Appendix A
Henselizations

In this appendix, I have gathered some facts about Henselizations that can be found scattered in the literature (some sources dealing more extensively with Henselizations are [70,71,77,106]). Hensel observed that solving an equation over the p-adics can be reduced to finding a root in the residue field, provided this root is simple. This property, now known as Hensel's Lemma—and a ring satisfying it, is called *Henselian*—, extends easily to any complete local ring; see Theorem A.1.1. Although any Noetherian local ring admits a uniquely defined, smallest complete overring, its completion—which inherits many of the good properties of the original ring, and in particular is Henselian—, the process introduces transcendental elements. The Henselization of a local ring is much closer to it than its completion, since it is a direct limit of finite etale extensions. As Eisenbud remarks

> "... [i]t can thus be used to give the same microscopic view of a variety as the completion, but without passing out of the category of algebraic varieties."
>
> [27, p. 186]

The main objective of this appendix is to give a direct construction of the Henselization which, to my knowledge, never appeared in print.[1]

A.1 Hensel's Lemma

A very important algebraic tool in studying local properties of a variety, or equivalently, properties of Noetherian local rings, is the completion $\widehat{R}$ of a Noetherian local ring R. It is again a Noetherian local ring, which inherits many of the properties of the original ring, and in fact, there is natural homomorphism $R \to \widehat{R}$, which is flat and unramified (recall that the latter means that the maximal ideal of R extends to the maximal ideal of its completion $\widehat{R}$). Whereas there is no

[1] Jan Denef, who was my promotor at the time, suggested the construction to me in 1981, which I then subsequently worked out and wrote up as part of my license thesis [87].

known classification of arbitrary Noetherian local rings, we do have many structure theorems, due mostly to Cohen, for complete Noetherian local rings. In particular, the equal characteristic complete regular local rings are completely classified by their residue field k and their dimension d: any such ring is isomorphic to the power series ring $k[[\xi_1,\dots,\xi_d]]$. Also extremely useful is the fact that we have an analogue of Noether normalization for complete Noetherian local domains: any such ring admits a regular subring over which it is finite. Another nice property of complete local rings is the following formal version of Newton's method for finding approximate roots.

Theorem A.1.1 (Hensel's Lemma). *Let $(R,\mathfrak{m})$ be a complete local ring with residue field k. Let $f \in R[t]$ be a monic polynomial in the single variable t, and let $\bar{f} \in k[t]$ denote its reduction modulo $\mathfrak{m}R[t]$. For every simple root $u \in k$ of $\bar{f} = 0$, we can find $a \in R$ such that $f(a) = 0$ and u is the image of a in k.*

Proof. Let $a_1 \in R$ be any lifting of u. Since $\bar{f}(u) = 0$, we get $f(a_1) \equiv 0 \mod \mathfrak{m}$. We will define elements $a_n \in R$ recursively such that $f(a_n) \equiv 0 \mod \mathfrak{m}^n$ and $a_n \equiv a_{n-1} \mod \mathfrak{m}^{n-1}$ for all $n > 1$. Suppose we already defined $a_1,\dots,a_n$ satisfying the above conditions. Consider the Taylor expansion

$$f(a_n + t) = f(a_n) + f'(a_n)t + g_n(t)t^2 \tag{A.1}$$

where $g_n \in R[t]$ is some polynomial. Since the image of a_n in k is equal to u, and since $\bar{f}'(u) \neq 0$ by assumption, $f'(a_n)$ does not lie in $\mathfrak{m}$ whence is a unit, say, with inverse u_n. Define $a_{n+1} := a_n - u_n f(a_n)$. Substituting $t = -u_n f(a_n)$ in (A.1), we get

$$f(a_{n+1}) \in (u_n f(a_n))^2 R \subseteq \mathfrak{m}^{2n},$$

as required.

To finish the proof, note that the sequence a_n is by construction Cauchy, and hence by assumption admits a limit $a \in R$ (whose residue is necessarily again equal to u). By continuity, $f(a)$ is equal to the limit of the $f(a_n)$ whence is zero. $\square$

There are sharper versions of this result, where the root in the residue field need not be simple (see [27, Theorem 7.3]), or even involving systems of equations (see [13, §4.6]; but see also the next section).

A local ring satisfying the hypothesis of the above theorem is normally called a *Henselian* ring, although we will deviate from that practice in the next section. For some equivalent definitions, we refer once more to the literature [70, 71, 77, 106]. From a model-theoretic point of view, it is more convenient to work with Henselian local rings than with complete ones, since they form a first-order definable class (as is clear from the defining condition).

As with completion, there exists a 'smallest' Henselian overring. More precisely, for each Noetherian local ring R, there exists a Noetherian local R-algebra $R^\sim$, its *Henselization*, satisfying the following universal property: any local homomorphism $R \to H$ with H a Henselian local ring, factors uniquely through an R-algebra homomorphism $R^\sim \to H$. Below, we will show the existence of such a

Henselization by giving a concrete construction of $R^{\sim}$. Note that Theorem A.1.1 and the universal property imply that $R^{\sim}$ is a subring of $\widehat{R}$, and in particular, the latter is the completion of the former.

A.2 Construction of the Henselization

Let $(R,\mathfrak{m})$ be a Noetherian local ring. By a *Hensel system* over R of size N, we mean a pair $(\mathscr{H},\mathbf{a})$ consisting of a system $\mathscr{H}(t)$ of N polynomial equations $h_1,\dots,h_N \in R[t]$ in the N unknowns $t := (t_1,\dots,t_N)$, and an approximate solution $\mathbf{a}$ modulo $\mathfrak{m}$ in R (meaning that $h_i(\mathbf{a}) \equiv 0 \mod \mathfrak{m}$ for all i), such that associated Jacobian matrix

$$\operatorname{Jac}(\mathscr{H}) := \begin{pmatrix} \partial h_1/\partial t_1 & \partial h_1/\partial t_2 & \dots & \partial h_1/\partial t_N \\ \partial h_2/\partial t_1 & \partial h_2/\partial t_2 & \dots & \partial h_2/\partial t_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial h_N/\partial t_1 & \partial h_N/\partial t_2 & \dots & \partial h_N/\partial t_N \end{pmatrix} \tag{A.2}$$

evaluated at $\mathbf{a}$ is invertible over R, that is to say, the *Jacobian determinant* $\det(\operatorname{Jac}(\mathscr{H}))$ evaluated at $\mathbf{a}$ is a unit in R. We express the latter condition also by saying that $\mathbf{a}$ is a *non-singular* approximate solution. An N-tuple $\mathbf{s}$ in some local R-algebra S is called a *solution* of the Hensel system $(\mathscr{H},\mathbf{a})$, if it is a solution of the system $\mathscr{H}$ and $\mathbf{s} \equiv \mathbf{a} \mod \mathfrak{m}S$. Note that $(\mathscr{H},\mathbf{s})$ is then a Hensel system over S, and therefore, we sometimes call $\mathscr{H}$ a Hensel system, without mentioning the (approximate) non-singular solution. A Hensel system of size $N = 1$ is just a Hensel equation together with a solution in the residue field, as in the statement of Hensel's lemma. In fact, R satisfies Hensel's lemma if and only if any Hensel system over R has a solution in R. The proof of this equivalence is not that easy (one can give for instance a proof using standard etale extensions as in [70]).

Instead, we alter out definition by calling a local ring R *Henselian*, if any Hensel system (of any size) over R has a solution in R. In conclusion, being Henselian in the new sense implies that in the old sense, and the converse also holds, but is harder to prove. An easy modification of the proof of Theorem A.1.1, left to the reader, shows that complete local rings are Henselian in this new sense. In fact, using multivariate Taylor expansion, we obtain the following stronger version.

A.2.1 *Any Hensel system $(\mathscr{H},\mathbf{a})$ over R admits a unique solution in the completion $\widehat{R}$.* □

We call an element $r \in \widehat{R}$ a *Hensel element* (over R) if there exists a Hensel system $(\mathscr{H},\mathbf{a})$ over R such that r is the first entry of the unique solution of this system in $\widehat{R}$. We will express this by saying that *$\mathscr{H}$ is a Hensel system for r*. Note that if $\mathbf{r} = (r_1,\dots,r_N)$ is a solution of a Hensel system $\mathscr{H}$ over R, then any r_i is a Hensel element. This is true by definition for r_1. For $i > 1$, let $\mathscr{H}'$ be obtained by interchanging the unknowns t_1 and t_i, as well as, h_1 with h_i. It follows that $\mathscr{H}'$ is a Hensel system for $(r_i,r_2,\dots,r_{i-1},r_1,r_{i+1},\dots,r_N)$, showing that r_i is a Hensel element.

Let $R^\sim$ be the subset of $\widehat{R}$ of all Hensel elements. For given Hensel elements r and r', we construct from their associated Hensel systems $(\mathscr{H}(t),\mathbf{a})$ and $(\mathscr{H}'(t'),\mathbf{a}')$ of size N and N' respectively, a new Hensel system for $r+r'$ as follows: let $N'' := N+N'+1$, let t'' be the N''-tuple of unknowns (u,t,t'), with u a single variable, and consider the system $\mathscr{H}''$ of N'' equations in t'' given by the equation $u = t_1 + t_1'$, and the systems $\mathscr{H}(t)$ and $\mathscr{H}'(t')$. One checks that $(\mathscr{H}'', a_1 + a_1', \mathbf{a}, \mathbf{a}')$ is a Hensel system—since its Jacobian determinant is the product of the Jacobian determinants of $\mathscr{H}$ and $\mathscr{H}'$—whose unique solution in $\widehat{R}$ has first entry equal to $r+r'$, showing that the latter is again a Hensel element. The same argument can be used to prove that the product of Hensel elements is again a Hensel element. With little effort one actually shows:

A.2.2 *The collection of all Hensel elements is a local ring $R^\sim$ with maximal ideal $\mathfrak{m}R^\sim$. Moreover, $R^\sim$ is Henselian, with completion equal to $\widehat{R}$.*

Indeed, let $\mathfrak{m}^\sim := \mathfrak{m}\widehat{R} \cap R^\sim$. To show that $R^\sim$ is local with maximal ideal $\mathfrak{m}^\sim$, it suffices to show that any element $r \in R^\sim$ not in $\mathfrak{m}^\sim$ is a unit in $R^\sim$. Since r does not belong to $\mathfrak{m}\widehat{R}$, it has an inverse in $\widehat{R}$. Using an auxiliary variable u and the equation $t_1 u = 1$, it is now not hard to show that $1/r$ is an Hensel element. In particular, R, $R^\sim$ and $\widehat{R}$ all have the same residue field k. To prove that $R^\sim$ is Henselian, we must verify the multivariate Hensel lemma, that is to say, let $(\tilde{\mathscr{H}}(t),\mathbf{a})$ be a Hensel system over $R^\sim$. Since $R^\sim$ and R have the same residue field, we may choose $\mathbf{a}$ in R. By A.2.1, there exists a unique solution $\mathbf{r}$ over $\widehat{R}$ of this Hensel system. Remains to show that $\mathbf{r}$ has its entries already in $R^\sim$, and to this end, it suffices by the above discussion to construct a Hensel system over R of which $\mathbf{r}$ is part of a solution.

Let $\mathbf{s} = (s_1,\dots,s_d)$ be the tuple of coefficients in $R^\sim$ of the equations $\tilde{\mathscr{H}}$ (listed in a fixed order), and let $\mathscr{H}(t,u)$ be obtained from $\tilde{\mathscr{H}}$ by replacing each of these coefficients by a new variable u_i, so that $\tilde{\mathscr{H}}(t) = \mathscr{H}(t,\mathbf{s})$. For each s_i, choose $b_i \in R$ such that $s_i \equiv b_i \mod \mathfrak{m}\widehat{R}$. Let $(\mathscr{H}_i(u_i,t_i),(b_i,\mathbf{c}_i))$ be a Hensel system for each Hensel element s_i, with t_i a finite tuple of auxiliary unknowns and $\mathbf{c}_i$ a tuple of the corresponding length in R, for $i = 1,\dots,d$. One easily checks that the system $\mathscr{G}$ in the unknowns $t,u_1,t_1,\dots,u_d,t_d$ at the tuple $\mathbf{c} := (\mathbf{a},b_1,\mathbf{c}_1,\dots,b_d,\mathbf{c}_d)$ given by $\mathscr{H}$ and all $\mathscr{H}_i$ is a Hensel system, since the Jacobian determinant of $(\mathscr{G},\mathbf{c})$ is the product of the Jacobian determinants of $(\mathscr{H},\mathbf{a})$ and the $(\mathscr{H}_i,(b_i,\mathbf{c}_i))$. By A.2.1, the unique solution of this Hensel system in $\widehat{R}$ must be of the form $(\mathbf{r},s_1,\mathbf{r}_1,\dots,s_d,\mathbf{r}_d)$, for some $\mathbf{r}_i$ in $\widehat{R}$, showing that $\mathbf{r} \in R^\sim$. □

It is unfortunately less easy to prove that $R^\sim$ is also Noetherian, and we postpone the discussion until after we proved our main result:

Theorem A.2.3. *The ring $R^\sim$ satisfies the universal property of Henselization: any Henselian local R-algebra S admits a unique structure of $R^\sim$-algebra.*

Proof. We need to show that there exists a (unique) R-algebra homomorphism $R^\sim \to S$. Given $r \in R^\sim$, let $(\mathscr{H},\mathbf{a})$ be a Hensel system admitting a solution with

first entry r. Since $\mathbf{a}$ is an approximate solution of $\mathscr{H}$ in R, it remains so in S. By (the revised) definition of Henselian, the approximate solution $\mathbf{a}$ lifts uniquely to a solution $\mathbf{s}$ in S. We define the image of r in S now as the first entry of this solution $\mathbf{s}$. Uniqueness guarantees firstly that this is an R-algebra homomorphism, an secondly that it is unique. □

Returning to the issue of Noetherianity, we will use the local flatness criterion discussed §3.3.6. We start with the flatness of the Henselization:

Proposition A.2.4. *For any ideal $I \subseteq R$, the Henselization of R/I is isomorphic to $R^{\sim}/IR^{\sim}$. Moreover, $R \to R^{\sim}$ is faithfully flat, whence a scalar extension, and $R^{\sim}$ is ind-Noetherian.*

Proof. Let $S := R/I$. It is not hard to show that any homomorphic image of a Henselian local ring is again Henselian. Hence $R^{\sim}/IR^{\sim}$ is Henselian, and the universal property of Henselizations then yields a unique homomorphism $S^{\sim} \to R^{\sim}/IR^{\sim}$. The composition of this homomorphism with $R^{\sim}/IR^{\sim} \to \widehat{R}/I\widehat{R}$ is injective, since the latter is the completion of S. Hence $S^{\sim} \to R^{\sim}/IR^{\sim}$ must also be injective. To prove surjectivity, let $r \in R^{\sim}$ and let $\mathscr{H}$ be a Hensel system for r. The reduction modulo I of this Hensel system therefore has a unique solution in $S^{\sim}$, and by uniqueness, the first entry of this solution must map to the image of r in $R^{\sim}/IR^{\sim}$. This proves the first assertion, and in particular that $I\widehat{R} \cap R^{\sim} = IR^{\sim}$, for any ideal $I \subseteq R$. The second assertion then follows from the flatness of $R \to \widehat{R}$ and Corollary 3.3.15. Since $R \to R^{\sim}$ is unramified by A.2.2, it is therefore a scalar extension (see §3.2.3).

So remains to show that $R^{\sim}$ is ind-Noetherian (defined after Corollary 3.3.22). Let $\mathbf{x}$ be a finite tuple in $R^{\sim}$. As already remarked before, we can find a Hensel system $\mathscr{H}(t)$ over R such that $\mathbf{x}$ is part of its unique solution. Hence, if $S_{\mathbf{x}}$ is the localization of $R[t]/(\mathscr{H})$ with respect to the ideal generated by $\mathfrak{m}$, then $\mathbf{x}$ is already a tuple in $S_{\mathbf{x}}$. It follows from the construction of $R^{\sim}$, that $S_{\mathbf{x}}^{\sim} = R^{\sim}$. In particular, $S_{\mathbf{x}} \to R^{\sim}$ is a scalar extension by what we just proved, and $R^{\sim}$ is the direct limit of the $S_{\mathbf{x}}$. □

Theorem A.2.5. *The Henselization of a Noetherian local ring is again Noetherian.*

Proof. It suffices to show that $R^{\sim} \to \widehat{R}$ is faithfully flat, since $\widehat{R}$ is Noetherian. To obtain flatness, it suffices in view of Corollary 3.3.25 and Proposition A.2.4 to show that $\mathrm{Tor}_1^{R^{\sim}}(\widehat{R}, k) = 0$, where k is the residue field of R. To this end, let

$$R^m \to R^n \to R \to k \to 0 \tag{A.3}$$

be an exact sequence. By Proposition A.2.4, tensoring with $R^{\sim}$ yields an exact sequence

$$(R^{\sim})^m \to (R^{\sim})^n \to R^{\sim} \to k \to 0.$$

By definition, $\mathrm{Tor}_1^{R^{\sim}}(\widehat{R}, k)$ is the homology of the complex obtained from tensoring this exact sequence with $\widehat{R}$, that is to say, of the complex

$$(\widehat{R})^m \to (\widehat{R})^n \to \widehat{R} \to k \to 0.$$

However, this latter complex is actually exact since it is obtained from tensoring (A.3) with the flat extension $\widehat{R}$, showing that $\operatorname{Tor}_1^{R^\sim}(\widehat{R},k) = 0$. $\square$

A.3 Etale Proto-grade

We conclude with a proto-graded version of the previous construction by constructing a proto-grading on the Henselization of a proto-graded Noetherian local ring $(R,\mathfrak{m})$, and giving conditions under which this proto-grading is Noetherian and faithfully flat. Define a proto-grading on $R^\sim$ by the condition that a Hensel element $y \in R^\sim$ has proto-grade at most n if it admits a Hensel system $(\mathscr{H},\mathbf{u})$ of length $N \leq n$, in which all polynomials have degree at most n, and all coefficients as well as all entries of $\mathbf{u}$ have proto-grade at most n.

A.3.1 *This yields a proto-grading on $R^\sim$, called the* etale proto-grading *on $R^\sim$, extending the proto-grading on R. Moreover, $R \to R^\sim$ is a morphism of proto-graded rings.*

The fact that this is a proto-grading follows from the proof that $R^\sim$ is a ring, since we explicitly constructed Hensel systems for sums, products, and inverses (of units). Since $t-a$ is a Hensel system of $a \in R$, its etale proto-grade is equal to its proto-grade in R, and the last assertion is now immediate. $\square$

The following result enables us to calculate protopowers:

Proposition A.3.2. *If R is a proto-graded Noetherian local ring and $R^\sim$ is viewed with its etale proto-grading extending the proto-grading on R, then we have an isomorphism*

$$(R^\sim)_\flat \cong (R_\flat)^\sim.$$

Proof. A special instance of the above isomorphism is the fact that if R is Henselian, then so is $R_\flat$. We prove this first, and so, let $(\mathscr{H},\mathbf{u})$ be a Hensel system over $R_\flat$ of proto-grade at most n, say. Choose approximations $\mathscr{H}_w$ and $\mathbf{u}_w$ over R of proto-grade at most n, with respective ultraproduct $\mathscr{H}$ and $\mathbf{u}$. By Łoś' Theorem, almost all $(\mathscr{H}_w,\mathbf{u}_w)$ are Hensel systems. Since R is Henselian by assumption, these Hensel systems have a (unique) solution $\mathbf{x}_w$. By definition, the $\mathbf{x}_w$ have etale proto-grade at most n, and hence their ultraproduct $\mathbf{x}$ lies in $R_\flat$. By Łoś' Theorem, $\mathbf{x}$ is then a solution of the Hensel system $(\mathscr{H},\mathbf{u})$.

Let R now be an arbitrary proto-graded Noetherian local ring. The embedding $R \to R^\sim$ induces an embedding $R_\flat \to (R^\sim)_\flat$. By our previous argument, $(R^\sim)_\flat$ is Henselian, whence by the universal property of a Henselization, we have a unique $R_\flat$-algebra embedding $(R_\flat)^\sim \to (R^\sim)_\flat$. To see that this is surjective, let x be an element in $(R^\sim)_\flat$, say of etale proto-grade at most n. Choose an approximation $x_w \in R^\sim$ of proto-grade at most n. Hence, almost each x_w is the first entry of the

unique solution $\mathbf{x}_w$ of a Hensel system $(\mathscr{H}_w, \mathbf{u}_w)$ over R of proto-grade at most n. Since the ultraproduct $(\mathscr{H}, \mathbf{u})$ of the $(\mathscr{H}_w, \mathbf{u}_w)$ is a Hensel system of proto-grade at most n, whence defined over $R_\flat$, the ultraproduct $\mathbf{x}$ of the $\mathbf{x}_w$ is a solution of etale proto-grade at most n, belonging therefore to $(R_\flat)^\sim$. Since x is its first entry, $x \in (R_\flat)^\sim$, as we wanted to show. □

Theorem A.3.3. *If R is a local ring with a Noetherian proto-grading, then the etale proto-grading on $R^\sim$ is also Noetherian. If R is moreover regular and the proto-grading on R is faithfully flat, then the etale proto-grading on $R^\sim$ is also faithfully flat.*

Proof. The first assertion follows from Proposition A.3.2 and Theorem A.2.5. To prove the second assertion, assuming that $(R, \mathfrak{m})$ is moreover regular, we first show that $R_\flat$ is also regular, by induction on the dimension d of R. Since the proto-grade is faithfully flat, $(R/I)_\flat = R_\flat/IR_\flat$ for all ideals $I \subseteq R$ by 9.1.7. Applied to $I = x_1R$, where $\mathfrak{m} = (x_1, \dots, x_d)R$, we have by induction that $(R/x_1R)_\flat = R_\flat/x_1R_\flat$ is regular, whence so is $R_\flat$, since x_1 is $R_\flat$-regular (as $R \to R_\flat$ is flat). Since $R_\flat \to (R_\flat)^\sim$ is a scalar extension by Proposition A.2.4, also $(R_\flat)^\sim$ is regular by 3.2.14. Hence, in view of Proposition A.3.2, we proved that $(R^\sim)_\flat$ is regular. Since $(x_1, \dots, x_d)$ is an $(R^\sim)_\natural$-regular sequence by the flatness of $R^\sim \to (R^\sim)_\natural$ (using Theorem A.2.5 and Corollary 3.3.3), the Cohen-Macaulay criterion for flatness (Theorem 3.3.9) together with Proposition 3.3.8, yields the desired flatness of $(R^\sim)_\flat \to (R^\sim)_\natural$. □

Let k be a field and ξ a finite tuple of indeterminates. For simplicity, we denote the Henselization of the localization of $k[\xi]$ with respect to the variables also by $k[\xi]^\sim$. A power series $f \in k[[\xi]]$ is called *algebraic* if it is a root of a non-zero polynomial in one variable with coefficients in $k[\xi]$. We denote the subring of algebraic power series by $k[[\xi]]^{\mathrm{alg}}$. The following result is well-known (see, for instance, [3, 77]).

A.3.4 *For any field k, the ring $k[[\xi]]^{alg}$ is equal to the Henselization $k[\xi]^\sim$ of $k[\xi]_\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal generated by the indeterminates.*

In particular, viewing $k[\xi]$ with its affine proto-grade given by degree (see (9.1.1.i)), we get an etale proto-grade on $k[[\xi]]^{\mathrm{alg}}$: an algebraic power series f has proto-grade at most n, if there exists a Hensel system in $k[\xi, t]$ for f of size at most n, such that the total degree of each polynomial in the system is at most n. Theorem 9.2.11, in conjunction with Theorem A.3.3 and Corollary 9.2.4, applied to this etale proto-grade on the ring of algebraic power series, immediately yields:

Theorem A.3.5. *For each pair (n, m) there exists a uniform bound $n' := n'(n, m)$ with the property that if k is an arbitrary field, $R := k[[\xi]]^{alg}$ the ring of algebraic power series with ξ an m-tuple of indeterminates, and $I := (f_1, \dots, f_s)R$ an ideal generated by elements f_i of etale proto-grade at most n, then I is generated by at most n' of the f_i, and its module of syzygies is generated by n' syzygies with entries of proto-grade at most n'. Moreover, if $f \in I$ has etale proto-grade at most n, then there exist algebraic power series g_i of etale proto-grade at most n' such that $f = g_1f_1 + \cdots + f_sg_s$.* □

In fact, we can now even give a non-linear version (as already mentioned, ideal membership amounts to solving a linear equation), which also extends Theorem 7.1.10 to arbitrary coefficients over the algebraic power series ring (and a similar argument using Theorem 7.1.6 would then also yield a uniform analogue of the next result).

Theorem A.3.6. *For each pair (n,m) there exists a uniform bound $n' := n'(n,m)$ with the property that if k is an arbitrary field, $R := k[[\xi]]^{alg}$ the ring of algebraic power series with ξ an m-tuple of indeterminates, and if $f_1 = \cdots = f_s = 0$, with $f_i \in R[t]$, is a polynomial system of $s \leq n$ equations in at most n unknowns t, of (extended) etale proto-grade at most n and admitting a solution in formal power series, then it admits an algebraic solution of etale proto-grade at most n'.*

Proof. Suppose no such bound holds for the pair (m,n), yielding for each w, a counterexample over a ring of algebraic power series $R_w := K_w[[\xi]]^{\mathrm{alg}}$, consisting of an n-tuple of equations $\mathbf{f}_w$ in $R_w[t]$ of extended etale proto-grade at most n, and a solution $\mathbf{y}_w$ in $K_w[[\xi]]$ of this system of equations (viewed as a system of equations in the unknowns t), but no such solution in algebraic power series of etale proto-grade at most w exists. Let K be the ultraproduct of the K_w, and let $\mathbf{y}_\natural$, $\mathbf{f}_\natural$, and $S_\natural$ be the respective ultraproducts of the $\mathbf{y}_w$, $\mathbf{f}_w$ and $K_w[[\xi]]$. By definition of the extended etale proto-grade (see 9.1.2), the $\mathbf{f}_w$ are polynomials of degree at most n with coefficients of etale proto-grade at most n, and hence their ultraproduct $\mathbf{f}_\natural$ lies in $K[[\xi]]^{\mathrm{alg}}[t]$. Moreover, by Łoś' Theorem, $\mathbf{y}_\natural$ is a solution of this system of equations in $S_\natural$, whence in $S_\sharp \cong K[[\xi]]$, by Proposition 7.1.8. Hence, by Theorem 7.1.5, we can find a solution $\mathbf{z}$ of this system of equations in $K[[\xi]]^{\mathrm{alg}}$. Let N be its etale proto-grade. By Proposition A.3.2, the ring $K[[\xi]]^{\mathrm{alg}}$ is just the protoproduct of the R_w, and hence by Łoś' Theorem, the approximations of $\mathbf{z}$ yield a counterexample for any w bigger than N. □

Appendix B
Boolean Rings

We mentioned Boolean rings in our sheaf-theoretic construction of an ultraproduct, Theorem 2.6.4. In this appendix, we generalize the notion of a Boolean ring to an n-Boolean ring, replacing the condition that all elements are idempotent by the condition that they are all n-potent (see below for definitions). The Stone Representation Theorem gives a description of Boolean rings in terms of power set rings. We give a proof, using the embedding theorems into ultrapowers from §7.1.3, in the more general context of n-Boolean rings, recovering in particular Henkin's version [41] of the Stone Representation Theorem [110]. In §B.3, we prove a similar result for an ω-Boolean ring, that is to say, a ring in which all elements are potent (but possibly of unbounded potency).[1] Not surprisingly, to control this unboundedness, we have to use protoproducts instead of ultraproducts. The last section then deals with a generalization studied already by Chacron, Bell, et al. ([19, 20, 12]), to wit, periodic rings.

B.1 n-Boolean Rings

A ring is called *torsion* if it has positive characteristic, that is to say, if $d = 0$ in R for some $0 \neq d \in \mathbb{N}$ (not necessarily prime). Any such ring admits a decomposition as a finite direct sum of rings of prime power characteristic, its so-called *primary components*:

B.1.1 *If R has torsion, say, of characteristic $p_1^{e_1} \cdots p_s^{e_s}$, with p_i distinct primes and $e_1 \geq 1$, then $R \cong R_1 \oplus \cdots \oplus R_s$, with the characteristic of R_i equal to $p_i^{e_i}$. If R is reduced, then all $e_i = 1$.*

Indeed, for each i, let d_i be the product of all $q_j := p_j^{e_j}$ except for $i = j$ and put $R_i := d_i R$. Since $R_i \cong R/\operatorname{Ann}_R(d_i)$, we may view it as a ring of characteristic q_i.

[1] This type of rings already occurs in [33], without any special name; elsewhere they are referred to as *J-rings*, after Jacobson [60], who proved that they are always commutative. In [1], the name n-Boolean refers to a different generalization of Boolean rings.

Since the d_i are relatively prime, there exist $r_i \in \mathbb{N}$ such that $1 = r_1d_1 + \cdots + r_sd_s$. For any $x \in R$, let $x_i := r_id_ix$. It follows that $x_i \in R_i$, and $x = x_1 + \cdots + x_n$. Since $x_ix_j = 0$ for $i \neq j$, the result follows readily. □

Recall that an element x in a ring R is called *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$. By the *reduction*, R_{red}, of a ring we mean the residue ring $R/\mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of R, that is to say, the ideal of all nilpotent elements. Let $n \geq 2$. We say that an element x in a (commutative) ring R is *n-potent*, if $x^n = x$. The least such $n \geq 2$ is called the *potency* of x. Instead of 2-potent, one usually says *idempotent*. We will call an element *potent* if it is n-potent for some n, that is to say, if it has finite potency. We define the *potency* of an ideal as the maximal potency of its members. In order to discuss potencies, the following terminology will be helpful. We say that *m pre-divides n* if $m-1$ divides $n-1$, for $m,n \in \mathbb{N}$. We may also express this by saying that m is a *pre-divisor* of n, or that n is a *pre-multiple* of m. Similarly, we define the *least common pre-multiple* and the *greatest common pre-divisor* of m and n in the obvious way. Note that 2 pre-divides any number, whereas 1 only pre-divides itself.

Lemma B.1.2. *Let R be a ring, $m,n \geq 2$ integers, and $x \in R$.*

B.1.2.i. *If x is m-potent, then it is n-potent for any pre-multiple n of m.*
B.1.2.ii. *If x is n-potent, and m pre-divides n then $x^{\frac{n-1}{m-1}}$ is m-potent.*
B.1.2.iii. *If x is both m-potent and n-potent, then it is also d-potent, where d is the greatest common pre-divisor of m and n.*

Proof. Let $(m-1)d = n-1$, whence $n = m + (d-1)(m-1)$. Multiplying $x^m = x$ with x^{m-1} yields $x^{2m-1} = x^m = x$. Continuing in this way, we get $x^n = x$. To prove (B.1.2.ii), observe that

$$(x^d)^m = x^{n-1+d} = x^n \cdot x^{d-1} = x \cdot x^{d-1} = x^d.$$

To prove (B.1.2.iii), suppose $m \leq n$ and we induct on n. If $m = n$, we are done, so we may assume $m < n$. Multiplying $x = x^m$ with x^{n-m} we get $x^{n-m+1} = x^n = x$. Since the greatest common pre-divisor of $(n-m+1)$ and m is also d, we are done by induction. □

By an *n-Boolean* ring B, we mean a ring in which every element is n-potent. Hence a 2-Boolean ring is just a Boolean ring, and any Boolean ring is n-Boolean for any n. If p is prime, then $\mathbb{Z}/p\mathbb{Z}$ is p-Boolean, since it is invariant under the Frobenius. Let us call a ring B *properly n-Boolean* if it is n-Boolean, but not m-Boolean for any $m < n$, that is to say, if B contains an element of potency n. It follows from Galois theory that any finite field $\mathbb{F}_q$ is properly q-Boolean. Immediately from Lemma B.1.2, we get:

Corollary B.1.3. *Let B be a ring, and $m,n \geq 2$.*

B.1.3.i. *If B is m-Boolean, then it is n-Boolean for any pre-multiple n of m.*
B.1.3.ii. *If B is both m-Boolean and n-Boolean, then it is also d-Boolean, where d is the greatest common pre-divisor of m and n.* □

Note that, in general, the characteristic of an n-Boolean ring can be larger than n. For instance, $\mathbb{Z}/6\mathbb{Z}$ is a 3-Boolean ring, and more generally, $\mathbb{Z}/2p\mathbb{Z}$ is p-Boolean, for every odd prime number p. We also observe that $\mathbb{Z}/15\mathbb{Z}$ and $\mathbb{Z}/30\mathbb{Z}$ are 5-Boolean but not 3-Boolean. To determine the possible characteristics, let $\alpha(n)$ be the greatest common divisor of all $k^n - k$, for $k \in \mathbb{N}$. For instance, $\alpha(2) = 2$, $\alpha(3) = 6$, $\alpha(4) = 2$, and $\alpha(5) = 30$. Note that by Fermat's Little Theorem, $\alpha(p)$ is divisible by p if p is prime, but from the previous examples, it is clear that in general it will be bigger. We will compute it in Corollary B.1.9 below, but for now we observe:

Lemma B.1.4. *If B is an n-Boolean ring, then its characteristic is a divisor of $\alpha(n)$. Moreover, $\alpha(n)$ is always square-free.*

Proof. Let d be the characteristic of B. For each k, since $k^n - k$ is zero in B, it must be divisible by d. To see the last assertion, we check that $\alpha(n)$ has p-adic order at most one, that is to say, is not divisible p^2, for any prime p. However, this is immediate since $\alpha(n)$ is a divisor of $p^n - p$, which has p-adic order one. □

In a Boolean ring B, one defines a partial order as follows: $a \leq b$ if $ab = a$, for $a, b \in B$. An element which is minimal among the non-zero elements of B is called an *atom*. Note that any multiple of an atom a is either zero or a itself, so that a is an atom if and only if aB has cardinality two. We call a Boolean ring *atomless*, if it has no atoms. In case B is a power set ring $\mathscr{P}(W)$ (see Example B.2.3), the order is given by inclusion, and so, atoms are precisely singletons. Moreover, the set of finite subsets is an ideal in B (by (B.1.5.x) below) whose residue ring is an example of an atomless Boolean ring.

Unlike Boolean rings, we cannot define a partial order on an n-Boolean ring B, for $n > 2$. Instead, we look at ideals with respect to inclusion. We call an ideal *atomic*, if it is a minimal non-zero ideal. The sum of all atomic ideals is called the *ideal of finite elements*. We call an element $x \in B$ an *atom* if it is idempotent and generates an atomic ideal. Note that this definition agrees with the older one in Boolean rings. In the next result, we gathered some basic facts about n-Boolean rings (recall that a prime ideal is called an *associated prime* if it is of the form $\operatorname{Ann}(x)$).

Proposition B.1.5. *Let B be an n-Boolean ring, $x, y \in B$ elements, $I \subseteq B$ an ideal, and $\mathfrak{p} \in \operatorname{Spec} B$ a prime ideal.*

- B.1.5.i. *$x^i B = xB$ for all $i > 0$, and any ideal is idempotent, that is to say, $I = I^2$;*
- B.1.5.ii. *x is idempotent if and only if it is of the form y^{n-1};*
- B.1.5.iii. *$xB \cap yB = xyB$ and $xB + yB = (x^{n-1} + y^{n-1} - x^{n-1}y^{n-1})B$. In particular, every finitely generated ideal is principal;*
- B.1.5.iv. *the annihilator of xB is equal to $(1 - x^{n-1})B$. It is a prime ideal if and only if xB is atomic;*
- B.1.5.v. *x is a unit in B if and only if $x^{n-1} = 1$ if and only if x is not a zero-divisor;*

B.1.5.vi. *any residue ring of B is n-Boolean. In particular, I is radical, whence equal to an intersection of prime ideals;*

B.1.5.vii. $\mathfrak{p}$ *is maximal with residue field a finite field of size at most n, isomorphic to $B_\mathfrak{p}$. In particular, B and any of its subrings have (Krull) dimension zero (one says that B is* hereditarily zero-dimensional*);*

B.1.5.viii. *if I is atomic, then it contains at most n elements, any non-zero element is a generator, and I contains a unique atom. Moreover, I with its induced addition and multiplication is isomorphic to a finite field $\mathbb{F}_q$, for some prime power $q = p^m$ pre-dividing n (with p dividing $\alpha(n)$), and q is the potency of I;*

B.1.5.ix. *any two distinct atoms are orthogonal, that is to say, their product is equal to zero;*

B.1.5.x. $\mathfrak{p}$ *is associated if and only if it is finitely generated, whence principal, if and only if it does not contain all finite elements. The ideal of finite elements is generated by all atoms and defines the closed subset of* $\operatorname{Spec} B$ *consisting of all non-principal prime ideals;*

B.1.5.xi. *there exists a non-principal prime ideal if and only if B is infinite. In particular, B is Noetherian if and only if B is finite if and only if* 1 *is a finite element;*

B.1.5.xii. *any B-module is flat, that is to say, B is absolutely flat, whence von Neumann regular.*

Proof. The inclusion $x^iB \subseteq xB$ is immediate. To prove the other inclusion, choose k such that $n^k \geq i$, and observe that $x = x^{n^k - i} \cdot x^i \in x^iB$, proving the first part of (B.1.5.i). Similarly, $I^2 \subseteq I$ and, for the converse, if $x \in I$ then $x \in x^2B \subseteq I^2$ by the first assertion. If x is idempotent, then $x = x^{n-1}$, and the converse of (B.1.5.ii) follows immediately from (B.1.2.ii) with $m = 2$. To prove (B.1.5.iii), let $z \in xB \cap yB$. If we write $z = ax = by$, then $z = ax = ax^n = byx^{n-1} \in xyB$. The second equality follows from the identity

$$(x^{n-1} + y^{n-1} - x^{n-1}y^{n-1})x = x + xy^{n-1} - xy^{n-1} = x$$

and the analogous identity for y. One direction in the first assertion of (B.1.5.iv) is clear since $x(1 - x^{n-1}) = x - x = 0$, so assume $ax = 0$. Hence $a = a(1 - x^{n-1}) \in (1 - x^{n-1})B$. This also proves (B.1.5.v), for if x is a unit, then $0 = \operatorname{Ann}(x) = (1 - x^{n-1})B$. To prove the direct inclusion in the second assertion in (B.1.5.iv), assume $\operatorname{Ann}(x)$ is prime. Given a non-zero ideal aB contained in xB, we need to show that $x \in aB$. By (B.1.5.iii), we have $axB = aB \cap xB = aB$, showing that $a \notin \operatorname{Ann}(x)$. Since $a(1 - a^{n-1}) = 0$ and $\operatorname{Ann}(x)$ is prime, $1 - a^{n-1} \in \operatorname{Ann}(x)$, showing that $x = xa^{n-1} \in aB$. Conversely, suppose xB is atomic, and we need to show that if a and b do not belong to $\operatorname{Ann}(x)$ then neither does their product. From $ax \neq 0$ and $bx \neq 0$ and the fact that xB is atomic, we get $axB = xB = bxB$ and hence $abxB = xB \neq 0$, so that, a fortiori, $abx \neq 0$.

The first assertion in (B.1.5.vi) is clear. Applying the observation that an n-Boolean ring is reduced to B/I, shows that I is radical. Since $\bar{B} := B/\mathfrak{p}$ is n-Boolean,

any element in $\bar{B}$ satisfies the equation $\xi^n - \xi = 0$. However, in a domain, this equation can have at most n solutions, showing that $\bar{B}$ has cardinality at most n. As any finite domain is a field, $\mathfrak{p}$ is maximal. If $x \in \mathfrak{p}B_{\mathfrak{p}}$, then $1 - x^{n-1}$ is a unit in $B_{\mathfrak{p}}$ killing x by (B.1.5.iv), showing that $\mathfrak{p}B_{\mathfrak{p}} = 0$ and hence $B_{\mathfrak{p}} \cong B/\mathfrak{p}$, thus completing the proof of (B.1.5.vii). To prove (B.1.5.viii), suppose I is atomic. Since I must be principal, it is isomorphic to $F := B/\operatorname{Ann}(I)$. Since $\operatorname{Ann}(I)$ is maximal by (B.1.5.iv) and (B.1.5.vii), the cardinality of I is at most n by (B.1.5.vii) and F is a field. If x is a non-zero element in I, then the inclusion $xB \subseteq I$ must be an equality, proving the second assertion in (B.1.5.viii). In particular, $y := x^{n-1}$ is a non-zero idempotent in I by (B.1.5.ii), whence an atom. To show that y is the unique atom in I, suppose z is another non-zero idempotent in I. By what we just proved, $yB = zB$, so that $y = az$ and $z = by$, for some $a, b \in B$. Multiplying the first equality with z, we get $zy = az^2 = az = y$, and similarly, multiplying the second with y yields $zy = by^2 = by = z$, and hence $z = y$. To prove the last assertion in (B.1.5.viii), let q be the cardinality of the field F (recall that $q = p^m$ for some m, with p the characteristic of F). Let $I^\times := I - \{0\}$ and let $x \in I^\times$. We argued that $y := x^{n-1}$ is the unique atom of I. Multiplying with x, we get $xy = x^n = x$. It is now easy to check that $I^\times$ is a multiplicative group with unit element y and that the isomorphism $I = yB \cong F$ sending $x = xy$ to the residue of x in F yields a group isomorphism between $I^\times$ and the multiplicative group of F. Since the latter is cyclic, so is the former. In particular, there exists $x \in I^\times$ such that the powers of x generate $I^\times$. Since x has therefore potency q, so does I. If q does not pre-divide n, then x has potency at most $d < q$ by Lemma B.1.2, with d the greatest common pre-divisor of q and n, contradiction. Finally, since the characteristic of B divides $\alpha(n)$ by Lemma B.1.4, so must p, concluding the proof of (B.1.5.viii). To prove (B.1.5.ix), let x and y be distinct atoms. By (B.1.5.viii), they generate different atomic ideals $xB \neq yB$, and hence their intersection, equal to xyB by (B.1.5.iii), must be a proper subideal of either atomic ideal, whence equal to zero, showing that $xy = 0$.

The first equivalence in (B.1.5.x) is immediate by (B.1.5.iii) and (B.1.5.iv). To prove the second, let zB be an arbitrary atomic ideal. If $\mathfrak{p}$ is not an associated prime, then $z\mathfrak{p}$ cannot be zero, lest $\mathfrak{p}$ is contained in $\operatorname{Ann}(z)$ whence by maximality, equal to it. Let $a \in \mathfrak{p}$ be such that $az \neq 0$. Since zB is atomic, $zB = azB$ and hence belongs to $\mathfrak{p}$. Conversely, if every atomic ideal is contained in $\mathfrak{p}$, then $\mathfrak{p}$ cannot be associated, since otherwise by the above, $\mathfrak{p} = \operatorname{Ann}(x) = (1 - x^{n-1})B$ with xB atomic, and so $1 = x + (1 - x^{n-1}) \in \mathfrak{p}$, contradiction. The last assertion in (B.1.5.x) is then clear from the above and (B.1.5.viii). By (B.1.5.x), in order to prove (B.1.5.xi), it suffices to show that 1 is a finite element if and only if B is finite, if and only if every ideal is principal. If $1 = a_1 + \cdots + a_s$ is a sum of atomic elements a_i, then $B = a_1B + \cdots + a_sB$. Since each a_iB is finite by (B.1.5.viii), so is B. Assume next that B is finite, then any ideal is finitely generated, whence principal by (B.1.5.iii). Finally, if every maximal ideal is principal, then the ideal of finite elements must be the unit ideal by (B.1.5.x). This cycle of implications concludes the proof of (B.1.5.xi). Finally, (B.1.5.xii) follows from (B.1.5.vii) since each localization at a prime ideal is a field. Alternatively, by (B.1.5.iii), any finitely

generated ideal $I \subseteq B$ is principal, say, of the form xB, and $B \cong xB \oplus B/xB$ by (B.1.5.iv). In particular, B/I, being a direct summand of B, is projective, whence flat, and hence $\mathrm{Tor}_1^B(B/I,\cdot)$ is identical zero. Absolute flatness now follows from Theorem 3.1.5. □

Remark B.1.6. Gilmer [33] showed that condition (B.1.5.xii) together with the first assertion of (B.1.5.vii) implies in turn that B is n-Boolean.

Corollary B.1.7. *Any embedding of n-Boolean rings is faithfully flat.*

Proof. Let $B \to C$ be an injective homomorphism of n-Boolean rings. The flatness of this homomorphism follows from (B.1.5.xii), so remains to show that C is non-degenerated. Suppose not, so that there exists a proper ideal $I \subseteq B$ such that $IC = C$. It follows that there must already exist a finitely generated ideal I with this property. Since I is principal by (B.1.5.iii), its generator a must be a unit in C. In particular, $a^{n-1} = 1$ in C by (B.1.5.v). Since $B \to C$ is injective, already $a^{n-1} = 1$ in B, showing that a is a unit in B. □

Corollary B.1.8. *Let $B \to C$ be an injective homomorphism of n-Boolean rings. Then an ideal $I \subseteq B$ is principal if and only if its image IC in C is.*

Proof. For the non-trivial direction, assume $I = yC$ for some $y \in C$. Hence there exist $a_1, \dots, a_n \in I$ such that y is a linear combination of these elements, that is to say, belongs to $(a_1, \dots, a_n)C$. By (B.1.5.iii), this ideal is generated by a single element a belonging to I. Hence $IC = aC$ and therefore by faithful flatness (Corollary B.1.7), we get $I = IC \cap B = aC \cap B = aB$. □

Corollary B.1.9. *For each n, the number $\alpha(n)$ is equal to the product of all prime numbers p pre-dividing n.*

Proof. If p pre-divides n, then we can write $n-1 = (p-1)d$ for some d. Hence $k^n \equiv (k^{p-1})^d \cdot k \equiv k \mod p$ for each k, by Fermat's Little Theorem, showing that p divides each $k^n - k$, whence $\alpha(n)$.

Conversely, if p divides $\alpha(n)$, then by construction, $\mathbb{Z}/p\mathbb{Z}$ is n-Boolean. Let d be the greatest common pre-divisor of p and n. Hence, $\mathbb{Z}/p\mathbb{Z}$ is d-Boolean, by Corollary B.1.3. Since $\mathbb{Z}/p\mathbb{Z}$ is properly p-Boolean, we must have $p = d$, as we wanted to show. Since $\alpha(n)$ is square-free by Lemma B.1.4, we are done. □

Immediately from B.1.1 and the fact that an n-Boolean ring is reduced, we get:

Corollary B.1.10. *Any n-Boolean ring is a finite direct sum of n-Boolean rings of prime characteristic.* □

Proposition B.1.11. *Let n be even. Any n-Boolean ring has characteristic* 2. *In particular, if n is not a power of two, then there are no properly n-Boolean rings.*

Proof. By Corollary B.1.9, since $n-1$ is odd, the only prime number p pre-dividing n is $p = 2$. □

For each n, let $\mathscr{B}_n$ be the collection of all finite fields whose cardinality predivides n. Note that any field in $\mathscr{B}_n$ is n-Boolean by Corollary B.1.3.

Theorem B.1.12. *For each n, a finite n-Boolean ring B is a direct sum of fields belonging to $\mathscr{B}_n$.*

Proof. Let $a_1,\dots,a_s$ be the atoms of B. By (B.1.5.xi), any element is a linear combination of the a_i, and by (B.1.5.ix), any two are orthogonal. In other words, B, as a ring, is isomorphic to the direct product $a_1B \oplus \dots \oplus a_sB$, and by (B.1.5.viii), each direct summand is a field belonging to $\mathscr{B}_n$. □

Remark B.1.13. The number of atoms, whence the number of direct summands, is equal to the length $\ell(B)$ of the Artinian ring B.

Corollary B.1.14. *A finite p-Boolean ring B of characteristic p, for p a prime number, is isomorphic as a ring to $\mathbb{F}_p^s$, where $s = \ell(B)$. More generally, a finite p^m-Boolean ring of characteristic p is isomorphic to a finite direct sum of fields of characteristic p.*

Proof. The second assertion is immediate from Theorem B.1.12. For the first, observe that the only field in $\mathscr{B}_p$ of characteristic p is $\mathbb{F}_p$, since p is the maximal cardinality by (B.1.5.viii), so that the result follows from Remark B.1.13. □

Together with Corollary B.1.10, this reproves the main theorem in [66].

B.2 Stone Representation Theorem

Given a fixed ring U, we say that a ring R is *U-like*, if every finitely generated subring of R embeds, as a ring, in a finite direct product U^s. This definition applies to the current situation as follows. Define the *universal n-Boolean* ring $\mathbb{B}_n$ as the direct sum of all fields in $\mathscr{B}_n$. Immediately from Theorem B.1.12 and Corollary B.1.7, we get:

B.2.1 *If B is a finite n-Boolean ring, then there exists a faithfully flat embedding $B \to \mathbb{B}_n^s$, where $s = \ell(B)$.* □

This embedding is in general not unique, since finite fields which are not prime fields have non-trivial isomorphisms. However, if p is prime, then the isomorphism between a finite p-Boolean ring B of characteristic p and $\mathbb{F}_p^l$, with $l = \ell(B)$, is canonical up to a permutation of the factors, since the atoms of B are unique and $\mathbb{F}_p$ has no non-trivial automorphisms.

Corollary B.2.2. *A ring is n-Boolean if and only if it is $\mathbb{B}_n$-like.*

Proof. Suppose B is n-Boolean. Let V be a finitely generated subring of B. Since V is in particular Noetherian, it is finite by (B.1.5.xi), and hence, by B.2.1, embeds in some direct product $\mathbb{B}_n^s$, showing that B is $\mathbb{B}_n$-like. Conversely, suppose B is $\mathbb{B}_n$-like, and take some $x \in B$. Let V be the subring of B generated by x. By assumption, it is a subring of $\mathbb{B}_n^s$ for some s, whence is n-Boolean. In particular, x is n-potent, showing that B is n-Boolean. □

Table B.1 Cardinalities of n-Boolean fields

n	$q = p^m$ pre-divisor of n
3	$2, 3$
5	$2, 3, 5$
7	$2, 3, 4, 7$
9	$2, 3, 5, 9$
11	$2, 3, 11$
13	$2, 3, 4, 5, 7, 13$
15	$2, 3, 8$
17	$2, 3, 5, 9, 17$
19	$2, 3, 4, 7, 19$
21	$2, 3, 5, 11$
23	$2, 3, 23$
25	$2, 3, 4, 5, 7, 9, 13, 25$
27	$2, 3, 27$

Table B.1 calculates the cardinalities of the finite fields in $\mathscr{B}_n$ for some odd values of n. Note that if n is odd, then 2 and 3 are always present in this list, as they pre-divide any odd number. The case $n = 13$ shows nonetheless that even if we assume that 2 and 3 are invertible, there can still be characteristics other than n, even if n is prime. Comparing $n = 5$ with $n = 25$ shows that more characteristics can appear when we take powers. In comparison, the list for 125 is $q = 2, 3, 5, 32, 125$.

Given a ring C, recall that we previously denoted an infinite Cartesian product of C by C_∞. Since we will work over various index sets, we amend this notation by as follows: the Cartesian power over the index set X will be denoted $C_{\infty(X)}$. Note that $C_{\infty(X)}$ can be identified with the ring of all maps $f\colon X \to C$, with addition and multiplication given component-wise.

Example B.2.3. We already remarked that for a set X, the power set ring $\mathscr{P}(X)$, with addition given by the symmetric difference and multiplication by intersection is a Boolean ring. We may view it as a Cartesian power $(\mathbb{F}_2)_{\infty(X)}$, by letting C be the two-element field $\mathbb{F}_2$, identifying a subset with its characteristic function.

If C is U-like, then so is any Cartesian power of C. Note that an ultrapower of a Cartesian power is no longer a Cartesian power, but we do have:

Lemma B.2.4. *Let U be a finite ring. For each set X, the ultrapower of the Cartesian power $U_{\infty(X)}$ embeds into the Cartesian power $U_{\infty(X_\natural)}$, where $X_\natural$ is the corresponding ultrapower of X.*

Proof. Note that since U is finite, it is equal to its own ultrapower. Let $C_\natural$ be the ultrapower of $C := U_{\infty(X)}$. Viewing C as the collection of maps $X \to U$, given $f \in C_\natural$, choose maps $f_w\colon X \to U$ with ultraproduct equal to f. Define $f_\natural\colon X_\natural \to U$ as follows. For $x \in X_\natural$, choose approximations $x_w \in X$ of x, and let $f_\natural(x)$ be the ultraproduct (in U) of the elements $f_w(x_w)$. Put differently, $f_\natural(x)$ is the unique value in U equal to almost all $f_w(x_w)$. The assignment $f \mapsto f_\natural$ yields a map $C_\natural \to$

$U_{\infty(X_\natural)}$. Since addition and multiplication are defined componentwise, one easily checks that $C_\natural \to U_{\infty(X_\natural)}$ is a homomorphism. Suppose $f \in C_\natural$ is not identical zero, whence neither are almost all f_w. In particular, for almost each w, there exists $x_w \in X$ such that $f_w(x_w) \neq 0$. It follows that $f_\natural(x) \neq 0$, where x is the ultraproduct of the x_w, proving that $f_\natural$ is non-zero, and hence $C_\natural \to U_{\infty(X_\natural)}$ is injective, as we wanted to show. $\square$

Theorem B.2.5 (Stone Representation). *Let U be a finite ring. A ring is U-like if and only if it is a subring of a Cartesian power of U. More precisely, B is U-like if and only if it admits an embedding into $U_{\infty(\mathbb{N}_\natural)}$, for some ultrapower $\mathbb{N}_\natural$ of $\mathbb{N}$.*

Proof. One direction is immediate since a subring of a U-like ring is U-like. For the opposite direction, assume B is U-like. We will show, using Remark 7.1.2, that there exists an embedding of B into some ultrapower $C_\natural$ of $C := U_{\infty(\mathbb{N})}$. Since $C_\natural$ is a subring of $U_{\infty(\mathbb{N}_\natural)}$ by Lemma B.2.4, this completes the proof. We want to verify the validity of (7.1.2.ii). Let V be a finitely generated subring of B. Hence V embeds in some finite direct product U^s, whence into C, showing that (7.1.2.ii) holds, whence also (7.1.2.iii). $\square$

The finiteness of U is important here, and we will see in the next section how in certain instances, we can circumvent this restriction.

Theorem B.2.6. *Let $n \geq 2$. A ring is n-Boolean if and only if it is a subring of a Cartesian power of the universal n-Boolean ring $\mathbb{B}_n$. More precisely, B is n-Boolean if and only if it admits a faithfully flat embedding into $\mathbb{B}_{n\infty(\mathbb{N}_\natural)}$, for some ultrapower $\mathbb{N}_\natural$ of $\mathbb{N}$.*

If $q = p^m$ is a prime power, then a ring of characteristic p is q-Boolean if and only if it is a subring of a Cartesian power of $\mathbb{F}_q$.

Proof. The first assertion is immediate by Theorem B.2.5 with $U = \mathbb{B}_n$, and Corollary B.2.2. Faithful flatness follows from Corollary B.1.7. The last assertion follows from the fact that the only q-Boolean rings of characteristic p are the subfields of $\mathbb{F}_q$ by finite field theory (such a field must be a subfield of the field of invariants of the q-Frobenius map acting on $\mathbb{F}_p^{\text{alg}}$, and this field of invariants is precisely $\mathbb{F}_q$). $\square$

The special case when $n = 2$, yields a version of the Stone Representation Theorem for Boolean rings a la Henkin [41], since $\mathbb{B}_2 = \mathbb{F}_2$.

Theorem B.2.7 (Stone Representation). *For each Boolean ring B, there exists an ultrapower $\mathbb{N}_\natural$ of $\mathbb{N}$ and a faithfully flat embedding $B \to \mathscr{P}(\mathbb{N}_\natural)$.* $\square$

B.3 ω-Boolean Rings

We say that a ring B is *ω-Boolean* if each element is potent (with possibly different potency). In particular, n-Boolean rings are ω-Boolean. For an example of

an ω-Boolean ring that for no n is n-Boolean, take the algebraic closure $\mathbb{F}_p^{\mathrm{alg}}$ of $\mathbb{F}_p$: every element is p^m-potent for some m, but m can be arbitrarily large. Unlike n-Boolean rings, ω-Boolean rings are no longer closed under taking ultraproducts. For instance, by Theorem 2.4.3, the ultraproduct of the ω-Boolean fields $\mathbb{F}_p^{\mathrm{alg}}$ is equal to $\mathbb{C}$, which clearly fails to be ω-Boolean. In fact, already any infinite Cartesian product of ω-Boolean rings is no longer ω-Boolean.

Lemma B.3.1. *For an ω-Boolean ring B, the following properties hold:*

B.3.1.i. *B is torsion;*
B.3.1.ii. *if B is finitely generated, then it is n-Boolean, for some n. In fact, a finite ring is ω-Boolean if and only if it is reduced, if and only if it is n-Boolean, for some n;*
B.3.1.iii. *any ideal in B is radical, any prime ideal is maximal, each localization at a prime ideal is a field, and B is hereditarily zero-dimensional;*
B.3.1.iv. *B is von Neumann regular, and any injective homomorphism between ω-Boolean rings is faithfully flat;*

Proof. Since $2^n - 2$ is zero in B, for some $n > 1$, we get (B.3.1.i). Let d be the characteristic of B. Since each generator has finite potency, it is integral over $\mathbb{Z}/d\mathbb{Z}$, and hence B is finite as a $\mathbb{Z}/d\mathbb{Z}$-module, whence finite. If n is the least common pre-multiple of all m, where m runs over all the finitely many potencies of elements in B, then B is n-Boolean by (B.1.2.i), proving the first assertion in (B.3.1.ii). We already observed that an ω-Boolean ring must be reduced, so suppose B is reduced and finite. In particular, it is Artinian, and hence a direct sum of local Artinian rings. Since the latter are reduced, they must be (finite) fields, whence, in particular, ω-Boolean. The first assertion in (B.3.1.iii) is clear, as $\bar{B} := B/I$ is ω-Boolean, whence reduced, for any ideal $I \subseteq B$. If I is prime, so that $\bar{B}$ is a domain, it must have prime characteristic, say p. Since any element is potent whence algebraic over $\mathbb{F}_p$, it is contained in $\mathbb{F}_p^{\mathrm{alg}}$. The result now follows since any subring of $\mathbb{F}_p^{\mathrm{alg}}$ is a field. By the same argument as in (B.1.5.vii), any localization of B at a prime ideal is then also a field, showing that B is absolutely flat, proving the first half of (B.3.1.iv). Suppose C is a degenerated ω-Boolean B-algebra. Hence it must already be degenerated over a finitely generated subalgebra $V \subseteq B$. Therefore, V cannot be a subring of C by Corollary B.1.7, whence a fortiori, neither can B, proving (B.3.1.iv). □

By B.1.1, a ring is ω-Boolean if and only if its primary components are. In analogy with our previous nomenclature, we say that a ring is *finite-like* (also called *locally finite*), if every finitely generated subring is finite, or, equivalently, if it is a direct limit of finite rings. By a *subquotient* of a ring A, we mean any ring of the form C/I with $C \subseteq A$ a subring and $I \subseteq C$ an ideal.

Corollary B.3.2. *For a ring B, the following are equivalent:*

B.3.2.i. *B is ω-Boolean;*
B.3.2.ii. *B is reduced and finite-like;*

B.3.2.iii. *any subquotient of B is reduced;*
B.3.2.iv. *any ideal in any subring of B is idempotent.*

Proof. Let us first show that in arbitrary ring R, every ideal is idempotent if and only if every ideal is radical. For the direct implication, let I be an ideal and $x^2 \in I$. Since $xR = x^2R$ is idempotent, we get $x \in I$, proving that I is radical. For the converse, let I be any ideal and $x \in I$. Hence $x^2 \in I^2$, and since I^2 is radical, we get $x \in I^2$, showing that $I \subseteq I^2$. This then also proves that (B.3.2.iii) and (B.3.2.iv) are equivalent. Applying either of the last two conditions to the prime subring of B, that is to say, the subring generated by 1, it is clear that B must be torsion and reduced in any of the four cases. Taking primary components, we may therefore assume that R has characteristic p. The equivalence of (B.3.2.i) and (B.3.2.ii) is immediate from (B.3.1.ii). Implication (B.3.2.i) $\Rightarrow$ (B.3.2.iii) is immediate from (B.3.1.iii). To prove the converse, let $x \in B$ and let V be the subring generated by x, that is to say, the image of $\mathbb{F}_p[\xi]$ in B under the map sending ξ to x. Since $\mathbb{F}_p[\xi]$ does not satisfy (B.3.2.iii), the kernel of this map must be non-zero. This implies that x is integral over $\mathbb{F}_p$. Therefore, V is finite, whence ω-Boolean by (B.3.1.ii), showing that x is potent. □

As already observed, it suffices to study ω-Boolean rings of prime characteristic. In that case, we can be more explicit:

Lemma B.3.3. *Any element in an ω-Boolean ring B of characteristic p is p^m-potent for some m. More generally, a potent element in a reduced ring R of characteristic p is p^m-potent, for some m.*

Proof. We only need to show the second assertion since B is reduced. Let V be the $\mathbb{F}_p$-subalgebra of R generated by x. Since x is potent, V is integral over $\mathbb{F}_p$, whence finite. Since it is reduced, it is isomorphic to a direct sum of fields, necessarily of characteristic p. Hence, for $q = p^m$ sufficiently large, all these fields are q-Boolean, whence so is their direct sum V, and hence x is q-potent. □

Example B.3.4. Without the assumption that R is reduced, the second assertion is false. For instance, in the ring $R := \mathbb{F}_3[\xi]/\xi^2\mathbb{F}_3[\xi]$, the element $\xi + 1$ is 4-potent, but for any power q of 3, we have $(\xi+1)^q = \xi^q + 1 = 1$.

Corollary B.3.5. *A ring of characteristic p is ω-Boolean if and only if it is $\mathbb{F}_p^{alg}$-like.*

Proof. One direction is clear, since $\mathbb{F}_p^{\mathrm{alg}}$ is ω-Boolean. Conversely, if B is ω-Boolean, then any finitely generated subring is q-potent for some power q of p by (B.3.1.ii) and Lemma B.3.3, and hence, by B.2.1, embeds in a finite direct product $(\mathbb{F}_p^{\mathrm{alg}})^s$. □

Lemma B.3.3 suggests the following proto-grading. On a reduced ring R of characteristic p, we define a pre-proto-grading, called the *potency proto-grade*, by the condition that x has proto-grade at most n, if it is p^s-potent for some $s \leq n$.

Note that by (B.1.2.i), an element has potency proto-grade at most n if it is $p^{n!}$-potent, since p^s pre-divides $p^{n!}$ for any $s \leq n$. To verify that this constitutes a pre-proto-grading, observe that if x is p^m-potent and y is p^n-potent, then $x+y$ and xy are $q := p^{mn}$-potent. Indeed, q is a pre-multiple of both p^m and p^n, and hence x and y are both q-potent by (B.1.2.i). Therefore, $x+y$ and xy have proto-grade at most mn since $(x+y)^q = x^q + y^q = x+y$ and $(xy)^q = x^q y^q = xy$. The subring of all elements of finite potency proto-grade is called the *potency subring* of R, and is denoted $\omega(R)$. It follows immediately from Lemma B.3.3 that $\omega(R)$ is the largest ω-Boolean subring of R. Using B.1.1 and the fact that the sum of mutually orthogonal potent elements is again potent, we may generalize all this to any reduced torsion ring R:

B.3.6 *In a reduced torsion ring, the potent elements form a subring $\omega(R)$.* □

Since rings of characteristic p are uniformly proto-graded with respect to potency, we can define their protoproduct, and we have:

B.3.7 *The protoproduct $R_\flat$ of rings R_w of characteristic p is equal to the potency subring $\omega(R_\natural)$ of their ultraproduct $R_\natural$.* □

For instance, the protopower of $\mathbb{F}_p^{\mathrm{alg}}$ is $\mathbb{F}_p^{\mathrm{alg}}$ itself, since the collection of elements of proto-grade at most n in $\mathbb{F}_p^{\mathrm{alg}}$ is $\mathbb{F}_{p^n}$. Using this observation, one easily shows:

B.3.8 *For each set X, the potency subring of the Cartesian power $(\mathbb{F}_p^{alg})_{\infty(X)}$ is the direct limit of the Cartesian powers $(\mathbb{F}_{p^n})_{\infty(X)}$, for $n \to \infty$.* □

Theorem B.3.9. *A ring B of characteristic p is ω-Boolean if and only if there is a faithfully flat map from B into the direct limit of the $(\mathbb{F}_{p^n})_{\infty(\mathbb{N}_\natural)}$, where $\mathbb{N}_\natural$ is some ultrapower of $\mathbb{N}$.*

Proof. One direction is again clear. We can imitate the proof of Theorem B.2.5, to obtain an embedding of B into an ultrapower $C_\natural$ of $C := (\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N})}$, since any finitely generated subring is p^n-Boolean, for some n, by (B.3.1.ii). However, since a homomorphism sends potent elements to potent elements, the image of the embedding $B \to C_\natural$ must lie in $\omega(C_\natural)$, that is to say, B embeds into the protopower $C_\flat$. So, in view of B.3.8 and the fact that potent elements are sent to potent elements, it remains to show that $C_\flat$ embeds into the Cartesian power $(\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N}_\natural)}$. Let $f \in C_\flat$, having proto-grade at most N. Hence there exist approximations $f_w \in (\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N})}$ of f of proto-grade at most N, that is to say, with $q := p^N$, almost all $f_w \in (\mathbb{F}_q)_{\infty(\mathbb{N})}$. By Lemma B.2.4, we can then view f as an element in $(\mathbb{F}_q)_{\infty(\mathbb{N}_\natural)}$, and this is clearly a subring of $(\mathbb{F}_p^{\mathrm{alg}})_{\infty(\mathbb{N}_\natural)}$, as we wanted to show. □

B.4 Periodic Rings

It follows from Example B.3.4 that in an arbitrary (non-reduced) ring, the set of potent elements is in general not closed under addition. To circumvent this problem, we generalize the notion of potency: an element x in a ring R is called *periodic*, if there exist $0 < m < n$ such that $x^n = x^m$, that is to say, if the multiplicative set of all powers of x is finite. Potent and nilpotent elements are periodic, and these are essentially the source of all periodic elements, at least in torsion rings:

Lemma B.4.1. *In a torsion ring R, an element is periodic if and only if it is a sum of a nilpotent and a potent element if and only if its image in R_{red} is potent.*

Proof. Let R be a torsion ring. Since the sum of potent, nilpotent or periodic orthogonal elements is again of the same respective type, we may reduce to the primary case by B.1.1, and hence assume that R has characteristic p^m, with p prime. Let x be a periodic element in R. Assume first that R is reduced (whence of characteristic p). We want to show that x is potent. Let V be the subalgebra generated by x. Since x satisfies some equation $\xi^{i+j} - \xi^i$ for $i, j > 0$, it is integral over $\mathbb{F}_p$. Therefore, V is finite, and hence n-Boolean by (B.3.1.ii), and, in fact, q-Boolean for some power q of p by Lemma B.3.3, concluding the proof in the reduced case. Assume next that R is arbitrary, and let $R_{\mathrm{red}} := R/\mathfrak{n}$ be the reduction of R, where $\mathfrak{n}$ is the nilradical of R. Since the image of x is periodic in R_{red}, whence q-potent for some power $q := p^e$, by the reduced case, we have $x - x^q = a$ for some $a \in \mathfrak{n}$. So remains to show that x^q is potent. It is well-known that the p-adic order of the binomial $\binom{q}{r}$ is equal to e minus the p-adic order of r. Hence, if the p-adic order of r is at most $e - m$, then $\binom{q}{r}$ is zero in R. In other words, in R, we have an identity

$$(\xi + \zeta)^q = \xi^q + \zeta^{p^{e-m+1}} f(\xi, \zeta) \tag{B.1}$$

for some polynomial $f \in \mathbb{Z}[\xi, \zeta]$. Increasing q if necessary (by taking some premultiple, see (B.1.2.i)), we may assume $a^{p^{e-m+1}} = 0$, and hence

$$x^q = (x^q + a)^q = (x^q)^q + a^{p^{e-m+1}} f(x, a) = x^{q^2},$$

by (B.1), showing that x^q is potent.

For the converse, we are left with showing that if $x = y + a$ is the sum of a potent y and a nilpotent a, then it is periodic. Applying the direct application to the potent whence periodic element y, we can write it as $y = y^q + b$ such that y^q is q-potent and b is nilpotent. Taking q sufficiently large, we may assume that the p^{e-m+1}-th powers of a, $a + b$, and b are all zero. Hence, by (B.1) applied to $x = y^q + a + b$, we get $x^q = y^{q^2}$. Since $y^q = y^{q^2}$ by assumption, we get $x^q = y^q$ and taking q-th powers gives $x^{q^2} = y^{q^2} = y^q = x^q$, showing that x is periodic. $\square$

We generalize the definition of $\omega(R)$ to an arbitrary torsion ring R as the collection of all periodic elements; this agrees with our previous definition for reduced rings by Lemma B.4.1. Using the criterion from Lemma B.4.1 in conjunction with B.3.6, we get:

Corollary B.4.2. *In a torsion ring R, the periodic elements form a subring* $\omega(R)$. □

The example $\mathbb{Q}[[\xi]]/\xi^2\mathbb{Q}[[\xi]]$ shows that the previous results are false in non-torsion rings: the sum of the potent element 1 and the nilpotent element ξ is not periodic. Following Chacron [19], we call a ring R *periodic* if all of its elements are periodic. Clearly, ω-Boolean rings are periodic. Since $2^n - 2^m = 0$ in R, for some $m \neq n$, a periodic ring must have torsion. In view of the primary decomposition given by B.1.1, it suffices therefore to study periodic rings of prime power characteristic. The following is the Stone Representation Theorem for (commutative) periodic rings (some equivalencies were already proven in [33]).

Theorem B.4.3. *For a ring R of prime power characteristic $q = p^m$, the following are equivalent:*

B.4.3.i. *R is periodic;*
B.4.3.ii. *the reduction R_{red} of R is ω-Boolean;*
B.4.3.iii. *every element of R is the sum of a potent and a nilpotent element;*
B.4.3.iv. *R is finite-like;*
B.4.3.v. *R is integral over $\mathbb{Z}/q\mathbb{Z}$;*
B.4.3.vi. *R is hereditarily zero-dimensional;*
B.4.3.vii. *the reduction R_{red} of R embeds into the direct limit of all $(\mathbb{F}_{p^n})_{\infty(\mathbb{N}_\natural)}$, where $\mathbb{N}_\natural$ is some ultrapower of $\mathbb{N}$.*

Proof. The equivalence of (B.4.3.i) and (B.4.3.iii) is given by Lemma B.4.1. The equivalence of (B.4.3.ii) and (B.4.3.vii) is given by Theorem B.3.9. Since periodic elements are integral, (B.4.3.i) implies (B.4.3.v). The equivalence of (B.4.3.v) and (B.4.3.iv), is clear since any finitely generated integral subring is finite over $\mathbb{Z}/q\mathbb{Z}$, whence finite. Since finite rings are zero-dimensional, this then also shows (B.4.3.v) $\Rightarrow$ (B.4.3.vi). Assume next (B.4.3.vi), and let B be a finitely generated subring of R. Since B is Noetherian and by assumption zero-dimensional, it is Artinian, and hence of finite length over $\mathbb{Z}/q\mathbb{Z}$. Therefore, B itself is finite, proving (B.4.3.iv). If R is finite-like, then so is its reduction R_{red}, and hence the latter is ω-Boolean by Corollary B.3.2, proving (B.4.3.iv) $\Rightarrow$ (B.4.3.ii). Finally, the implication (B.4.3.ii) $\Rightarrow$ (B.4.3.iii) is given by Lemma B.4.1. □

Remark B.4.4. Without requiring q to be a prime power, all conditions, except the last one, (B.4.3.vii), are still equivalent by B.1.1. It follows from (B.4.3.v) that in an arbitrary ring R of characteristic d, the subring $\omega(R)$ of periodic elements is equal to the integral closure of $\mathbb{Z}/d\mathbb{Z}$ in R.

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Index

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Edited by J.-M. Morel, F. Takens, B. Teissier, P.K. Maini

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Addresses:

Professor J.-M. Morel, CMLA,
École Normale Supérieure de Cachan,
61 Avenue du Président Wilson,
94235 Cachan Cedex, France
E-mail: Jean-Michel.Morel@cmla.ens-cachan.fr

Professor F. Takens, Mathematisch Instituut,
Rijksuniversiteit Groningen, Postbus 800,
9700 AV Groningen, The Netherlands
E-mail: F.Takens@rug.nl

Professor B. Teissier,
Institut Mathématique de Jussieu,
UMR 7586 du CNRS,
Équipe "Géométrie et Dynamique",
175 rue du Chevaleret,
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