Depending on the task at hand this information can be anything between highly useful and absolutely unnecessary. In any case the error estimate is dominated by the worst error estimate of the four components. Since the exact solution for the system is $(x+1, x^{1/2}/\Gamma(3/2), 1, 0)^{\mathrm{T}}$ we see that the second, third and fourth components of the exact solution have smooth derivatives of order $1/2$; thus they may be approximated with order $O(h^{3/2})$ according to Theorem C.4. The first component is smooth itself; it allows an application of Corollary C.8 that gives an $O(h)$ error estimate on the full interval $[0,T]$ and an $O(h^{3/2})$ estimate on each interval of the form $[\varepsilon, T]$ with $\varepsilon > 0$. Since the latter case covers the problem considered in Table C.3, we have agreement between theoretical and numerical results.

In order to explain the weaknesses of this concept, we look at a second example, already considered in [45] and [35].

Example C.3. Solve the nonlinear three-term equation

$$D_{*0}^{1.455} y(x) = -x^{0.1} \frac{E_{1.545}(-x)}{E_{1.445}(-x)} \mathrm{e}^x y(x) D_{*0}^{0.555} y(x) + \mathrm{e}^{-2x} - [D_{*0}^1 y(x)]^2 \qquad \text{(C.27)}$$

for $0 \le x \le 1$, equipped with the initial conditions $y(0) = 1$ and $y'(0) = -1$, with the same algorithm.

The exact solution of this problem is $y(x) = \exp(-x)$. When applying our idea to this equation, we first need to calculate the order γ of the new system as described in Theorem 8.1. In our case the result is $\gamma = 1/200$, and hence the dimension of the resulting system is $N = 1.455/\gamma = 291$ – a rather large number. In a first attempt we have tried to solve the system with the Adams–Bashforth–Moulton scheme as

Table C.3 Bagley–Torvik equation solved with Adams method

Step size	Numerical solution	Error	Estimated order of convergence
0.5	6.15131473519232	−0.15131473519232	
0.25	6.04684102179946	−0.04684102179946	1.69
0.125	6.01602947553912	−0.01602947553912	1.55
0.0625	6.00562770408881	−0.00562770408881	1.51

Table C.4 Numerical results for Example C.3 (system with $N = 291$, $\gamma = 0.005$) using the standard PECE-type Adams algorithm

Step size	Maximal error	Run time
$1/200$	0.3904	101.2 s
$1/400$	0.2193	368.4 s
$1/800$	0.1164	1358.0 s
$1/1{,}600$	0.0600	5017.4 s

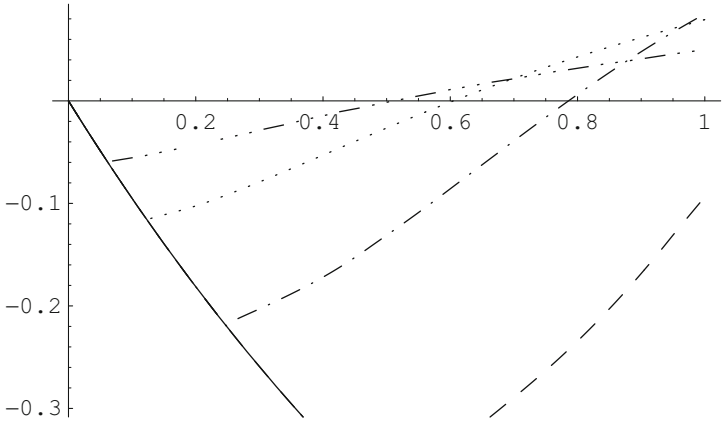

Fig. C.1 Approximation errors for Example C.3 with $h = 1/100$ (*solid line*), $h = 1/200$ (*dashed line*), $h = 1/400$ (*dot–dashed line*), $h = 1/800$ (*dotted line*), $h = 1/1{,}600$ (*dashed and triple-dotted line*)

above. The resulting errors are reported in Table C.4 and Fig. C.1. For the purpose of comparison with later methods we have also included information about the run time of the algorithm on a standard 500 MHz Pentium PC in double precision arithmetic.

It is clearly seen that the results for $h = 1/100$ and $h = 1/200$ are totally unacceptable. There is a simple explanation for this phenomenon which becomes evident when one takes a look at the numerical solution of the initial value problem and not at the approximation error: In each case the first 99 steps of the algorithm do not change the first component of the approximate solution. In other words we get stuck at the initial value instead of following the exact solution. The reason for this behaviour can be found in the structure of the function F on the right-hand side of the system (C.26) and of its initial condition: The first component (index 0) is y_0, the component with index 200 ($= 1/\gamma$) is y_0' ($\neq 0$), and all the components in between vanish. The interaction of the Adams–Bashforth–Moulton method with the function F now implies that the non-zero component is propagated by one row in each predictor step and another row in each corrector step, so in the first 99 steps only a total of 198 zeros are added to the initial value of the leading component of the numerical solution. The last (199th) zero is then used in the predictor of the 100th step, and the corrector of step 100 is actually the first operation where a non-zero entry reaches the first component of the solution. Thus we always have an initial

interval of $2/\gamma - 1$ steps where the numerical solution is constant before it can start to make progress towards the exact solution. Notice that in our Bagley–Torvik example above we had $\gamma = 1/2$, and so (since here $2/\gamma - 1 = 3$) the effect is negligible. This means that, if one wants to keep the structure of the algorithm and the uniform step size, then the only way to reduce this effect is a drastic reduction of the step size h, essentially by the rule $h = O(\gamma)$. A look at Fig. C.1 confirms this observation.

Let us briefly summarize what we have achieved so far: The approach described above has the advantage of producing a system of equations with a very simple structure. As a consequence of this structure, numerical schemes for this system can be implemented on a parallel computer architecture in a rather efficient way. However we have also seen some disadvantages:

(a) The method only works in the case of commensurate multi-term equations (if $n_k \leq 1$) or under the even more restrictive assumption that all the n_j are rational (if $n_k > 1$)
(b) The dimension of the system can be very large (depending on the precise values of the parameters of the original equation); this can lead to very long run times on sequential machines
(c) The structure of the initial conditions of the new system can be problematic for some types of numerical algorithms; in particular we may be forced to use excessively small step sizes

In order to overcome these problems (at least partially), we propose two possible strategies. The first one, taken from [45], is a slight refinement of the idea used so far that works in two stages. We recall that the problem is mainly due to the large dimension of the system, and this in turn is a consequence of the size of the greatest common divisor of the orders of the differential operators in the given system. Therefore we introduce a stage of preliminary manipulations before actually starting the numerical algorithm.

This first stage consists of replacing the given initial value problem (C.24) by a new differential equation

$$D_{*0}^{\tilde{n}_k}\tilde{y}(x) = f(x, \tilde{y}(x), D_{*0}^{\tilde{n}_1}\tilde{y}(x), D_{*0}^{\tilde{n}_2}\tilde{y}(x), \ldots, D_{*0}^{\tilde{n}_{k-1}}\tilde{y}(x)) \tag{C.28}$$

with identical initial conditions (C.24b). We thus perturb the orders of the differential operators, but all other parameters of the given problem (the function f on the right-hand side and the initial conditions) remain unchanged.

The essence of this idea is that, according to Theorem 8.8, the exact solution $\tilde{y}$ of this new initial value problem and the exact solution y of the original problem differ only by

$$\|y - \tilde{y}\|_\infty = O\left(\max_{j=1,2,\ldots,k} |n_j - \tilde{n}_j|\right). \tag{C.29}$$

Here by $\|\cdot\|_\infty$ we denote the Chebyshev norm taken over a suitable finite interval $[0, T]$, say, where both problems have a solution.

In order to exploit the capabilities of this approach, we need to choose the new parameters $\tilde{n}_1, \ldots, \tilde{n}_k$ in such a way that they have the following three properties:

(a) $\tilde{n}_1, \ldots, \tilde{n}_k \in \mathbb{Q}$
(b) The least common multiple of the denominators of $\tilde{n}_1, \ldots, \tilde{n}_k$ is small
(c) $\max_j |n_j - \tilde{n}_j|$ is small

Here condition (a) asserts that a conversion of (C.28) to a single-term system (as described in Chap. 8) is possible. Specifically, since only the new values $\tilde{n}_j$ enter the later stages of the scheme, such a conversion is always possible, without any restrictions on the original values n_j. The purpose of condition (b) is to keep the dimension N of this system small (remember that in Theorem 8.1 we had seen that essentially $N = Mn_k$ where M is the least common multiple mentioned in condition (b)), which – according to our initial idea – was the main point of the concept. Condition (c) finally makes sure that the error introduced by this perturbation remains small, cf. (C.29).

It must be noted of course that there is a conflict between conditions (b) and (c): In many cases it will be possible to improve the approximation required in (c) at the price of increasing the least common multiple mentioned in (b). A proper compromise must be found in this case. It seems to be impossible however to state a generally valid strategy for the solution of this conflict; a good compromise will likely depend on the specific parameters of the equation under consideration.

This completes the first stage of the algorithm. At the end of this stage we have found a new initial value problem that consists of the perturbed differential equation (C.28) together with the original (unperturbed) initial conditions (C.24b).

The second stage of the algorithm is then the stage where the initial value problem that was constructed in stage 1 will be solved numerically. In practice we will first use the approach described at the beginning of this section: We convert the new initial value problem into a single-term system, and then we will solve this system numerically (for example by means of the Adams method).

Example C.4. Construct an approximate solution for the problem from Example C.3 by the two-stage strategy outlined above.

As a first attempt to solve the problem with our refined method, we approximate (C.27) by

$$D_{*0}^{1.5}\tilde{y}(x) = -x^{0.1}\frac{E_{1.545}(-x)}{E_{1.445}(-x)}\mathrm{e}^{x}\tilde{y}(x)D_{*0}^{0.5}\tilde{y}(x) + \mathrm{e}^{-2x} - [D_{*0}^{1}\tilde{y}(x)]^2, \tag{C.30}$$

convert (C.30) to a three-dimensional system of order $\gamma = 0.5$, and solve this system numerically with the Adams method in its standard form using various step sizes. The results are described in Table C.5.

Table C.5 Numerical results of first approximation ($N = 3$, $\gamma = 0.5$)

Step size	Maximal error	Run time
1/10	0.136	0.07 s
1/20	0.124	0.18 s
1/40	0.118	0.56 s

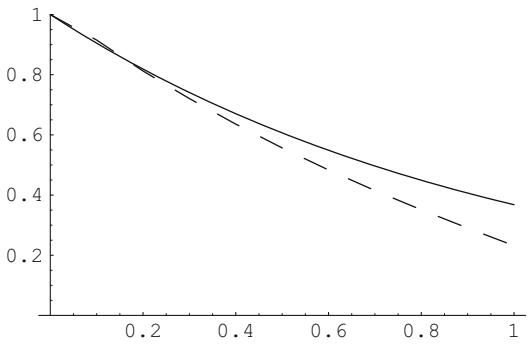

Fig. C.2 Exact solution and first approximation ($N = 3$, $\gamma = 0.5$, step size $h = 0.1$)

We can see that there is almost no improvement when we change the step size from $1/20$ to $1/40$. This indicates that the error of the Adams scheme (i.e. the error introduced in the second stage) is already very small compared to the error of the first stage (i.e. the error introduced by perturbing the differential equation). Therefore there is no need to look for an improved scheme for the solution of this simple system. Note in particular (see Fig. C.2) that even the crudest of these three approximations (the dashed line) gives a *qualitatively* correct picture of the exact solution (the solid line). Certainly this cannot be said to be true for our plain and simple first approach discussed above.

In order to obtain a better approximation with our method we must now reduce the error of stage 1, i.e. we need to introduce smaller perturbations in the orders of the differential operators. We thus try to approximate the given equation (C.27) not by (C.30) but by

$$D^{1.45}_{*0}\tilde{y}(x) = -x^{0.1}\frac{E_{1.545}(-x)}{E_{1.445}(-x)}\mathrm{e}^{x}\tilde{y}(x)D^{0.55}_{*0}\tilde{y}(x) + \mathrm{e}^{-2x} - [D^{1}_{*0}\tilde{y}(x)]^2 \tag{C.31}$$

and proceed as above. Consequently we find that we have to solve a 29-dimensional system of order 0.05 numerically. This task is (in particular due to the nature of the initial conditions) much more difficult than the previous one, and therefore we need to put more effort into the numerical scheme. For the moment we interpret this requirement as a demand for a smaller step size; an alternative will be considered later. The results are given in Table C.6.

For the purpose of comparison with the previous example we have included the case of a step size of $1/40$. As can be seen by comparing Tables C.5 and C.6, the error is much larger now than it was before. The reason is the problem that we mentioned above: Since the dimension of the system has been increased, the numerical solution needs more time to get away from the initial value. An even more obvious picture of the situation appears when we look at the graphical data provided in Fig. C.3. Here again the solid line is the exact solution, the other lines correspond to the numerical solutions (dashed line: $h = 1/40$; dash-dotted line: $h = 1/100$;

Table C.6 Numerical results of second approximation ($N = 29$, $\gamma = 0.05$)

Step size	Maximal error	Run time
$1/40$	0.2015	0.9 s
$1/100$	0.0861	5.5 s
$1/200$	0.0440	21.5 s
$1/400$	0.0222	82.4 s

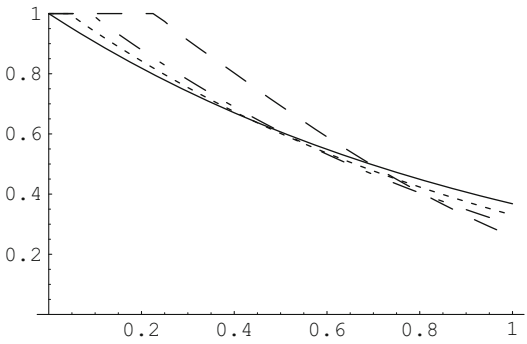

Fig. C.3 Exact solution and second approximation ($N = 29$, $\gamma = 0.05$, various step sizes)

dotted line: $h = 1/200$). We thus have to say that the graph for $h = 1/40$ does *not* give a qualitatively correct picture of the true solution.

As pointed out above, we will sometimes be forced to choose the parameters in such a way that the dimension of the system is larger than desirable. In this context the present 29-dimensional system may be considered to be such a case. That means that also the number of zeros in the initial condition of the resulting system is larger than one would like it to be, which forces us to use a very small step size.

This is where our second strategy mentioned above comes into play as a possible alternative. Specifically, it may be useful to replace the plain PECE structure by a P(EC)$^\mu$E method (i.e. by introducing additional corrector iterations) as described in Theorem C.5. This allows for a quicker propagation of the non-zero elements, and it may be possible to avoid the use of excessively small step sizes. We shall provide a numerical example now. This flexibility in the number of corrector steps is actually one of the main reasons why we suggest the Adams scheme and not, e.g., the method of [34]. Recall that, as derived in Remark C.4, by (for example) doubling the number of corrector iterations, we essentially leave the computational complexity unchanged.

If we would use the other option and reduce the step size by a factor of two, then the run time would increase by a factor of four because the complexity of the algorithm is $O(h^{-2})$. Both approaches would reduce the size of the initial interval where the numerical solution gets stuck at the initial value by a factor of $1/2$.

The data obtained by our P(EC)$^\mu$E approach are given in Table C.7. Note that the data of Table C.6 correspond to this method with $\mu = 1$.

It is clearly seen that there is a significant advantage in this approach: By choosing $\mu = 10$ and $h = 1/40$ for example, we obtain an absolute error that is about 25% smaller than in the case $\mu = 1$ and $h = 1/200$, and at the same time the run time is

Table C.7 Numerical results of second approximation ($N = 29$, $\gamma = 0.05$) as in (C.31) with P(EC)$^{\mu}$E algorithm

Number μ of corrector iterations	Step size	Maximal error	Run time
10	$1/40$	0.03175	4.9 s
10	$1/100$	0.01174	28.6 s
20	$1/40$	0.00989	9.3 s
20	$1/100$	0.00379	55.0 s

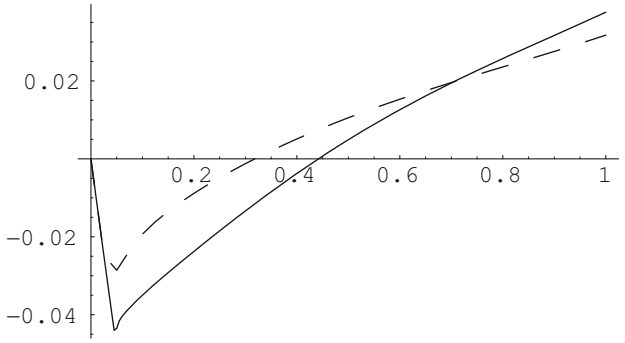

Fig. C.4 Errors for second approximation ($N = 29$, $\gamma = 0.05$) as in (C.31) with various combinations of step size and number of corrector steps

75% shorter. The reason is the following. In the case $\mu = 1$ the numerical solution gets stuck at the initial value for a rather long interval. At the end of this interval the true solution has moved away significantly from the initial value, and here the error attains its maximum. Over the remainder of the interval $[0, 1]$ the numerical solution then has to creep towards the exact solution, and the error gets smaller. If we choose a larger value for μ, we make the problematic initial interval smaller, and therefore we also diminish the error attained over this interval. This is apparent from Fig. C.4 where we have compared the absolute errors for $\mu = 1$, $h = 1/200$ (solid line) and $\mu = 10$, $h = 1/40$ (dashed line).

In this example one can of course now apply the idea of using many corrector steps also to the given equation (C.27) itself. This is equivalent to skipping stage 1 of our two-stage process. It is clear that the run times of the plain PECE scheme (see Table C.4) are not competitive. Therefore we once again revert to the P(EC)$^{\mu}$E structure with larger values for μ and larger step sizes as before. Some results are stated in Table C.8.

Comparing Tables C.4 and C.8 we once again find a significant run time advantage in the P(EC)$^{\mu}$E method as compared to the PECE method without losing accuracy, but even the approximations obtained by the faster P(EC)$^{\mu}$E approach are less accurate and more time consuming than the results presented in Table C.7 where we had used a simpler differential equation system.

Table C.8 Numerical results for unperturbed equation ($N = 291$, $\gamma = 0.005$), with P(EC)$^{\mu}$E algorithm

Number μ of corrector iterations	Step size	Maximal error	Run time
10	1/100	0.16473	117.3 s
10	1/200	0.08607	446.8 s
20	1/100	0.08607	222.8 s
20	1/200	0.04400	811.0 s

Based on our theoretical considerations here and in Sect. C.1 and on heuristical arguments coming from the numerical results, we now give a complete description of a possible algorithm for the approximate solution of the initial value problem (C.24). The algorithm will follow the basic ideas outlined above. The fundamental concept is that we assume a bound on the complexity to be given (expressed in terms of the least common multiple of the denominators of the orders $\tilde{n}_j$) and that we try to achieve a high accuracy in the solution without exceeding the complexity limit.

Specifically, we assume that the user specifies a parameter $M \in \mathbb{N}$ which we interpret as an upper bound for the least common multiple of the denominators of $\tilde{n}_1, \ldots, \tilde{n}_k$. Since the dimension N of the system that we shall construct in stage 2 of the algorithm is given by $N = M\tilde{n}_k \approx Mn_k$, this data gives us an upper bound on the dimension and hence an upper bound on the arithmetic complexity.

We begin by constructing the perturbations required for the first stage. This is very simple; for $j = 1, 2, \ldots, k$ we only have to set $\tilde{n}_j := \alpha_j / M$ where $\alpha_j \in \mathbb{N}$ is chosen to be the natural number closest to Mn_j (i.e. $\alpha_j = \lfloor Mn_j + 0.5 \rfloor$). In this way we make sure that, for every single j, the quantity $|\tilde{n}_j - n_j|$ is minimized under the condition that the least common multiple of the denominators of $\tilde{n}_1, \ldots, \tilde{n}_k$ is bounded by M. This essentially completes the first stage.

The second stage begins by rewriting the perturbed equations as a system of order $\gamma = 1/M$ and dimension N as described in Theorem 8.1. This system is solved by the P(EC)$^{\mu}$E scheme indicated above. To avoid the problems caused by the large number of zeros in the new initial condition, we choose the parameter μ in a way that depends on the number of zeros (i.e. on M); specifically we set $\mu := M$ as suggested in [36]. Note that it follows from our considerations that it is neither necessary nor helpful to introduce additional flexibility by choosing different values for the parameter μ in each step. The choice that we propose here is sufficient to avoid the problems caused by the (possibly) large number of zeros in the initial condition. Choosing μ larger than this would not give a better order of accuracy, so there is no point in doing that (cf. the considerations on (C.32) below). Choosing μ smaller (permanently or temporarily) would mean that the problem cannot be avoided totally, so one would have to assume a deterioration of the approximation quality, but on the other hand it would not lead to a significantly faster algorithm because the arithmetic complexity of the entire scheme is (asymptotically) independent of μ.

Another advantage of the P(EC)$^\mu$E scheme with the choice of μ indicated above can be explained by a look at Theorem C.5: The algorithm converges to the true solution of the perturbed equation with an error of

$$\max_{j=0,1,\ldots,N} |y(t_j) - y_h(t_j)| = O(h^p) \quad \text{where} \quad p = \min(2, 1+\gamma\mu). \tag{C.32}$$

Since here we have $\mu = M = 1/\gamma$ by construction, this means that in the proposed scheme we actually have $p = 2$ in every case, so we find slightly better convergence behaviour than in the simple PECE approach; indeed this is the maximum order than one can possibly obtain by an algorithm that uses the approximation method underlying our scheme.

This approach is particularly useful when one is looking for a computationally inexpensive but still reasonably accurate approximation. In many applications this will be what is desired because often one needs to solve a great number of such initial value problems whose solutions are then required as input data for other problems. Additionally, high accuracy is frequently impossible to obtain anyway because the given data (in particular the orders n_j of the differential operators) are something like material constants known only up to a certain (usually moderate) precision.

Conversion to Multi-Order Systems

An alternative approach has been suggested in [59]. Two variants are possible; we begin with the one that is simpler to describe and discuss the other one later.

The basic idea is to use a completely different transformation of the given initial value problem into the form of a system of fractional differential equations. Essentially this amounts to replacing the path that uses Theorem 8.1 or 8.2 that we had used above by the transformation to a multi-order system according to Theorem 8.9 or Theorem 8.10. We begin with the former and, as above, assume that the original initial value problem is given in the form

$$D_{*0}^{n_k} y(x) = f(x, y(x), D_{*0}^{n_1} y(x), D_{*0}^{n_2} y(x), \ldots, D_{*0}^{n_{k-1}} y(x)) \tag{C.33a}$$

(where $0 < n_1 < n_2 < \ldots < n_k$) with a suitable function f and initial conditions

$$y^{(j)}(0) = y_0^{(j)}, \qquad j = 0, 1, \ldots, \lceil n_k \rceil - 1. \tag{C.33b}$$

However we now assume (without loss of generality) that additionally we have

$$\{1, 2, \ldots, \lceil n_k \rceil - 1\} \subset \{n_1, n_2, \ldots, n_k\}. \tag{C.34}$$

This implies $n_j - n_{j-1} \le 1$ for all j. Consider now the differences $d_j := n_j - n_{j-1}$ for $j = 1, 2, 3, \ldots, k$, where we have defined $n_0 := 0$. We then introduce the new functions

$$y_0 := y, \qquad y_j := D_{*0}^{d_j} y_{j-1} \quad (j = 1, 2, \ldots, k),$$

such that

$$\begin{aligned} y_1 &= D_{*0}^{d_1} y_0 = D_{*0}^{n_1-n_0} y_0 = D_{*0}^{n_1} y, \\ y_2 &= D_{*0}^{d_2} y_1 = D_{*0}^{n_2-n_1} D_{*0}^{n_1} y = D_{*0}^{n_2} y, \\ &\vdots \\ y_k &= D_{*0}^{d_k} y_{k-1} = D_{*0}^{n_k-n_{k-1}} D_{*0}^{n_{k-1}} y = D_{*0}^{n_k} y. \end{aligned}$$

For the derivation of these identities one can proceed as in the proof of Lemma 3.13; the key point is that – due to our assumption (C.34) – we never jump across an integer when moving from n_{j-1} to n_j. Thus we may rewrite (C.33a) in the form

$$\begin{aligned} D_{*0}^{d_1} y_0 &= y_1, \\ D_{*0}^{d_2} y_1 &= y_2, \\ &\vdots \\ D_{*0}^{d_{k-1}} y_{k-2} &= y_{k-1}, \\ D_{*0}^{d_k} y_{k-1} &= f(x, y_0(x), y_1(x), y_2(x), \dots, y_{k-1}(x)). \end{aligned} \tag{C.35a}$$

We have found the required system of differential equations; the corresponding initial values obviously have to be

$$y_j(0) = \begin{cases} y_0^{(n_j)} & \text{if } n_j \in \mathbb{N}_0, \\ 0 & \text{else} \end{cases} \tag{C.35b}$$

in view of the fact that $y_j = D_{*0}^{n_j} y$, Lemma 3.11 and the given initial values (C.33b). A comparison of this multidimensional initial value problem with its counterpart (C.25) constructed above reveals a number of substantial differences even though formally they are equivalent in the sense that the first components of the solutions of the two problems coincide:

(a) The dimension of the new system is k (a small number in typical applications), independent of the values of the n_j; we had seen above (see, e.g., Example C.3) that the other approach could give rise to systems of very large dimension even if k was small.

(b) The number of zeros in the initial condition (C.33b) relates to the number of zeros in its counterpart (C.25b) in the same way as the dimensions.

(c) The structure of the left-hand side of the new system (C.35a) is much more complicated than it was in the old system (C.25a).

(d) The formal structures of the right-hand sides of the two system do not differ from each other at all.

It turns out that, in view of the considerations with respect to the approach using single-order systems, the first two points mentioned above indicate the capability of the new approach to avoid the potentially serious problems encountered in the first approach. On the other hand, the third item reveals that we have to pay a certain price for this improvement: Instead of requiring an approximation for only one differential operator we now need to work with operators of order $d_1, d_2, \ldots, d_k$. These orders may or may not coincide with each other.

Example C.5. We rewrite the equations from Examples C.2 and C.3 as systems of equations according to the ideas outlined above.

For the Bagley–Torvik equation

$$Ay''(x) = -BD_{*0}^{3/2}y(x) - Cy(x) + C(x+1), \qquad y(0) = y'(0) = 1,$$

from Example C.2, we have

$$n_1 = 1, \quad n_2 = \frac{3}{2} \quad \text{and} \quad n_3 = 2,$$

such that

$$d_1 = 1, \quad d_2 = \frac{1}{2} \quad \text{and} \quad d_3 = \frac{1}{2}.$$

The resulting system thus is three-dimensional and reads

$$\begin{aligned} D_{*0}^{1}y_0(x) &= y_1(x), \\ D_{*0}^{1/2}y_1(x) &= y_2(x), \\ D_{*0}^{1/2}y_2(x) &= -\frac{B}{A}y_2(x) - \frac{C}{A}y_0(x) + \frac{C}{A}(x+1) \end{aligned}$$

with initial conditions

$$y_0(0) = y_1(0) = 1 \quad \text{and} \quad y_2(0) = 0.$$

A comparison with Example C.2 shows that the differences for this simple example are small: The dimension is reduced by one, and two different fractional derivatives appear on the left-hand side of the system.

In the other example, the equation was

$$\begin{aligned} D_{*0}^{1.455}y(x) = -x^{0.1}\frac{E_{1.545}(-x)}{E_{1.445}(-x)}\exp(x)y(x)D_{*0}^{0.555}y(x) \\ + \exp(-2x) - [D_{*0}^{1}y(x)]^2, \end{aligned}$$

with initial conditions $y(0) = 1$ and $y'(0) = -1$. Here the new approach uses the parameters

$$n_1 = 0.555, \quad n_2 = 1 \quad \text{and} \quad n_3 = 1.455,$$

such that

$$d_1 = 0.555, \quad d_2 = 0.445 \quad \text{and} \quad d_3 = 0.455.$$

Once again we obtain a three-dimensional system; this time it has the form

$$\begin{aligned}
D_{*0}^{0.555} y_0(x) &= y_1(x), \\
D_{*0}^{0.445} y_1(x) &= y_2(x), \\
D_{*0}^{0.455} y_2(x) &= -x^{0.1} \frac{E_{1.545}(-x)}{E_{1.445}(-x)} \exp(x) y_0(x) y_1(x) + \exp(-2x) - [y_2(x)]^2
\end{aligned}$$

with initial conditions

$$y_0(0) = 1, \quad y_1(0) = 0 \quad \text{and} \quad y_2(0) = -1.$$

Now the difference to the single-order system approach is enormous: The dimension of the system is reduced from 291 to 3, but of course we now have to work with three different differential operators.

Before we come to the question for a suitable numerical scheme for systems of this structure, let us briefly introduce a small modification of the idea presented so far that may lead to a slightly more efficient scheme. This is motivated by the observation from the two examples above that integer order differential operators that are local by nature are decomposed into two (or more) non-local fractional differential operators: In the Bagley–Torvik example we have $y'' = D_{*0}^{1/2} y_2 = D_{*0}^{1/2} D_{*0}^{1/2} y_1 = D_{*0}^{1/2} D_{*0}^{1/2} D_{*0}^{1} y_0$, and so in any approximation method the possibility to save time by making use of the locality is lost. A similar decomposition $y_2 = y' = D_{*0}^{0.445} y_1 = D_{*0}^{0.445} D_{*0}^{0.555} y_0$ is used in the other example. It would thus be preferable to use the alternative systems

$$\begin{aligned}
D_{*0}^{1} y_0(x) &= y_1(x), \\
D_{*0}^{1/2} y_1(x) &= y_2(x), \\
D_{*0}^{1} y_1(x) &= -\frac{B}{A} y_2(x) - \frac{C}{A} y_0(x) + \frac{C}{A}(x+1)
\end{aligned}$$

with initial conditions

$$y_0(0) = y_1(0) = 1 \quad \text{and} \quad y_2(0) = 0$$

for the Bagley–Torvik problem and

$$\begin{aligned}
D_{*0}^{0.555} y_0(x) &= y_1(x), \\
D_{*0}^{1} y_0(x) &= y_2(x), \\
D_{*0}^{0.455} y_2(x) &= -x^{0.1} \frac{E_{1.545}(-x)}{E_{1.445}(-x)} \exp(x) y_0(x) y_1(x) + \exp(-2x) - [y_2(x)]^2
\end{aligned}$$

with initial conditions

$$y_0(0) = 1, \quad y_1(0) = 0 \quad \text{and} \quad y_2(0) = -1$$

for the other example. In this way we simplify the structure somewhat because some of the fractional differential operators on the left-hand side can be replaced by integer-order operators. This modification is equivalent to using Theorem 8.10 instead of Theorem 8.9 in our multi-order system approach.

For the numerical solution of these systems of equations one can then use a numerical scheme for scalar fractional differential equations for each component separately. Of course one needs to take into account that the individual equations now will typically not be of the same order, so numerical methods for different orders must be used. In general, there do not seem to be any advantages in using differently constructed methods for the individual equations; rather one would usually prefer to use just one class of numerical schemes and merely change the orders of the algorithms as prescribed by the orders of the differential operators on the left-hand side of the system.

In either of the two multi-order system approaches, Edwards et al. [59] have investigated the use of the formula of [34] for the numerical solution of the resulting system of equations; later results [65] indicate that our Adams–Bashforth–Moulton method is likely to be more efficient. As far as the error is concerned, it turns out that the behaviour of the entire scheme is dominated by the component with the worst behaviour.

A comparison with the single-order system approach shows that the multi-order system approach is always applicable (there are no number-theoretic restrictions on the orders $n_1, \ldots, n_k$), and it will in many cases lead to a system with a considerably smaller dimension. However the structure of the left-hand side becomes more complicated.

The numerical experiments of Ford and Connolly [66] indicate that the conversion to a multi-order system via Theorem 8.10 tends to be the computationally least efficient of the approaches presented here. They found the conversion to a multi-order system via Theorem 8.9 and the approach using the transformation to single-order systems by means of Theorem 8.1 or 8.2 to be usually preferable. Which of these ideas works best seems to depend on the particular problem under consideration, so a generally valid advice cannot be given.

Exercise

Exercise C.1. Give an explicit proof of Theorem C.2.

Appendix D
Useful Results from Analysis

In this chapter we collect some information on some concepts from Analysis that is useful in the remainder of the text.

D.1 Euler's Gamma Function

We begin with the Gamma function.

We recall the definition

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\, \mathrm{d}t$$

for $x > 0$. Elementary considerations from the theory of improper integrals reveal that the integral exists. Moreover, upon setting $x = 1$ we easily see

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}t = \lim_{z\to\infty} \int_0^z \mathrm{e}^{-t}\, \mathrm{d}t = \lim_{z\to\infty} [-\mathrm{e}^{-t}]_0^z = \lim_{z\to\infty} (1 - \mathrm{e}^{-z}) = 1. \qquad \text{(D.1)}$$

Additionally we may, for arbitrary $x > 0$, manipulate the integral in the definition of the Gamma function by means of a partial integration. This yields

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x \mathrm{e}^{-t}\, \mathrm{d}t = \lim_{z\to\infty, y\to 0+} \int_y^z t^x \mathrm{e}^{-t}\, \mathrm{d}t \\
&= \lim_{z\to\infty, y\to 0+} \left([-\mathrm{e}^{-t} t^x]_{t=y}^{t=z} + x \int_y^z t^{x-1} \mathrm{e}^{-t}\, \mathrm{d}t \right) \\
&= x \int_0^\infty t^{x-1} \mathrm{e}^{-t}\, \mathrm{d}t = x\Gamma(x).
\end{aligned}$$

We have thus shown

Theorem D.1 (Functional Equation for Γ). *If $x > 0$ then $x\Gamma(x) = \Gamma(x+1)$.*

Now we may prove the all important relation between the Gamma function and the factorial:

Proof (of Theorem 1.3). The proof uses mathematical induction. The induction basis ($n = 1$) reads $\Gamma(1) = 0! = 1$ which is true in view of (D.1). For the induction step, we use the functional equation and the induction hypothesis:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

as desired. □

There is one other important application of the functional equation of the Gamma function. We solve it for $\Gamma(x)$; it then reads

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \tag{D.2}$$

if $x > 0$. Now the expression on the right-hand side is meaningful not only if $x > 0$ but also in the case $-1 < x < 0$. Therefore we may use it as a definition for the left-hand side, i.e. for $\Gamma(x)$, in that case (which is not covered by the original definition because the defining integral is divergent for $x < 0$). Having done this extension, the Gamma function is also defined for $-1 < x < 0$, and we may return to (D.2) with x in that range. This allows us to extend the definition to $-2 < x < -1$. Proceeding in this manner, we find a definition for the Gamma function that can be applied for all $x \in \mathbb{R}$ with the exception of those for which $-x \in \mathbb{N}_0$.

As a consequence of these considerations, we find another important identity involving the Gamma function:

Theorem D.2. *Let $n \notin \mathbb{Z}$ and $k \in \mathbb{N}_0$. Then,*

$$(-1)^{k+1}\Gamma(n-k)\Gamma(k+1-n) = \Gamma(-n)\Gamma(n+1).$$

Another useful identity in this context is

Theorem D.3 (Reflection Formula for Γ). *Let $0 < x < 1$. Then,*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

It is also possible to find an alternative representation, due to Gauss, for the Gamma function. This representation actually holds for the extension indicated above. However, in practical calculations one frequently observes that the integral representation is easier to handle.

Theorem D.4 (Gauss' Product Formula for Γ). *Let $x \in \mathbb{R}$, $-x \notin \mathbb{N}_0$. Then,*

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)}.$$

The asymptotic behaviour of $\Gamma(x)$ as $x \to \infty$ is sometimes important; it can be described by the following result [2, Chapter 6].

Theorem D.5 (Stirling's Formula). *For $x \to \infty$ we have*

$$\Gamma(x+1) = \left(\frac{x}{\mathrm{e}}\right)^x \sqrt{2\pi x}(1+o(1)).$$

One last result that we shall mention explicitly is the following integral identity. We leave the proof as an exercise.

Theorem D.6. *Let $\alpha, \beta \in \mathbb{R}_+$. Then*

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and hence

$$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}\,\mathrm{d}t = x^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The integral in the first equation of Theorem D.6 is known as *Euler's integral of the first kind* or *Euler's Beta function* $B(\alpha,\beta)$.

More information on the Gamma function may be found, e.g., in the classical work of Artin [6] or in the usual reference works on special functions like [2, Chapter 6] or [62, Chapter I].

D.2 Fixed Point Theorems

The proofs of various existence and uniqueness theorems throughout this text have been based on classical theorems asserting existence or uniqueness of fixed points of certain operators.

The first of these theorems is the following generalization of Banach's fixed point theorem that we take from [189].

Theorem D.7 (Weissinger's Fixed Point Theorem). *Assume (U,d) to be a non-empty complete metric space, and let $\alpha_j \geq 0$ for every $j \in \mathbb{N}_0$ and such that $\sum_{j=0}^{\infty} \alpha_j$ converges. Furthermore, let the mapping $A : U \to U$ satisfy the inequality*

$$d(A^j u, A^j v) \leq \alpha_j d(u,v) \tag{D.3}$$

for every $j \in \mathbb{N}$ and every $u,v \in U$. Then, A has a uniquely determined fixed point u^*. Moreover, for any $u_0 \in U$, the sequence $(A^j u_0)_{j=1}^{\infty}$ converges to this fixed point u^*.

An immediate consequence is

Corollary D.8 (Banach's Fixed Point Theorem). *Assume (U,d) to be a non-empty complete metric space, let $0 \leq \alpha < 1$, and let the mapping $A : U \to U$ satisfy the inequality*

$$d(Au,Av) \leq \alpha d(u,v) \tag{D.4}$$

for every $u,v \in U$. Then, A has a uniquely determined fixed point u^*. Furthermore, for any $u_0 \in U$, the sequence $(A^j u_0)_{j=1}^{\infty}$ converges to this fixed point u^*.

Moreover we also used a slightly different result that asserts only the existence but not the uniqueness of a fixed point. Here we may work with weaker assumptions on the operator in question. A proof may be found, e.g., in [29].

Theorem D.9 (Schauder's Fixed Point Theorem). *Let (E,d) be a complete metric space, let U be a closed convex subset of E, and let $A : U \to U$ be a mapping such that the set $\{Au : u \in U\}$ is relatively compact in E. Then A has at least one fixed point.*

In this context we recall a definition:

Definition D.1. Let (E,d) be a metric space and $F \subseteq E$. The set F is called *relatively compact in E* if the closure of F is a compact subset of E.

A helpful classical result from Analysis in connection with such sets is as follows. The proof can be found in many standard textbooks, e.g. in [30, p. 30].

Theorem D.10 (Arzelà–Ascoli). *Let $F \subseteq C[a,b]$ for some $a < b$, and assume the sets to be equipped with the Chebyshev norm. Then, F is relatively compact in $C[a,b]$ if F is equicontinuous (i.e. for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $f \in F$ and all $x, x^* \in [a,b]$ with $|x - x^*| < \delta$ we have $|f(x) - f(x^*)| < \varepsilon$) and uniformly bounded (i.e. there exists a constant $C > 0$ such that $\|f\|_\infty \leq C$ for every $f \in F$).*

D.3 The Laplace Transform

The Laplace transform method is an extremely useful tool for the analysis of linear (fractional or classical) initial value problems. In particular, it allows us to replace a differential equation by an algebraic equation. We take the fundamental definition from the classical book of Doetsch [56] where the interested reader may find a comprehensive treatment of the Laplace transform.

Definition D.2. Let $f : [0,\infty) \to \mathbb{R}$ be given. The function F defined by

$$F(s) := \mathscr{L} f(s) := \int_0^\infty f(x) \mathrm{e}^{-sx} \, \mathrm{d}x$$

is called the *Laplace transform* of f whenever the integral exists.

It is rather simple to calculate the Laplace transform of some elementary functions.

Example D.1. (a) For $f(x) = \exp(ax)$ with $a \in \mathbb{R}$ we have $\mathscr{L}f(s) = 1/(s-a)$ whenever $s > a$.
(b) For $f(x) = x^k$ with $k > -1$ we find $\mathscr{L}f(s) = \Gamma(k+1)/s^{k+1}$ whenever $s > 0$.
(c) For $f(x) = \sin \omega x$ with $\omega > 0$ we have $\mathscr{L}f(s) = \omega/(s^2+\omega^2)$, again for $s > 0$.

We cite the most important rules for Laplace transforms.

Theorem D.11. *Assume the functions f_1, f_2 and f_3 to be given on $[0,\infty)$ and to be such that their Laplace transforms exist for all $s \geq s_0$ with some suitable $s_0 \in \mathbb{R}$. Then we have the following rules.*

(a) If $f_3 = a_1 f_1 + a_2 f_2$ with arbitrary real constants a_1 and a_2 then

$$\mathscr{L}f_3(s) = a_1 \mathscr{L}f_1(s) + a_2 \mathscr{L}f_2(s)$$

(linearity of the Laplace transform).

(b) If f_3 is the convolution *of f_1 and f_2, i.e. if*

$$f_3(x) = \int_0^x f_1(x-t) f_2(t)\,\mathrm{d}t,$$

then

$$\mathscr{L}f_3(s) = \mathscr{L}f_1(s) \cdot \mathscr{L}f_2(s)$$

(the convolution theorem*). In other words: The convolution in the original domain corresponds to the usual product in the Laplace domain.*

(c) If $f_3(x) = \int_0^x f_1(t)\,\mathrm{d}t$ then we have for $s > \max\{0, s_0\}$

$$\mathscr{L}f_3(s) = \frac{1}{s}\mathscr{L}f_1(s)$$

(the integration theorem*).*

(d) Let $m \in \mathbb{N}$. If $f_3 = D^m f_1$ is the mth derivative of f_1 then

$$\mathscr{L}f_3(s) = s^m \mathscr{L}f_1(s) - \sum_{k=1}^{m} s^{m-k} f_1^{(k-1)}(0)$$

(the differentiation theorem*).*

(e) Let $a > 0$ and $f_3(x) = f_1(ax)$. Then

$$\mathscr{L}f_3(s) = \frac{1}{a}\mathscr{L}f_1(s/a).$$

(f) Let $a \in \mathbb{R}$ and $f_3(x) = \mathrm{e}^{-ax} f_1(x)$. Then

$$\mathscr{L}f_3(s) = \mathscr{L}f_1(s+a).$$

(g) *Let $m \in \mathbb{N}$ and $f_3(x) = x^m f_1(x)$. Then*

$$\mathscr{L} f_3(s) = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}s^m} \mathscr{L} f_1(s).$$

(h) *Let $f_3(x) = f_1(x)/x$. Then*

$$\mathscr{L} f_3(s) = \int_s^\infty \mathscr{L} f_1(\sigma)\,\mathrm{d}\sigma.$$

(i) *Let $a \in \mathbb{R}$ and*

$$f_3(x) = \begin{cases} 0 & \text{for } x < a, \\ f_1(x-a) & \text{for } x \geq a. \end{cases}$$

Then

$$\mathscr{L} f_3(s) = \mathrm{e}^{-as} \mathscr{L} f_1(s).$$

Part (d), the differentiation theorem, is of particular interest to us. Specifically this result needed to be generalized to $m \notin \mathbb{N}$ with a suitable definition of the differential operator. We have dealt with this question in Theorem 7.1.

Of course it is not sufficient to have the Laplace transform; for practical work the inverse transform is required too. There are various ways to express this inverse; one possibility is contained in the following result. We refer to the standard books on Laplace transforms for details on the "suitable assumptions".

Theorem D.12. *Under suitable assumptions on f we have*

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \exp(sx) \mathscr{L} f(s)\,\mathrm{d}s.$$

Under certain conditions, the long-term behaviour of functions may also be expressed with the help of Laplace transforms, see [27, 76, 158] and the references cited therein:

Theorem D.13 (Final Value Theorem). *Assume that $\mathscr{L} f$ does not have any singularities in the closed right half-plane $\{s \in \mathbb{C} : \mathrm{Re}\, s \geq 0\}$, except for possibly a simple pole at the origin. Then,*

$$\lim_{x \to \infty} f(x) = \lim_{s \to 0+} s \mathscr{L} f(s).$$

Remark D.1. The condition on the singularities of $\mathscr{L} f$ is essential here: If $\mathscr{L} f$ has a pole with positive real part, then $f(x)$ is unbounded as $x \to \infty$, and if $\mathscr{L} f$ has a pole on the imaginary axis (but not at the origin) then f has persistent oscillations, so $\lim_{x \to \infty} f(x)$ does not exist either.

D.4 Hadamard's Finite-Part Integral

The integral $\int_a^b (x-a)^{-\mu} f(x)\,\mathrm{d}x$ is divergent for $\mu \geq 1$ whenever $f(a) \neq 0$. Nevertheless it is sometimes useful to assign a finite value to such integrals. This has been observed by Hadamard [86] in connection with solution methods for certain partial differential equations, and he introduced the following idea for the solution of this problem, known as the *finite-part integral*. We shall mainly require this concept for $\mu \notin \mathbb{N}$, and therefore we will restrict our attention to these values of μ. The consideration of integer values requires some small modifications.

The Hadamard finite-part of the integral (that we will, for the sake of simplicity, denote by the same symbol as the standard integral) is, roughly speaking, defined by a Taylor expansion of f at $x = a$ where the resulting singular integrals are defined by

$$\int_a^b (x-a)^{-\mu}\,\mathrm{d}x = \frac{1}{1-\mu}(b-a)^{1-\mu} \qquad (\mu > 1). \tag{D.5}$$

Essentially this means that we first replace the integral $\int_a^b (x-a)^{-\mu}\,\mathrm{d}x$ by the expression $\int_{a+\varepsilon}^b (x-a)^{-\mu}\,\mathrm{d}x$ for $\varepsilon > 0$. This is a convergent integral; its value is simply $(1-\mu)^{-1}[(b-a)^{1-\mu} - \varepsilon^{1-\mu}]$. Then we let $\varepsilon \to 0$. Of course the limit does not exist for $\mu \geq 1$, and so Hadamard suggested simply to ignore the unbounded contribution $\lim_{\varepsilon\to 0} \varepsilon^{1-\mu}/(1-\mu)$ and to assign the value of the remaining (finite) expression $(1-\mu)^{-1}(b-a)^{1-\mu}$ – hence the name "finite-part integral".

A precise way to define the finite-part integral is (for $\mu \notin \mathbb{N}$)

$$\begin{aligned}\int_a^b (x-a)^{-\mu} f(x)\,\mathrm{d}x := &\sum_{k=0}^{\lfloor\mu\rfloor-1} \frac{f^{(k)}(a)(b-a)^{k+1-\mu}}{(k+1-\mu)k!} \\ &+ \int_a^b (x-a)^{-\mu} R_{\lfloor\mu\rfloor-1}(x,a)\,\mathrm{d}x.\end{aligned} \tag{D.6a}$$

Here,

$$R_p(x,a) := \frac{1}{p!}\int_a^x (x-y)^p f^{(p+1)}(y)\,\mathrm{d}y \tag{D.6b}$$

is the remainder of the pth degree Taylor polynomial of f with expansion point a. It is well known that a sufficient condition for the existence of the integral (D.6a) is that $f \in C^s[a,b]$ with $\mu - 1 < s \in \mathbb{N}$. This is due to the fact that then the remainder term of the Taylor expansion has a zero at a whose order is so high that the singularity in the other factor in the last integral in (D.6a) is almost being cancelled; the remaining singularity is weak and integrable in the improper sense.

An alternative representation that is helpful for us can be taken from [61, eq. (A17)]:

Theorem D.14. *Let $\mu > 1$ but $\mu \notin \mathbb{N}$ and $m := \lceil \mu - 1 \rceil$. For $f \in C^m[a,b]$ we have*

$$\frac{1}{\Gamma(1-\mu)} \int_a^b (b-x)^{-\mu} f(x)\,\mathrm{d}x = \sum_{k=0}^{m-1} \frac{(b-a)^{k-\mu+1}}{\Gamma(k-\mu+2)} f^{(k)}(a) + J_a^{m-\mu+1} f^{(m)}(b).$$

We mention here the most important properties of the finite-part integral:

- In contrast to the classical Riemann or Lebesgue integral, the finite-part integral is not a positive functional, i.e. the inequality

$$\left| \int_a^b (x-a)^{-\mu} f(x)\,\mathrm{d}x \right| \le \int_a^b (x-a)^{-\mu} |f(x)|\,\mathrm{d}x$$

  is not true in general.
- The finite-part integral is a consistent extension of the concept of regular integrals, i.e. whenever the integral $\int_a^b (x-a)^{-\mu} f(x)\,\mathrm{d}x$ exists in the classical sense, then it also exists in the finite-part sense, and the two integrals have the same value.
- The finite-part integral is additive with respect to the union of integration intervals and invariant with respect to translation.
- The finite-part integral is linear.
- The usual change-of-variables rule remains valid if $\mu \notin \mathbb{N}$.

A very useful result on these integrals is as follows.

Theorem D.15. *Let $f \in C^k[a,b]$ for some $k \in \mathbb{N}_0$, and let $p < k$. Then, for $a < x < b$,*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)(x-t)^{-p}\,\mathrm{d}t = -p \int_a^x f(t)(x-t)^{-p-1}\,\mathrm{d}t.$$

We leave the proof as an exercise to the reader.

D.5 Approximation Theory

A well-known concept from approximation theory that we had to use in the proof of Theorem 2.25 was the *Bernstein polynomial*. A classical reference is the book of Lorentz [117]. The definition is

$$B_N[f](t) := \sum_{k=0}^{N} \binom{N}{k} t^k (1-t)^{N-k} f\left(\frac{k}{N}\right)$$

where $f : [0,1] \to \mathbb{R}$. The fundamental result that we require is a convergence theorem:

Theorem D.16. *Let $f \in C^\ell[0,1]$ for some $\ell \in \mathbb{N}_0$. Then, for all $\mu \in \{0,1,2,\ldots,\ell\}$, the sequence $(D^\mu B_N[f])_{N=1}^\infty$ converges uniformly towards $D^\mu f$.*

A proof may be found in [117, §1.8].

Exercises

Exercise D.1. Give a proof for Theorem D.6.

Exercise D.2. Give a proof for Theorem D.2.

Exercise D.3. Prove the relations stated in Example D.1.

Exercise D.4. Evaluate the finite-part integral $\int_0^1 x^{-\mu}\,\mathrm{d}x$ for $\mu > 1$.

Exercise D.5. Give a proof for Theorem D.15.

References

1. Abel, N.H.: Auflösung einer mechanischen Aufgabe. J. Reine Angew. Math. **1**, 153–157 (1826)
2. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions, 2nd printing with corrections. National Bureau of Standards, Washington (1964); republished by Dover, New York (1965)
3. Adomian, G.: Solving Frontier Problems of Physics: the Decomposition Method. Kluwer, Dordrecht (1994)
4. Agarwal, R.P., Benchohra, M., Hamani, S.: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973–1033 (2010)
5. Ahmad, W.M., El-Khazali, R.: Fractional-order dynamical models of love. Chaos Solitons Fractals **33**, 1367–1375 (2007)
6. Artin, E.: Einführung in die Theorie der Gammafunktion. Teubner, Leipzig (1931); English translation: The Gamma Function. Holt, Rinehart and Winston, New York (1964)
7. Audounet, J., Roquejoffre, J.-M.: An asymptotic fractional differential model of spherical flame. In: Matignon, D., Montseny, G. (eds.) Fractional Differential Systems: Models, Methods and Applications, pp. 15–27. SMAI, Paris (1998)
8. Audounet, J., Giovangigli, V., Roquejoffre, J.-M.: A threshold phenomenon in the propagation of a point source initiated flame. Physica D **121**, 295–316 (1998)
9. Bagley, R.L., Calico, R.A.: Fractional order state equations for the control of viscoelastically damped structures. J. Guid. Contr. Dynam. **14**, 304–311 (1991)
10. Bagley, R.L., Torvik, P.J.: On the existence of the order domain and the solution of distributed order equations – Part I. Int. J. Appl. Math. **2**, 865–882 (2000)
11. Bagley, R.L., Torvik, P.J.: On the existence of the order domain and the solution of distributed order equations – Part II. Int. J. Appl. Math. **2**, 965–987 (2000)
12. Bai, J., Feng, X.-C.: Fractional-order anisotropic diffusion for image denoising. IEEE Trans. Image Process. **16**, 2492–2502 (2007)
13. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus: Models and Numerical Methods. World Scientific, Singapore (2012)
14. Basset, A.B.: On the motion of a sphere in a viscous liquid. Philos. Trans. R. Soc. A **179**, 43–63 (1888)
15. Basset, A.B.: On the descent of a sphere in a viscous liquid. Q. J. Pure Appl. Math. **41**, 369–381 (1910)
16. Benchohra, M., Hamani, S., Ntouyas, S.K.: Boundary value problems for differential equations with fractional order. Surv. Math. Appl. **3**, 1–12 (2008)
17. Benson, D.A.: The fractional advection–dispersion equation: development and application. Ph.D. thesis, University of Nevada at Reno (1998)
18. Bonilla, B., Rivero, M., Trujillo, J.J.: On systems of linear fractional differential equations with constant coefficients. Appl. Math. Comput. **187**, 68–78 (2007)
19. Braß, H.: Quadraturverfahren. Vandenhoeck & Ruprecht, Göttingen (1977)

20. Brunner, H., van der Houwen, P.J.: The Numerical Solution of Volterra Equations. North-Holland, Amsterdam (1986)
21. Brunner, H., Pedas, A., Vainikko, G.: The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations. Math. Comput. **68**, 1079–1095 (1999)
22. Caponetto, R., Dongola, G., Fortuna, L., Petráš, I.: Fractional Order Systems: Modeling and Control Applications. World Scientific, River Edge, NJ (2010)
23. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent – II. Geophys. J. Roy. Astron. Soc. **13**, 529–539 (1967); reprinted in Fract. Calc. Appl. Anal. **11**, 4–14 (2008)
24. Caputo, M., Mainardi, F.: A new dissipation model based on memory mechanism. Pure Appl. Geophys. **91**, 134–147 (1971); reprinted in Fract. Calc. Appl. Anal. **10**, 310–323 (2007)
25. Caputo, M., Mainardi, F.: Linear models of dissipation in anelastic solids. Rivista del Nuovo Cimento **1**, 161–198 (1971)
26. Chatterjee, A.: Statistical origins of fractional derivatives in viscoelasticity. J. Sound Vib. **284**, 1239–1245 (2005)
27. Chen, J., Lundberg, K.H., Davison, D.E., Bernstein, D.S.: The final value theorem revisited – infinite limits and irrational functions. IEEE Contr. Syst. Mag. **27** (3), 97–99 (2007)
28. Chern, J.-T.: Finite element modeling of viscoelastic materials on the theory of fractional calculus. Ph.D. thesis, Pennsylvania State University (1993)
29. Collatz, L.: Funktionalanalysis und numerische Mathematik. Springer, Berlin (1968)
30. Corduneanu, C.: Principles of Differential and Integral Equations, 2nd edn. Chelsea Publ. Comp., New York (1977)
31. Cuesta, E., Finat Codes, J.: Image processing by means of a linear integro-differential equation. In: Hamza, M.H. (ed.) Visualization, Imaging, and Image Processing 2003, Paper 91. ACTA Press, Calgary (2003)
32. de Hoog, F., Weiss, R.: Asymptotic expansions for product integration. Math. Comput. **27**, 295–306 (1973)
33. Deng, W.: Short memory principle and a predictor-corrector approach for fractional differential equations. J. Comput. Appl. Math. **206**, 174–188 (2007)
34. Diethelm, K.: An algorithm for the numerical solution of differential equations of fractional order. Electron. Trans. Numer. Anal. **5**, 1–6 (1997)
35. Diethelm, K.: Predictor-corrector strategies for single- and multi-term fractional differential equations. In: Lipitakis, E.A. (ed.) Proceedings of the 5th Hellenic-European Conference on Computer Mathematics and Its Applications, pp. 117–122. LEA Press, Athens (2002)
36. Diethelm, K.: Efficient solution of multi-term fractional differential equations using P(EC)^{m}E methods. Computing **71**, 305–319 (2003)
37. Diethelm, K.: Smoothness properties of solutions of Caputo-type fractional differential equations. Fract. Calc. Appl. Anal. **10**, 151–160 (2007)
38. Diethelm, K.: An investigation of some nonclassical methods for the numerical approximation of Caputo-type fractional derivatives. Numer. Algorithms **47**, 361–390 (2008)
39. Diethelm, K.: Multi-term fractional differential equations, multi-order fractional differential systems and their numerical solution. J. Eur. Syst. Autom. **42**, 665–676 (2008)
40. Diethelm, K.: On the separation of solutions of fractional differential equations. Fract. Calc. Appl. Anal. **11**, 259–268 (2008)
41. Diethelm, K.: An improvement of a nonclassical numerical method for the computation of fractional derivatives. J. Vib. Acoust. **131**, 014502 (2009)
42. Diethelm, K., Ford, N.J.: Numerical solution methods for distributed order differential equations. Fract. Calc. Appl. Anal. **4**, 531–542 (2001)
43. Diethelm, K., Ford, N.J.: Analysis of fractional differential equations. J. Math. Anal. Appl. **265**, 229–248 (2002)
44. Diethelm, K., Ford, N.J.: Numerical solution of the Bagley–Torvik equation. BIT **42**, 490–507 (2002)
45. Diethelm, K., Ford, N.J.: Multi-order fractional differential equations and their numerical solution. Appl. Math. Comput. **154**, 621–640 (2004)

46. Diethelm, K., Ford, N.J.: Numerical analysis for distributed order differential equations. J. Comput. Appl. Math. **225**, 96–104 (2009)
47. Diethelm, K., Ford, N.J.: Volterra integral equations and fractional calculus: do neighbouring solutions intersect? J. Integr. Equ. Appl. (in press)
48. Diethelm, K., Ford, N.J., Freed, A.D.: A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynam. **29**, 3–22 (2002)
49. Diethelm, K., Ford, N.J., Freed A.D.: Detailed error analysis for a fractional Adams method. Numer. Algorithms **36**, 31–52 (2004)
50. Diethelm, K., Freed, A.D.: On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity. In: Keil, F., Mackens, W., Voß, H., Werther, J. (eds.) Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties, pp. 217–224. Springer, Heidelberg (1999)
51. Diethelm, K., Freed, A.D.: The FracPECE subroutine for the numerical solution of differential equations of fractional order. In: Heinzel, S., Plesser, T. (eds.) Forschung und wissenschaftliches Rechnen: Beiträge zum Heinz-Billing-Preis 1998, pp. 57–71. Gesellschaft für wissenschaftliche Datenverarbeitung, Göttingen (1999)
52. Diethelm, K., Luchko, Y.: Numerical solution of linear multi-term initial value problems of fractional order. J. Comput. Anal. Appl. **6**, 243–263 (2004)
53. Diethelm, K., Walz, G.: Numerical solution of fractional order differential equations by extrapolation. Numer. Algorithms **16**, 231–253 (1997)
54. Diethelm, K., Weilbeer, M.: A numerical approach for Joulin's model of a point source initiated flame. Fract. Calc. Appl. Anal. **7**, 191–212 (2004)
55. Dixon, J., McKee, S.: Weakly singular discrete Gronwall inequalities. Z. Angew. Math. Mech. **66**, 535–544 (1986)
56. Doetsch, G.: Anleitung zum praktischen Gebrauch der Laplace-Transformation und der Z-Transformation, 6th edn. Oldenbourg, München (1989)
57. Dokoumetzidis, A., Magin, R., Macheras, P.: A commentary on fractionalization of multicompartmental models. J. Pharmacokinet. Pharmacodyn. **37**, 203–207 (2010)
58. Dzherbashyan, M.M., Nersesian, A.B.: Fractional derivatives and the Cauchy problem for differential equations of fractional order. Izv. Akad. Nauk Arm. SSR, Mat. **3**, 3–29 (1968) (in Russian)
59. Edwards, J.T., Ford, N.J., Simpson, A.C.: The numerical solution of linear multi-term fractional differential equations: systems of equations. J. Comput. Appl. Math. **148**, 401–418 (2002)
60. Edwards, J.T., Roberts, J.A., Ford, N.J.: A comparison of Adomian's decomposition method and Runge–Kutta methods for approximate solution of some predator prey model equations. Numerical Analysis Report 309, Manchester Centre for Computational Mathematics (1997)
61. Elliott, D.: An asymptotic analysis of two algorithms for certain Hadamard finite-part integrals. IMA J. Numer. Anal. **13**, 445–462 (1993)
62. Erdelyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G. (eds.): Higher Transcendental Functions, vol. I. McGraw-Hill, New York (1953)
63. Erdelyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G. (eds.): Higher Transcendental Functions, vol. III. McGraw-Hill, New York (1955)
64. Fitzgerald, W.J., Leung, C.M.: A critical review of the Adomian method. Tech. Report CUED/F-INFENG/TR. 277. University of Cambridge (1996)
65. Ford, N.J., Connolly, J.A.: Comparison of numerical methods for fractional differential equations. Commun. Pure Appl. Anal. **5**, 289–307 (2006)
66. Ford, N.J., Connolly, J.A.: Systems-based decomposition schemes for the approximate solution of multi-term fractional differential equations. J. Comput. Appl. Math. **229**, 382–391 (2009)
67. Ford, N.J., Simpson, A.C.: The approximate solution of fractional differential equations of order greater than 1. In: Deville, M., Owens, R. (eds.) Proceedings of the 16th IMACS World Congress on Scientific Computation, Applied Mathematics, and Simulation, Paper 213-1. IMACS, New Brunswick (2000)

68. Freed, A.D., Diethelm, K.: Fractional calculus in biomechanics: a 3D viscoelastic model using regularized fractional-derivative kernels with application to the human calcaneal fat pad. Biomech. Model. Mechanobiol. **5**, 203–215 (2006)
69. Freed, A.D., Diethelm, K., Luchko, Y.: Fractional-order viscoelasticity (FOV): constitutive development using the fractional calculus (first annual report). Technical Memorandum 2002-211914, NASA Glenn Research Center, Cleveland (2002)
70. Galeone, L., Garrappa, R.: Fractional Adams–Moulton methods. Math. Comput. Simul. **79**, 1358–1367 (2008)
71. Gaul, L., Klein, P., Kempfle, S.: Damping description involving fractional operators. Mech. Syst. Signal Process. **5**, 81–88 (1991)
72. Gel'fand, I.M., Shilov, G.E.: Generalized Functions, vol. 1. Academic Press, New York (1964)
73. Gel'fond, A.O., Leont'ev, A.F.: On a generalization of the Fourier series. Mat. Sb. (N. S.) **29**, 477–500 (1951) (in Russian)
74. Gerasimov, A.N.: A generalization of linear laws of deformation and its application to the problems of internal friction. Prikl. Mat. Mekh. **12**, 251–260 (1948) (in Russian)
75. Glöckle, W.G., Nonnenmacher, T.F.: A fractional calculus approach to self-similar protein dynamics. Biophys. J. **68**, 46–53 (1995)
76. Gluskin, E.: Let us teach this generalization of the final-value theorem. Eur. J. Phys. **24**, 591–597 (2003)
77. Gorenflo, R.: Fractional calculus: some numerical methods. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 277–290. Springer, Wien (1997)
78. Gorenflo, R., De Fabritiis, G., Mainardi, F.: Discrete random walk models for symmetric Lévy-Feller diffusion processes. Physica A **269**, 79–89 (1999)
79. Gorenflo, R., Loutchko, J., Luchko, Y.: Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivative. Fract. Calc. Appl. Anal. **5**, 491–518 (2002); Corrections: Fract. Calc. Appl. Anal. **6**, 111–112 (2003)
80. Gorenflo, R., Mainardi, F.: Fractional oscillations and Mittag-Leffler functions. In: Proceedings of the International Workshop on the Recent Advances in Applied Mathematics (RAAM '96), pp. 193–208. Kuwait University, Department of Mathematics and Computer Science, Kuwait (1996)
81. Gorenflo, R., Mainardi, F.: Fractional calculus: integral and differential equations of fractional order. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 223–276. Springer, Wien (1997)
82. Gorenflo, R., Rutman, R.: On ultraslow and intermediate processes. In: Rusev, P., Dimovski, I., Kiryakova, V. (eds.) Transform Methods and Special Functions, pp. 61–81. Science Culture Technology, Singapore (1995)
83. Gorenflo, R., Vivoli, A.: Fully discrete random walks for space-time fractional diffusion equations. Signal Process. **83**, 2411–2420 (2003)
84. Gross, B.: On creep and relaxation. J. Appl. Phys. **18**, 212–221 (1947)
85. Grünwald, A.K.: Über "begrenzte" Derivationen und deren Anwendung. Z. Angew. Math. Phys. **12**, 441–480 (1867)
86. Hadamard, J.: Lectures on Cauchy's Problem in Linear Partial Differential Equations. Yale Univ. Press, New Haven (1923); reprinted by Dover, New York (1952)
87. Hadid, S.B., Ta'ani, A.A., Momani, S.M.: Some existence theorems on differential equations of generalized order through a fixed-point theorem. J. Fract. Calc. **9**, 45–49 (1996)
88. Hairer, E., Nørsett, S.P., Wanner, G.: Solving Ordinary Differential Equations I: Nonstiff Problems, 2nd edn. Springer, Berlin (1993)
89. Hairer, E., Wanner, G.: Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems. Springer, Berlin (1991)
90. Haubold, H.J., Mathai, A.M., Saxena, R.K.: Mittag-Leffler functions and their applications. arXiv:0909.0230v2 [math.CA] (2009)
91. He, J.H.: Approximate solution of non linear differential equations with convolution product nonlinearities. Comput. Methods Appl. Mech. Eng. **167**, 69–73 (1998)

92. He, J.H.: Variational iteration method – some recent results and new interpretations. J. Comput. Appl. Math. **207**, 3–17 (2007)
93. Heaviside, O.: Electromagnetic Theory, vol. II, 2nd reprint. Benn, London (1925)
94. Hilfer, R. (ed.): Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
95. Hille, E.: Lectures on Ordinary Differential Equations. Addison-Wesley, Reading (1969)
96. Hille, E.: Ordinary Differential Equations in the Complex Domain. Dover, New York (1997)
97. Jacob, N., Krägeloh, A.M.: The Caputo fractional derivative, Feller semigroups, and the fractional power of the first order derivative on $C_\infty(\mathbb{R}_0^+)$. Fract. Calc. Appl. Anal. **5**, 395–410 (2002)
98. Johnson, W.P.: The curious history of Faà di Bruno's formula. Am. Math. Mon. **109**, 217–234 (2002)
99. Joulin, G.: Point-source initiation of lean spherical flames of light reactants: an asymptotic theory. Combust. Sci. Technol. **43**, 99–113 (1985)
100. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
101. Kilbas, A.A., Trujillo, J.J.: Differential equations of fractional order. Methods, results and problems. I. Appl. Anal. **78**, 153–192 (2001)
102. Kilbas, A.A., Trujillo, J.J.: Differential equations of fractional order. Methods, results and problems. II. Appl. Anal. **81**, 435–493 (2002)
103. Kiryakova, V.: The multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus. Comput. Math. Appl. **59**, 1885–1895 (2010)
104. Kiryakova, V.: The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. Comput. Math. Appl. **59**, 1128–1141 (2010)
105. Klages, R., Radons, G., Sokolov, I.M. (eds.): Anomalous Transport: Foundations and Applications. Wiley-VCH, Weinheim (2008)
106. Kochubei, A.N.: Fractional differential equations: α-entire solutions, regular and irregular singularities. Fract. Calc. Appl. Anal. **12**, 135–158 (2009)
107. Koeller, R.C.: Polynomial operators, Stieltjes convolution, and fractional calculus in hereditary mechanics. Acta Mech. **58**, 251–264 (1986)
108. Krägeloh, A.M.: Feller semigroups generated by fractional derivatives and pseudo-differential operators. Ph.D. thesis, Universität Erlangen (2001)
109. Kress, R.: Linear Integral Equations, 2nd edn. Springer, New York (1999)
110. Lederman, C., Roquejoffre, J.-M., Wolanski, N.: Mathematical justification of a nonlinear integro-differential equation for the propagation of spherical flames. C. R. Math. Acad. Sci. Paris **334**, 569–574 (2002)
111. Le Mehauté, A., Tenreiro Machado, J.A., Trigeassou, J.C., Sabatier, J. (eds.): Fractional Differentiation and Its Applications. Ubooks, Neusäß (2005)
112. Letnikov, A.V.: Theory of differentiation with an arbitrary index. Mat. Sb. **3**, 1–66 (1868) (in Russian)
113. Liang, S., Jeffrey, D.J.: Comparison of homotopy analysis method and homotopy perturbation method through an evolution equation. Commun. Nonlinear Sci. Numer. Simulat. **14**, 4057–4064 (2009)
114. Liao, S.J.: Beyond Perturbation: Introduction to the Homotopy Analysis Method. Chapman & Hall/CRC Press, Boca Raton (2003)
115. Linz, P.: Analytical and Numerical Methods for Volterra Equations. SIAM, Philadelphia (1985)
116. Liouville J.: Memoire sur quelques questions de géometrie et de mécanique, et sur un nouveau gentre pour resoudre ces questions. J. Ecole Polytech. **13**, 1–69 (1832)
117. Lorentz, G.G.: Bernstein Polynomials, 2nd edn. Chelsea Publ. Comp., New York (1986)
118. Lu, J.-F., Hanyga, A.: Wave field simulation for heterogeneous porous media with singular memory drag force. J. Comput. Phys. **208**, 651–674 (2005)
119. Lu, J.G.: Chaotic dynamics of the fractional-order Lü system and its synchronization. Phys. Lett. A **354**, 305–311 (2006)

120. Lubich, C.: Runge-Kutta theory for Volterra and Abel integral equations of the second kind. Math. Comput. **41**, 87–102 (1983)
121. Lubich, C.: Discretized fractional calculus. SIAM J. Math. Anal. **17**, 704–719 (1986)
122. Lubinsky, D.S.: A survey of weighted polynomial approximation with exponential weights. Surv. Approx. Theory **3**, 1–105 (2007)
123. Luchko, Y., Gorenflo, R.: An operational method for solving fractional differential equations with the Caputo derivatives, Acta Math. Vietnamica **24**, 207–233 (1999)
124. Magin, R.: Fractional Calculus in Bioengineering. Begell House, Redding (2006)
125. Mainardi, F.: Fractional calculus: some basic problems in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 291–348. Springer, Wien (1997)
126. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity. Imperial College Press, London (2010)
127. Mainardi, F., Gorenflo, R.: On Mittag-Leffler-type functions in fractional evolution processes. J. Comput. Appl. Math. **118**, 283–299 (2000)
128. Mainardi, F., Raberto, M., Gorenflo, R., Scalas, E.: Fractional calculus and continuous-time finance II: the waiting-time distribution. Physica A **287**, 468–481 (2000)
129. Marchaud, A.: Sur les dérivées et sur les différences des fonctions de variables réelles. J. Math. **6**, 337–425 (1927)
130. Marks, R.J. II, Hall, M.W.: Differintegral interpolation from a bandlimited signal's samples. IEEE Trans. Acoust. Speech Signal Process. **29**, 872–877 (1981)
131. Mathai, A.M., Haubold, H.J.: Special Functions for Applied Scientists. Springer, New York (2008)
132. Matignon, D.: Représentations en variables d'état de modèles de guides d'ondes avec dérivation fractionnaire. Ph.D. thesis, Université Paris XI (1994)
133. Matignon, D., Montseny, G. (eds.): Fractional Differential Systems: Models, Methods, and Applications. SMAI, Paris (1998)
134. Metzler, R., Schick, W., Kilian, H.-G., Nonnenmacher, T.F.: Relaxation in filled polymers: a fractional calculus approach. J. Chem. Phys. **103**, 7180–7186 (1995)
135. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
136. Mittag-Leffler, G.M.: Une généralisation de l'intégrale de Laplace-Abel. C. R. Acad. Sci. Paris (Ser. II) **136**, 537–539 (1903)
137. Mittag-Leffler, G.M.: Sur la nouvelle fonction $E_\alpha(x)$. C. R. Acad. Sci. Paris (Ser. II) **137**, 554–558 (1903)
138. Momani, S.M.: Local and global uniqueness theorems on differential equations of noninteger order via Bihari's and Gronwall's inequalities. Rev. Téc. Fac. Ing. Univ. Zulia **23**, 66–69 (2000)
139. Momani, S., Odibat, Z.: Homotopy perturbation method for nonlinear partial differential equations of fractional order. Phys. Lett. A **365**, 345–350 (2007)
140. Nkamnang, A.R.: Diskretisierung von mehrgliedrigen Abelschen Integralgleichungen und gewöhnlichen Differentialgleichungen gebrochener Ordnung. Ph.D. thesis, Freie Universität Berlin (1999)
141. Nonnenmacher, T.F., Metzler, R.: On the Riemann–Liouville fractional calculus and some recent applications. Fractals **3**, 557–566 (1995)
142. Nutting, P.G.: A new general law of deformation. J. Franklin Inst. **191**, 679–685 (1921)
143. Nutting, P.G.: A general stress–strain–time formula. J. Franklin Inst. **235**, 513–524 (1943)
144. Odibat, Z.M.: Analytic study on linear systems of fractional differential equations. Comput. Math. Appl. **59**, 1171–1183 (2010)
145. Odibat, Z., Momani, S., Erturk, V.S.: Generalized differential transform method: application to differential equations of fractional order. Appl. Math. Comput. **197**, 467–477 (2008)
146. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic, New York (1974)
147. Olmstead, W.E., Handelsman, R.A.: Diffusion in a semi-infinite region with nonlinear surface dissipation. SIAM Rev. **18**, 275–291 (1976)

148. Perron, O.: Über Integration von gewöhnlichen Differentialgleichungen durch Reihen. Sitz.-Ber. Heidelberger Akademie der Wissenschaften A Math.-Nat. Klasse Abhandlung **2** (1919)
149. Perron, O.: Über Integration von gewöhnlichen Differentialgleichungen durch Reihen, II. Sitz.-Ber. Heidelberger Akademie der Wissenschaften A Math.-Nat. Klasse Abhandlung **8** (1919)
150. Perron, O.: Über Integration von gewöhnlichen Differentialgleichungen durch Reihen, III. Sitz.-Ber. Heidelberger Akademie der Wissenschaften A Math.-Nat. Klasse Abhandlung **12** (1919)
151. Perron, O.: Über Integration partieller Differentialgleichungen durch Reihen. Sitz.-Ber. Heidelberger Akademie der Wissenschaften A Math.-Nat. Klasse Abhandlung **9** (1920)
152. Podlubny, I.: Fractional-order systems and fractional-order controllers. Technical report UEF-03-94, Institute for Experimental Physics, Slovak Acad. Sci. (1994)
153. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
154. Podlubny, I.: Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. **5**, 367–386 (2002)
155. Podlubny, I., Dorcak, L., Misanek, J.: Application of fractional-order derivatives to calculation of heat load intensity change in blast furnace walls. Trans. Tech. Univ. Košice **5**, 137–144 (1995)
156. Popović, J.K., Atanacković, M.T., Pilipović, A.S., Rapaić, M.R., Pilipović, S., Atanacković, T.M.: A new approach to the compartmental analysis in pharmacokinetics: fractional time evolution of Diclofenac. J. Pharmacokinet. Pharmacodyn. **37**, 119–134 (2010)
157. Rabotnov, Yu.N.: Polzuchest Elementov Konstruktsii. Nauka, Moscow (1966); English translation: Creep Problems in Structural Members. North-Holland, Amsterdam (1969)
158. Rasof, B.: The initial- and final-value theorems in Laplace transform theory. J. Franklin Inst. **274**, 165–177 (1962)
159. Répaci, A.: Nonlinear dynamical systems: on the accuracy of Adomian's decomposition method. Appl. Math. Lett. **3**(4), 35–39 (1990)
160. Riesz, F., Sz.-Nagy, B.: Vorlesungen über Funktionalanalysis. Deutscher Verlag der Wissenschaften, Berlin (1956)
161. Roquejoffre, J.-M., Vázquez, J.-L.: Ignition and propagation in an integro-differential model for spherical flames. Discrete Contin. Dyn. Syst. Ser. B **2**, 379–387 (2002)
162. Ross, B.: The development of fractional calculus 1695–1900. Hist. Math. **4**, 75–89 (1977)
163. Rossikhin, Yu.A.: Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids. Appl. Mech. Rev. **63**, 010701 (2010)
164. Rossikhin, Yu.A., Shitikova, M.V.: Comparative analysis of viscoelastic models involving fractional derivatives of different orders. Fract. Calc. Appl. Anal. **10**, 111–121 (2007)
165. Rudin, W.: Principles of Mathematical Analysis. McGraw-Hill, New York (1953)
166. Sabatier, J., Agrawal, O.P., Tenreiro Machado, J.A. (eds.): Advances in Fractional Calculus. Springer, Dordrecht (2007)
167. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
168. Sard, A.: Integral representation of remainders. Duke Math. J. **15**, 333–345 (1948)
169. Scalas, E., Gorenflo, R., Mainardi, F.: Fractional calculus and continuous-time finance. Physica A **284**, 376–384 (2000)
170. Scalas, E., Gorenflo, R., Mainardi, F.: Uncoupled continuous-time random walks: analytic solution and limiting behaviour of the master equation. Phys. Rev. E **69**, 011107 (2004)
171. Schmidt, A., Gaul, L.: On a critique of a numerical scheme for the calculation of fractionally damped dynamical systems. Mech. Res. Commun. **33**, 99–107 (2006)
172. Scott Blair, G.W., Reiner, M.: The rheological law underlying the Nutting equation. Appl. Sci. Res. **2**, 225–234 (1951)
173. Sedletskij, A.M.: Asymptotic formulae for zeros of a function of Mittag-Leffler's type. Anal. Math. **20**, 117–132 (1994) (in Russian)
174. Seybold, H.J., Hilfer, R.: Numerical results for the generalized Mittag-Leffler functions. Fract. Calc. Appl. Anal. **8**, 127–139 (2005)

175. Seybold, H.J., Hilfer, R.: Numerical algorithm for calculating the generalized Mittag-Leffler function. SIAM J. Numer. Anal. **47**, 69–88 (2008)
176. Shaw, S., Warby, M.K., Whiteman, J.R.: A comparison of hereditary integral and internal variable approaches to numerical linear solid elasticity. In: Proceedings of the XIII Polish Conference on Computer Methods in Mechanics, Poznan (1997)
177. Singh, S.J., Chatterjee, A.: Galerkin projections and finite elements for fractional order derivatives. Nonlinear Dynam. **45**, 183–206 (2006)
178. Song, L., Xu, S.Y., Yang, J.Y.: Dynamical models of happiness with fractional order. Commun. Nonlinear Sci. Numer. Simulat. **15**, 616–628 (2010)
179. Strehmel, K., Weiner, R.: Numerik gewöhnlicher Differentialgleichungen. Teubner, Stuttgart (1995)
180. Taş, K., Tenreiro Machado, J.A., Baleanu, D. (eds.): Mathematical Methods in Engineering. Springer, Dordrecht (2007)
181. Tatari, M., Dehghan, M.: On the convergence of He's variational iteration method. J. Comput. Appl. Math. **207**, 121–128 (2007)
182. Tavazoei, M.S.: Comments on stability analysis of a class of nonlinear fractional-order systems. IEEE Trans. Circ. Syst. II **56**, 519–520 (2009)
183. Tavazoei, M.S., Haeri, M., Bolouki, S., Siami, M.: Stability preservation analysis for frequency-based methods in numerical simulation of fractional-order systems. SIAM J. Numer. Anal. **47**, 321–328 (2008)
184. Torvik, P.J., Bagley, R.L.: On the appearance of the fractional derivative in the behavior of real materials. J. Appl. Mech. **51**, 294–298 (1984)
185. Trinks, C., Ruge, P.: Treatment of dynamic systems with fractional derivatives without evaluating memory-integrals. Comput. Mech. **29**, 471–476 (2002)
186. Uchaikin, V.V.: Method of Fractional Derivatives. Artishok Publ. House, Ul'yanovsk (2008) (in Russian)
187. Verotta, D.: Fractional compartmental models and multi-term Mittag-Leffler response functions. J. Pharmacokinet. Pharmacodyn. **37**, 209–215 (2010)
188. Walz, G.: Asymptotics and Extrapolation. Akademie, Berlin (1996)
189. Weissinger, J.: Zur Theorie und Anwendung des Iterationsverfahrens. Math. Nachr. **8**, 193–212 (1952)
190. Williamson, J.H.: Lebesgue Integration. Holt, Rinehart and Winston, New York (1962)
191. Wiman, A.: Über den Fundamentalsatz in der Teorie der Funktionen $E_\alpha(x)$. Acta Math. **29**, 191–201 (1905)
192. Wiman, A.: Über die Nullstellen der Funktionen $E_\alpha(x)$. Acta Math. **29**, 217–234 (1905)
193. Woon, S.C.: Analytic continuation of operators. Applications: from number theory and group theory to quantum field and string theories. Rev. Math. Phys. **11**, 463–501 (1999)
194. Yuan, L., Agrawal, O.P.: A numerical scheme for dynamic systems containing fractional derivatives. J. Vib. Acoust. **124**, 321–324 (2002)

Index

Lecture Notes in Mathematics

For information about earlier volumes
please contact your bookseller or Springer
LNM Online archive: springerlink.com

Vol. 1817: E. Koelink, W. Van Assche (Eds.), Orthogonal Polynomials and Special Functions. Leuven 2002 (2003)
Vol. 1818: M. Bildhauer, Convex Variational Problems with Linear, nearly Linear and/or Anisotropic Growth Conditions (2003)
Vol. 1819: D. Masser, Yu. V. Nesterenko, H. P. Schlickewei, W. M. Schmidt, M. Waldschmidt, Diophantine Approximation. Cetraro, Italy 2000. Editors: F. Amoroso, U. Zannier (2003)
Vol. 1820: F. Hiai, H. Kosaki, Means of Hilbert Space Operators (2003)
Vol. 1821: S. Teufel, Adiabatic Perturbation Theory in Quantum Dynamics (2003)
Vol. 1822: S.-N. Chow, R. Conti, R. Johnson, J. Mallet-Paret, R. Nussbaum, Dynamical Systems. Cetraro, Italy 2000. Editors: J. W. Macki, P. Zecca (2003)
Vol. 1823: A. M. Anile, W. Allegretto, C. Ringhofer, Mathematical Problems in Semiconductor Physics. Cetraro, Italy 1998. Editor: A. M. Anile (2003)
Vol. 1824: J. A. Navarro González, J. B. Sancho de Salas, C^∞ – Differentiable Spaces (2003)
Vol. 1825: J. H. Bramble, A. Cohen, W. Dahmen, Multiscale Problems and Methods in Numerical Simulations, Martina Franca, Italy 2001. Editor: C. Canuto (2003)
Vol. 1826: K. Dohmen, Improved Bonferroni Inequalities via Abstract Tubes. Inequalities and Identities of Inclusion-Exclusion Type. VIII, 113 p, 2003.
Vol. 1827: K. M. Pilgrim, Combinations of Complex Dynamical Systems. IX, 118 p, 2003.
Vol. 1828: D. J. Green, Grbner Bases and the Computation of Group Cohomology. XII, 138 p, 2003.
Vol. 1829: E. Altman, B. Gaujal, A. Hordijk, Discrete-Event Control of Stochastic Networks: Multimodularity and Regularity. XIV, 313 p, 2003.
Vol. 1830: M. I. Gil', Operator Functions and Localization of Spectra. XIV, 256 p, 2003.
Vol. 1831: A. Connes, J. Cuntz, E. Guentner, N. Higson, J. E. Kaminker, Noncommutative Geometry, Martina Franca, Italy 2002. Editors: S. Doplicher, L. Longo (2004)
Vol. 1832: J. Azéma, M. Émery, M. Ledoux, M. Yor (Eds.), Séminaire de Probabilités XXXVII (2003)
Vol. 1833: D.-Q. Jiang, M. Qian, M.-P. Qian, Mathematical Theory of Nonequilibrium Steady States. On the Frontier of Probability and Dynamical Systems. IX, 280 p, 2004.
Vol. 1834: Yo. Yomdin, G. Comte, Tame Geometry with Application in Smooth Analysis. VIII, 186 p, 2004.
Vol. 1835: O.T. Izhboldin, B. Kahn, N.A. Karpenko, A. Vishik, Geometric Methods in the Algebraic Theory of Quadratic Forms. Summer School, Lens, 2000. Editor: J.-P. Tignol (2004)
Vol. 1836: C. Năstăsescu, F. Van Oystaeyen, Methods of Graded Rings. XIII, 304 p, 2004.
Vol. 1837: S. Tavaré, O. Zeitouni, Lectures on Probability Theory and Statistics. Ecole d'Eté de Probabilités de Saint-Flour XXXI-2001. Editor: J. Picard (2004)
Vol. 1838: A.J. Ganesh, N.W. O'Connell, D.J. Wischik, Big Queues. XII, 254 p, 2004.
Vol. 1839: R. Gohm, Noncommutative Stationary Processes. VIII, 170 p, 2004.
Vol. 1840: B. Tsirelson, W. Werner, Lectures on Probability Theory and Statistics. Ecole d'Eté de Probabilités de Saint-Flour XXXII-2002. Editor: J. Picard (2004)
Vol. 1841: W. Reichel, Uniqueness Theorems for Variational Problems by the Method of Transformation Groups (2004)
Vol. 1842: T. Johnsen, A. L. Knutsen, K_3 Projective Models in Scrolls (2004)
Vol. 1843: B. Jefferies, Spectral Properties of Noncommuting Operators (2004)
Vol. 1844: K.F. Siburg, The Principle of Least Action in Geometry and Dynamics (2004)
Vol. 1845: Min Ho Lee, Mixed Automorphic Forms, Torus Bundles, and Jacobi Forms (2004)
Vol. 1846: H. Ammari, H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements (2004)
Vol. 1847: T.R. Bielecki, T. Bjrk, M. Jeanblanc, M. Rutkowski, J.A. Scheinkman, W. Xiong, Paris-Princeton Lectures on Mathematical Finance 2003 (2004)
Vol. 1848: M. Abate, J. E. Fornaess, X. Huang, J. P. Rosay, A. Tumanov, Real Methods in Complex and CR Geometry, Martina Franca, Italy 2002. Editors: D. Zaitsev, G. Zampieri (2004)
Vol. 1849: Martin L. Brown, Heegner Modules and Elliptic Curves (2004)
Vol. 1850: V. D. Milman, G. Schechtman (Eds.), Geometric Aspects of Functional Analysis. Israel Seminar 2002-2003 (2004)
Vol. 1851: O. Catoni, Statistical Learning Theory and Stochastic Optimization (2004)
Vol. 1852: A.S. Kechris, B.D. Miller, Topics in Orbit Equivalence (2004)
Vol. 1853: Ch. Favre, M. Jonsson, The Valuative Tree (2004)
Vol. 1854: O. Saeki, Topology of Singular Fibers of Differential Maps (2004)
Vol. 1855: G. Da Prato, P.C. Kunstmann, I. Lasiecka, A. Lunardi, R. Schnaubelt, L. Weis, Functional Analytic Methods for Evolution Equations. Editors: M. Iannelli, R. Nagel, S. Piazzera (2004)
Vol. 1856: K. Back, T.R. Bielecki, C. Hipp, S. Peng, W. Schachermayer, Stochastic Methods in Finance, Bressanone/Brixen, Italy, 2003. Editors: M. Fritelli, W. Runggaldier (2004)
Vol. 1857: M. Émery, M. Ledoux, M. Yor (Eds.), Séminaire de Probabilités XXXVIII (2005)
Vol. 1858: A.S. Cherny, H.-J. Engelbert, Singular Stochastic Differential Equations (2005)
Vol. 1859: E. Letellier, Fourier Transforms of Invariant Functions on Finite Reductive Lie Algebras (2005)
Vol. 1860: A. Borisyuk, G.B. Ermentrout, A. Friedman, D. Terman, Tutorials in Mathematical Biosciences I. Mathematical Neurosciences (2005)

Vol. 1861: G. Benettin, J. Henrard, S. Kuksin, Hamiltonian Dynamics – Theory and Applications, Cetraro, Italy, 1999. Editor: A. Giorgilli (2005)
Vol. 1862: B. Helffer, F. Nier, Hypoelliptic Estimates and Spectral Theory for Fokker-Planck Operators and Witten Laplacians (2005)
Vol. 1863: H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms (2005)
Vol. 1864: K. Efstathiou, Metamorphoses of Hamiltonian Systems with Symmetries (2005)
Vol. 1865: D. Applebaum, B.V. R. Bhat, J. Kustermans, J. M. Lindsay, Quantum Independent Increment Processes I. From Classical Probability to Quantum Stochastic Calculus. Editors: M. Schürmann, U. Franz (2005)
Vol. 1866: O.E. Barndorff-Nielsen, U. Franz, R. Gohm, B. Kümmerer, S. Thorbjønsen, Quantum Independent Increment Processes II. Structure of Quantum Lévy Processes, Classical Probability, and Physics. Editors: M. Schürmann, U. Franz, (2005)
Vol. 1867: J. Sneyd (Ed.), Tutorials in Mathematical Biosciences II. Mathematical Modeling of Calcium Dynamics and Signal Transduction. (2005)
Vol. 1868: J. Jorgenson, S. Lang, $\mathrm{Pos}_n(\mathrm{R})$ and Eisenstein Series. (2005)
Vol. 1869: A. Dembo, T. Funaki, Lectures on Probability Theory and Statistics. Ecole d'Eté de Probabilités de Saint-Flour XXXIII-2003. Editor: J. Picard (2005)
Vol. 1870: V.I. Gurariy, W. Lusky, Geometry of Mntz Spaces and Related Questions. (2005)
Vol. 1871: P. Constantin, G. Gallavotti, A.V. Kazhikhov, Y. Meyer, S. Ukai, Mathematical Foundation of Turbulent Viscous Flows, Martina Franca, Italy, 2003. Editors: M. Cannone, T. Miyakawa (2006)
Vol. 1872: A. Friedman (Ed.), Tutorials in Mathematical Biosciences III. Cell Cycle, Proliferation, and Cancer (2006)
Vol. 1873: R. Mansuy, M. Yor, Random Times and Enlargements of Filtrations in a Brownian Setting (2006)
Vol. 1874: M. Yor, M. Émery (Eds.), In Memoriam Paul-Andr Meyer - Sminaire de Probabilits XXXIX (2006)
Vol. 1875: J. Pitman, Combinatorial Stochastic Processes. Ecole d'Et de Probabilits de Saint-Flour XXXII-2002. Editor: J. Picard (2006)
Vol. 1876: H. Herrlich, Axiom of Choice (2006)
Vol. 1877: J. Steuding, Value Distributions of *L*-Functions (2007)
Vol. 1878: R. Cerf, The Wulff Crystal in Ising and Percolation Models, Ecole d'Et de Probabilités de Saint-Flour XXXIV-2004. Editor: Jean Picard (2006)
Vol. 1879: G. Slade, The Lace Expansion and its Applications, Ecole d'Et de Probabilits de Saint-Flour XXXIV-2004. Editor: Jean Picard (2006)
Vol. 1880: S. Attal, A. Joye, C.-A. Pillet, Open Quantum Systems I, The Hamiltonian Approach (2006)
Vol. 1881: S. Attal, A. Joye, C.-A. Pillet, Open Quantum Systems II, The Markovian Approach (2006)
Vol. 1882: S. Attal, A. Joye, C.-A. Pillet, Open Quantum Systems III, Recent Developments (2006)
Vol. 1883: W. Van Assche, F. Marcellàn (Eds.), Orthogonal Polynomials and Special Functions, Computation and Application (2006)
Vol. 1884: N. Hayashi, E.I. Kaikina, P.I. Naumkin, I.A. Shishmarev, Asymptotics for Dissipative Nonlinear Equations (2006)
Vol. 1885: A. Telcs, The Art of Random Walks (2006)
Vol. 1886: S. Takamura, Splitting Deformations of Degenerations of Complex Curves (2006)
Vol. 1887: K. Habermann, L. Habermann, Introduction to Symplectic Dirac Operators (2006)
Vol. 1888: J. van der Hoeven, Transseries and Real Differential Algebra (2006)
Vol. 1889: G. Osipenko, Dynamical Systems, Graphs, and Algorithms (2006)
Vol. 1890: M. Bunge, J. Funk, Singular Coverings of Toposes (2006)
Vol. 1891: J.B. Friedlander, D.R. Heath-Brown, H. Iwaniec, J. Kaczorowski, Analytic Number Theory, Cetraro, Italy, 2002. Editors: A. Perelli, C. Viola (2006)
Vol. 1892: A. Baddeley, I. Bárány, R. Schneider, W. Weil, Stochastic Geometry, Martina Franca, Italy, 2004. Editor: W. Weil (2007)
Vol. 1893: H. Hanßmann, Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems, Results and Examples (2007)
Vol. 1894: C.W. Groetsch, Stable Approximate Evaluation of Unbounded Operators (2007)
Vol. 1895: L. Molnár, Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces (2007)
Vol. 1896: P. Massart, Concentration Inequalities and Model Selection, Ecole d'Été de Probabilités de Saint-Flour XXXIII-2003. Editor: J. Picard (2007)
Vol. 1897: R. Doney, Fluctuation Theory for Lévy Processes, Ecole d'Été de Probabilités de Saint-Flour XXXV-2005. Editor: J. Picard (2007)
Vol. 1898: H.R. Beyer, Beyond Partial Differential Equations, On linear and Quasi-Linear Abstract Hyperbolic Evolution Equations (2007)
Vol. 1899: Séminaire de Probabilités XL. Editors: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (2007)
Vol. 1900: E. Bolthausen, A. Bovier (Eds.), Spin Glasses (2007)
Vol. 1901: O. Wittenberg, Intersections de deux quadriques et pinceaux de courbes de genre 1, Intersections of Two Quadrics and Pencils of Curves of Genus 1 (2007)
Vol. 1902: A. Isaev, Lectures on the Automorphism Groups of Kobayashi-Hyperbolic Manifolds (2007)
Vol. 1903: G. Kresin, V. Maz'ya, Sharp Real-Part Theorems (2007)
Vol. 1904: P. Giesl, Construction of Global Lyapunov Functions Using Radial Basis Functions (2007)
Vol. 1905: C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations (2007)
Vol. 1906: T. Schuster, The Method of Approximate Inverse: Theory and Applications (2007)
Vol. 1907: M. Rasmussen, Attractivity and Bifurcation for Nonautonomous Dynamical Systems (2007)
Vol. 1908: T.J. Lyons, M. Caruana, T. Lévy, Differential Equations Driven by Rough Paths, Ecole d'Été de Probabilités de Saint-Flour XXXIV-2004 (2007)
Vol. 1909: H. Akiyoshi, M. Sakuma, M. Wada, Y. Yamashita, Punctured Torus Groups and 2-Bridge Knot Groups (I) (2007)
Vol. 1910: V.D. Milman, G. Schechtman (Eds.), Geometric Aspects of Functional Analysis. Israel Seminar 2004-2005 (2007)
Vol. 1911: A. Bressan, D. Serre, M. Williams, K. Zumbrun, Hyperbolic Systems of Balance Laws. Cetraro, Italy 2003. Editor: P. Marcati (2007)

Vol. 1912: V. Berinde, Iterative Approximation of Fixed Points (2007)
Vol. 1913: J.E. Marsden, G. Misiołek, J.-P. Ortega, M. Perlmutter, T.S. Ratiu, Hamiltonian Reduction by Stages (2007)
Vol. 1914: G. Kutyniok, Affine Density in Wavelet Analysis (2007)
Vol. 1915: T. Bıyıkoğlu, J. Leydold, P.F. Stadler, Laplacian Eigenvectors of Graphs. Perron-Frobenius and Faber-Krahn Type Theorems (2007)
Vol. 1916: C. Villani, F. Rezakhanlou, Entropy Methods for the Boltzmann Equation. Editors: F. Golse, S. Olla (2008)
Vol. 1917: I. Veselić, Existence and Regularity Properties of the Integrated Density of States of Random Schrdinger (2008)
Vol. 1918: B. Roberts, R. Schmidt, Local Newforms for GSp(4) (2007)
Vol. 1919: R.A. Carmona, I. Ekeland, A. Kohatsu-Higa, J.-M. Lasry, P.-L. Lions, H. Pham, E. Taflin, Paris-Princeton Lectures on Mathematical Finance 2004. Editors: R.A. Carmona, E. inlar, I. Ekeland, E. Jouini, J.A. Scheinkman, N. Touzi (2007)
Vol. 1920: S.N. Evans, Probability and Real Trees. Ecole d'Été de Probabilités de Saint-Flour XXXV-2005 (2008)
Vol. 1921: J.P. Tian, Evolution Algebras and their Applications (2008)
Vol. 1922: A. Friedman (Ed.), Tutorials in Mathematical BioSciences IV. Evolution and Ecology (2008)
Vol. 1923: J.P.N. Bishwal, Parameter Estimation in Stochastic Differential Equations (2008)
Vol. 1924: M. Wilson, Littlewood-Paley Theory and Exponential-Square Integrability (2008)
Vol. 1925: M. du Sautoy, L. Woodward, Zeta Functions of Groups and Rings (2008)
Vol. 1926: L. Barreira, V. Claudia, Stability of Nonautonomous Differential Equations (2008)
Vol. 1927: L. Ambrosio, L. Caffarelli, M.G. Crandall, L.C. Evans, N. Fusco, Calculus of Variations and Non-Linear Partial Differential Equations. Cetraro, Italy 2005. Editors: B. Dacorogna, P. Marcellini (2008)
Vol. 1928: J. Jonsson, Simplicial Complexes of Graphs (2008)
Vol. 1929: Y. Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes (2008)
Vol. 1930: J.M. Urbano, The Method of Intrinsic Scaling. A Systematic Approach to Regularity for Degenerate and Singular PDEs (2008)
Vol. 1931: M. Cowling, E. Frenkel, M. Kashiwara, A. Valette, D.A. Vogan, Jr., N.R. Wallach, Representation Theory and Complex Analysis. Venice, Italy 2004. Editors: E.C. Tarabusi, A. D'Agnolo, M. Picardello (2008)
Vol. 1932: A.A. Agrachev, A.S. Morse, E.D. Sontag, H.J. Sussmann, V.I. Utkin, Nonlinear and Optimal Control Theory. Cetraro, Italy 2004. Editors: P. Nistri, G. Stefani (2008)
Vol. 1933: M. Petkovic, Point Estimation of Root Finding Methods (2008)
Vol. 1934: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (Eds.), Séminaire de Probabilités XLI (2008)
Vol. 1935: A. Unterberger, Alternative Pseudodifferential Analysis (2008)
Vol. 1936: P. Magal, S. Ruan (Eds.), Structured Population Models in Biology and Epidemiology (2008)
Vol. 1937: G. Capriz, P. Giovine, P.M. Mariano (Eds.), Mathematical Models of Granular Matter (2008)
Vol. 1938: D. Auroux, F. Catanese, M. Manetti, P. Seidel, B. Siebert, I. Smith, G. Tian, Symplectic 4-Manifolds and Algebraic Surfaces. Cetraro, Italy 2003. Editors: F. Catanese, G. Tian (2008)
Vol. 1939: D. Boffi, F. Brezzi, L. Demkowicz, R.G. Durán, R.S. Falk, M. Fortin, Mixed Finite Elements, Compatibility Conditions, and Applications. Cetraro, Italy 2006. Editors: D. Boffi, L. Gastaldi (2008)
Vol. 1940: J. Banasiak, V. Capasso, M.A.J. Chaplain, M. Lachowicz, J. Miękisz, Multiscale Problems in the Life Sciences. From Microscopic to Macroscopic. Będlewo, Poland 2006. Editors: V. Capasso, M. Lachowicz (2008)
Vol. 1941: S.M.J. Haran, Arithmetical Investigations. Representation Theory, Orthogonal Polynomials, and Quantum Interpolations (2008)
Vol. 1942: S. Albeverio, F. Flandoli, Y.G. Sinai, SPDE in Hydrodynamic. Recent Progress and Prospects. Cetraro, Italy 2005. Editors: G. Da Prato, M. Rckner (2008)
Vol. 1943: L.L. Bonilla (Ed.), Inverse Problems and Imaging. Martina Franca, Italy 2002 (2008)
Vol. 1944: A. Di Bartolo, G. Falcone, P. Plaumann, K. Strambach, Algebraic Groups and Lie Groups with Few Factors (2008)
Vol. 1945: F. Brauer, P. van den Driessche, J. Wu (Eds.), Mathematical Epidemiology (2008)
Vol. 1946: G. Allaire, A. Arnold, P. Degond, T.Y. Hou, Quantum Transport. Modelling, Analysis and Asymptotics. Cetraro, Italy 2006. Editors: N.B. Abdallah, G. Frosali (2008)
Vol. 1947: D. Abramovich, M. Mariño, M. Thaddeus, R. Vakil, Enumerative Invariants in Algebraic Geometry and String Theory. Cetraro, Italy 2005. Editors: K. Behrend, M. Manetti (2008)
Vol. 1948: F. Cao, J-L. Lisani, J-M. Morel, P. Mus, F. Sur, A Theory of Shape Identification (2008)
Vol. 1949: H.G. Feichtinger, B. Helffer, M.P. Lamoureux, N. Lerner, J. Toft, Pseudo-Differential Operators. Quantization and Signals. Cetraro, Italy 2006. Editors: L. Rodino, M.W. Wong (2008)
Vol. 1950: M. Bramson, Stability of Queueing Networks, Ecole d'Eté de Probabilits de Saint-Flour XXXVI-2006 (2008)
Vol. 1951: A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia, A Non Linear Transfer Technique for Renorming (2009)
Vol. 1952: R. Mikhailov, I.B.S. Passi, Lower Central and Dimension Series of Groups (2009)
Vol. 1953: K. Arwini, C.T.J. Dodson, Information Geometry (2008)
Vol. 1954: P. Biane, L. Bouten, F. Cipriani, N. Konno, N. Privault, Q. Xu, Quantum Potential Theory. Editors: U. Franz, M. Schuermann (2008)
Vol. 1955: M. Bernot, V. Caselles, J.-M. Morel, Optimal Transportation Networks (2008)
Vol. 1956: C.H. Chu, Matrix Convolution Operators on Groups (2008)
Vol. 1957: A. Guionnet, On Random Matrices: Macroscopic Asymptotics, Ecole d'Eté de Probabilits de Saint-Flour XXXVI-2006 (2009)
Vol. 1958: M.C. Olsson, Compactifying Moduli Spaces for Abelian Varieties (2008)
Vol. 1959: Y. Nakkajima, A. Shiho, Weight Filtrations on Log Crystalline Cohomologies of Families of Open Smooth Varieties (2008)
Vol. 1960: J. Lipman, M. Hashimoto, Foundations of Grothendieck Duality for Diagrams of Schemes (2009)

Vol. 1961: G. Buttazzo, A. Pratelli, S. Solimini, E. Stepanov, Optimal Urban Networks via Mass Transportation (2009)
Vol. 1962: R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, Y. Xiao, A Minicourse on Stochastic Partial Differential Equations (2009)
Vol. 1963: W. Siegert, Local Lyapunov Exponents (2009)
Vol. 1964: W. Roth, Operator-valued Measures and Integrals for Cone-valued Functions and Integrals for Cone-valued Functions (2009)
Vol. 1965: C. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations (2009)
Vol. 1966: D. Deng, Y. Han, Harmonic Analysis on Spaces of Homogeneous Type (2009)
Vol. 1967: B. Fresse, Modules over Operads and Functors (2009)
Vol. 1968: R. Weissauer, Endoscopy for GSP(4) and the Cohomology of Siegel Modular Threefolds (2009)
Vol. 1969: B. Roynette, M. Yor, Penalising Brownian Paths (2009)
Vol. 1970: M. Biskup, A. Bovier, F. den Hollander, D. Ioffe, F. Martinelli, K. Netočný, F. Toninelli, Methods of Contemporary Mathematical Statistical Physics. Editor: R. Kotecký (2009)
Vol. 1971: L. Saint-Raymond, Hydrodynamic Limits of the Boltzmann Equation (2009)
Vol. 1972: T. Mochizuki, Donaldson Type Invariants for Algebraic Surfaces (2009)
Vol. 1973: M.A. Berger, L.H. Kauffmann, B. Khesin, H.K. Moffatt, R.L. Ricca, De W. Sumners, Lectures on Topological Fluid Mechanics. Cetraro, Italy 2001. Editor: R.L. Ricca (2009)
Vol. 1974: F. den Hollander, Random Polymers: École d'Été de Probabilités de Saint-Flour XXXVII – 2007 (2009)
Vol. 1975: J.C. Rohde, Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication (2009)
Vol. 1976: N. Ginoux, The Dirac Spectrum (2009)
Vol. 1977: M.J. Gursky, E. Lanconelli, A. Malchiodi, G. Tarantello, X.-J. Wang, P.C. Yang, Geometric Analysis and PDEs. Cetraro, Italy 2001. Editors: A. Ambrosetti, S.-Y.A. Chang, A. Malchiodi (2009)
Vol. 1978: M. Qian, J.-S. Xie, S. Zhu, Smooth Ergodic Theory for Endomorphisms (2009)
Vol. 1979: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (Eds.), Śeminaire de Probablitiés XLII (2009)
Vol. 1980: P. Graczyk, A. Stos (Eds.), Potential Analysis of Stable Processes and its Extensions (2009)
Vol. 1981: M. Chlouveraki, Blocks and Families for Cyclotomic Hecke Algebras (2009)
Vol. 1982: N. Privault, Stochastic Analysis in Discrete and Continuous Settings. With Normal Martingales (2009)
Vol. 1983: H. Ammari (Ed.), Mathematical Modeling in Biomedical Imaging I. Electrical and Ultrasound Tomographies, Anomaly Detection, and Brain Imaging (2009)
Vol. 1984: V. Caselles, P. Monasse, Geometric Description of Images as Topographic Maps (2010)
Vol. 1985: T. Linß, Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems (2010)
Vol. 1986: J.-P. Antoine, C. Trapani, Partial Inner Product Spaces. Theory and Applications (2009)
Vol. 1987: J.-P. Brasselet, J. Seade, T. Suwa, Vector Fields on Singular Varieties (2010)
Vol. 1988: M. Broué, Introduction to Complex Reflection Groups and Their Braid Groups (2010)
Vol. 1989: I.M. Bomze, V. Demyanov, Nonlinear Optimization. Cetraro, Italy 2007. Editors: G. di Pillo, F. Schoen (2010)
Vol. 1990: S. Bouc, Biset Functors for Finite Groups (2010)
Vol. 1991: F. Gazzola, H.-C. Grunau, G. Sweers, Polyharmonic Boundary Value Problems (2010)
Vol. 1992: A. Parmeggiani, Spectral Theory of Non-Commutative Harmonic Oscillators: An Introduction (2010)
Vol. 1993: P. Dodos, Banach Spaces and Descriptive Set Theory: Selected Topics (2010)
Vol. 1994: A. Baricz, Generalized Bessel Functions of the First Kind (2010)
Vol. 1995: A.Y. Khapalov, Controllability of Partial Differential Equations Governed by Multiplicative Controls (2010)
Vol. 1996: T. Lorenz, Mutational Analysis. A Joint Framework for Cauchy Problems *In* and *Beyond* Vector Spaces (2010)
Vol. 1997: M. Banagl, Intersection Spaces, Spatial Homology Truncation, and String Theory (2010)
Vol. 1998: M. Abate, E. Bedford, M. Brunella, T.-C. Dinh, D. Schleicher, N. Sibony, Holomorphic Dynamical Systems. Cetraro, Italy 2008. Editors: G. Gentili, J. Guenot, G. Patrizio (2010)
Vol. 1999: H. Schoutens, The Use of Ultraproducts in Commutative Algebra (2010)
Vol. 2000: H. Yserentant, Regularity and Approximability of Electronic Wave Functions (2010)
Vol. 2001: T. Duquesne, O.E. Barndorff-Nielson, O. Reichmann, J. Bertoin, K.-I. Sato, J. Jacod, C. Schwab, C. Klüppelberg, Lévy Matters I (2010)
Vol. 2002: C. Pötzsche, Geometric Theory of Discrete Nonautonomous Dynamical Systems (2010)
Vol. 2003: A. Cousin, S. Crépey, O. Guéant, D. Hobson, M. Jeanblanc, J.-M. Lasry, J.-P. Laurent, P.-L. Lions, P. Tankov, Paris-Princeton Lectures on Mathematical Finance 2010. Editors: R.A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, J.A. Scheinkman, N. Touzi (2010)
Vol. 2004: K. Diethelm, The Analysis of Fractional Differential Equations (2010)

Recent Reprints and New Editions

Vol. 1702: J. Ma, J. Yong, Forward-Backward Stochastic Differential Equations and their Applications. 1999 – Corr. 3rd printing (2007)
Vol. 830: J.A. Green, Polynomial Representations of GL_n, with an Appendix on Schensted Correspondence and Littelmann Paths by K. Erdmann, J.A. Green and M. Schoker 1980 – 2nd corr. and augmented edition (2007)
Vol. 1693: S. Simons, From Hahn-Banach to Monotonicity (Minimax and Monotonicity 1998) – 2nd exp. edition (2008)
Vol. 470: R.E. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. With a preface by D. Ruelle. Edited by J.-R. Chazottes. 1975 – 2nd rev. edition (2008)
Vol. 523: S.A. Albeverio, R.J. Høegh-Krohn, S. Mazzucchi, Mathematical Theory of Feynman Path Integral. 1976 – 2nd corr. and enlarged edition (2008)
Vol. 1764: A. Cannas da Silva, Lectures on Symplectic Geometry 2001 – Corr. 2nd printing (2008)

LECTURE NOTES IN MATHEMATICS

Springer

Edited by J.-M. Morel, F. Takens, B. Teissier, P.K. Maini

Editorial Policy (for the publication of monographs)

1. Lecture Notes aim to report new developments in all areas of mathematics and their applications - quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome.

   Monograph manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They may be based on specialised lecture courses. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for doctoral theses to be accepted for the Lecture Notes series, though habilitation theses may be appropriate.

2. Manuscripts should be submitted either to Springer's mathematics editorial in Heidelberg, or to one of the series editors. In general, manuscripts will be sent out to 2 external referees for evaluation. If a decision cannot yet be reached on the basis of the first 2 reports, further referees may be contacted: The author will be informed of this. A final decision to publish can be made only on the basis of the complete manuscript, however a refereeing process leading to a preliminary decision can be based on a pre-final or incomplete manuscript. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter, a bibliography and several sample chapters.

   Authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer refereeing times and nevertheless unclear referees' recommendations, making further refereeing of a final draft necessary.

   Authors should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

3. Manuscripts should in general be submitted in English. Final manuscripts should contain at least 100 pages of mathematical text and should always include

   – a table of contents;
   – an informative introduction, with adequate motivation and perhaps some historical remarks: it should be accessible to a reader not intimately familiar with the topic treated;
   – a subject index: as a rule this is genuinely helpful for the reader.

   For evaluation purposes, manuscripts may be submitted in print or electronic form, in the latter case preferably as pdf- or zipped ps-files. Lecture Notes volumes are, as a rule, printed digitally from the authors' files. To ensure best results, authors are asked to use the LaTeX2e style files available from Springer's web-server at:

   ftp://ftp.springer.de/pub/tex/latex/svmonot1/ (for monographs).

Additional technical instructions, if necessary, are available on request from: lnm@springer.com.

4. Careful preparation of the manuscripts will help keep production time short besides ensuring satisfactory appearance of the finished book in print and online. After acceptance of the manuscript authors will be asked to prepare the final LaTeX source files (and also the corresponding dvi-, pdf- or zipped ps-file) together with the final printout made from these files. The LaTeX source files are essential for producing the full-text online version of the book (see www.springerlink.com/content/110312 for the existing online volumes of LNM).

   The actual production of a Lecture Notes volume takes approximately 12 weeks.

5. Authors receive a total of 50 free copies of their volume, but no royalties. They are entitled to a discount of 33.3% on the price of Springer books purchased for their personal use, if ordering directly from Springer.

6. Commitment to publish is made by letter of intent rather than by signing a formal contract. Springer-Verlag secures the copyright for each volume. Authors are free to reuse material contained in their LNM volumes in later publications: a brief written (or e-mail) request for formal permission is sufficient.

Addresses:

Professor J.-M. Morel, CMLA,
École Normale Supérieure de Cachan,
61 Avenue du Président Wilson, 94235 Cachan Cedex, France
E-mail: Jean-Michel.Morel@cmla.ens-cachan.fr

Professor F. Takens, Mathematisch Instituut,
Rijksuniversiteit Groningen, Postbus 800,
9700 AV Groningen, The Netherlands
E-mail: F.Takens@math.rug.nl

Professor B. Teissier, Institut Mathématique de Jussieu,
UMR 7586 du CNRS, Équipe "Géométrie et Dynamique",
175 rue du Chevaleret
75013 Paris, France
E-mail: teissier@math.jussieu.fr

For the "Mathematical Biosciences Subseries" of LNM:

Professor P.K. Maini, Center for Mathematical Biology,
Mathematical Institute, 24-29 St Giles,
Oxford OX1 3LP, UK
E-mail: maini@maths.ox.ac.uk

Springer, Mathematics Editorial I, Tiergartenstr. 17
69121 Heidelberg, Germany,
Tel.: +49 (6221) 487-8259
Fax: +49 (6221) 4876-8259
E-mail: lnm@springer.com