Appendix A
Discrete Inequalities

Throughout, let $\mathbb{I}$ be a discrete interval.

A.1 Generalized Exponential Function

We naturally equip the set of sequences $a : \mathbb{I}' \to \mathbb{F}$ with the operations $+, \cdot$ yielding an algebra. For real-valued $a, b : \mathbb{I}' \to \mathbb{R}$ we write $a \leq b$, if $a(k) \leq b(k)$ is satisfied for all $k \in \mathbb{I}'$; correspondingly one defines $a < b$ and the uniform difference

$$a \ll b \quad :\Leftrightarrow \quad 0 < \lfloor b - a \rfloor := \inf_{k \in \mathbb{I}'} (b(k) - a(k)),$$

as well as the supremum $\lceil a \rceil := \sup_{k \in \mathbb{I}'} a(k)$. In addition, intervals are defined as

$$[a, b] := \{c : \mathbb{I}' \to \mathbb{R} | \, a \leq c \leq b\}, \qquad (a, b) := \{c : \mathbb{I}' \to \mathbb{R} | \, a \ll c \ll b\}$$

and similarly for half-open intervals. Merely for notational reasons, it is advantageous to introduce

Definition A.1.1. Let $k, \kappa \in \mathbb{I}$ and $a : \mathbb{I}' \to \mathbb{F}$. The *generalized exponential function* $e_a : \left\{(k, \kappa) \in \mathbb{I}^2 : \, \kappa \leq k\right\} \to \mathbb{F}$ is given by the product

$$e_a(k, \kappa) := \prod_{n=\kappa}^{k-1} a(n) \quad \text{for all } \kappa \leq k.$$

In case, $a(n) \neq 0$ for $n \in \{k, \ldots, \kappa - 1\}$ we extend e_a to $\mathbb{I}^2$ by

$$e_a(k, \kappa) := \prod_{n=k}^{\kappa-1} a(n)^{-1} \quad \text{for all } k < \kappa.$$

The next result is elementary and merely an observation for later reference.

Proposition A.1.2 (properties of e_a). *Let $k, l, \kappa \in \mathbb{I}$ and $a, b : \mathbb{I}' \to \mathbb{F}$. Then one has $e_1(k, l) = 1$ and the following holds:*

(a) Semigroup property

$$e_a(k, l)e_a(l, \kappa) = e_a(k, \kappa) \quad \textit{for all } \kappa \leq l \leq k \tag{A.1a}$$

and (A.1a) *holds for all $k, l, \kappa \in \mathbb{I}$, if $a(n) \neq 0$ for all $n \in \mathbb{I}'$.*

(b) Multiplication theorem

$$e_a(k, l)e_b(k, l) = e_{ab}(k, l) \quad \textit{for all } l \leq k \tag{A.1b}$$

and (A.1b) *holds for all $k, l \in \mathbb{I}$, if $a(n) \neq 0$ for all $n \in \mathbb{I}'$.*

(c) In case $\alpha \in \mathbb{Z}$ and $a(n) \neq 0$ for $n \in \{l, \dots, k-1\}$ it is

$$e_a(k, l)^\alpha = e_{a^\alpha}(k, l) \quad \textit{for all } l \leq k \tag{A.1c}$$

and (A.1c) *holds for all $k, l \in \mathbb{I}$, if $a(n) \neq 0$ for all $n \in \mathbb{I}'$.*

(d) For $\mathbb{F} = \mathbb{R}$ one has the monotonicity theorem

$$0 < a \leq b \quad \Rightarrow \quad \begin{cases} e_a(k, l) \leq e_b(k, l) & \textit{for all } l \leq k, \\ e_b(k, l) \leq e_a(k, l) & \textit{for all } k \leq l. \end{cases}$$

Proof. We omit the elementary proof. □

Lemma A.1.3 (asymptotic behavior of e_a). *Let $\kappa \in \mathbb{I}$, $\gamma > 0$ and $a : \mathbb{I}' \to \mathbb{F}$.*

(a) If $\mathbb{I}$ is unbounded above and there exists a $K > 1$ with $|a(k)| \leq \left(1 - \frac{1}{k}\right)^\gamma$ for all $k \geq K$, then $\lim_{k\to\infty} e_a(k, \kappa) = 0$.

(b) If $\mathbb{I}$ is unbounded below, $a(k) \neq 0$ for $k < \kappa$ and there exists a $K < 1$ with $|a(k)| \geq \left(1 - \frac{1}{k}\right)^\gamma$ for all $k < K$, then $\lim_{k\to-\infty} e_a(k, \kappa) = 0$.

Remark A.1.4. (1) Obviously, assertion (a) holds in case $0 \leq a \ll 1$, whereas (b) is fulfilled for $1 \ll a$. In both situations one obtains even exponential convergence.

(2) It is an easy consequence of Lemma A.1.3 that the following implications hold true for functions $a, b : \mathbb{I} \to \mathbb{R}$ with $0 < a \ll b$:

$$\limsup_{k\to\infty} \tfrac{b(k)}{k} < \lfloor b - a \rfloor \quad \Rightarrow \quad \lim_{k\to\infty} e_{\frac{a}{b}}(k, \kappa) = 0,$$
$$\limsup_{k\to-\infty} -\tfrac{a(k)}{k} < \lfloor b - a \rfloor \quad \Rightarrow \quad \lim_{k\to-\infty} e_{\frac{b}{a}}(k, \kappa) = 0.$$

(3) If for every $\varepsilon > 0$ there exists an $N = N(\varepsilon) \geq 0$ such that

$$|e_a(k, \kappa)| \leq \varepsilon \quad \text{for all } k, \kappa \in \mathbb{I},\ k - \kappa \geq N,$$

then there exist $C \geq 1$, $\alpha \in (0, 1)$ such that $|e_a(k, \kappa)| \leq C\alpha^{k-\kappa}$ for all $\kappa \leq k$.

Proof. In both cases we employ Proposition A.1.2(a). However, since assertion (a) can be shown analogously, we only prove (b). This follows from

$$\begin{aligned} |e_a(\kappa,k)| &\overset{(A.1a)}{=} |e_a(\kappa,K)|\,|e_a(K,k)| \geq |e_a(\kappa,K)| \prod_{n=k}^{K-1} \left(\tfrac{n-1}{n}\right)^\gamma \\ &= |e_a(\kappa,K)| \left(\tfrac{k-1}{K-1}\right)^\gamma \xrightarrow[k\to-\infty]{} \infty \quad \text{for all } \kappa \in \mathbb{I}, \end{aligned}$$

since we have $\lim_{k\to-\infty} |e_a(k,\kappa)| = \lim_{k\to-\infty} |e_a(\kappa,k)|^{-1} = 0$. □

Lemma A.1.5. *Let $k_1, k_2, k, \kappa \in \mathbb{I}$ with $k_1 \leq k_2$. If $a, b : \mathbb{I}' \to \mathbb{R}$ are positive sequences, then the following holds:*

(a) In case $a \ll b$ one has

$$\sum_{n=k_1}^{k_2-1} e_a(k,n+1) e_b(n,\kappa) \leq \frac{e_a(k,\kappa)}{\lfloor b-a \rfloor} \left[e_{\frac{b}{a}}(k_2,\kappa) - e_{\frac{b}{a}}(k_1,\kappa) \right]. \tag{A.1d}$$

(b) In case $b \ll a$ one has

$$\sum_{n=k_1}^{k_2-1} e_a(k,n+1) e_b(n,\kappa) \leq \frac{e_a(k,\kappa)}{\lfloor a-b \rfloor} \left[e_{\frac{b}{a}}(k_1,\kappa) - e_{\frac{b}{a}}(k_2,\kappa) \right]. \tag{A.1e}$$

Proof. Let $k_1, k_2, k, \kappa \in \mathbb{I}$ with $k_1 \leq k_2$. We derive a preparatory identity, which yields from elementary properties of the exponential function in Proposition A.1.2:

$$\begin{aligned} \sum_{n=k_1}^{k_2-1} e_a(k,n+1) e_b(n,\kappa) &\overset{(A.1a)}{=} e_a(k,\kappa) \sum_{n=k_1}^{k_2-1} e_a(\kappa,n+1) e_b(n,\kappa) \\ &\overset{(A.1b)}{=} e_a(k,\kappa) \sum_{n=k_1}^{k_2-1} \frac{e_{\frac{b}{a}}(n,\kappa)}{a(n)} = e_a(k,\kappa) \sum_{n=k_1}^{k_2-1} \frac{e_{\frac{b}{a}}(n+1,\kappa) - e_{\frac{b}{a}}(n,\kappa)}{b(n)-a(n)}. \end{aligned} \tag{A.1f}$$

(a) From (A.1f) one immediately gets by "telescopic summation" that

$$\begin{aligned} \sum_{n=k_1}^{k_2-1} e_a(k,n+1) e_b(n,\kappa) &\overset{(A.1f)}{\leq} \frac{e_a(k,\kappa)}{\lfloor b-a \rfloor} \sum_{n=k_1}^{k_2-1} \left[e_{\frac{b}{a}}(n+1,\kappa) - e_{\frac{b}{a}}(n,\kappa) \right] \\ &= \frac{e_a(k,\kappa)}{\lfloor b-a \rfloor} \left[e_{\frac{b}{a}}(k_2,\kappa) - e_{\frac{b}{a}}(k_1,\kappa) \right]. \end{aligned}$$

(b) This yields analogously to (a) from (A.1f). □

A.2 Gronwall Inequalities

We present a sufficiently general discrete version of the Gronwall lemma.

Proposition A.2.1 (Gronwall inequality). *Let $\kappa \in \mathbb{I}$ be given.*

(a) If the sequences $a, b, u : \mathbb{I}_\kappa^+ \to \mathbb{R}$ satisfy $b(k) \geq 0$ and

$$u(k) \leq a(k) + \sum_{l=\kappa}^{k-1} b(l)u(l) \quad \textit{for all } \kappa \leq k, \tag{A.2a}$$

then one has the explicit estimate for all $\kappa \leq k$,

$$u(k) \leq e_{1+b}(k,\kappa)a(\kappa) + \sum_{l=\kappa}^{k-1} e_{1+b}(k, l+1)\left[a'(l) - a(l)\right]. \tag{A.2b}$$

(b) If the sequences $a, b, u : \mathbb{I}_\kappa^- \to \mathbb{R}$ satisfy $b(k) \in [0,1)$ and

$$u(k) \leq a(k) + \sum_{l=k}^{\kappa-1} b(l)u(l) \quad \textit{for all } k \leq \kappa,$$

then one has the explicit estimate

$$u(k) \leq e_{1-b}(k,\kappa)a(\kappa) + \sum_{l=k}^{\kappa-1} e_{1-b}(k, l+1)\left[a(l) - a'(l)\right] \quad \textit{for all } k < \kappa.$$

Remark A.2.2. Inequality (A.2b) is the best possible, in the sense that equality in (A.2a) implies equality in (A.2b).

Proof. Since the assertion (a) can be shown similarly we restrict to the proof of the slightly more involved assertion (b). We set for abbreviation

$$c(k) := a(k) + \sum_{l=k}^{\kappa-1} b(l)u(l) \quad \text{for all } k \leq \kappa$$

and according to our assumptions one has $u(k) \leq c(k)$, which yields

$$c(k) - c'(k) = a(k) - a'(k) + b(k)u(k) \leq a(k) - a'(k) + b(k)c(k),$$

hence, one has $c(k) \leq (1 - b(k))^{-1} (c'(k) + a(k) - b'(k))$ for all $k < \kappa$. By mathematical induction in backward time we get

$$c(k) \leq e_{1-b}(k, \kappa)c(\kappa) + \sum_{l=k}^{\kappa-1} e_{1-b}(k, l+1) \left[a(l) - a'(l)\right] \quad \text{for all } k < \kappa$$

and due to $c(\kappa) = a(\kappa)$, $u(k) \leq c(k)$ this leads to the assertion. □

Proposition A.2.3 (uniform Gronwall inequality). *Let $\kappa \in \mathbb{I}$. If $a, b, u : \mathbb{I}_\kappa^+ \to \mathbb{R}$ are sequences satisfying*

$$u'(k) \leq a(k)u(k) + b(k) \quad \textit{for all } \kappa \leq k, \tag{A.2c}$$

then the following holds:

(a) In case $a(k) \geq 0$ one has

$$u(k) \leq e_a(k, \kappa)u(\kappa) + \sum_{l=\kappa}^{k-1} e_a(k, l+1)b(l) \quad \textit{for all } \kappa \leq k.$$

(b) In case $a(k) \geq 1$ and if there exist reals $\alpha_1, \alpha_2, \alpha_3 \geq 0$ and $N \in \mathbb{Z}_0^+$ with

$$\alpha_1 := \sup_{k \geq \kappa} e_a(k+N, k), \quad \alpha_2 := \sup_{k \geq \kappa} \sum_{n=k}^{k+N} \frac{b(n)}{a(n)}, \quad \alpha_3 := \sup_{k \geq \kappa} \sum_{n=k}^{k+N} u(n),$$

then one has $u(n) \leq \alpha_1 \left(\alpha_2 + \frac{\alpha_3}{N+1}\right)$ for all $\kappa \leq k$ and $n \in [k, k+N]_{\mathbb{Z}}$.

Proof. The proof of (a) is an easy induction and omitted. To deduce (b) we have

$$e_a(\kappa, l+1)u'(l) - e_a(\kappa, l)u(l) = e_a(\kappa, l)\left(\frac{u'(l)}{a(l)} - u(l)\right) \overset{\text{(A.2c)}}{\leq} e_a(\kappa, l)\frac{b(l)}{a(l)}$$

for all $l \geq \kappa$ and "telescopic" summation yields (note $a(k) \geq 1$)

$$e_a(\kappa, n)u(n) - e_a(\kappa, m)u(m) \leq \sum_{l=m}^{n-1} e_a(\kappa, l)\frac{b(l)}{a(l)} \leq e_a(\kappa, m)\sum_{l=m}^{n-1} \frac{b(l)}{a(l)}$$

for all $n \geq m \geq \kappa$ which, in turn, implies

$$u(n) \leq e_a(n, m)\left(u(m) + \sum_{l=m}^{n-1} \frac{b(l)}{a(l)}\right) \quad \text{for all } n \geq m \geq \kappa.$$

Hence, we obtain $u(n) \leq \alpha_1(u(m) + \alpha_2)$ for all $m \geq \kappa$ and $n \in [m, m+N]_{\mathbb{Z}}$. The estimate leads to

$$
\begin{aligned}
(N+1)u(n) = \sum_{m=k}^{k+N} u(n) &\leq \alpha_1 \left(\sum_{m=k}^{k+N} u(m) + (N+1)\alpha_2 \right) \\
&\leq \alpha_1 \left[\alpha_3 + (N+1)\alpha_2 \right]
\end{aligned}
$$

and division by $N+1$ gives the result. □

A.3 Remarks

Generalized exponential function: Our use of the generalized exponential function is modeled after its counterpart from the calculus on time scales (cf. [204]). In order to avoid certain uniformity assumptions in discretization theory it turned out advantageous to have time-varying growth rates $a(k)$, i.e., to work with $e_a(k,l)$ instead of α^{k-l}; this has been demonstrated, for instance, in [246].

Gronwall inequalities: Our Proposition A.2.1 extends previous work in [17], [20, Lemma 2.1] and more general versions of the discrete Gronwall inequality can be found in [3, pp. 184–192, Sect. 4.1] or [175, p. 1ff, Chap. 1]. Various versions of the Gronwall lemma and its application to discretizations of parabolic problems are presented in [143]. The uniform Gronwall inequality from Proposition A.2.3 plays a crucial role to study the behavior of dissipativity properties under discretization. The continuous version of Proposition A.2.3(b) is due to [453, Lemma 5.1] and similar variants can be found in [125, Lemma 8.2] or [142, Appendix 2].

The analysis of discretizations for Volterra integral equations with weakly singular kernels requires adapted discrete Gronwall inequalities, which might be considered as discrete counterparts to the Gronwall-Henry inequality (see [432, p. 625, Lemma D.4]). We refer to [45, 120] for corresponding results and to the monograph [67] for a survey.

Appendix B
Fixed Point and Inversion Theorems

In order to introduce measures of noncompactness, we follow an axiomatic path (cf. [35]) and focus on properties needed below. For this, let X be a complete metric space. A mapping $\chi : 2^X \to [0, \infty]$ is called *measure of noncompactness* on X, if the following conditions are met for $A, B \subseteq X$:

(c_0) A is bounded $\Leftrightarrow \chi(A) < \infty$.
(c_1) *(Regularity)* A is relatively compact $\Leftrightarrow \chi(A) = 0$.
(c_2) *(Invariance under closure)* $\chi(A) = \chi(\mathrm{cl}_X A)$ and, if X is a Banach space, then $\chi(A) = \chi(\mathrm{co}_X A)$ *(invariance under convex hull)*.
(c_3) *(Semi-additivity)* $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$ and, if X is a Banach space, then $\chi(A + B) \leq \chi(A) + \chi(B)$ *(algebraic semi-additivity)*.

From these axioms, one deduces the properties (cf. [35, p. 19]):

(c_4) *(Monotonicity)* $A \subseteq B \Rightarrow \chi(A) \leq \chi(B)$.
(c_5) *(Kuratowski property)* If $A_k \subseteq X$, $k \in \mathbb{N}$, are closed bounded sets with $A_k \subseteq A_l$ for $l \leq k$ and $\lim_{k\to\infty} \chi(A_k) = 0$, then for every sequence $k_n \to \infty$ in $\mathbb{N}$ and $u_n \in A_{k_n}$ there is a convergent subsequence of $(u_n)_{n\in\mathbb{N}}$.

Moreover, if $X = X_1 \times X_2$ is the cartesian product of two metric spaces X_1, X_2 with χ_1, χ_2 denoting corresponding measures of noncompactness, then

(c_6) $\chi(A_1 \times A_2) = \psi(\chi_1(A_1), \chi_2(A_2))$ for $A_1 \subseteq X_1$, $A_2 \subseteq X_2$ defines a measure of noncompactness on X,

where $\psi : [0, \infty)^2 \to [0, \infty)$ is a convex function satisfying $\psi(x_1, x_2) = 0$ if and only if $x_1 = x_2 = 0$ (cf. [40, p. 14, Theorem 3.3.2]). In functional analysis, the following three measures of noncompactness are commonly used:

Example B.0.1. The *Hausdorff measure of noncompactness* is defined as

$$\alpha(A) := \inf\left\{\rho > 0 \,\middle|\, \exists N \in \mathbb{N} : \exists x_1, \ldots, x_N \in X : A \subseteq \bigcup_{n=1}^{N} B_\rho(x_n, X)\right\},$$

the *Kuratowski measure of noncompactness* is given by

$$\beta(A) := \inf\left\{\rho > 0 \,\middle|\, \begin{array}{l} \exists N \in \mathbb{N} : \exists A_1, \ldots, A_N \subseteq X \text{ with} \\ \operatorname{diam}_X A_n \leq \rho :\ A \subseteq \bigcup_{n=1}^{N} A_n \end{array}\right\}$$

and the *separation measure of noncompactness* reads as

$$\gamma(A) := \sup\left\{\rho > 0 \,\middle|\, \rho = \inf_{m \neq n} d_X(x_m, x_n) \text{ with a sequence } (x_n) \text{ in } A\right\}.$$

Since a ball of radius ρ has diameter at most 2ρ we have

$$\alpha(A) \leq \gamma(A) \leq \beta(A) \leq 2\alpha(A) \quad \text{for all } A \subseteq X \tag{B.0a}$$

(see [35, p. 26, Remark 3.2]). Moreover, it is shown in [35, p. 17ff, Chap. 2] that α, β, γ are measures of noncompactness satisfying (c_0)–(c_5); in addition one has

$$\alpha(A) \leq \gamma(A) \leq \beta(A) \leq \operatorname{diam}_X A \quad \text{for all } A \subseteq X. \tag{B.0b}$$

B.1 Contractive Mappings

This section centers around contractions depending on parameters, and the behavior of corresponding fixed points under varying parameters. As prototype result we get

Theorem B.1.1 (uniform C^0-contraction principle). *Let X be a complete metric space and Y be a set. If a mapping $T : X \times Y \to X$ satisfies*

$$\operatorname{lip}_1 T < 1,$$

then there exists a unique function $x^* : Y \to X$ with $T(x^*(y), y) \equiv x^*(y)$ on Y and, if Y is a first countable topological space, one has:

(a) If $T(x, \cdot)$ is continuous for all $x \in X$, then $x^* \in C(X, Y)$.
(b) If Y is metrizable and $\operatorname{lip}_2 T < \infty$, *then* $\operatorname{lip} x^* \leq \frac{\operatorname{lip}_2 T}{1 - \operatorname{lip}_1 T}$.

Proof. In view of Banach's fixed point theorem (see, for example, [295, p. 361, Lemma 1.1]) the existence of a unique function $x^* : Y \to X$ is clear.

(a) We only have to verify the continuity of x^*. Since Y is first countable, it suffices to verify sequential continuity. To this end let $(y_n)_{n \in \mathbb{N}}$ be a sequence in Y converging to some arbitrarily given $y_0 \in Y$. Then we have

$$\begin{aligned} d(x^*(y_n), x^*(y_0)) &= d(T(x^*(y_n), y_n), T(x^*(y_0), y_0)) \\ &\leq d(T(x^*(y_n), y_n), T(x^*(y_0), y_n)) + d(T(x^*(y_0), y_n), T(x^*(y_0), y_0)) \\ &\leq \operatorname{lip}_1 T\, d(x^*(y_n), x^*(y_0)) + d(T(x^*(y_0), y), T(x^*(y_0), y_0)) \end{aligned}$$

for all $n \in \mathbb{N}$ and consequently

$$d(x^*(y_n), x^*(y_0)) \leq (1 - \operatorname{lip}_1 T)^{-1} d(T(x^*(y_0), y_n), T(x^*(y_0), y_0). \quad \text{(B.1a)}$$

The continuity of T in the second argument implies our claim for $n \to \infty$.

(b) This immediately follows from (B.1a). □

Let X, Y be complete metric spaces equipped with respective measures of noncompactness χ_X, χ_Y. A mapping $f : X \to Y$ is said to fulfill a *Darbo condition*, if there exists a real $k \geq 0$ such that

$$\chi_Y(f(B)) \leq k\chi_X(B) \quad \text{for all } B \subseteq X \text{ bounded;} \quad \text{(B.1b)}$$

the smallest possible so-called *Darbo constant* k is denoted by $\operatorname{dar} f$. In case of a compact mapping f one has $\operatorname{dar} f = 0$. Furthermore, for the measures of noncompactness from Example B.0.1 it is $\operatorname{dar} f \leq \operatorname{lip} f$ and we will see in Remark C.2.2 that strict inequality can hold.

Corollary B.1.2. *Let Y be metrizable and complete.*

(a) If $\operatorname{lip}_2 T < \infty$, then $\operatorname{dar} x^* \leq \frac{\operatorname{lip}_2 T}{1 - \operatorname{lip}_1 T}$.

(b) If every mapping $T(x_0, \cdot) : Y \to X$, $x_0 \in x^*(Y)$, is bounded, then also the fixed point mapping $x^* : Y \to X$ is bounded. Moreover, if T additionally fulfills a Darbo condition with $\operatorname{dar} T \leq 1$, then $\operatorname{dar} x^* \leq \operatorname{dar} T$.

Proof. (a) From Theorem B.1.1(b) we know that the fixed point function x^* is bounded and therefore [35, p. 39, Example 5] implies our claim.

(b) We initially show that x^* maps bounded subsets of Y into bounded sets of X. For this, choose a fixed $y_0 \in Y$ and as in (B.1a) we deduce

$$d(x^*(y), x^*(y_0)) \leq (1 - \operatorname{lip}_1 T)^{-1} d(T(x^*(y_0), y), T(x^*(y_0), y_0) \quad \text{for all } y \in Y.$$

Hence, our assumptions guarantee $\sup_{y \in B} d(T(x^*(y_0), y), T(x^*(y_0), y_0)) < \infty$ for every bounded subset $B \subseteq Y$, and consequently x^* is bounded.

Our property (c_6) implies that $\chi(B_1 \times B_2) := \max\{\chi_X(B_1), \chi_Y(B_2)\}$, where $B_1 \subseteq X$, $B_2 \subseteq Y$, defines a measure of noncompactness on the product space $X \times Y$. Suppose $B \subseteq Y$ is bounded. By assumption, there exists a $k \in [0, 1]$ with

$$\chi_X(T(B_1, B)) \leq k\chi(B_1 \times B) \quad \text{for all } B_1 \subseteq X \text{ bounded,}$$

and since $x^*(B) \subseteq X$ is bounded, we can deduce (note $k \leq 1$) that

$$\begin{aligned} \chi_X(x^*(B)) &= \chi_X(T(x^*(B), B)) \leq k\chi(x^*(B) \times B) \\ &= k \max\{\chi_X(x^*(B)), \chi_Y(B)\} = k\chi_Y(B). \end{aligned}$$

This yields our assertion $\operatorname{dar} x^* \leq k$. □

Lemma B.1.3. *Let X, Y be metric spaces, Z be a first countable topological space and suppose a mapping $T : X \times Z \to Y$ satisfies $\sup_{n\in\mathbb{N}} \operatorname{lip} T(\cdot, z_n) < \infty$ for all convergent sequences $(z_n)_{n\in\mathbb{N}}$ in Z. If $T(x, \cdot) : Z \to Y$ is continuous for all $x \in X$, then T is continuous itself.*

Proof. For sequences $(x_n, z_n)_{n\in\mathbb{N}}$ in $X \times Z$ with limit (x_0, z_0) we get

$$\begin{aligned} d(T(x_n, z_n), T(x_0, z_0)) &\le d(T(x_n, z_n), T(x_0, z_n)) + d(T(x_0, z_n), T(x_0, z_0)) \\ &\le \sup_{n\in\mathbb{N}} \operatorname{lip} T(\cdot, z_n)\, d(x_n, x_0) + d(T(x_0, z_n), T(x_0, z_0)) \end{aligned}$$

for all $n \in \mathbb{N}$. By assumption, both terms on the right-hand side tend to 0 in the limit $n \to \infty$. Since Z is first countable, this implies our claim. □

Corollary B.1.4. *Let X, Y be metric spaces, X be complete and Z be a set. If the mappings $f : X \times Z \to Y$, $g : Y \times Z \to X$ satisfy $\operatorname{lip}_1 f \operatorname{lip}_1 g < 1$, then:*

(a) For each $z \in Z$ there exist unique $x^*(z) \in X$, $y^*(z) \in Y$ satisfying the identities $y^*(z) \equiv f(x^*(z), z)$, $x^*(z) \equiv g(y^*(z), z)$ on Z.
(b) If Z is a first countable topological space and $f(x, \cdot) : Z \to Y$, $g(y, \cdot) : Z \to X$ are continuous for each $y \in Y$, $x \in X$, then $x^* : Z \to X$, $y^* : Z \to Y$ are continuous.

Proof. We define the mapping $T : X \times Z \to X$ by $T(x, z) := g(f(x, z), z)$.

(a) Due to $\operatorname{lip}_1 f \operatorname{lip}_1 g < 1$ we have $\operatorname{lip}_1 T < 1$ and Theorem B.1.1 implies the existence of a unique fixed point $x^*(z) \in X$ of $T(\cdot, z)$ for all $z \in Z$. The claim follows, if we set $y^*(z) := f(x^*(z), z)$.

(b) Our Lemma B.1.3 guarantees that g is continuous, which implies that also $T(x, \cdot)$, $x \in X$, is continuous. Theorem B.1.1(a) implies the continuity $x^* : Z \to X$ and, thanks to Lemma B.1.3, also $y^* : Z \to Y$ is continuous. □

From now on, let X, Y be Banach spaces.

Theorem B.1.5 (uniform C^m-contraction principle). *Let $m \in \mathbb{Z}_0^+$ and suppose both $U \subseteq X$, $V \subseteq Y$ are open sets. If a mapping $T : \operatorname{cl} U \times V \to \operatorname{cl} U$ satisfies*

$$\operatorname{lip}_1 T < 1, \qquad\qquad T \in C^m(U \times V, X),$$

then there exists a unique $x^* \in C^m(V, U)$ with $T(x^*(y), y) \equiv x^*(y)$ on V.

Proof. See [87, p. 25, Theorem 2.2]. □

Corollary B.1.6. *Let $m \in \mathbb{Z}_0^+$, Z be a Banach space and $U \subseteq X$, $V \subseteq Z$, $W \subseteq Y$ be open subsets. If mappings $f : \operatorname{cl} U \times V \to W$, $g : W \times V \to \operatorname{cl} U$ satisfy $\operatorname{lip}_1 f \operatorname{lip}_1 g < 1$ and $f \in C^m(U \times V, Y)$, $g \in C^m(W \times V, X)$, then there exist unique C^m-functions $x^* : V \to X$, $y^* : V \to Y$ so that $y^*(z) \equiv f(x^*(z), z)$, $x^*(z) \equiv g(y^*(z), z)$ holds on V.*

Proof. We define the mapping $T : \operatorname{cl} U \times V \to \operatorname{cl} U$ by $T(x, z) := g(f(x, z), z)$. Then Theorem B.1.5 is applicable to T yielding a unique C^m-function x^*. By assumption, also y^* given by $y^*(z) = f(x^*(z), z)$ is of class C^m. □

B.2 Compact Mappings

In this section, we present existence theorems for solutions of fixed point or further nonlinear problems involving a compact operator. This compactness can be weakened using the notion of a measure of noncompactness $\chi : 2^X \to [0, \infty]$ as introduced on p. 351. Then a mapping $T : X \to X$ on a Banach space X is called χ-*set contraction*, if it fulfills a Darbo condition with $\operatorname{dar} T \in [0, 1)$, and we denote T as χ-*condensing*, if $\chi(T(B)) < \chi(B)$ for any bounded set $B \subseteq X$ for which $T(B)$ is bounded and $\chi(B) > 0$. Examples for χ-set contractions (or χ-condensing maps) are compact or contracting mappings.

Theorem B.2.1 (Darbo fixed point theorem). *Let C be a nonempty bounded, closed and convex subset of a Banach space X. If $T : C \to C$ is a continuous χ-set contraction, then there exists a fixed point of T.*

Proof. See [180, p. 133, (C.3)]. □

Theorem B.2.2 (Schauder fixed point theorem). *Let C be a nonempty bounded, closed and convex subset of a normed space X. If a continuous mapping $T : C \to C$ has relatively compact image $T(C) \subseteq X$, then there exists a fixed point of T.*

Proof. For Schauder's theorem see [345, p. 470], while the generalization to merely normed spaces can be done using [345, p. 472]. □

The following result is helpful to find zeros of nonlinear equations.

Proposition B.2.3. *Let B be a closed ball in an inner product space X. If a continuous map $T : B \to X$ has relatively compact image $T(B) \subseteq X$ and*

$$\Re \langle T(x), x \rangle \neq 0 \quad \textit{for all } x \in \operatorname{bd} B,$$

then there exists a $x_0 \in B$ such that $T(x_0) = 0$.

Proof. We assume the contrary, i.e., $T(x) \neq 0$ for all $x \in B$. The continuous mapping $\Re \langle T(\cdot), \cdot \rangle : X \to \mathbb{R}$ does not change sign on $\operatorname{bd} B$ and we suppose it is negative. Then the mapping $\bar{T} : B \to \operatorname{bd} B$, $\bar{T}(x) := \frac{r}{\|T(x)\|} T(x)$ is well-defined and continuous, where $r > 0$ denotes the radius of B. Moreover, it is easily seen that $\bar{T}(B)$ is relatively compact. Thus, by Theorem B.2.2 there exists a fixed point $x^* = \frac{r}{\|T(x^*)\|} T(x^*)$ and we arrive at the contradiction

$$0 < r^2 = \langle x^*, x^* \rangle = \frac{r}{\|T(x^*)\|} \langle T(x^*), x^* \rangle = \frac{r}{\|T(x^*)\|} \Re \langle T(x^*), x^* \rangle < 0.$$

In the remaining case where $\Re \langle T(\cdot), \cdot\rangle : X \to \mathbb{R}$ is positive, the same contradiction can be derived using $\bar{T}(x) := -\frac{r}{\|T(x)\|}T(x)$. □

With the use of asymptotic fixed point theorems one can get rid of the restriction to self-mappings of bounded sets into itself. Here we focus on the Kuratowski measure of noncompactness β:

Theorem B.2.4. *If a continuous map $T : X \to X$ on a Banach space X is*

(i) *β-condensing and*
(ii) Compact dissipative*, i.e., there is a bounded set $B \subseteq X$ such that, for any compact set $K \subseteq X$, there is an $N = N(K) \in \mathbb{Z}_0^+$ such that $T^n(K) \subseteq B$ for all $n \geq N$,*

then there exists a fixed point of T.

Proof. See [344] or [196, Theorem 7]. □

B.3 Global Inverse Function Theorems

Let X, Y, Z be sets. Given a mapping $f : X \times Y \to Z$, provided $f(\cdot, y)$, $y \in Y$, is bijective, we denote the mapping $(z, y) \mapsto f(\cdot, y)^{-1}(z)$ by $f_1^{-1} : Z \times Y \to X$.

The next result is applicable to contractive perturbations of the identity:

Theorem B.3.1 (Lipschitz inverse function theorem). *Let X be a complete metric space, Y be a set and Z be a metric linear space. If two mappings $f, g : X \times Y \to Z$ are such that $f_1^{-1} : Z \times Y \to X$ exists and one has*

$$\operatorname{lip}_1 f_1^{-1} \operatorname{lip}_1 g < 1,$$

then also the sum $f(\cdot, y) + g(\cdot, y) : X \to Z$ is invertible with

$$\operatorname{lip}_1 (f+g)_1^{-1} \leq \frac{\operatorname{lip}_1 f_1^{-1}}{1 - \operatorname{lip}_1 f_1^{-1} \operatorname{lip}_1 g}. \tag{B.3a}$$

Moreover, the mapping $(f+g)_1^{-1} : Z \times Y \to X$ satisfies:

(a) *Provided Y is a first countable topological space and the mappings f_1^{-1}, $g(x, \cdot)$, $x \in X$, are continuous, then also $(f+g)_1^{-1}$ is continuous.*
(b) *Provided Y is metrizable with $\operatorname{lip}_2 f_1^{-1}, \operatorname{lip}_2 g < \infty$, then*

$$\operatorname{lip}_2 (f+g)_1^{-1} \leq \frac{\operatorname{lip}_1 f_1^{-1} \operatorname{lip}_2 g + \operatorname{lip}_2 f_1^{-1}}{1 - \operatorname{lip}_1 f_1^{-1} \operatorname{lip}_1 g}.$$

(c) *Provided X, Z are Banach spaces, Y an open subset of a Banach space and f_1^{-1}, g are of class C^m, then $(f+g)_1^{-1} \in C^m(Z \times Y, X)$.*

Proof. In the first part we suppress the dependence of f, g on the fixed parameter $y \in Y$. We define a mapping $T : X \times Z \to X$ by $T(x, z) := f^{-1}(z - g(x))$ and obtain $\operatorname{lip}_1 T \le \operatorname{lip} f^{-1} \operatorname{lip} g < 1$. Thus, by Theorem B.1.1 there exists a unique function $x^* : Z \to X$ such that $x^*(z)$ is the unique fixed point of $f^{-1}(z - g(\cdot))$ for all $z \in Z$. Because of the equivalence

$$x = x^*(z) \quad \Leftrightarrow \quad x = f^{-1}(z - g(x)) \quad \Leftrightarrow \quad z = f(x) + g(x)$$

the fixed point mapping x^* is the promised inverse of $f + g$. In addition, due to the estimate $\operatorname{lip}_2 T \le \operatorname{lip} f^{-1}$ one also gets (B.3a) from the triangle inequality.
Now define $T : X \times Z \times Y \to X$ by $T(x, z, y) := f_1^{-1}(z - g(x, y), y)$:

(a) The mapping T fulfills the assumptions of Theorem B.1.1(a), yielding that the fixed point mapping $x^* : Y \times Z \to X$ is continuous.

(b) Follows from Theorem B.1.1(b) since $\operatorname{lip}_{(2,3)} T \le \operatorname{lip}_1 f_1^{-1} \operatorname{lip}_2 g + \operatorname{lip}_2 f_1^{-1}$.

(c) Is finally a direct consequence of Theorem B.1.5. □

Theorem B.3.2. *Let X be a normed space. If a mapping $T : X \to X$ is completely continuous and there exists a $\gamma > 0$ such that*

$$\gamma \|x - \bar{x}\| \le \|x - T(x) - \bar{x} + T(\bar{x})\| \quad \textit{for all } x, \bar{x} \in X, \tag{B.3b}$$

then $I_X - T : X \to X$ is bijective with $\operatorname{lip}(I_X - T)^{-1} \le \gamma^{-1}$.

Proof. See [180, p. 130, Corollary (8.6)], where the Lipschitz condition for the inverse of $I - T$ follows immediately from (B.3b). □

Our next inverse function theorem guarantees that a strongly monotone Lipschitz mapping is bijective with Lipschitzian inverse.

Theorem B.3.3. *Let X be a Hilbert space and Y be a set. Suppose that a mapping $T : X \times Y \to X$ fulfills $\operatorname{lip}_1 T < \infty$ and there exists a $\gamma > 0$ such that*

$$\gamma \|x - \bar{x}\|^2 \le \Re \langle T(x, y) - T(\bar{x}, y), x - \bar{x} \rangle \quad \textit{for all } x, \bar{x} \in X \tag{B.3c}$$

and $y \in Y$, then $T(\cdot, y)$ is a Lipeomorphism with $\operatorname{lip}_1 T_1^{-1} \le \gamma^{-1}$ and the inverse function $T_1^{-1} : X \times Y \to X$ satisfies:

(a) Provided Y is a first countable topological space and $T(x, \cdot)$, $x \in X$, is continuous, then also T_1^{-1} is continuous.
(b) Provided Y is metrizable with $\operatorname{lip}_2 T < \infty$, then $\operatorname{lip}_2 T_1^{-1} \le \frac{2(1+\operatorname{lip}_2 T)}{\gamma}$.
(c) Provided X is a Banach space, Y an open subset of a Banach space and T is of class C^m, then $T_1^{-1} \in C^m(X \times Y, X)$.

Remark B.3.4. Given a smooth mapping $T(\cdot, y) \in C^1(X, X)$ it is a consequence of the mean value theorem (see [295, p. 342, Corollary 4.3]) that the coercivity condition (B.3c) is implied by $\gamma \|x\|^2 \le \Re \langle D_1 T(\xi, y)x, x \rangle$ for $x, \xi \in X$, $y \in Y$.

Proof. In the beginning, we suppress the dependence of T on the fixed parameter $y \in Y$. Above all we have $\operatorname{lip} T > 0$, since otherwise T is constant and (B.3c) cannot hold. We begin the proof by showing that $T : X \to X$ is bijective, i.e., we show that for each $\xi \in X$ there exists a unique $x \in X$ with $T(x) = \xi$. For an arbitrary $\varepsilon > 0$, this in turn is equivalent to the existence of a unique fixed point of the mapping $\Phi_\xi(x) := \varepsilon\xi - \varepsilon T(x) + x$. We obtain from (B.3c) that

$$\begin{aligned}\|\Phi_\xi(x) - \Phi_\xi(\bar{x})\|^2 &= \|x - \bar{x}\|^2 + \varepsilon^2 \|T(x) - T(\bar{x})\|^2 - 2\varepsilon \langle T(x) - T(\bar{x}), x - \bar{x}\rangle \\ &\le \left[1 + \varepsilon^2(\operatorname{lip} T)^2\right] \|x - \bar{x}\|^2 - 2\varepsilon \langle T(x) - T(\bar{x}), x - \bar{x}\rangle \\ &\le \left[1 + \varepsilon(\varepsilon(\operatorname{lip} T)^2 - 2\gamma)\right] \|x - \bar{x}\|^2 \quad \text{for all } x, \bar{x} \in X\end{aligned}$$

and it is easy to see that the function $\ell(\varepsilon) := 1 + \varepsilon(\varepsilon(\operatorname{lip} T)^2 - 2\gamma)$ achieves its global minimum $1 - \frac{\gamma^2}{(\operatorname{lip} T)^2}$ at $\varepsilon_0 := \frac{\gamma}{(\operatorname{lip} T)^2}$. The above estimate guarantees $\ell(\varepsilon) \ge 0$ for all $\varepsilon > 0$ and in case $\varepsilon = \varepsilon_0$ one has $\operatorname{lip} \Phi_\xi \le \sqrt{\ell(\varepsilon_0)} = \sqrt{1 - \frac{\gamma^2}{(\operatorname{lip} T)^2}} \in [0, 1)$. Hence, Banach's fixed point theorem (see [295, p. 361, Lemma 1.1]) yields the existence of a unique fixed point $x^*(\xi) \in X$ of Φ_ξ and thus $x^* = T^{-1}$. To establish a Lipschitz condition on this inverse, we pick $x, \bar{x} \in X$ and $\xi, \bar{\xi} \in X$ with $\xi = T(x)$, $\bar{\xi} = T(\bar{x})$ and get from the Cauchy-Schwarz inequality

$$\begin{aligned}\left\|T^{-1}(\xi) - T^{-1}(\bar{\xi})\right\|^2 &= \|x - \bar{x}\|^2 \overset{\text{(B.3c)}}{\le} \gamma^{-1}\Re \langle T(x) - T(\bar{x}), x - \bar{x}\rangle \\ &\le \gamma^{-1}\Re \left\langle \xi - \bar{\xi}, x - \bar{x}\right\rangle \le \gamma^{-1} \left\|\xi - \bar{\xi}\right\| \|x - \bar{x}\| \\ &= \gamma^{-1} \left\|\xi - \bar{\xi}\right\| \left\|T^{-1}(\xi) - T^{-1}(\bar{\xi})\right\|,\end{aligned}$$

which implies $\operatorname{lip} T^{-1} \le \gamma^{-1}$ and we are done.

Now the mapping $\Phi : X \times X \times Y \to X$ is given by $\Phi(x, \xi, y) := \varepsilon\xi - \varepsilon T(x, y) + x$:

(a) By assumption, $\Phi(\cdot, \xi, y)$ is a uniform contraction in $(\xi, y) \in X \times Y$ and Φ fulfills the continuity conditions required in Theorem B.1.1(a).

(b) We set $\varepsilon = \gamma/(\operatorname{lip}_1 T)^2$ and the above estimate for Φ_ξ yields

$$\operatorname{lip}_1 \Phi \le \sqrt{1 - \frac{\gamma^2}{(\operatorname{lip} T)^2}}, \qquad \operatorname{lip}_{(2,3)} \Phi \le \frac{\gamma}{(\operatorname{lip} T)^2}(1 + \operatorname{lip}_2 T).$$

Having this at hand, we apply Theorem B.1.1(b) to Φ and obtain assertion (b).

(c) Follows from Theorem B.1.5. □

Theorem B.3.5. *Let $m \in \mathbb{N}$, X be a Hilbert space and Z be an open subset of a Banach space. If a C^m-mapping $T : X \times Z \to X$ satisfies that for every $z \in Z$ there exists a continuous function $\omega_z : [0, \infty) \to (0, \infty)$ with $\int_0^\infty \frac{ds}{\omega_z(s)} = \infty$ and*

$$|\Re\langle D_1T(x,z)y,y\rangle| \geq \omega_z(\|x\|)\,\|y\|^2 \quad \textit{for all } x,y \in X, \tag{B.3d}$$

then $T(\cdot,z): X \to X$*,* $z \in Z$*, is a global* C^m*-diffeomorphism and moreover the inverse* $T_1^{-1}: X \times Z \to X$ *is of class* C^m*.*

Proof. The mapping $g: X^2 \times Z \to X$, $g(x,y,z) = T(x,z) - y$ is of class C^m. Thanks to $D_1g(x,y,z) = D_1T(x,z)$ we can deduce from [390, Corollary 3.7] that $T_z := T(\cdot,z): X \to X$ is a global C^1-diffeomorphism for all $z \in Z$. Moreover, from differentiating the identity $x \equiv T_z(T_z^{-1}(x))$ on X one obtains $DT_z(x) \in GL(X)$ for all $x \in X$ and the local inverse function theorem (cf., for example, [295, p. 361, Theorem 1.2]) ensures that T_z is a C^m-diffeomorphism. On the other hand, the inverse $T_1^{-1}: X \times Z \to X$ satisfies the identity $g(T_1^{-1}(y,z),y,z) \equiv 0$ and by the implicit function theorem (see [295, p. 364, Theorem 2.1, p. 361, Theorem 1.2]) we know that T_1^{-1} is a C^m-mapping. □

Our final result is a global version of a tool due to [104, Lemma 1].

Theorem B.3.6. *Let* X, Z *denote Banach spaces and suppose that the mappings* $L \in L(X,Z)$*,* $F: X \to Z$ *satisfy:*

(i) For a subspace $S \subseteq X$ *the map* $L_S := L|_S$ *is one-to-one and* $\operatorname{im} L_S$ *is closed.*
(ii) $F(0) = 0$*,* $F(X) \subseteq \operatorname{im} L_S$ *and* $\operatorname{lip} F < \infty$*.*

If the nonlinearity F fulfills the estimate

$$\left\|L_S^{-1}\right\| \operatorname{lip} F < 1, \tag{B.3e}$$

then the following holds true:

(a) There exists a Lipeomorphism $\Psi: \ker L \to \{x \in X : Lx = F(x)\}$ *with*

$$\operatorname{lip}\Psi \leq \left(1 - \left\|L_S^{-1}\right\| \operatorname{lip} f\right)^{-1}, \qquad \operatorname{lip}\Psi^{-1} \leq 1 + \left\|L_S^{-1}\right\| \operatorname{lip} f. \tag{B.3f}$$

(b) Under the assumption $M := \sup_{x\in X}\|F(x)\| < \infty$ *the Lipeomorphism* Ψ *is "near identity" in the sense of*

$$\|\Psi(y) - y\|_X \leq \left\|L_S^{-1}\right\| M \quad \textit{for all } y \in \ker L,$$
$$\left\|\Psi^{-1}(x) - x\right\|_Z \leq \left\|L_S^{-1}\right\| M \quad \textit{for all } x \in \Psi(\ker L).$$

Proof. Define $Y := \ker L$ and $T: X \times Y \to X$ by $T(x,y) := L_S^{-1}F(x) + y$.

(a) Thanks to our assumptions we have

$$\operatorname{lip}_1 T \leq \left\|L_S^{-1}\right\| \operatorname{lip} f \overset{\text{(B.3e)}}{<} 1, \qquad\qquad \operatorname{lip}_2 T = 1$$

and by Theorem B.1.1 there exists a unique mapping $\Psi: Y \to X$ satisfying the identity $T(\Psi(y),y) \equiv \Psi(y)$ on Y and the left estimate in (B.3f). Moreover, $\Psi(y)$ solves the equation $Lx = F(x)$. On the other hand, it is straight forward to see that

Ψ is one-to-one, while Ψ being onto can be seen as in [104, p. 435, Remark (b)]. Given $\Psi(y_i) = x_i$, $i = 1, 2$, its inverse function satisfies

$$\left\|\Psi^{-1}(x_1) - \Psi^{-1}(x_2)\right\| = \|y_1 - y_2\| = \left\|x_1 - L_S^{-1}f(x_1) - x_2 + L_S^{-1}T(x_2)\right\|$$
$$\leq \left(1 + \left\|L_S^{-1}\right\| \operatorname{lip} f\right) \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in X,$$

which implies the remaining estimate in (B.3f).

(b) Let $x \in \ker L$ and $y \in \Psi(\ker L)$. The first estimate is an immediate consequence of the identity $\Psi(y) - y \equiv L_S^{-1}F(\Psi(y))$ on $\ker L$ and in order to deduce the second relation, simply set $y = \Psi^{-1}(x)$ into the first one. □

Corollary B.3.7. *Let $m \in \mathbb{N}$. If $F \in C^m(X, Z)$ holds, then $\Psi : \ker L \to X$ is m-times continuously differentiable.*

Proof. Use Theorem B.1.5 instead of Theorem B.1.1 in the above proof. □

B.4 Remarks

Comprehensive introductions to measures of noncompactness can be found in various texts on nonlinear analysis, and in particular in [35, 40], [280, p. 187] or [5, p. 9ff]. Connections to the essential spectrum of an operator can be found in [192, p. 14, Lemma 2.3.3] or [465, p. 497, Proposition 11.9].

Contractive mappings: The uniform contraction principle from Theorem B.1.1 or B.1.5 appears in a variety of references (e.g., [87, p. 25, Theorem 2.2] or [180, 201, 465]) and turned out to be a very convenient tool in many applications. For instance, Theorem B.1.5 forms the basis to derive the implicit function theorem. However, in various cases the differentiability assumption in Theorem B.1.5 is too strong – in particular in applications to show smoothness of center and pseudo-stable/-unstable manifolds. Here, substitution operators become only differentiable, if one continuously embeds their ranges in a larger space. Such generalizations of Theorem B.1.5 to scales of Banach spaces can be found in [205, 409, 410, 457].

Compact mappings: The Schauder fixed point theorem is fundamental in nonlinear analysis (cf., e.g., [180]). Our more general version Theorem B.2.2 is valid in normed spaces and taken from [345, p. 472]. In Banach spaces, Theorem B.2.1 clearly implies Theorem B.2.2. The following Proposition B.2.3 generalizes a classical result due to Poincaré (cf. [95, p. 58, Lemma 7.2]) to the infinite-dimensional setting. Conditions guaranteeing uniqueness in Schauder's theorem are due to [249] and a generalization to set contractions comes from [441]. The asymptotic fixed point result in Theorem B.2.4 was discovered independently in [196, 344].

Global inverse function theorems: We refer to [1, p. 138, Exercise 2.5K] for a local version of the Lipschitz inverse function theorem given in Theorem B.3.1.

A prototype global inversion theorem is due to Banach-Mazur (cf. [465, p. 174, Theorem 4.G]) and states that proper mappings are globally invertible, if and only if they are local homeomorphisms. The result from Theorem B.3.3 can be interpreted as a nonlinear version of the Lax–Milgram theorem (see, e.g., [432, p. 86, Lemma 36.5]), which also inspired its proof. Our Theorem B.3.5 is a corollary of a more general inverse function result. It belongs in the framework of the Hadamard–Levy theorem (cf. [1, p. 131, 2.5.17]) and a survey on such results can be found in [102, 111, 389, 390]. On the other hand, both (B.3c) and (B.3d) can be interpreted as strong monotonicity conditions and an overview of surjectivity results under such assumptions is given in [161, 339].

We remark that coercivity conditions like given in (B.3c) or (B.3d) are sufficient for injectivity (cf. [180, p. 75, (B.1)]). On the other hand, there exists an array of conditions for a mapping to be surjective; among them we mention Minty's theorem (cf. [180, p. 76, (B.7)]), quasi-boundedness conditions (cf. [180, p. 125, Theorem (5.6)]) or Schauder domain invariance (cf. [180, p. 130, Corollary (8.6)]).

Finally, our Theorem B.3.6 is a global and quantitative version of a result due to [104, Lemma 1].

Appendix C
Smooth Mappings and Extensions

C.1 Differentiability

Let X, Y, Z be Banach spaces and $U \subseteq X$, $V \subseteq Y$ be nonempty open subsets.

It is an elementary consequence of the mean value inequality (cf. [295, p. 342, Corollary 4.3]) that C^1-functions satisfy a global Lipschitz condition on each convex compact set. Now we consider a somehow inverse situation.

Proposition C.1.1. *If* $f : U \to Y$ *is differentiable and* $\operatorname{lip} f < \infty$*, then one has* $\|Df(x)\|_{L(X,Y)} \leq \operatorname{lip} f$ *for all* $x \in U$.

Proof. Let $\varepsilon > 0$ and $x \in U$. Due to the differentiability of $f : U \to Y$ there exists a $\delta = \delta(\varepsilon) > 0$ with $\|f(x+h) - f(x) - Df(x)h\|_Y \leq \varepsilon \|h\|_X$ for all $h \in \dot{B}_\delta(0)$, and

$$\begin{aligned}\|Df(x)h\|_Y &\leq \|f(x+h) - f(x)\|_Y + \left\|-\big(f(x+h) - f(x) - Df(x)h\big)\right\|_Y \\ &\leq (\operatorname{lip} f + \varepsilon)\|h\|_X \quad \text{for all } h \in \dot{B}_\delta(0).\end{aligned}$$

Setting $z := \frac{h}{\|h\|_X}$ implies $\|Df(x)z\|_Y \leq \operatorname{lip} f + \varepsilon$ for all $z \in \operatorname{bd} B_1(0)$ and thus

$$\|Df(x)\|_{L(X,Y)} = \sup_{\|z\|_X = 1} \|Df(x)z\|_Y \leq \operatorname{lip} f + \varepsilon \quad \text{for all } x \in U.$$

Since $\varepsilon > 0$ was arbitrary, this gives the assertion. □

The following result implies that completely continuous and differentiable mappings have compact derivatives.

Proposition C.1.2. *If* $f : U \to Y$ *is differentiable in* $x_0 \in U$ *and* $\operatorname{dar} f < \infty$*, then one has* $\operatorname{dar} Df(x_0) \leq \operatorname{dar} f$.

Proof. Let $\varepsilon > 0, x_0 \in U$ and suppose χ_X, χ_Y denote measures of noncompactness on X, Y, respectively. Due to the differentiability of f there exists a $\delta = \delta(\varepsilon) > 0$ with $\|f(x+h) - f(x) - Df(x)h\|_Y \leq \varepsilon \|h\|_X$ for all $h \in B_\delta(0)$. Now let us

assume that $C \subseteq X$ is a nonempty bounded set. We can find a $\rho > 0$ such that $C \subseteq \bar{B}_\rho(0)$ and so $\lambda C \subseteq \bar{B}_\delta(0)$ for $\lambda = \delta/\rho$. Therefore, from (c_3) we have

$$\chi_Y\left(Df(x_0)(\lambda C)\right) \le \chi_Y\left(f(x_0 + \lambda C)\right) + 2\varepsilon\delta \le \operatorname{dar} f\, \chi_X(\lambda C) + 2\varepsilon\delta,$$

which implies $\chi_Y\left(Df(x_0)C\right) \le \operatorname{dar} f \chi_X(C) + 2\varepsilon\rho$ and passing to the one-sided limit $\varepsilon \searrow 0$ yields our claim. □

Theorem C.1.3 (chain rule). *Let $m \in \mathbb{N}$. If $f : U \to Z$, $g : V \to Y$ are of class C^m with $g(V) \subseteq U$, then also the composition $f \circ g : V \to Z$ is m-times continuously differentiable and for $l \in \{1, \ldots, m\}$, $x \in V$ the derivatives possess representations as a so-called* partially unfolded derivative tree

$$D^l(f \circ g)(x) = \sum_{j=0}^{l-1} \binom{l-1}{j} D^j\left[Df(g(x))\right] \cdot D^{l-j}g(x) \tag{C.1a}$$

and as a so-called totally unfolded derivative tree

$$\begin{aligned} &D^l(f \circ g)(x)x_1 \cdots x_l \\ &= \sum_{j=1}^{l} \sum_{(N_1,\ldots,N_j)\in P_j^<(l)} D^j f(g(x)) D^{\#N_1} g(x) x_{N_1} \cdots D^{\#N_j} g(x) x_{N_j} \end{aligned} \tag{C.1b}$$

for any $x_1, \ldots, x_l \in X$.

Proof. A proof of (C.1a) follows by an easy induction argument (cf. [435, p. 266, Satz B.3]), while (C.1b) is shown in [408, Theorem 2]. □

C.2 Smooth Norms and Extensions

Let X, Y be normed spaces. The existence of retractions or smooth bump functions on X is of crucial importance to deduce local results from global ones via extending functions. The next tool is sufficient in a Lipschitz setting.

Lemma C.2.1. *The so-called* radial retraction $r_X : X \to \bar{B}_1(0, X)$,

$$r_X(x) := \begin{cases} x & \text{for } \|x\| \le 1, \\ \frac{1}{\|x\|}x & \text{for } \|x\| > 1 \end{cases}$$

satisfies $\operatorname{lip} r_X \in [1, 2]$ *and for an inner product space X one has* $\operatorname{lip} r_X = 1$.

Remark C.2.2. Although [321, p. 83, Proposition 5.7] works with the Kuratowski measure of noncompactness β, only the properties (c_2) and (c_4) are needed to deduce

$$\operatorname{dar} r_X = \begin{cases} 0 & \text{for } \dim X < \infty, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. In a general normed space X it is clear that $\operatorname{lip} r_X \geq 1$ holds. On the other side, $\operatorname{lip} r_X \leq 2$ is shown for example in [9, p. 262, Lemma (19.8)]. The assertion for an inner product space is due to [123]. □

Example C.2.3. (1) It is $\operatorname{lip} r_{C(\Omega)} = 2$, where Ω is a compact subset of a locally compact space (see [243, Corollary 4]).

(2) For every $p \in (1, \infty)$ one has $\operatorname{lip} r_{\ell^p(\mathbb{N})} < 2$, whereas $\operatorname{lip} r_{\ell^p(\mathbb{N})} = 2$ for $p \in \{1, \infty\}$ (see [243, Corollary 4]). Moreover, it is shown in [242] that

$$\operatorname{lip} r_{\ell^p(\mathbb{N},\mathbb{R}^2)} = \max_{t\in[0,1]} \frac{(1 + t^{p(p-1)})^{1/p}(1 + t^q)^{1/q}}{1 + t^p}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

(3) Let $k \in \mathbb{N}$, $p \in [1, \infty]$. On the Lebesgue space $L^p(\Omega)$ we get $\operatorname{lip} r_{L^p(\Omega)} < 2$ for $p \in (1, \infty)$ and $\operatorname{lip} r_{L^p(\Omega)} = 2$ in the limit cases $p \in \{1, \infty\}$. The same holds in the Sobolev spaces $W^{k,p}(\Omega)$ (cf. [243, Corollary 4]) and results for such spaces of functions with values in Banach spaces can also be found in [243].

The next proposition easily follows from the Kirszbraun–Valentine extension theorem (cf. [455, Corollary]). For an alternative proof in the case of nonexpanding mappings see also [396, Theorem 5].

Proposition C.2.4. *Let X, Y be Hilbert spaces. If $S \subseteq X$ a nonempty set, then for any function $F : S \to Y$ with $\operatorname{lip} F < \infty$ there exists an extension $\bar{F} : X \to Y$ with $F(x) \equiv \bar{F}(x)$ on S and $\operatorname{lip} \bar{F} = \operatorname{lip} F$.*

Proof. Define the mapping $T : S \times Y \to X \times Y$ by $T(x, y) := (0, F(x))$, which satisfies $\operatorname{lip} T \leq \operatorname{lip} F$. Then [455, Corollary] yields a global extension $\bar{T} : X \times Y \to X \times Y$ of T with $\bar{T}(x, y) = T(x, y)$ for all $(x, y) \in S \times Y$ and $\operatorname{lip} \bar{T} \leq \operatorname{lip} T$. If $\Pi_2 : X \times Y \to Y$ is the projection $\Pi_2(x, y) := y$, then $\bar{F}(x) := \Pi_2\bar{T}(x, 0)$ satisfies the assertion. □

A simpler, yet more quantitative and explicit situation occurs when extending functions defined on balls in general linear spaces X, Y. For this, given a neighborhood $\Omega \subseteq X \times Y$ of 0, it is convenient to introduce the condition

$$\bar{B}_\rho(0, X) \times \bar{B}_\rho(0, Y) \subseteq \Omega \tag{$B_\Omega(\rho)$}$$

needed in the following

Proposition C.2.5 (Lipschitzian extension). *Let (Z, d) be a metric space and let $\Omega \subseteq X \times Y$ denote a neighborhood of 0. If $F : \Omega \to Z$ is a locally Lipschitzian mapping, i.e., for each $r > 0$ fulfilling $(B_\Omega(r))$ one has*

$$\ell_i(r) := \operatorname{lip}_i F|_{B_r(0,X)\times B_r(0,Y)} < \infty \quad \textit{for all } i = 1, 2,$$

then for $\rho > 0$ satisfying ($B_\Omega(\rho)$) there exists a mapping $F^\rho : X \times Y \to Z$ with the following properties:

(a) $F^\rho(x, y) = F(x, y)$ for all $x \in B_\rho(0, X)$, $y \in B_\rho(0, Y)$.
(b) One has the global Lipschitz estimates

$$\operatorname{lip}_1 F^\rho \leq \ell_1(\rho) \operatorname{lip} r_X, \qquad \operatorname{lip}_2 F^\rho \leq \ell_2(\rho) \operatorname{lip} r_Y.$$

Remark C.2.6. If the mapping F satisfies a Darbo condition, then Remark C.2.2 and [35, p. 39, Proposition 5.3(b)] implies

$$\operatorname{dar} F^\rho \leq \begin{cases} 0 & \text{for } \dim X \times Y < \infty, \\ \operatorname{dar} F & \text{otherwise.} \end{cases}$$

Proof. Choose a $\rho > 0$ such that ($B_\Omega(\rho)$) holds and define the extension

$$F^\rho : X \times Y \to Z, \qquad F^\rho(x, y) := F\big(\rho r_X\big(\tfrac{x}{\rho}\big), \rho r_Y\big(\tfrac{y}{\rho}\big)\big).$$

By definition of the radial retractions r_X, r_Y, the arguments of F are contained in the ball $B_\rho(0) \times B_\rho(0) \subseteq \Omega$ and F^ρ is defined on $X \times Y$. Moreover, assertion (a) is valid. In order to establish assertion (b), we choose $x, \bar{x} \in X$ and obtain

$$\begin{aligned} d(F^\rho(x, y), F^\rho(\bar{x}, y)) &= d\Big(F\big(\rho r_X\big(\tfrac{x}{\rho}\big), \rho r_Y\big(\tfrac{y}{\rho}\big)\big), F\big(\rho r_X\big(\tfrac{\bar{x}}{\rho}\big), \rho r_U\big(\tfrac{y}{\rho}\big)\big)\Big) \\ &\leq \ell_1(\rho)\rho \left\| r_X\big(\tfrac{x}{\rho}\big) - r_X\big(\tfrac{\bar{x}}{\rho}\big) \right\| \leq \ell_1(\rho) \operatorname{lip} r_X \|x - \bar{x}\| \end{aligned}$$

for all $y \in Y$, which yields the first claimed global Lipschitz-estimate for F^ρ. Since the second one follows along the same lines, the proof is finished. □

Our next aim is to deduce a counterpart to the previous Proposition C.2.5 for differentiable functions. Thus, from now on suppose X, Y are Banach spaces. In a first step it is important to have information about differentiability properties of norms on Banach spaces. Here, the following notion is appropriate:

Definition C.2.7. We say $(X, \|\cdot\|)$ is a C^m*-Banach space*, $m \in \mathbb{N} \cup \{\infty\}$, provided there exists a norm $\|\cdot\|^*$ equivalent to $\|\cdot\|$ such that the norm mapping

$$n_X : X \setminus \{0\} \to [0, \infty), \qquad n_X(x) := \|x\|^*$$

is m-times continuously differentiable.

Example C.2.8. Let $\mathbb{I}$ be an unbounded discrete interval. Given a nonempty infinite set Ω we consider spaces of real-valued functions or sequences.

(1) $\ell^1(\mathbb{I})$ canonically equipped with $\|x\| := \sum_{k \in \mathbb{I}} |x_k|$ is not a C^1-Banach space (see [151, p. 314, Fact 10.5] and [284, p. 158, 14.11(1)]).

(2) $\ell^\infty(\mathbb{I})$ with the natural norm $\|x\| := \sup_{k\in\mathbb{I}}|x_k|$ is not a C^1-Banach space (see [151, p. 347, 10.4]).

(3) $c_0(\mathbb{I})$ is the space of sequences $x : \mathbb{I} \to \mathbb{R}$ such that $\{t \in \mathbb{I} : |x(t)| > \varepsilon\}$ is finite for every $\varepsilon > 0$. The norm $\|x\| := \sup_{t\in\mathbb{I}}|x(t)|$ makes it a C^∞-Banach space (cf. [116, p. 189, Theorem 1.5]). However, the derivative $Dn_{c_0(\Omega)}$ of the norm mapping is not globally Lipschitz [284, p. 164, Corollary 15.8]).

(4) $C[0,1]$ equipped with norm $\|x\| := \sup_{t\in[0,1]}|x(t)|$ is not a C^1-Banach space (see [151, p. 314, Fact 10.5] and [284, p. 158, 14.11(1)]).

(5) We equip the Lebesgue space $L^p(\Omega)$, $p \geq 1$, with its canonical norm and obtain (see [116, p. 184, Theorem 1.1 and p. 222, Corollary 4.11]):

- If $p \in \mathbb{N}$ is even, then $L^p(\Omega)$ is a C^∞-Banach space. More precisely, the norm mapping $n_{L^p(\Omega)}$ is C^∞ off 0, $D^{p-1}n^p_{L^p(\Omega)}$ is continuous, $D^p n^p_{L^p(\Omega)}$ is constant and $D^{p+1}n^p_{L^p(\Omega)}$ vanishes identically.
- If $p \in \mathbb{N}$ is odd, then $L^p(\Omega)$ is a C^{p-1}-Banach space. More precisely, $n_{L^p(\Omega)}$ is C^{p-1} off 0 and $D^{p-1}n^p_{L^p(\Omega)}$ is locally Lipschitz; in addition, every infinite-dimensional $L^p(\Omega)$ is not a C^p-Banach space.
- If $p \notin \mathbb{N}$, then $L^p(\Omega)$ is a $C^{[p]}$-Banach space. More precisely, $n_{L^p(\Omega)}$ is $C^{[p]}$ and $D^{[p]}n^p_{L^p(\Omega)}$ is $(p-[p])$-Hölder; moreover, for every infinite-dimensional $L^p(\Omega)$ there exists no norm equivalent to $n_{L^p(\Omega)}$ whose $[p]$-th order derivative is locally α-Hölder with $\alpha > p - [p]$.

For norms it is sufficient to show differentiability on spheres only.

Proposition C.2.9. *If there exists an $r > 0$ such that the norm mapping n_X is differentiable in every $x \in \operatorname{bd} B_r(0)$, then X is a C^1-Banach space with*

$$\|Dn_X(x)\|_{L(X,\mathbb{R})} \leq 1 \quad \textit{for all } x \in X \setminus \{0\}. \tag{C.2a}$$

Proof. Due to its positive homogeneity, the norm mapping n_X is differentiable off $\{0\}$ (cf. [151, p. 242]) and [151, p. 244, Corollary 8.5] implies that n_X is continuously differentiable in every $x \in X \setminus \{0\}$. By the lower triangle inequality one has $\operatorname{lip} n_X \leq 1$, thus, Proposition C.1.1 implies (C.2a). □

Proposition C.2.10. *Hilbert spaces are C^∞-Banach spaces.*

Proof. The norm on a Hilbert space X is a composition of C^∞-functions, namely $q : X \to \mathbb{R}$, $q(x) := \langle x, x\rangle$ and the square root function $\sqrt{\cdot} : (0,\infty) \to \mathbb{R}$. □

Proposition C.2.11. *Finite-dimensional spaces are C^∞-Banach spaces.*

Proof. Let X be a linear space over $\mathbb{F}$ with basis $\{e_1,\ldots,e_d\}$, $d \in \mathbb{N}$. For vectors $x, y \in X$ with representations $x = \sum_{i=1}^d \xi_i e_i$, $y = \sum_{i=1}^d \eta_i e_i$ and unique coefficients $\xi_i, \eta_i \in \mathbb{F}$ we observe that $\langle x, y\rangle := \sum_{i=1}^d \xi_i\overline{\eta_i}$ defines an inner product on X. Since all norms on finite-dimensional linear spaces are equivalent (cf. [295, p. 38, Corollary 3.14]), Proposition C.2.10 yields the assertion. □

Next we provide criteria for the existence of differentiable norms.

Proposition C.2.12. *Let $m \in \mathbb{N}$ and S be a set.*

(a) If X' is separable, then X is a C^1-Banach space, and the converse holds on separable spaces X.
(b) If X' is a C^1-Banach space, then X is reflexive.
(c) If $c_0(S) \to X \to Y$ is a short exact sequence[1] *and Y admits a C^m-norm, then X is a C^m-Banach space.*
(d) If there exists a continuous injection $X \to Y$ with closed range, and if Y admits a C^m-norm, then X is a C^m-Banach space.

Proof. (a) See [116, p. 50, Theorem 3.1(i) and p. 51, Corollary 3.3].

(b) See [151, p. 244, Theorem 8.6].

(c) See [284, p. 143, Proposition 13.17].

(d) Suppose $T \in L(X,Y)$ is one-to-one, i.e., $\ker T = \{0\}$. Then the mapping $\|T\cdot\|_Y$ is a C^m-norm on X. It remains to establish that $\|\cdot\|^* := \|T\cdot\|_Y$ is equivalent to the given norm on X. We obtain $\|x\|^* \le \|T\|_{L(X,Y)} \|x\|_X$, $x \in X$, and in order to establish the converse inequality, we observe that $T : X \to \operatorname{im} T$ is bijective with $T^{-1} \in L(\operatorname{im} T, X)$ (cf. [244, p. 167, 5.21]) and thus we eventually obtain to estimate $\|x\|_X \le \left\|T^{-1}\right\|_{L(\operatorname{im} T,X)} \|Tx\|_Y$. □

Example C.2.13. Let $\Omega \subseteq \mathbb{R}^N$ be a domain satisfying the cone condition. If the reals $p, q \in [1,\infty)$ and $k \in \mathbb{N}$ fulfill one of the conditions

- $kq \le N$ and $q \le p \le \frac{Nq}{N-kq}$, or
- $kq = N$ and $q \le p < \infty$,

then the Sobolev space $W^{k,q}(\Omega)$ inherits its smoothness properties from $L^p(\Omega)$ (cf. Example C.2.8(5)); this is a consequence of Proposition C.2.12(d) and the Sobolev embedding theorem (cf. [432, p. 608, Theorem B.2]).

Using the following result one obtains that smoothness properties for spaces of real-valued functions can be lifted to spaces of $\mathbb{R}^N$-valued functions.

Proposition C.2.14. *Let $m \in \mathbb{Z}_0^+$. The product $X \times Y$ of C^m-Banach spaces X, Y is also C^m w.r.t. the norm $\|(x,y)\|_{X\times Y} := \sqrt{\|x\|_X^2 + \|y\|_Y^2}$.*

Proof. See [284, p. 143, Proposition 13.17(1)]. □

The following result prepares smooth extensions. It addresses bump functions and provides a kind of optimality in their Lipschitz constant (cf. Fig. C.1).

Lemma C.2.15 (bump functions). *For every real $s > 1$ there exists a function $\vartheta \in C^\infty(\mathbb{R})$ such that $\vartheta(t) \equiv 1$ on $(-\infty, 1]$, $\vartheta(t) \in [0,1]$ for $t \in [1,2]$, $\vartheta(t) \equiv 0$ on $[2,\infty)$ and $D\vartheta(t) \in [-s, 0]$ for $t \in \mathbb{R}$, as well as $t\vartheta(t) \in [0,s]$ for all $t \ge 0$.*

[1] The space $c_0(S)$ is the closure of all functions on S with finite support in the Banach space of all bounded functions on S equipped with the supremum norm.

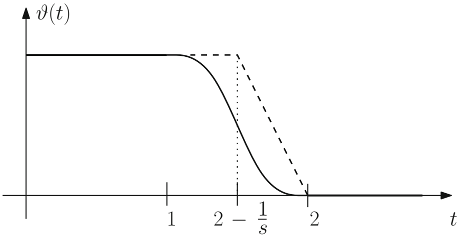

Fig. C.1 The bump function from Lemma C.2.15

Proof. For reals $r > 0$ consider the bump function $\omega_r : \mathbb{R} \to \mathbb{R}$,

$$\omega_r(t) := \begin{cases} \exp\left(-\frac{r}{1-4t^2}\right) & \text{for } |t| < \frac{1}{2}, \\ 0 & \text{for } |t| \geq \frac{1}{2} \end{cases}$$

of class C^∞ (cf. [1, p. 94]). Then $\vartheta_r : \mathbb{R} \to \mathbb{R}$, $\vartheta_r(t) := \int_{-\infty}^t \omega_r / \int_{-\infty}^\infty \omega_r$ is an increasing C^∞-function with $\vartheta_r(t) = 0$ for $t \leq -\frac{1}{2}$, $\vartheta_r(t) = 1$ for $t \geq \frac{1}{2}$ and the derivative $D\vartheta_r(t) = \omega_r(t)/\int_{-\infty}^\infty \omega_r$. From the properties of ω_r we see that $\min_{t\in\mathbb{R}} D\vartheta_r(t) = 0$ and $m(r) := \max_{t\in\mathbb{R}} D\vartheta_r(t) = \exp(-r)/\int_{-\infty}^\infty \omega_r$. It is not difficult to prove that $m : (0, \infty) \to \mathbb{R}$ is a strictly increasing continuous function with $\lim_{r\searrow 0} m(r) = 1$. Thus, for every $s > 1$ there exists a $r^* > 0$ such that $m(r^*) \leq s$, and therefore $D\vartheta_{r^*}(t) \in [0, s]$ for all $t \in \mathbb{R}$. In conclusion, the function ϑ given by $\vartheta(t) := \vartheta_{r^*}(\frac{3}{2} - t)$ satisfies the assertions.

Yet, it remains the establish the final estimate. By construction, the minimal slope of ϑ is greater or equal than $-s$. Due to $\vartheta(2) = 0$, this yields $\vartheta(t) \leq -s(t-2)$ for all $t \in \left[2 - \frac{1}{s}, 2\right]$ (see Fig. C.1). Thus, it is $t\vartheta(t) = st(2-t) \leq 2 - \frac{1}{s}$ for all $t \in \left[2 - \frac{1}{s}, 2\right]$ and since also $t\vartheta(t) \leq t \leq 2 - \frac{1}{s}$ holds for $t \in \left[0, 2 - \frac{1}{s}\right]$, we have deduced the desired inequality $t\vartheta(t) \leq 2 - \frac{1}{s}$ for all $t \geq 0$. Having this at hand, the elementary estimate $2 - \frac{1}{s} \leq s$, $s > 1$, yields our claim. □

Proposition C.2.16 (cut-off functions). *If X is a C^m-Banach space, then for all reals $\rho > 0$ and $s > 1$ there exists a C^m-function $\chi_\rho : X \to \mathbb{R}$ such that $\chi_\rho(x) \equiv 1$ for $\|x\| \leq \rho$, $\chi_\rho(x) \in (0, 1)$ for $\|x\| \in (\rho, 2\rho)$, $\chi_\rho(x) \equiv 0$ for $\|x\| \geq 2$ and $\|D\chi_\rho(x)\| \leq \frac{s}{\rho}$, as well as $x\chi_\rho(x) \in \bar{B}_{s\rho}(0, X)$ for all $x \in X$.*

Proof. Given $\rho > 0$ we define $\chi_\rho : X \to \mathbb{R}$ by $\chi_\rho(x) := \vartheta\left(\frac{\|x\|}{\rho}\right)$ with a bump function ϑ from Lemma C.2.15. In a neighborhood of 0 we have $\chi_\rho(x) \equiv 1$. Outside this set, by assumption, χ_ρ is the composition of C^m-mappings and therefore of class C^m. The bound for the derivative follows from the chain rule in Theorem C.1.3 yielding

$$\|D\chi_\rho(x)\| \leq \frac{1}{\rho}\left|D\vartheta\left(\frac{\|x\|}{\rho}\right)\right|\left\|Dn\left(\frac{\|x\|}{\rho}\right)\right\| \overset{\text{(C.2a)}}{\leq} \frac{s}{\rho} \quad \text{for all } x \in X$$

using Lemma C.2.15. It is a consequence of the final estimate in Lemma C.2.15 that $\|\chi_\rho(x)x\| = \rho\vartheta\left(\frac{\|x\|}{\rho}\right)\frac{\|x\|}{\rho} \leq \rho s$ for all $x \in X$ holds and we are done. □

After these preparations we finally arrive at a smooth version of Proposition C.2.5.

Proposition C.2.17 (*C^m*-extension). *Let Z be a Banach space, suppose that X, Y are C^m-Banach spaces and let $\Omega \subseteq X \times Y$ be an open neighborhood of 0. Provided $F : \Omega \to Z$ is a C^m-mapping so that for each $r > 0$ with $(B_\Omega(r))$ one has*

$$\ell_i(r) := \operatorname{lip}_i F|_{B_r(0,X)\times B_r(0,Y)} < \infty \quad \textit{for all } i = 1, 2,$$

then for reals $s > 1$ and $\rho > 0$ satisfying $(B_\Omega(s\rho))$ there exists a C^m-mapping $F^\rho : X \times Y \to Z$ with the following properties:

(a) $F^\rho(x, y) = F(x, y)$ for all $x \in B_\rho(0, X)$, $y \in B_\rho(0, Y)$.
(b) One has the global Lipschitz estimates

$$\operatorname{lip}_1 F^\rho \leq (1 + 2s)\ell_1(\rho), \qquad \operatorname{lip}_2 F^\rho \leq (1 + 2s)\ell_2(\rho).$$

(c) If the derivatives $D^n F$, $n \in \{0, \dots, m\}$, are bounded on $B_{s\rho}(0, X) \times B_{s\rho}(0, Y)$, then the same holds for $F^\rho : X \times Y \to Z$.

Remark C.2.18. In case F satisfies a Darbo condition, then Remark C.2.2 in combination with [35, p. 39, Proposition 5.3(b)] implies

$$\operatorname{dar} F^\rho \leq (1 + 2s)\begin{cases} 0 & \text{for } \dim X \times Y < \infty, \\ \operatorname{dar} F & \text{otherwise.} \end{cases}$$

In particular, this means that compactness of F persists when passing over to the modified mapping F^ρ; this observation also applies to Remark C.2.6 above.

Proof. For a given $s > 1$ choose a $\rho > 0$ such that the inclusion $(B_\Omega(s\rho))$ holds. Thanks to the final inclusion in Proposition C.2.16 this guarantees $\chi_\rho(x)x \in \bar{B}_{s\rho}(0)$ and also $(\chi_\rho(x)x, \chi_\rho(y)y) \in \Omega$ for all $x \in X$, $y \in Y$. Therefore, we can define the extension $F^\rho : X \times Y \to Z$ by $F^\rho(x, y) := F(\chi_\rho(x)x, \chi_\rho(y)y)$, which is of class C^m and satisfies assertions (a) and (c). In order to establish the remaining claim (b), we consider the C^m-function $\theta_\rho : X \to X$, $\theta_\rho(x) := \chi_\rho(x)x$. By the product rule (cf. [295, p. 336]), Proposition C.2.16 and Lemma C.2.15 one has the estimate

$$\|D\theta_\rho(x)\| \leq \|D\chi_\rho(x)x\| + |\chi_\rho(x)| \leq \frac{s}{\rho}\|x\| + |\chi_\rho(x)| \leq 2s + 1 \quad \text{for all } x \in X$$

and consequently $\operatorname{lip}\theta_\rho \leq 2s + 1$ by the mean value inequality (cf. [295, p. 342, Corollary 4.3]). This yields the Lipschitz estimate

$$\|F^\rho(x, y) - F^\rho(\bar{x}, y)\| \leq \ell_1(s\rho)\|\theta_\rho(x) - \theta_\rho(\bar{x})\| \leq (1 + 2s)\ell_1(s\rho)\|x - \bar{x}\|$$

for all $x, \bar{x} \in X$, $y \in Y$ and the claimed condition for $\mathrm{lip}_1 F^\rho$. Similarly, one shows the second Lipschitz estimate yielding $\mathrm{lip}_2 F^\rho$. □

C.3 Remarks

Differentiability: Due to the lack of a reference for Proposition C.1.1, we gave a direct proof. Yet, Proposition C.1.2 is from [5, p. 25, Theorem 1.5.9] and somewhat illustrates parallel proof strategies for assertions on Lipschitz and Darbo conditions.

The higher order chain rule from Theorem C.1.3 is an important technical tool needed in various contexts, e.g., to prove smoothness properties of invariant manifolds. On the one hand it is necessary to rigorously compute the higher order derivatives of compositions of maps, the so-called "derivative tree". It turned out to be advantageous to use two different representations of the derivative tree, namely a "totally unfolded derivative tree" (in Taylor approximations or to compute explicit bounds of higher order derivatives) and besides a "partially unfolded derivative tree" to elaborate certain induction arguments in a recursive way. On the other hand, one needs explicit versions of the chain rule to get Taylor approximations of invariant manifolds (see [383]). Unfortunately, but due to its complexity, certain versions of the chain rule found in the literature need a little care. Nonetheless, a variety of different formulations for the higher order chain rule from Theorem C.1.3 can be found in [407, 408].

Smooth norms and extensions: Estimates for the Lipschitz constants of the radial retraction (also called *radial projection*) appeared for the first time in [123]. Indeed, the minimal Lipschitz constant 1 turned out to be a characteristic quantity to characterize inner product spaces (cf. [112]).

The problem of finding Banach spaces which admit a smooth norm found a certain attention over the last decades. Thus, a comprehensive approach to this topic can be found in the monographs [116], [151, Chaps. 8–10], [284] or the book [451]. Here, also deeper questions on the geometry of Banach spaces are addressed. Referring to Example C.2.8 it is particularly interesting, whether such smoothness results also hold for norms in spaces of vector-valued (say Bochner-integrable) functions. This issue is discussed in [451, pp. 25–26, Theorem 2.2.9]. For a brief outline on smooth norms see [1, pp. 385–388, 5.5B].

Finally, classes of Orlicz spaces with a C^m-norm can be found in [317]. It is shown in [167, Applications (ii)] that Λ-spaces (closures of C^∞ in the Hölder-topology) admit a norm that is C^∞ away from the origin.

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