Appendix A
Tools

A.1 The Lemma of Gronwall and its Generalizations

Gronwall's estimate plays a key role whenever the growth of a function is bounded by linear terms of the function itself. Such a bound of the growth can be described by an integral inequality or a differential inequality.
First we consider the estimate resulting from an integral inequality. It is very popular indeed for continuous functions and thus can be found in many standard textbooks such as [10, 92, 181]. Subsequent Proposition A.1, however, provides a similar estimate (almost everywhere) for any nonnegative function that is merely Lebesgue integrable.

Proposition 1 (Lemma of Gronwall : Integral version**).**
Let $\psi,\, g \in L^1([a,b],\mathbb{R}),\ f \in C^0([a,b])$ *satisfy* $\psi(\cdot),\, f(\cdot) \geq 0$ *and*

$$\psi(t) \;\leq\; g(t) + \int_a^t f(s)\, \psi(s)\, ds \qquad \text{for } \mathscr{L}^1\text{-almost every } t \in [a,b].$$

Then, for $\mathscr{L}^1$*-almost every* $t \in [a,b]$,

$$\psi(t) \;\leq\; g(t) + \int_a^t e^{\mu(t)-\mu(s)}\, f(s)\, g(s)\, ds$$

with $\mu(t) := \int_a^t f(s)\, ds.$
Assuming in addition that $g(\cdot)$ *is upper semicontinuous and that* $\psi(\cdot)$ *is lower semicontinuous or monotone, then this inequality holds for any* $t \in \,]a,b[$.

Proof. The function $\varphi : [a,b] \longrightarrow \mathbb{R},\ t \longmapsto \int_a^t f(s)\, \psi(s)\, ds$ is absolutely continuous and satisfies for almost every $t \in [a,b]$ (since $f(\cdot) \geq 0$)

$$\varphi'(t) \;=\; f(t)\, \psi(t) \;\leq\; f(t)\, g(t) + f(t)\, \varphi(t).$$

Thus, $t \longmapsto e^{-\mu(t)}\,\varphi(t)$ is also absolutely continuous and has the weak derivative

$$\tfrac{d}{dt}\,\big(e^{-\mu(t)}\,\varphi(t)\big) \;=\; e^{-\mu(t)}\,\big(\varphi'(t) - f(t)\,\varphi(t)\big) \;\le\; e^{-\mu(t)}\,f(t)\,g(t).$$

Now we obtain for any $t \in [a,b]$

$$\begin{aligned} e^{-\mu(t)}\,\varphi(t) &\le e^{-\mu(a)}\,\varphi(a) + \int_a^t e^{-\mu(s)}\quad f(s)\,g(s)\;ds \\ \varphi(t) &\le \qquad\quad 0 \;+ \int_a^t e^{\mu(t)-\mu(s)}\,f(s)\,g(s)\;ds \end{aligned}$$

and this estimate implies the assertion for Lebesgue-almost every $t \in [a,b]$.

Now suppose that $g(\cdot)$ is upper semicontinuous and that $\psi(\cdot)$ is lower semicontinuous or monotone. Then for every $t \in]a,b[$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $]a,b[$ such that $t_n \longrightarrow t \quad (n \longrightarrow \infty)$ and

$$\begin{aligned} \psi(t) &\le \limsup_{n\to\infty}\; \psi(t_n), \\ \psi(t_n) &\le g(t_n) \;+\; \int_a^{t_n} e^{\mu(t_n)-\mu(s)} f(s)\,g(s)\;ds \end{aligned}$$

for each $n \in \mathbb{N}$. As an easy consequence, we obtain

$$\begin{aligned} \psi(t) &\le \limsup_{n\to\infty}\;\Big(g(t_n) + \int_a^{t_n} e^{\mu(t_n)-\mu(s)}\,f(s)\,g(s)\;ds\Big) \\ &\le \qquad\quad g(t) \;+ \int_a^t e^{\mu(t)-\mu(s)}\,f(s)\,g(s)\;ds. \end{aligned}$$

$\square$

This integral version of Gronwall's Lemma now leads to a subdifferential version which has two new aspects: First, the nonnegative function $\psi(\cdot)$ does not have be continuous, but just lower semicontinuous (as in [130]). Second, the hypothesis about an affine linear bound of the upper Dini derivative is not required in the whole time interval, but just at Lebesgue-almost every time. The proof is based on a connection to Proposition A.1 by means of a nondecreasing auxiliary function (in combination with Fatou's Lemma):

Proposition 2. *Let $\psi : [a,b] \longrightarrow \mathbb{R}$ and $f, g \in C^0([a,b],\mathbb{R})$ satisfy $f(\cdot), g(\cdot) \ge 0$ and*

$$\begin{aligned} 0 \;\le\; \psi(t) &\le \limsup_{h\downarrow 0}\; \psi(t-h), && \textit{for every } t \in\,]a,\,b], \\ \psi(t) &\ge \limsup_{h\downarrow 0}\; \psi(t+h), && \textit{for every } t \in [a,\,b[, \end{aligned}$$

$$\limsup_{h\downarrow 0}\; \tfrac{\psi(t+h)-\psi(t)}{h} \;\le\; f(t)\cdot \limsup_{h\downarrow 0}\; \psi(t-h) \;+\; g(t) \quad \textit{for almost every } t \in\,]a,\,b[.$$

Then, for every $t \in [a,b]$, the function $\psi(\cdot)$ fulfills the upper estimate

$$\psi(t) \;\le\; \psi(a)\cdot e^{\mu(t)} \;+\; \int_a^t e^{\mu(t)-\mu(s)}\,g(s)\,ds$$

with $\mu(t) := \int_a^t f(s)\,ds.$

Proof. Obviously, the auxiliary function $\xi : [a,b] \longrightarrow \mathbb{R}_0^+,\ t \longmapsto \sup_{[a,t]} \psi(\cdot)$ is nonnegative and nondecreasing. The second assumption about $\psi(\cdot)$ implies the continuity of $\xi(\cdot)$. Furthermore, it satisfies for $\mathscr{L}^1$-almost every $t \in]a,b[$

$$\limsup_{h \downarrow 0} \frac{\xi(t+h) - \xi(t)}{h} \le f(t) \cdot \xi(t) + g(t).$$

Indeed, choose any $t \in]a,b[$ for which the third assumption about ψ is satisfied. Then for any $\delta > 0$, there exists some $h_0 \in]0, b-t[$ such that for all $h \in]0, h_0]$,

$$\begin{aligned} \frac{\psi(t+h)-\psi(t)}{h} &\le f(t) \cdot \xi(t) + g(t) + \delta \\ \text{i.e.} \qquad \psi(t+h) &\le \big(f(t) \cdot \xi(t) + g(t) + \delta\big) \cdot h + \psi(t) \\ &\le \big(f(t) \cdot \xi(t) + g(t) + \delta\big) \cdot h + \xi(t). \end{aligned}$$

Hence, $\xi(t+h) = \max\big\{\xi(t), \sup_{[t,t+h]} \psi(\cdot)\big\}$ fulfills this estimate for all $h \in]0, h_0]$:

$$\begin{aligned} \xi(t+h) &\le \big(f(t) \cdot \xi(t) + g(t) + \delta\big) \cdot h + \xi(t) \\ \frac{\xi(t+h)-\xi(t)}{h} &\le f(t) \cdot \xi(t) + g(t) + \delta. \end{aligned}$$

As $\delta > 0$ was chosen arbitrarily, we obtain the claimed estimate for the upper Dini derivative of $\xi(\cdot)$ at t.

In particular, the continuous function $\xi(\cdot)$ is bounded in the compact interval $[a,b]$ and thus, so is $\psi(\cdot)$. The auxiliary function

$$[a,b[\longrightarrow \mathbb{R}_0^+, \quad t \longmapsto \limsup_{h \downarrow 0} \frac{\xi(t+h) - \xi(t)}{h}$$

is Lebesgue-measurable and bounded Lebesgue-almost everywhere. The well-known Lemma of Fatou implies for every $T \in [a,b[$

$$\limsup_{h \downarrow 0} \int_0^T \frac{\xi(t+h) - \xi(t)}{h}\, dt \le \int_0^T \limsup_{h \downarrow 0} \frac{\xi(t+h) - \xi(t)}{h}\, dt$$

and thus lays the basis for estimating $\xi(T) - \xi(0)$:

$$\begin{aligned} \limsup_{h \downarrow 0} \int_0^T \frac{\xi(t+h) - \xi(t)}{h}\, dt &= \limsup_{h \downarrow 0} \frac{1}{h} \cdot \Big(\int_0^T \xi(t+h)\, dt - \int_0^T \xi(t)\, dt\Big) \\ &= \limsup_{h \downarrow 0} \frac{1}{h} \cdot \Big(\int_T^{T+h} \xi(t)\, dt - \int_0^h \xi(t)\, dt\Big) \\ &= \xi(T) - \xi(0) \end{aligned}$$

due to the continuity of $\xi(\cdot)$. Now we obtain an estimate for $\xi(T)$ for *every* $T \in [a,b[$

$$\xi(T) - \xi(0) \le \int_0^T \limsup_{h \downarrow 0} \frac{\xi(t+h) - \xi(t)}{h}\, dt \le \int_0^T \big(f(t) \cdot \xi(t) + g(t)\big)\, dt.$$

Finally, the claim results from Proposition A.1. □

Remark 3. 1. This subdifferential version of Gronwall's Lemma also holds if $f, g : [a,b[\longrightarrow \mathbb{R}_0^+$ are only upper semicontinuous (instead of continuous). The proof is based on upper approximations of $f(\cdot)$, $g(\cdot)$ by continuous functions.

2. The condition $\limsup\limits_{h \downarrow 0} \frac{\psi(t+h)-\psi(t)}{h} \le f(t) \cdot \psi(t) + g(t)$ (supposed in the widespread forms of Gronwall's Lemma) is stronger than the third assumption of Proposition A.2 due to the semicontinuity condition $\psi(t) \le \limsup\limits_{h \downarrow 0} \psi(t-h)$.

A similar statement holds with limits inferior replacing the limits superior — under the additional assumption, however, that the growth condition is fulfilled at *every* time (instead of $\mathscr{L}^1$-almost every time). The proof presented by the author in [130] is based on a simple indirect argument and thus, it is completely independent of the integral version in Proposition A.1:

Proposition 4. *Let $\psi : [a,b] \longrightarrow \mathbb{R}$ and $f, g \in C^0([a,b], \mathbb{R})$ satisfy $f(\cdot) \ge 0$ and*

$$
\begin{aligned}
0 \le \psi(t) &\le \liminf_{h \downarrow 0} \psi(t-h), && \text{for every } t \in]a, b], \\
\psi(t) &\ge \liminf_{h \downarrow 0} \psi(t+h), && \text{for every } t \in [a, b[, \\
\liminf_{h \downarrow 0} \frac{\psi(t+h)-\psi(t)}{h} &\le f(t) \cdot \liminf_{h \downarrow 0} \psi(t-h) + g(t) && \text{for every } t \in]a, b[.
\end{aligned}
$$

Then, for every $t \in [a,b]$, the function $\psi(\cdot)$ fulfills the upper estimate

$$
\psi(t) \le \psi(a) \cdot e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)} \, g(s) \, ds
$$

with $\mu(t) := \int_a^t f(s) \, ds.$

Proof. Let $\delta > 0$ be arbitrarily small. The proof is based on comparing ψ with the auxiliary function $\varphi_\delta : [a,b] \longrightarrow \mathbb{R}$ that uses $\psi(a) + \delta$, $g(\cdot) + \delta$ instead of $\psi(a)$, $g(\cdot)$:

$$
\varphi_\delta(t) := \big(\psi(a) + \delta\big) \, e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)} \, (g(s) + \delta) \, ds.
$$

Then, $\varphi_\delta'(t) = f(t) \, \varphi_\delta(t) + g(t) + \delta$ in $[a,b[$,
$\varphi_\delta(s_n) > \psi(s_n)$ for some sequence $s_n \downarrow a$.

Assume now that there exists some $t_0 \in]a,b]$ such that $\varphi_\delta(t_0) < \psi(t_0)$. Setting

$$
t_1 := \inf \big\{ t \in [a, t_0] \bigm| \varphi_\delta(\cdot) < \psi(\cdot) \text{ in } [t, t_0] \big\} \ge s_1 > a,
$$

we conclude $t_1 < t_0$ from the condition $\psi(t_0) \le \liminf\limits_{h \downarrow 0} \psi(t_0 - h)$ and the continuity of $\varphi_\delta(\cdot)$. Moreover, $\varphi_\delta(t_1) = \psi(t_1)$ is a consequence of

$$
\begin{aligned}
\varphi_\delta(t_1) &= \lim_{h \downarrow 0} \varphi_\delta(t_1 - h) \ge \liminf_{h \downarrow 0} \psi(t_1 - h) \ge \psi(t_1), \\
\varphi_\delta(t_1) &= \lim_{h \downarrow 0} \varphi_\delta(t_1 + h) \le \liminf_{h \downarrow 0} \psi(t_1 + h) \le \psi(t_1).
\end{aligned}
$$

Thus, the definition of t_1 implies

$$\begin{aligned}
\liminf_{h\downarrow 0} \frac{\varphi_\delta(t_1+h)-\varphi_\delta(t_1)}{h} &\le \liminf_{h\downarrow 0} \frac{\psi(t_1+h)-\psi(t_1)}{h} \\
\varphi'_\delta(t_1) &\le f(t_1)\cdot \liminf_{h\downarrow 0} \quad \psi(t_1-h) + g(t_1) \\
f(t_1)\,\varphi_\delta(t_1)+g(t_1)+\delta &\le f(t_1)\cdot \limsup_{h\downarrow 0} \ \varphi_\delta(t_1-h) + g(t_1) \\
&\le f(t_1)\cdot \varphi_\delta(t_1) \qquad\qquad + g(t_1)
\end{aligned}$$

— a contradiction. Finally, $\varphi_\delta(\cdot) \ge \psi(\cdot)$ for any $\delta > 0$. □

A.2 Filippov's Theorem for Differential Inclusions

According to the well-known convention, we define the solutions to a differential inclusion in the sense of Carathéodory as it is described e.g. in [15, 19]. The Theorem of Filippov represents the counterpart of the Cauchy-Lipschitz Theorem about ordinary differential equations.

Definition 5. Let $\widetilde{F} : [0,T]\times\mathbb{R}^N \leadsto \mathbb{R}^N$ be a set-valued map.
A curve $x : [0,T] \longrightarrow \mathbb{R}^N$ is called a *solution* to the differential inclusion $x'(\cdot) \in \widetilde{F}(\cdot,x(\cdot))$ a.e. if $x(\cdot)$ is absolutely continuous and its (weak) derivative $x'(\cdot)$ satisfies $x'(t) \in \widetilde{F}(t,x(t))$ for Lebesgue-almost every $t \in [0,T]$.

The *reachable set* of $\widetilde{F}$ and a nonempty initial set $M \subset \mathbb{R}^N$ at time $t \in [0,T]$ contains the points $x(t)$ of all solutions $x(\cdot)$ of $x'(\cdot) \in \widetilde{F}(\cdot,x(\cdot))$ a.e. starting in M, i.e.

$$\vartheta_{\widetilde{F}}(t,M) \ := \ \Big\{ x(t)\in\mathbb{R}^N \ \Big| \ \begin{array}{l} x(\cdot) \ \in \ W^{1,1}([0,t],\mathbb{R}^N), \quad x(0)\in M, \\ x'(\cdot) \ \in \ \widetilde{F}(\cdot,x(\cdot)) \ \ \mathscr{L}^1\text{-almost everywhere in } [0,t] \end{array} \Big\}.$$

Theorem 6 (Generalized Theorem of Filippov).
Let $\mathscr{O}$ be a relatively open subset of $[0,T]\times\mathbb{R}^N$. Take a set-valued map $\widetilde{F} : \mathscr{O} \leadsto \mathbb{R}^N$, an arc $y(\cdot) \in W^{1,1}([0,T],\mathbb{R}^N)$, a point $\eta \in \mathbb{R}^N$ and $\delta \in \,]0,\infty]$ such that

$$\mathscr{N}(y,\delta) := \bigcup_{0\le t\le T} \{t\}\times\mathbb{B}_\delta(y(t)) \subset \mathscr{O}.$$

Assume that

(*i*) $\widetilde{F}(t,z) \neq \emptyset$ *is closed for every* $(t,z) \in \mathscr{N}(y,\delta)$ *and* Graph $\widetilde{F}$ *is* $\mathscr{L}^1 \times \mathscr{B}^N$ *measurable,*

(*ii*) *there exists* $k(\cdot) \in L^1([0,T])$ *such that* $\widetilde{F}(t,z_1) \subset \widetilde{F}(t,z_2) + k(t)\,|z_1 - z_2| \cdot \mathbb{B}_1$ *for all* $z_1, z_2 \in \mathbb{B}_\delta(y(t))$ *and Lebesgue-almost every* $t \in [0,T]$.

Suppose further

$$e^{\|k\|_{L^1}} \cdot \Big(|\eta - y(0)| + \int_0^T \operatorname{dist}\Big(y'(t),\ \widetilde{F}(t,y(t))\Big)\, dt \Big) \ \le\ \delta.$$

Then there exists a solution $x(\cdot) \in W^{1,1}([0,T],\mathbb{R}^N)$ *of* $x'(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ *a.e. satisfying* $x(0) = \eta$ *and*

$$\|x - y\|_{L^\infty} \le |\eta - y(0)|\ e^{\|k\|_{L^1}} + \int_0^T e^{\int_t^T k(s)\,ds}\ \operatorname{dist}\Big(y'(t),\ \widetilde{F}(t,y(t))\Big)\ dt\,.$$

Now assume that (*i*) *and* (*ii*) *are replaced by the stronger hypotheses:*

(*i'*) $\widetilde{F}(t,z) \neq \emptyset$ *is convex and compact for every* $(t,z) \in \mathscr{N}(y,\delta)$,

(*ii'*) *there exist* $\omega(\cdot) : [0,\infty[\longrightarrow [0,\infty[$ *and* $k_\infty \in]0,\infty[$ *such that* $\lim_{h \downarrow 0} \omega(h) = 0$,

$$\widetilde{F}(t_1,z_1) \ \subset\ \widetilde{F}(t_2,z_2) + \Big(k_\infty\,|z_1 - z_2| + \omega(|t_1 - t_2|)\Big)\ \mathbb{B}_1$$

for all $(t_1,z_1),\ (t_2,z_2) \in \mathscr{N}(y,\delta)$.

If $y(\cdot)$ *is continuously differentiable, then the solution* $x(\cdot)$ *can be chosen as a continuously differentiable function too.*

Proof is given in [180, Theorem 2.4.3], for example.

For applying Filippov's Theorem to *compact* reachable sets in $\mathbb{R}^N$, we combine some global properties of a set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ of space and time and coin the new term "Filippov continuous". It reflects the gist of the feature "measurable/Lipschitz" defined in [19, Definition 9.5.1] – but in a more detailed formulation.

Definition 7. A set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is called *Filippov continuous* if it satisfies the following conditions:

1.) all values of $\widetilde{F}$ are nonempty closed subsets of $\mathbb{R}^N$,
2.) Graph $\widetilde{F} \subset [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$ belongs to $\mathscr{L}^1 \otimes \mathscr{L}^N \otimes \mathscr{B}^N$,
3.) $\widetilde{F}$ has at most linear growth, i.e. $\displaystyle\sup_{(t,x) \in [0,T] \times \mathbb{R}^N}\ \sup_{v \in \widetilde{F}(t,x)} \frac{|v|}{|x|+|t|+1} < \infty$.
4.) there is $\lambda(\cdot) \in L^1([0,T],\mathbb{R})$ such that at Lebesgue-almost every time $t \in [0,T]$, the set-valued map $\widetilde{F}(t,\cdot) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is $\lambda(t)$-Lipschitz w.r.t. $d\!l$.

Here $\mathscr{L}^N$ consists of all Lebesgue subsets of $\mathbb{R}^N$ and, $\mathscr{B}^N$ denotes the set of all Borel subsets of $\mathbb{R}^N$. Condition (2.) is equivalent to the measurability of the set-valued map $\widetilde{F}$ according to Characterization Theorem A.75 (on page 489) below. Furthermore, the linear growth condition (3.) implies first that all values of $\widetilde{F}$ are

compact and second that Gronwall's Lemma provides locally uniform bounds for solutions to the corresponding nonautonomous differential inclusion.
These conditions are slightly stronger than the assumptions of Theorem A.6. Indeed, Theorem A.6 does not assume the linear growth condition (3.) and, Lipschitz continuity with respect to space is supposed only locally. These distinctions result from different emphases: Theorem A.6 focuses on spatially local aspects of existence of solutions to a differential inclusion. We, however, aim for conclusions about reachable sets in the whole Euclidean space. The additional linear growth condition (3.), for example, is to ensure that we can restrict our geometric considerations to compact neighborhoods of compact initial sets.

Proposition 8 (Invariance Theorem). *Let $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be Filippov continuous. Assume the nonempty closed set $K \subset \mathbb{R}^N$ to satisfy*

$$F(t,x) \ \subset\ T_K(x) \qquad \textit{for every } x \in K \textit{ and } \mathscr{L}^1\textit{-almost every } t \in [0,T].$$

with $T_K(x) \subset \mathbb{R}^N$ denoting the contingent cone of K at x in the sense of Bouligand.

Then every solution $x(\cdot) \in W^{1,1}([t_1,t_2],\mathbb{R}^N)$ to the differential inclusion $x'(\cdot) \in \widetilde{F}(\cdot,x(\cdot))$ a.e. with $[t_1,t_2] \subset [0,T]$ and $x(t_1) \in K$ has all its values in K.

Proof. It adapts the standard proof of [14, Theorem 5.3.4] that deals with autonomous differential inclusions.
Every solution $x(\cdot) \in W^{1,1}([t_1,t_2],\mathbb{R}^N)$ of $x'(\cdot) \in \widetilde{F}(\cdot,x(\cdot))$ a.e. is even Lipschitz continuous due to the linear growth condition on $\widetilde{F}$ (and Gronwall's Lemma). The auxiliary distance function $\delta : [t_1,t_2] \longrightarrow \mathbb{R},\ t \longmapsto \operatorname{dist}\big(x(t), K\big)$ is Lipschitz continuous. Whenever $x(\cdot)$ and $\delta(\cdot)$ are differentiable at time $t \in [t_1,t_2]$, it satisfies with a projection point $y_t \in K$ of $x(t)$ (i.e. $|x(t) - y_t| = \operatorname{dist}(x(t),K)$) and any $v \in \mathbb{R}^N$

$$\begin{aligned}
\delta'(t) \ &\le\ \liminf_{h\downarrow 0}\ \tfrac{1}{h} \cdot \big(\operatorname{dist}(x(t+h),K)\ -\ |x(t)-y_t|\big)\\
&\le\ \liminf_{h\downarrow 0}\ \tfrac{1}{h} \cdot \quad \operatorname{dist}\Big(y_t + \int_t^{t+h} x'(s)\,ds,\ K\Big)\\
&\le\ \liminf_{h\downarrow 0}\ \tfrac{1}{h} \cdot \Big(\operatorname{dist}\big(y_t + h\,v,\ K\big)\ +\ \Big|h\,v - \int_t^{t+h} x'(s)\,ds\Big|\Big)\\
&\le\ \liminf_{h\downarrow 0}\ \tfrac{1}{h} \cdot \quad \operatorname{dist}\big(y_t + h\,v,\ K\big)\ +\ \ \big|v - x'(t)\big|.
\end{aligned}$$

Selecting now $v \in \widetilde{F}(t,y_t)$ with $|x'(t) - v| \le d\!l(\widetilde{F}(t,x(t)),\, \widetilde{F}(t,y_t))$, we conclude from $\widetilde{F}(t,y_t) \subset T_K(y_t)$ and the $\lambda(t)$-Lipschitz continuity of $\widetilde{F}(t,\cdot)$ the estimate

$$\delta'(t) \ \le\ 0\ +\ d\!l(\widetilde{F}(t,x(t)),\, \widetilde{F}(t,y_t)) \ \le\ \lambda(t)\,|x(t)-y_t| \ =\ \lambda(t)\,\delta(t)$$

for $\mathscr{L}^1$-almost every $t \in [t_1,t_2]$. According to Gronwall's Lemma (Proposition A.2), $\delta(0) = 0$ implies $\delta(\cdot) \equiv 0$ and thus, every value $x(t)$ belongs to the closed set K.

□

A.3 Scorza-Dragoni Theorem and Applications to Reachable Sets

The classical theorem of Scorza-Dragoni [167] can be extended to functions between metric spaces as shown by Ricceri and Villani. A so-called Carathéodory function depends on two arguments, namely "time" (in a topological space like $\mathbb{R}$) and "state" (in a metric space). By definition, it is measurable with respect to time and continuous with respect to state. The key point of Scorza-Dragoni is to guarantee continuity with respect to both arguments on "almost" the whole domain in the following sense:

Proposition 9 ([160, Theorem 1]). *Let S be a compact Hausdorff topological space, μ a Radon measure on S and X,Y metric spaces. Suppose X to be separable.*

Then every Carathéodory function $g : S \times X \longrightarrow Y$ satisfies the so-called Scorza-Dragoni property, i.e. for every $\varepsilon > 0$, there exists a closed subset $S_\varepsilon \subset S$ with $\mu(S \setminus S_\varepsilon) < \varepsilon$ such that the restriction $g|_{S_\varepsilon \times X}$ is continuous.

Now this proposition can be regarded as a counterpart of well-known Lusin's Theorem (relating measurability to continuity almost everywhere) – but now for functions with two arguments.
In 1977 Jarnik and Kurzweil published an extension of the Scorza-Dragoni Theorem to set-valued maps which are measurable in time and upper semicontinuous in space [96]:

Proposition 10 ([83, Corollary 2.2], [96]). *Let X be a separable metric space. Suppose that $\widetilde{F} : [0,T] \times X \rightsquigarrow \mathbb{R}^N$ has convex closed values and for $\mathscr{L}^1$-almost all $t \in [0,T]$, $\widetilde{F}(t,\cdot)$ is upper semicontinuous. Assume that $\widetilde{F}$ is measurably bounded, i.e. there is a measurable function $\beta : [0,T] \longrightarrow \mathbb{R}$ such that for $\mathscr{L}^1$-almost all $t \in [0,T]$ and every $x \in X$, $|\widetilde{F}(t,x)|_\infty \leq \beta(t)$.*

Then there exists a set-valued map $\widehat{F} : [0,T] \times X \rightsquigarrow \mathbb{R}^N$ with closed convex values satisfying the following conditions:

1. *For $\mathscr{L}^1$-almost all $t \in [0,T]$ and for all $x \in X$, $\widehat{F}(t,x) \subset \widetilde{F}(t,x)$.*
2. *For every measurable set $\Lambda \subset [0,T]$ and any measurable maps $u : \Lambda \longrightarrow X$, $v : \Lambda \longrightarrow \mathbb{R}^N$ with $v(\cdot) \in \widetilde{F}(\cdot, u(\cdot))$ $\mathscr{L}^1$-a.e. in Λ, we have $v(\cdot) \in \widehat{F}(\cdot, u(\cdot))$ a.e.*
3. *For any $\varepsilon > 0$, there is a closed set $J_\varepsilon \subset [0,T]$ such that $\mathscr{L}^1([0,T] \setminus J_\varepsilon) < \varepsilon$ and $\widehat{F}|_{J_\varepsilon \times X}$ is upper semicontinuous.*

This proposition provides a useful tool for investigating nonautonomous differential inclusions with set-valued maps being measurable in time and upper semicontinuous in space. Indeed, it bridges the gap to differential inclusions with upper semicontinuous right-hand side. Motivated by the nomenclature of Aubin in [14], we introduce the following abbreviating term for this type of set-valued maps:

Definition 11. A set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $(t,x) \mapsto \widetilde{F}(t,x)$ is called *nonautonomous Marchaud map* if it has the following properties:

1. $\widetilde{F}$ is nontrivial (i.e. Graph $\widetilde{F} \neq \emptyset$),
2. $\widetilde{F}(t,\cdot)$ is upper semicontinuous for Lebesgue-almost every $t \in [0,T]$,
3. $\widetilde{F}(\cdot,x)$ is measurable for every $x \in \mathbb{R}^N$,
4. $\widetilde{F}$ has compact convex values and
5. there exists $\mu(\cdot) \in L^1([0,T])$ such that $\widetilde{F}(t,x) \subset \mu(t)\,(1+|x|)\,\mathbb{B}$ for all $x \in \mathbb{R}^N$ and Lebesgue-almost every $t \in [0,T]$.

Such a Scorza-Dragoni type theorem also holds for set-valued maps being continuous with respect to space at Lebesgue-almost every time. Frankowska, Plaskacz and Rzeżuchowski concluded the following version from their counterpart of Proposition A.10 by means of a single-valued parameterization [83]. Alternatively, it can be regarded as a special case of Proposition A.9 with values in the metric space $Y := (\mathscr{K}(\mathbb{R}^N), dl)$.

Proposition 12 ([83, Theorem 2.4]). *Let the set-valued map* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $(t,x) \mapsto \widetilde{F}(t,x)$ *have nonempty compact values, be measurable with respect to t and continuous with respect to x.*
Then for every $\varepsilon > 0$, *there exists a closed set* $J_\varepsilon \subset [0,T]$ *with* $\mathscr{L}^1([0,T] \setminus J_\varepsilon) < \varepsilon$ *for which the restriction* $\widetilde{F}|_{J_\varepsilon \times \mathbb{R}^N}$ *is continuous.*

Applications to Reachable Sets: Integral Funnel Equation

Considering a nonautonomous differential inclusion, the set-valued map on its right-hand side provides a first-order approximation of the reachable set starting in an arbitrary point. For various nonautonomous differential inclusions with continuous right-hand side, this result is well-known as *integral funnel equation* due to papers of Kurzhanski, Filippova, Panasyuk, Tolstonogov and others (e.g. [110, 152]).

In [83], Frankowska, Plaskacz and Rzeżuchowski extended such approximating results to differential inclusions whose right-hand sides are just measurable in time. Their detailed estimates of the Hausdorff distances, however, are formulated for an arbitrary initial point in space (rather than initial sets). Now we verify that these estimates hold even locally uniformly in space and time:

Proposition 13. *Let the set-valued map* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *satisfy*
1. $\widetilde{F}$ *has nonempty closed convex values,*
2. for $\mathscr{L}^1$*-almost all* $t \in [0,T]$, *the map* $\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $x \mapsto \widetilde{F}(t,x)$ *is continuous,*
3. for every $x \in \mathbb{R}^N$, *the map* $[0,T] \rightsquigarrow \mathbb{R}^N$, $t \mapsto \widetilde{F}(t,x)$ *is measurable,*
4. there exists $\mu(\cdot) \in L^1([0,T])$ *with* $\left|\widetilde{F}(t,x)\right|_\infty \leq \mu(t)$ *for all* $x \in \mathbb{R}^N$ *and a.e. t.*

Then, there exists a set $J \subset [0,T]$ *of full Lebesgue measure (i.e.* $\mathscr{L}^1([0,T] \setminus J) = 0$*) such that for every* $t \in J$ *and* $K \in \mathscr{K}(\mathbb{R}^N)$,

$$\tfrac{1}{h} \cdot dl\Big(\vartheta_{\widetilde{F}(t+\cdot,\cdot)}(h, K),\ \bigcup_{x \in K} \big(x + h \cdot \widetilde{F}(t,x)\big)\Big) \longrightarrow 0 \qquad \text{for } h \downarrow 0.$$

Proof consists of subsequent Corollary A.15 and Lemma A.16 focusing on the Pompeiu-Hausdorff excesses

$$h \longmapsto \operatorname{dist}\Big(\vartheta_{\widetilde{F}(t+\cdot,\cdot)}(h, K), \qquad \bigcup_{x \in K} \big(x + h \cdot \widetilde{F}(t,x)\big)\Big),$$
$$h \longmapsto \operatorname{dist}\Big(\bigcup_{x \in K} \big(x + h \cdot \widetilde{F}(t,x)\big),\ \ \vartheta_{\widetilde{F}(t+\cdot,\cdot)}(h, K),\Big)$$

respectively. Indeed, the subsequent inclusions are locally uniform with respect to the initial point $x \in K$ and small time $h > 0$.

Lemma 14. *Let* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *be a nonautonomous Marchaud map with nonempty (compact convex) values.*
Then there exists a set $J \subset [0,T]$ *of full measure (i.e.* $\mathscr{L}^1([0,T] \setminus J) = 0$*) with the following property: For every* $t_0 \in J$, $x_0 \in \mathbb{R}^N$ *and* $\varepsilon \in]0,1[$, *there are* $t_1 > 0$ *and* $\delta > 0$ *satisfying for all* $x \in \mathbb{B}_\delta(x_0)$, $h \in]0,t_1[$.

$$\vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,x) \ \subset\ x + h\ \big(\widetilde{F}(t_0,x_0) + \varepsilon\,\mathbb{B}\big).$$

Applying this result to every time $t_0 \in J \subset [0,T]$ at which $\widetilde{F}(t,\cdot) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is continuous in addition, we obtain directly:

Corollary 15. *Under the assumptions of Proposition A.13, there exists a subset* $J \subset [0,T]$ *of full measure (i.e.* $\mathscr{L}^1([0,T] \setminus J) = 0$*) with the following property: For every* $t_0 \in J$, $x_0 \in \mathbb{R}^N$ *and* $\varepsilon \in]0,1[$, *there are* $t_1 > 0$ *and* $\delta > 0$ *satisfying*

$$\vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,x) \ \subset\ x + h\ \big(\widetilde{F}(t_0,x) + 2\,\varepsilon\,\mathbb{B}\big)$$

for all $x \in \mathbb{B}_\delta(x_0)$, $h \in]0,t_1[$. □

Before proving Lemma A.14 in detail, we formulate the opposite inclusion correctly. This completes the proof of Proposition A.13.

Lemma 16. *Under the assumptions of Proposition A.13, there exists a subset* $J \subset [0,T]$ *of full measure (i.e.* $\mathscr{L}^1([0,T] \setminus J) = 0$*) with the following property: For every* $t_0 \in J$, $x_0 \in \mathbb{R}^N$ *and* $\varepsilon \in]0,1[$, *there are* $t_1 > 0$ *and* $\delta > 0$ *satisfying*

$$x + h\,\widetilde{F}(t_0,x) \ \subset\ \vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,x) \ +\ \varepsilon\,h\,\mathbb{B}$$

for all $x \in \mathbb{B}_\delta(x_0)$, $h \in]0,t_1[$.

Finally we now discuss the missing proofs of Lemmas A.14 and A.16:

Proof (of Lemma A.14). It follows the same arguments of [83, Lemma 2.6] and thus uses the basic idea of Rzeżuchowski in [166].
Let $\widehat{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ denote the set-valued map according to Scorza-Dragoni type Proposition A.10. For any $\gamma > 0$, there exists a closed subset $\widetilde{J}_\gamma \subset [0,T]$ with $\mathscr{L}^1([0,T] \setminus \widetilde{J}_\gamma) < \gamma$ such that $\widehat{F}|_{\widetilde{J}_\gamma \times \mathbb{R}^N}$ is upper semicontinuous and

$$\text{Graph}\, \widehat{F}|_{\widetilde{J}_\gamma \times \mathbb{R}^N} \subset \text{Graph}\, \widetilde{F}.$$

Now let $J_\gamma \subset \widetilde{J}_\gamma$ denote the set of density points of $\widetilde{J}_\gamma$ that are also Lebesgue points of $\mu(\cdot) \cdot \chi_{[0,T] \setminus \widetilde{J}_\gamma}(\cdot) : [0,T] \longrightarrow \mathbb{R}$. It satisfies $\mathscr{L}^1(J_\gamma) = \mathscr{L}^1(\widetilde{J}_\gamma)$ because Lebesgue points of each Lebesgue-integrable function always have full Lebesgue measure [189, Theorem 1.3.8] and thus, in particular, density points of any measurable set also have full Lebesgue measure.

For arbitrary $t_0 \in J_\gamma$, $x_0 \in \mathbb{R}^N$ and $\varepsilon \in]0,1]$, the upper semicontinuity of $\widehat{F}|_{J_\gamma \times \mathbb{R}^N}$ and the construction of J_γ provide $r, \delta, t_1 > 0$ satisfying for every $t \in [t_0,\, t_0 + t_1]$

$$\begin{cases} \widehat{F}\big(J_\gamma \cap [t_0,t],\, \mathbb{B}_r(x_0)\big) \subset \widehat{F}(t_0,x_0) + \frac{\varepsilon}{3}\,\mathbb{B} \subset \widetilde{F}(t_0,x_0) + \frac{\varepsilon}{3}\,\mathbb{B}, \\ \vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}\big(t-t_0,\, \mathbb{B}_\delta(x_0)\big) \subset x_0 + r\,\mathbb{B}, \\ \frac{\mathscr{L}^1([t_0,t] \cap \widetilde{J}_\gamma)}{t-t_0}\; \widetilde{F}(t_0,x_0) \subset \widetilde{F}(t_0,x_0) + \frac{\varepsilon}{3}\,\mathbb{B}, \\ \frac{1}{t-t_0} \displaystyle\int_{[t_0,t] \setminus \widetilde{J}_\gamma} \mu(s)\, ds \le \frac{\varepsilon}{3} \cdot (1 + |x_0| + r)^{-1}. \end{cases}$$

Then for any $x \in \mathbb{B}_\delta(x_0)$ and $h \in [0,\, t_1]$, we obtain

$$\begin{aligned} &\vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,x) - x \subset \\ &\subset \int_{[t_0,t_0+h] \cap \widetilde{J}_\gamma} \widehat{F}\big(s,\, \mathbb{B}_r(x_0)\big)\, ds + \int_{[t_0,t_0+h] \setminus \widetilde{J}_\gamma} \widehat{F}\big(s,\, \mathbb{B}_r(x_0)\big)\, ds \\ &\subset \mathscr{L}^1([t_0,t_0+h] \cap \widetilde{J}_\gamma) \cdot \Big(\widetilde{F}(t_0,x_0) + \tfrac{\varepsilon}{3}\,\mathbb{B}\Big) + \int_{[t_0,t_0+h] \setminus \widetilde{J}_\gamma} \mu(s)\,(1+|x_0|+r)\, ds \cdot \mathbb{B} \\ &\subset h\,\Big(\widetilde{F}(t_0,x_0) + \tfrac{\varepsilon}{3}\,\mathbb{B} + \tfrac{\varepsilon}{3}\,\mathbb{B}\Big) + \tfrac{\varepsilon}{3}\, h\, \mathbb{B} \\ &= h\,\big(\widetilde{F}(t_0,x_0) + \varepsilon\,\mathbb{B}\big). \end{aligned}$$

□

Proof (of Lemma A.16). Choosing $\gamma > 0$ arbitrarily small, Proposition A.12 (on page 447) provides a closed subset $\widetilde{J}_\gamma \subset [0,T]$ with $\mathscr{L}^1([0,T] \setminus \widetilde{J}_\gamma) < \gamma$ such that the set-valued restriction $\widetilde{F}|_{\widetilde{J}_\gamma \times \mathbb{R}^N}$ is continuous.
As in the proof of Lemma A.14, let $J_\gamma \subset \widetilde{J}_\gamma$ denote the set of density points of $\widetilde{J}_\gamma$ that are Lebesgue points of $\mu(\cdot) \cdot \chi_{[0,T] \setminus \widetilde{J}_\gamma}(\cdot) \in L^1([0,T])$ in addition. It also satisfies

$$\mathscr{L}^1(J_\gamma) = \mathscr{L}^1(\widetilde{J}_\gamma) > T - \gamma.$$

For arbitrary $t_0 \in J_\gamma$, $x_0 \in \mathbb{R}^N$ and $\varepsilon \in]0,1]$, the continuity of $\widetilde{F}|_{J_\gamma \times \mathbb{R}^N}$ and the construction of J_γ guarantee parameters $r, \delta, t_1 \in]0,1]$ successively such that for every $t \in [t_0,\, t_0 + t_1] \cap J_\gamma$, $x \in \mathbb{B}_\delta(x_0)$, $y \in \mathbb{B}_r(x_0)$

$$\begin{cases} d\!l\big(\widetilde{F}(t,y),\, \widetilde{F}(t_0,x_0)\big) \;\le\; \frac{\varepsilon}{8} \\ x \,+\, (t-t_0)\cdot \widetilde{F}(t_0,x) \;\subset\; x_0 + r\,\mathbb{B}, \\ \frac{\mathscr{L}^1([t_0,t]\setminus \widetilde{J}_\gamma)}{t-t_0}\;\; \widetilde{F}(t_0,x_0) \;\subset\; \frac{\varepsilon}{4}\,\mathbb{B}, \\ \frac{1}{t-t_0}\displaystyle\int_{[t_0,t]\setminus \widetilde{J}_\gamma} \mu(s)\; ds \;\le\; \frac{\varepsilon}{4} \\ \delta + \displaystyle\int_{[t_0,t]} \mu(s)\; ds \;\le\; r. \end{cases}$$

Choose now any $x \in \mathbb{B}_\delta(x_0)$ and $v \in \widetilde{F}(t_0,x)$. We want to verify for all $h \in [0,t_1]$

$$x \,+\, h\,v \;\in\; \vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,\,x) \,+\, \varepsilon\, h\, \mathbb{B}.$$

Since all values of $\widetilde{F}$ are assumed to be convex, the projection of v on $\widetilde{F}(\cdot,\cdot)$

$$[0,T]\times\mathbb{R}^N \;\rightsquigarrow\; \mathbb{R}^N,\;\; (t,y) \;\mapsto \Pi_{\widetilde{F}(t,y)}(v) \stackrel{\text{Def.}}{=} \big\{ w \in \widetilde{F}(t,y) \;\big|\; \mathrm{dist}(v,\, \widetilde{F}(t,y)) = |w-v| \big\}$$

is single-valued and thus denoted by $f : [0,T]\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$.
Moreover, $f(\cdot,y) : [0,T] \longrightarrow \mathbb{R}^N$ is measurable for every $y \in \mathbb{R}^N$ due to Proposition A.80 (on page 490). Whenever $\widetilde{F}(t,\cdot) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is continuous, its composition with the projection mapping is upper semicontinuous in the sense of Painlevé-Kuratowski according to [162, Proposition 4.9] and thus, the single-valued function $f(t,\cdot) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous. As a consequence, f is a Carathéodory function in $[0,T]\times\mathbb{R}^N$ with the time-dependent absolute bound $\mu(\cdot) \in L^1([0,T])$ and, its restriction $f|_{J_\gamma\times\mathbb{R}^N}$ is continuous because $\widetilde{F}|_{J_\gamma\times\mathbb{R}^N}$ is continuous.
There exists an absolutely continuous solution $y(\cdot) : [t_0, t_0+t_1] \longrightarrow \mathbb{R}^N$ to the ordinary differential equations $y'(\cdot) = f\big(\cdot, y(\cdot)\big)$ a.e. with $y(t_0) = x$. Then, $y(\cdot)$ solves the differential inclusion $y'(\cdot) \in \widetilde{F}(\cdot, y(\cdot))$ a.e. and satisfies for all $h \in [0,t_1]$

$$\begin{aligned} &\big|x + h\,v \;-\; y(t_0+h)\big| \\ &\le \int_{[t_0,t_0+h]\cap J_\gamma} \big|v - f(s,y(s))\big|\; ds \;\;+ \int_{[t_0,t_0+h]\setminus J_\gamma} \big(|v| \,+\, \mu(s)\big)\; ds \\ &\le \int_{[t_0,t_0+h]\cap J_\gamma} \mathrm{dist}\big(v,\, \widetilde{F}(s,\,y(s))\big)\; ds \;+ \int_{[t_0,t_0+h]\setminus J_\gamma} \big(|v| \,+\, \mu(s)\big)\; ds \\ &\le \qquad 2\,\tfrac{\varepsilon}{8}\cdot h \qquad\qquad + \qquad 2\,\tfrac{\varepsilon}{4}\cdot h \,+\, \tfrac{\varepsilon}{4}\cdot h \quad = \varepsilon\cdot h. \qquad \square \end{aligned}$$

This proof of Lemma A.16 is quite easy to adapt to the following statement whose autonomous counterpart is used for verifying Proposition 1.68 (2.) (on page 70):

Lemma 17. *In addition to the assumptions of Proposition A.13, let $K \subset \mathbb{R}^N$ be a nonempty compact subset and $R > 0$.*

Then there exists a subset $J \subset [0,T]$ of full Lebesgue measure such that for every $t_0 \in J$, $x_0 \in K$ and $\varepsilon \in]0,1[$, there are $t_1, \delta > 0$ with

$$x \,+\, h\cdot\widetilde{F}(t_0,x) \,+\, h\cdot\big(T_K^C(x_0)\cap\mathbb{B}_R\big) \;\subset\; \vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,K) \,+\, 2\,\varepsilon\, h\,\mathbb{B}$$

for all $x \in \mathbb{B}_\delta(x_0)\cap K,\;\; h \in]0,t_1[$.

The contribution of the circatangent cone $T_K^C(x_0)$ to this modification is summarized in the next lemma:

Lemma 18. *Let K, M be nonempty compact subsets of $\mathbb{R}^N$.*
For each point $x_0 \in K$ and every $\varepsilon > 0$, there exists a radius $\rho > 0$ such that

$$\operatorname{dist}(y + h\,w,\ K) - \operatorname{dist}(y,\ K) \ \le\ \varepsilon\, h$$

is satisfied for all $w \in T_K^C(x_0) \cap M$, $h \in [0, \rho]$ and $y \in \mathbb{R}^N$ with $|y - x_0| \le \rho$.

Proof (of Lemma A.18). Equivalently to Definition 1.63 (on page 68), a vector $v \in \mathbb{R}^N$ belongs to the circatangent cone $T_K^C(x)$ in $x_0 \in K$ if and only if for every $\varepsilon > 0$, there exists a radius $\rho(x_0, \varepsilon, v) > 0$ with

$$\operatorname{dist}(y + h\,v,\ K) - \operatorname{dist}(y,\ K) \ \le\ \tfrac{\varepsilon}{2}\, h$$

for all $h \in [0, \rho]$ and $y \in \mathbb{R}^N$ with $|y - x_0| \le \rho$. This is easy to prove indirectly by means of the projection on the compact set $K \subset \mathbb{R}^N$ – similarly to the morphological analogue in Lemma 5.39 (on page 427, see also the Clarke's "original" definition of tangents via "generalized directional derivative" in [46, § 2]).
In particular, all vectors $w \in \mathbb{B}_{\frac{\varepsilon}{2}}(v) \subset \mathbb{R}^N$ have in common:

$$\operatorname{dist}(y + h\,w,\ K) - \operatorname{dist}(y,\ K) \ \le\ \varepsilon\, h$$

for every $h \in [0, \rho]$ and $y \in \mathbb{R}^N$ with $|y - x_0| \le \rho$ due to the triangle inequality. Hence, the radius $\rho > 0$ can be chosen locally uniformly with respect to $v \in T_K^C(x_0)$, i.e. for every compact $M \subset \mathbb{R}^N$ and $\varepsilon > 0$, there is $\rho = \rho(x_0, \varepsilon, M) > 0$ with

$$\operatorname{dist}(y + h\,w,\ K) - \operatorname{dist}(y,\ K) \ \le\ \varepsilon\, h$$

for all $w \in M \cap T_K^C(x_0)$, $h \in [0, \rho]$ and $y \in \mathbb{R}^N$ with $|y - x_0| \le \rho$. $\square$

Proof (of Lemma A.17). Fix any $\gamma > 0$ and construct closed subsets $J_\gamma \subset \widetilde{J}_\gamma \subset [0, T]$ as in the proof of Lemma A.16.
For arbitrary $t_0 \in J_\gamma$, $x_0 \in K$ and $\varepsilon \in \,]0, 1]$, the continuity of $\widetilde{F}|_{J_\gamma \times \mathbb{R}^N}$, the selection of J_γ and Lemma A.18 (in addition now) provide $r, \delta, t_1 \in \,]0, 1]$ successively such that for every $t \in [t_0, t_0 + t_1] \cap J_\gamma$, $h \in [0, t_1]$, $x \in \mathbb{B}_\delta(x_0)$, $y \in \mathbb{B}_r(x_0)$ and $w \in T_K^C(x_0) \cap \mathbb{B}_R$,

$$\begin{cases} \operatorname{dist}(y + h\,w,\ K) - \operatorname{dist}(y, K) \ \le\ \varepsilon\, h \\ d\!\!l\big(\widetilde{F}(t, y),\ \widetilde{F}(t_0, x_0)\big) \ \le\ \frac{\varepsilon}{8} \\ x + h \cdot \widetilde{F}(t_0, x) \ \subset\ x_0 + r\,\mathbb{B}, \\ \frac{\mathscr{L}^1([t_0, t] \setminus \widetilde{J}_\gamma)}{t - t_0}\ \widetilde{F}(t_0, x_0) \ \subset\ \frac{\varepsilon}{4}\,\mathbb{B}, \\ \frac{1}{t - t_0} \displaystyle\int_{[t_0, t] \setminus \widetilde{J}_\gamma} \mu(s)\ ds \ \le\ \frac{\varepsilon}{4} \\ \delta + h\,(R + \varepsilon) + \displaystyle\int_{[t_0, t]} \mu(s)\ ds \ \le\ r. \end{cases}$$

For all $x \in \mathbb{B}_\delta(x_0)$, $v \in \widetilde{F}(t_0,x)$, $w \in T_K^C(x_0) \cap \mathbb{B}_R$ and $h \in [0,t_1]$, we are now to check

$$x + h\,(v+w) \;\in\; \vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,K) + 2\,\varepsilon\,h\,\mathbb{B}.$$

As in the proof of Lemma A.16, the projection of v on $\widetilde{F}(\cdot,\cdot)$

$$[0,T]\times\mathbb{R}^N \rightsquigarrow \mathbb{R}^N,\;\; (t,y) \mapsto \Pi_{\widetilde{F}(t,y)}(v) \stackrel{\text{Def.}}{=} \left\{w \in \widetilde{F}(t,y) \;\middle|\; \operatorname{dist}(v, \widetilde{F}(t,y)) = |w-v|\right\}$$

induces a single-valued Carathéodory function $f : [0,T]\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$.
The new essential aspect is to base the comparison on an absolutely continuous solution $y(\cdot) : [t_0,t_0+t_1] \longrightarrow \mathbb{R}^N$ that does not start in x, but in a possibly different point of K:
Choose $z = z(x,h,w) \in K$ with $|x+h\,w-z| = \operatorname{dist}(x+h\,w,K) \le \varepsilon\,h$. In particular, $|z-x_0| \le \delta + h\,(R+\varepsilon) < r$. Then there exists an absolutely continuous solution $y(\cdot) : [t_0,t_0+t_1] \longrightarrow \mathbb{R}^N$ to the ordinary differential equations $y'(\cdot) = f\big(\cdot, y(\cdot)\big)$ a.e. with $y(t_0) = z \in K$. $y(\cdot)$ has all values in $\mathbb{B}_r(x_0)$ and solves the differential inclusion $y'(\cdot) \in \widetilde{F}(\cdot, y(\cdot))$ a.e. again (but depends now on x,h,w).

The comparison with $t \mapsto x+h\,w+(t-t_0)\,v$ at time t_0+h leads to the estimate

$$\begin{aligned}
&\operatorname{dist}\big(x+h\,(v+w),\; \vartheta_{\widetilde{F}(t_0+\cdot,\cdot)}(h,K)\big) \;\le\; \big|x+h\,(v+w) \,-\, y(t_0+h)\big|\\
&\le\; |x+h\,w-z| \;+\; \int_{[t_0,t_0+h]\cap J_\gamma} \big|v-f(s,y(s))\big|\;ds \;+\; \int_{[t_0,t_0+h]\setminus J_\gamma} \big(|v|+\mu(s)\big)\,ds\\
&\le\; \varepsilon\,h \;+\; \int_{[t_0,t_0+h]\cap J_\gamma} \operatorname{dist}\big(v,\,\widetilde{F}(s,y(s))\big)\,ds \;+\; \int_{[t_0,t_0+h]\setminus J_\gamma} \big(|v|+\mu(s)\big)\,ds\\
&\le\; \varepsilon\,h \;+\; 2\,\tfrac{\varepsilon}{8}\cdot h \;+\; 2\,\tfrac{\varepsilon}{4}\cdot h + \tfrac{\varepsilon}{4}\cdot h\\
&\le\; 2\,\varepsilon\,h.
\end{aligned}$$

□

A.4 Relaxation Theorem of Filippov-Ważewski for Differential Inclusions

The so-called Relaxation Theorem bridges the gap between a differential inclusion

$$x'(\cdot) \;\in\; \widetilde{F}(\cdot, x(\cdot))$$

and its *relaxed* counterpart with (pointwise) convexified values on the right-hand side, i.e.,

$$y'(\cdot) \;\in\; \overline{\text{co}}\;\widetilde{F}(\cdot, y(\cdot)).$$

In particular, it provides sufficient conditions on the set-valued map $\widetilde{F} : [0,T]\times\mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ which make the additional assumption of convex values dispensable in regard to compact reachable sets.

Theorem 19 (Relaxation Theorem of Filippov-Ważewski). *Suppose for the set-valued map* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *and the curve* $y(\cdot) \in W^{1,1}([0,T], \mathbb{R}^N)$:

(*1.*) *the values* $\widetilde{F}$ *are nonempty closed subsets of* $\mathbb{R}^N$*,*

(*2.*) *for every* $x \in \mathbb{R}^N$, $F(\cdot, x) : [0,T] \rightsquigarrow \mathbb{R}^N$ *is measurable,*

(*3.*) *there exist* $\rho > 0$ *and* $\lambda(\cdot) \in L^1([0,T], \mathbb{R}_0^+)$ *such that for* $\mathscr{L}^1$*-almost every* $t \in [0,T]$*, the restriction* $F(t,\cdot)\big|_{\mathbb{B}_\rho(y(t))} : \mathbb{B}_\rho(y(t)) \rightsquigarrow \mathbb{R}^N$ *is* $\lambda(t)$*-Lipschitz continuous w.r.t.* $d\!l$*,*

(*4.*) *there is* $\mu(\cdot) \in L^1([0,T])$ *with* $\big|\widetilde{F}(t, y(t))\big|_\infty \le \mu(t)$ *for* $\mathscr{L}^1$*-almost every* t*.*

(*5.*) $[0,T] \longrightarrow \mathbb{R}$, $t \longmapsto \operatorname{dist}\big(y'(t),\ \widetilde{F}(t, y(t))\big)$ *is Lebesgue-integrable,*

(*6.*) $e^{\|k\|_{L^1}} \cdot \displaystyle\int_0^T \operatorname{dist}\Big(y'(t),\ \widetilde{F}(t, y(t))\Big)\, dt \ \le\ \rho$,

(*7.*) $y'(t) \ \in\ \overline{\mathrm{co}}\, \widetilde{F}(t,\, y(t))$ *for* $\mathscr{L}^1$*-almost every* $t \in [0,T]$.

Then for every $\delta > 0$*, there exists a solution* $x(\cdot) \in W^{1,1}([0,T], \mathbb{R}^N)$ *to the differential inclusion* $x'(\cdot) \in F(\cdot, x(\cdot))$ *a.e. satisfying* $x(0) = y(0)$ *and* $\|x(\cdot) - y(\cdot)\|_{L^\infty} \le \delta$*.*

Proof is given in [80, Theorem 1.36], for example, as a consequence of Filippov's Theorem A.6 and an appropriate selection principle. The autonomous counterpart and its proof can be found in [15, Theorem 2.4.2].

Aubin and Frankowska have already pointed out a well-known consequence in [19, Theorem 10.4.4]:

Corollary 20. *In addition to the hypotheses of Relaxation Theorem A.19 with* $\rho = \infty$*, assume that* $R(\cdot) \in L^1([0,T])$ *satisfies* $\widetilde{F}(t,x) \subset R(t)\, \mathbb{B}$ *for every* $x \in \mathbb{R}^N$ *and a.e.* t*.*

Then the solutions to the differential inclusion $x'(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ *a.e. are dense in the set of solutions to the relaxed inclusion* $y'(\cdot) \in \overline{\mathrm{co}}\, \widetilde{F}(\cdot, y(\cdot))$ *a.e. with respect to the supremum norm.* □

Considering now reachable sets of differential inclusions, we obtain

Corollary 21. *Let* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *be Filippov continuous* (*according to Definition A.7 on page 444*).

Then, $\vartheta_{\widetilde{F}}(t,K) = \vartheta_{\overline{\mathrm{co}}\,\widetilde{F}}(t,K)$ *for every* $K \in \mathscr{K}(\mathbb{R}^N)$ *and* $t \in [0,T]$.

Proof. Relaxation Theorem A.19 implies

$$\overline{\vartheta_{\widetilde{F}}(t,M)} \ = \ \overline{\vartheta_{\overline{\mathrm{co}}\,\widetilde{F}}(t,M)}$$

for every nonempty (not necessarily closed) subset $M \subset \mathbb{R}^N$ and any $t \in [0,T]$.
In addition, the reachable set $\vartheta_{\widetilde{F}}(t,K) \subset \mathbb{R}^N$ is closed as a consequence of Filippov's Theorem A.6 (on page 443). Finally, $\overline{\mathrm{co}}\ \widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has Filippov continuity in common with $\widetilde{F}$ and thus, $\vartheta_{\overline{\mathrm{co}}\,\widetilde{F}}(t,K) \subset \mathbb{R}^N$ is also closed. □

A.5 Regularity of Reachable Sets of Differential Inclusions

In this section, we focus on the boundary of reachable sets of differential inclusions. Adjoint arcs are used for describing the time-dependent limiting normal cones. They serve as tools for sufficient conditions on the differential inclusion for preserving smooth boundaries shortly, for example.
First we prove in Proposition A.34 that $C^{1,1}$ boundaries are preserved for short times. Then according to Proposition A.36, the same hypothesis guarantees that the evolution of smooth sets is reversible in time. Afterwards, the conditions on the Hamiltonian function $\mathscr{H}_F$ are supposed to be stronger for guaranteeing that points evolve into sets of positive erosion (see Proposition A.41). Finally, we estimate the maximal shrinking of exterior or interior balls and focus on exterior tusks.

Definition 22. For any set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \leadsto \mathbb{R}^N$, the support function

$$\begin{aligned} \mathscr{H}_{\widetilde{F}} : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N &\longmapsto \mathbb{R} \\ (t,x,p) &\longmapsto \sigma\big(p, \widetilde{F}(t,x)\big) \stackrel{\text{Def.}}{=} \sup \big\{ \langle p, v \rangle \;\big|\; v \in \widetilde{F}(t,x) \big\} \end{aligned}$$

is called (*upper*) *Hamiltonian* of $\widetilde{F}$.

A.5.1 Normal Cones and Compact Sets: Definitions and Notation

This section serves mainly the purpose of clarifying the notation in regard to normal cones and summarizing some features of compact subsets of $\mathbb{R}^N$.

Definition 23. Let $C \subset \mathbb{R}^N$ be a nonempty closed set.
A vector $\eta \in \mathbb{R}^N$, $\eta \neq 0$, is said to be a *proximal normal vector* to C at $x \in C$ if there exists $\rho > 0$ with $\mathbb{B}_\rho(x + \rho \frac{\eta}{|\eta|}) \cap C = \{x\}$.
The supremum of all ρ with this property is called *proximal radius* of C at x in direction η. The cone of all proximal normal vectors is called the *proximal normal cone* to C at x and is abbreviated as $N_C^P(x)$.

The so-called *limiting normal cone* $N_C(x)$ to C at x consists of all vectors $\eta \in \mathbb{R}^N$ that can be approximated by sequences $(\eta_n)_{n \in \mathbb{N}}$, $(x_n)_{n \in \mathbb{N}}$ satisfying

$$x_n \longrightarrow x, \quad \eta_n \longrightarrow \eta, \qquad x_n \in C, \quad \eta_n \in N_C^P(x_n),$$

i.e. $$N_C(x) \stackrel{\text{Def.}}{=} \operatorname{Limsup}_{\substack{y \longrightarrow x \\ y \in C}} N_C^P(y) \qquad \text{(in the sense of Painlevé-Kuratowski).}$$
As a further abbreviation, we set ${}^\flat N_C(x) := N_C(x) \cap \mathbb{B} = \big\{ v \in N_C(x) \,\big|\, |v| \leq 1 \big\}$.

Convention. In the following we restrict ourselves to normal directions at boundary points, i.e. strictly speaking, Graph N_C and Graph ${}^\flat N_C$ are the abbreviations of Graph $N_C|_{\partial C}$ and Graph ${}^\flat N_C|_{\partial C}$, respectively.

Lemma 24 ([47, Lemma 6.4]). *For a nonempty closed subset $M \subset \mathbb{R}^N$, assume $\eta \in N^P_{\overline{\mathbb{R}^N \setminus M}}(x) \setminus \{0\}$ and $N_M^P(x) \neq \{0\}$. Then, $N^P_{\overline{\mathbb{R}^N \setminus M}}(x) = -N_M^P(x) = \mathbb{R}_0^+ \, \eta$.*

Definition 25. $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ abbreviates the set of all nonempty compact N-dimensional $C^{1,1}$ submanifolds of $\mathbb{R}^N$ with boundary.

A closed subset $C \subset \mathbb{R}^N$ is said to have *positive erosion of radius* $\rho > 0$ if for each $r \in]0, \rho[$, there exists a closed set $M \subset \mathbb{R}^N$ with

$$\begin{cases} C \;=\; \{x \in \mathbb{R}^N \,|\, \mathrm{dist}(x, M) \;\leq r\}, \\ M \;=\; \{x \in C \;\;\,|\, \mathrm{dist}(x, \partial C) \geq r\}. \end{cases}$$

$\mathcal{K}_\circ^\rho(\mathbb{R}^N)$ consists of all sets with positive erosion of radius $\rho > 0$ and, set $\quad \mathcal{K}_\circ(\mathbb{R}^N) := \bigcup_{\rho > 0} \mathcal{K}_\circ^\rho(\mathbb{R}^N)$.

Definition 26 (Sets of positive reach [79], [55, Definition 4.7.1]**).**
A nonempty set $M \subset \mathbb{R}^N$ is said to have *positive reach* if there exists $h > 0$ such that the projection $\Pi_M(x) \overset{\text{Def.}}{=} \big\{y \in \overline{M} \,\big|\, |x-y| = \mathrm{dist}(x, M)\big\}$ is single-valued for every $x \in \mathbb{R}^N$ with $\mathrm{dist}(x, M) < h$. The maximum $h > 0$ for which this property holds is called the *reach* of M.

Remark 27. The morphological term "erosion" is motivated by the fact that a set $C = \overline{C^\circ} \subset \mathbb{R}^N$ has positive erosion if and only if the closure $\overline{\mathbb{R}^N \setminus C}$ of its complement has positive reach. This implies a collection of interesting regularity properties presented (for closed subsets of a Hilbert space) in [47, 48, 159]. Here we summarize some of the features for subsets of $\mathbb{R}^N$:

Proposition 28 **(**[47], [48, Theorem 4.1], [55, Theorem 4.7.1], [159, Theorem 4.1]**).**
Given a nonempty closed subset $M \subset \mathbb{R}^N$, the following conditions are equivalent:

(1.) *M has positive reach $\geq \rho > 0$,*

(2.) $\mathrm{dist}(\cdot, M)$ *belongs to* $C^{1,1}_{\mathrm{loc}}\big(\{0 < \mathrm{dist}(\cdot, M) < \rho\}\big)$,

(3.) $\mathrm{dist}(\cdot, M)$ *belongs to* $C^{1}_{\mathrm{loc}}\big(\{0 < \mathrm{dist}(\cdot, M) < \rho\}\big)$,

(4.) $\Pi_M(x) \subset M$ *is single-valued for all points* $x \in \mathbb{R}^N$ *with* $0 < \mathrm{dist}(x, M) < \rho$,

(5.) $\Pi_M(x) \subset M$ *is single-valued for all* $x \in \mathbb{R}^N$ *with* $0 < \mathrm{dist}(x, M) < \rho$ *and,* Π_M *belongs to* $C^{0,1}_{\mathrm{loc}}\big(\{0 < \mathrm{dist}(\cdot, M) < \rho\}\big)$,

(6.) $\mathrm{dist}(\cdot, M)^2$ *belongs to* $C^{1,1}_{\mathrm{loc}}\big(\{0 < \mathrm{dist}(\cdot, M) < \rho\}\big)$,

(7.) *for every* $r \in]0, \rho[$, *all points* $x \in \mathbb{R}^N$ *with* $0 < \mathrm{dist}(x, M) < r$ *satisfy*
$$\mathrm{dist}(x,\, M) \;+\; \mathrm{dist}\big(x,\, \mathbb{R}^N \setminus \mathbb{B}_r(M)^\circ\big) \;=\; r,$$

(8.) *for any* $r \in]0, \rho[$, $\big\{\mathrm{dist}(\cdot\,,\, \overline{\mathbb{R}^N \setminus \mathbb{B}_r(M)}) \geq r\big\} \;=\; M$,

(9.) *every proximal normal vector* $\neq 0$ *at any* $x \in \partial M$ *has proximal radius* $\geq \rho$,

(10.) *for any* $r \in]0, \rho[$, *each* $x \in \mathbb{R}^N$ *with* $\mathrm{dist}(x, M) = r$ *satisfies* $N^P_{\mathbb{B}_r(M)}(x) \neq \{0\}$,

(11.) $N_M(\cdot) \cap \mathbb{B}^\circ_\rho$ *is hypermonotone, i.e. whenever* $x_1, x_2 \in M$ *and* $v_k \in N_M(x_k)$ *with* $|v_k| < \rho$ $(k = 1, 2)$, *then* $\;(v_1 - v_2) \cdot (x_1 - x_2) \;\geq\; -\,|x_1 - x_2|^2$,

(12.) $\mathrm{dist}(y - x,\, T_M(x)) \;\leq\; \frac{1}{2\rho}\, |y - x|^2$ *for any* $y, x \in M$ *(global Shapiro property).*

Corollary 29 ([48, Corollary 4.15]).
Every nonempty closed set $M \subset \mathbb{R}^N$ with positive reach $\geq \rho > 0$ fulfills:

(a) *the proximal, limiting and Clarke normal cone coincide at each point $x \in M$,*

(b) *for every $r \in]0,\rho[$ and each point $x \in \mathbb{R}^N$ with $\operatorname{dist}(x,M) = r$,*
$$N^P_{\mathbb{B}_r(M)}(x) \;=\; N_{\mathbb{B}_r(M)}(x) \;=\; \mathbb{R}^+_0 \cdot \big(x - p(x)\big)$$
where $p(x) \in M$ is the unique closest point to x in M,

(c) *for every $r \in]0,\rho[$, the topological boundary of $\big\{\operatorname{dist}(\cdot,M) \leq r\big\}$ is a $C^{1,1}$ submanifold of codimension 1 in $\mathbb{R}^N$.*

A.5.2 Adjoint Arcs for Evolving Normal Cones to Reachable Sets

The so-called Hamilton condition is known under very mild assumptions using the tools of nonsmooth functions. First we quote the version of Vinter's monograph [180]. Applying these results to proximal balls leads to a necessary condition on boundary points of reachable sets and their proximal normal vectors. Approximating sequences then lay the basis for extending this result to limiting normal vectors in subsequent Proposition A.32. In particular, it is formulated only for Hamiltonian functions with continuous partial derivatives $\partial_x \mathscr{H}_{\widetilde{F}}, \partial_y \mathscr{H}_{\widetilde{F}}$ because we exploit the regularity of solutions to ordinary differential equations in the next sections.

Proposition 30 (Extended Hamilton Condition).
Let $x(\cdot) \in W^{1,1}([S,T],\mathbb{R}^N)$ be a local minimizer (with respect to perturbations in $W^{1,1}([0,T],\mathbb{R}^N)$) of the problem
$$\begin{array}{ll} g(y(S),y(T)) \longrightarrow \min & \\ \textit{over} \;\; y(\cdot) \in W^{1,1}([S,T],\mathbb{R}^N) \textit{ satisfying} & \\ \quad y'(t) \in \widetilde{F}(t,y(t)) & \textit{for Lebesgue-almost every } t \in [S,T], \\ \quad (y(S),\, y(T)) \in C \subset \mathbb{R}^N \times \mathbb{R}^N. & \end{array}$$
Assume also that

($G1$) *g is locally Lipschitz continuous;*

($G2$)$'$ *$\widetilde{F}(t,x) \neq \emptyset$ is convex for each (t,x), $\widetilde{F}$ is $\mathscr{L}^{1+N} \times \mathscr{B}^N$ measurable, and* $\operatorname{Graph} \widetilde{F}(t,\cdot)$ *is closed for each $t \in [S,T]$.*

Suppose, furthermore, that either of the following hypotheses is satisfied:

(a) *There exist $k \in L^1([S,T])$ and $\varepsilon > 0$ such that for almost every t*
$$\widetilde{F}(t,x_1) \cap \big(x'(t) + \varepsilon\, k(t)\, \mathbb{B}\big) \;\subset\; \widetilde{F}(t,x_2) + k(t)\,|x_1 - x_2|\,\mathbb{B}$$
for all $x_1,x_2 \in \mathbb{B}_\varepsilon(x(t))$.

(b) *There exist $k \in L^1([S,T])$, $\overline{K} > 0$ and $\varepsilon > 0$ such that the following two conditions are satisfied for almost every $t \in [S,T]$ and all $x_1,x_2 \in \mathbb{B}_\varepsilon(x(t))$*
$$\widetilde{F}(t,x_1) \cap \big(x'(t) + \varepsilon\,\mathbb{B}\big) \subset \widetilde{F}(t,x_2) + k(t)\,|x_1 - x_2|\,\mathbb{B},$$
$$\inf\big\{\,|v - x'(t)| \;\big|\; v \in \widetilde{F}(t,x_1)\big\} \leq \overline{K}\,|x_1 - x(t)|\,.$$

Then there exist an arc $p(\cdot) \in W^{1,1}([S,T],\mathbb{R}^N)$ *and a constant* $\lambda \geq 0$ *such that*

$$
\begin{array}{rl}
(i) & (p(\cdot),\, \lambda) \neq (0,0), \\
(ii) & p'(t) \in co\,\big\{\, \eta \in \mathbb{R}^N \,\big|\, (\eta, p(t)) \in N_{\mathrm{Graph}\,\widetilde{F}(t,\,\cdot\,)}(x(t),\, x'(t)) \,\big\} \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t \\
(iii) & \big(p(S),\, -p(T)\big) \in \lambda\, \partial^L g(x(S), x(T)) + N_C(x(S), x(T)).
\end{array}
$$

Condition (ii) *implies*

$$
\begin{array}{rll}
(iv) & p(t) \cdot x'(t) = \sup\, \big(p(t) \cdot \widetilde{F}(t, x(t))\big) & \textit{for } \mathscr{L}^1\textit{-a.e. } t \\
(v) & p'(t) \in co\,\big\{ -q \in \mathbb{R}^N \,\big|\, (q, x'(t)) \in \partial^L \mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)|_{(x(t),p(t))} \big\} & \textit{for } \mathscr{L}^1\textit{-a.e. } t.
\end{array}
$$

Proof is presented in [180, Theorem 7.7.1], for example.

Remark 31. This adjoint $p(\cdot)$ also satisfies $|p'(t)| \leq k(t)\, |p(t)|$ for almost every t as an immediate consequence of statement (ii) and the so-called *Mordukhovich criterion* (see e.g. [162, Theorem 9.40]).

Proposition 32. *Suppose for the set-valued map* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$

1. $\widetilde{F}(\cdot)$ *is measurable with nonempty convex compact values,*
2. *for* $\mathscr{L}^1$*-almost every* $t \in [0,T]$, $\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)$ *is continuously differentiable in* $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$,
3. *there exists* $k(\cdot) \in L^1([0,T]$ *such that for* $\mathscr{L}^1$*-almost every* $t \in [0,T]$, $\|\partial_{(x,p)} \mathscr{H}_{\widetilde{F}}(x,p)\| \leq k(t) \cdot (1 + |x| + |p|)$ *for all* $x, p \in \mathbb{R}^N$, $|p| > 1$.

Let $K \in \mathscr{K}(\mathbb{R}^N)$ *be any initial set and* $t_0 > 0$.

For every boundary point $x_0 \in \partial\, \vartheta_{\widetilde{F}}(t_0, K)$ *and normal* $\nu \in N_{\vartheta_{\widetilde{F}}(t_0,K)}(x_0) \setminus \{0\}$, *there exist a solution* $x(\cdot) \in W^{1,1}([0,t_0],\mathbb{R}^N)$ *and its adjoint* $p(\cdot) \in W^{1,1}([0,t_0],\mathbb{R}^N)$ *with*

$$
\begin{cases}
x'(t) = \frac{\partial}{\partial p}\, \mathscr{H}_{\widetilde{F}}(t,\, x(t),\, p(t)) \in \widetilde{F}(t, x(t)), & x(t_0) = x_0, \quad x(0) \in \partial K, \\
p'(t) = -\frac{\partial}{\partial x}\, \mathscr{H}_{\widetilde{F}}(t,\, x(t),\, p(t)), & p(t_0) = \nu, \quad p(0) \in N_K(x(0)).
\end{cases}
$$

□

A.5.3 Hamiltonian System Helps Preserving $C^{1,1}$ Boundaries Shortly

Definition 33. For a set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, the standard hypothesis $(\widetilde{\mathscr{H}})$ comprises the following conditions on $\mathscr{H}_{\widetilde{F}}(t,x,p) := \sup\ p \cdot \widetilde{F}(t,x)$

1. $\widetilde{F}$ is measurable and has nonempty compact convex values,
2. $\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot) : \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \longrightarrow \mathbb{R}$ is continuously differentiable for every t,
3. for every $R > 1$, there exists $\lambda_R(\cdot) \in L^1([0,T])$ such that the derivative of $\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)$ restricted to $\mathbb{B}_R \times (\mathbb{B}_R \setminus \overset{\circ}{\mathbb{B}}_{\frac{1}{R}})$ is $\lambda_R(t)$-Lipschitz continuous for Lebesgue-almost every $t \in [0,T]$,
4. there is $k_{\widetilde{F}} \in L^1([0,T])$ such that for a.e. $t \in [0,T]$ and all $x, p \in \mathbb{R}^N$ $(|p| \geq 1)$,
$$\left\| \partial_{(x,p)}\, \mathscr{H}_{\widetilde{F}}(t,x,p) \right\|_{\mathrm{Lin}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})} \ \leq\ k_{\widetilde{F}}(t) \cdot (1 + |x| + |p|).$$

Proposition 34. *Assume standard hypothesis $(\widetilde{\mathscr{H}})$ for $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. For every initial compact set $K \in \mathscr{K}_{C^{1,1}}(\mathbb{R}^N)$, there exist $\tau = \tau(\widetilde{F},K) > 0$ and $\rho = \rho(\widetilde{F},K) > 0$ such that $\vartheta_{\widetilde{F}}(t,K)$ is also a N-dimensional $C^{1,1}$ submanifold of $\mathbb{R}^N$ with boundary for all $t \in [0,\tau]$ and, its radius of curvature is $\geq \rho$ at every boundary point. In particular, $\vartheta_{\widetilde{F}}(t,K)$ has both positive reach and erosion.*

The proof of Proposition A.34 is based on the following lemma:

Lemma 35. *Suppose for $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and the Hamiltonian system*
$$\begin{cases} y'(t) = \ \ \frac{\partial}{\partial q} H(t, y(t), q(t)), & y(0) = y_0 \\ q'(t) = -\frac{\partial}{\partial y} H(t, y(t), q(t)), & q(0) = \psi(y_0) \end{cases} \qquad (*)$$
the following properties:

1. *$H(t,\cdot,\cdot)$ is differentiable for every $t \in [0,T]$,*
2. *for every $R > 0$, there exists $k_R \in L^1([0,T])$ such that the derivative of $H(t,\cdot,\cdot)$ is $k_R(t)$-Lipschitz continuous on $\mathbb{B}_R \times \mathbb{B}_R$ for $\mathscr{L}^1$-almost every t,*
3. *ψ is locally Lipschitz continuous,*
4. *every solution $(y(\cdot), q(\cdot))$ to the Hamiltonian system $(*)$ can be extended to $[0,T]$ and depends continuously on the initial data in the following sense: Let each $(y_n(\cdot), q_n(\cdot))$ be a solution satisfying $y_n(t_n) \longrightarrow z_0$, $q_n(t_n) \longrightarrow q_0$ for some $t_n \longrightarrow t_0$, $z_0, q_0 \in \mathbb{R}^N$.*
   Then $(y_n(\cdot), q_n(\cdot))_{n \in \mathbb{N}}$ converges uniformly to a solution $(y(\cdot), q(\cdot))$ to the Hamiltonian system with $y(t_0) = z_0$, $q(t_0) = q_0$.

For a compact set $K \subset \mathbb{R}^N$ and $t \in [0,T]$, define
$$M_t^{\mapsto}(K) := \left\{ (y(t), q(t)) \,\middle|\, (y(\cdot), q(\cdot)) \text{ solves system } (*),\ y_0 \in K \right\} \subset \mathbb{R}^N \times \mathbb{R}^N.$$
Then there exist $\delta > 0$ and $\lambda > 0$ such that $M_t^{\mapsto}(K)$ is the graph of a λ-Lipschitz continuous function for every $t \in [0,\delta]$.

Proof (of Lemma A.35). It is based on the indirect proof of [80, Lemma 5.5] about the same Hamiltonian system with $y(T) = y_T,\ q(T) = q_T$ given (without mentioning the uniform Lipschitz constant λ explicitly).

Suppose that the claim is false. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $]0,T]$ with $t_n \longrightarrow 0$ such that either $M_{t_n}^{\mapsto}(K)$ is not the graph of a Lipschitz function or the corresponding Lipschitz constants converge to ∞. In both cases, we can find distinct solutions $(y_n^1(\cdot), q_n^1(\cdot)),\ (y_n^2(\cdot), q_n^2(\cdot)),\ n \in \mathbb{N},$ to the Hamiltonian system $(*)$ with

$$\varepsilon_n := \frac{|\, y_n^1(t_n) - y_n^2(t_n)\,|}{|\, q_n^1(t_n) - q_n^2(t_n)\,|} \longrightarrow 0 \qquad \text{for } n \longrightarrow \infty.$$

Assumption (4.) and $K \in \mathscr{K}(\mathbb{R}^N)$ imply $\bigcup_{0\le t\le T} M_t^{\mapsto}(K) \subset \mathbb{B}_R \times \mathbb{B}_R$ for some $R > 0$.

Assumption (2.) provides the estimate

$$\begin{aligned}
&|y_n^1(t) - y_n^2(t)|\\
&\le \quad |y_n^1(t_n) - y_n^2(t_n)| + \int_t^{t_n} k_R(s)\,\Big(|y_n^1(s) - y_n^2(s)| + |q_n^1(s) - q_n^2(s)|\Big)\,ds\\
&\le \varepsilon_n\, |q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s)\,\Big(|y_n^1(s) - y_n^2(s)| + |q_n^1(s) - q_n^2(s)|\Big)\,ds
\end{aligned}$$

for all $t \in [0,t_n]$, and the integral version of Gronwall's inequality (Proposition A.1) leads to a constant $C_1 > 0$ (independent of n) with

$$|y_n^1(t) - y_n^2(t)| \le C_1\Big(\varepsilon_n\, |q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s)\ |q_n^1(s) - q_n^2(s)|\ ds\Big).$$

Due to $\sup_n \varepsilon_n < \infty$, we obtain a constant $C_2 > 0$ such that for all $n \in \mathbb{N},\ t \in [0,t_n]$,

$$\begin{aligned}
&|q_n^1(t) - q_n^2(t)|\\
&\le \quad |q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s)\Big(|y_n^1(s) - y_n^2(s)| + |q_n^1(s) - q_n^2(s)|\Big)ds\\
&\le C_2\,\Big(|q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s)\ |q_n^1(s) - q_n^2(s)|\ ds\Big).
\end{aligned}$$

As a consequence of Gronwall's Proposition A.1 again, there is a constant $C_3 > 0$ (independent of n) with $|q_n^1(t) - q_n^2(t)| \le C_3\, |q_n^1(t_n) - q_n^2(t_n)|$ for all $n,\ t \in [0,t_n]$. In particular,

$$\varepsilon_n' := \sup_{0\le t\le t_n} \frac{|\,y_n^1(t) - y_n^2(t)\,|}{|\,q_n^1(t_n) - q_n^2(t_n)\,|} \le C_1\,\Big(\varepsilon_n + C_3 \int_0^{t_n} k_R(s)\ ds\Big) \overset{n\to\infty}{\longrightarrow} 0.$$

Similarly we get a constant $C_4 = C_4(\|k_R\|_{L^1}) > 0$ fulfilling

$$|q_n^1(t_n) - q_n^2(t_n)| \le C_4\, |q_n^1(0) - q_n^2(0)| = C_4\, |\psi(y_n^1(0)) - \psi(y_n^2(0))|$$

for all $n \in \mathbb{N}$ sufficiently large. Indeed, for all $t \in [0,t_n]$, assumption (2.) ensures

$$\begin{aligned}
&|q_n^1(t) - q_n^2(t)|\\
&\le |q_n^1(0) - q_n^2(0)| + \int_0^t k_R(s)\,\Big(\quad |y_n^1(s) - y_n^2(s)\,| + |q_n^1(s) - q_n^2(s)|\Big)\,ds\\
&\le |q_n^1(0) - q_n^2(0)| + \int_0^t k_R(s)\,\Big(\varepsilon_n'\, |q_n^1(t_n) - q_n^2(t_n)| + |q_n^1(s) - q_n^2(s)|\Big)\,ds
\end{aligned}$$

and Gronwall's inequality (Proposition A.1) provides $C_5 = C_5(\|k_R\|_{L^1}) > 0$ such

that for every $n \in \mathbb{N}$,

$$|q_n^1(t_n) - q_n^2(t_n)| \le \tfrac{C_5}{2}\, |q_n^1(0) - q_n^2(0)| + \mathrm{const}(\|k_R\|_{L^1})\ \varepsilon_n'\ |q_n^1(t_n) - q_n^2(t_n)|.$$

Due to $\varepsilon_n' \longrightarrow 0$, we obtain $|q_n^1(t_n) - q_n^2(t_n)| \le C_5\, |q_n^1(0) - q_n^2(0)|$ for all $n \in \mathbb{N}$ large enough. Finally,

$$\begin{aligned} \frac{|\psi(y_n^1(0)) - \psi(y_n^2(0))|}{|y_n^1(0) - y_n^2(0)|} &= \frac{|q_n^1(0) - q_n^2(0)|}{|q_n^1(t_n) - q_n^2(t_n)|} \cdot \frac{|q_n^1(t_n) - q_n^2(t_n)|}{|y_n^1(0) - y_n^2(0)|} \\ &\ge \quad \frac{1}{C_5} \quad \cdot \quad \frac{1}{\varepsilon_n'} \\ &\longrightarrow \quad \infty \qquad \text{for } n \longrightarrow \infty \end{aligned}$$

— contradicting the local Lipschitz continuity of ψ at each joint cluster point of $(y_n^1(0))_{n\in\mathbb{N}}$ and $(y_n^2(0))_{n\in\mathbb{N}}$ in K. □

Proof (of Proposition A.34). Assuming that $K \in \mathscr{K}(\mathbb{R}^N)$ is a N-dimensional $C^{1,1}$ submanifold of $\mathbb{R}^N$ with boundary, the *exterior* unit normal vectors to K (restricted to ∂K) can be extended to a Lipschitz continuous function $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$. Choosing some cut-off function $\varphi \in C^\infty([0,\infty[, [0,1])$ with $\varphi|_{[0,\frac14]} \equiv 0$, $\varphi|_{[\frac12,\infty[} \equiv 1$, $H(t,x,p) := \mathscr{H}_{\widetilde{F}}(t,x,p) \cdot \varphi(|p|)$ satisfies condition (1.), (2.), (4.) of Lemma A.35 due to standard hypothesis $(\widetilde{\mathscr{H}})$.

For arbitrary $x_0 \in \partial K$, consider the differential equations

$$\begin{cases} x'(t) = \ \ \frac{\partial}{\partial p} H(t, x(t), p(t)), & x(0) = x_0, \\ p'(t) = -\frac{\partial}{\partial x} H(t, x(t), p(t)), & p(0) = \psi(x_0). \end{cases} \qquad (**)$$

Due to $|\psi(\cdot)| = 1$ on ∂K and $H \in C^{1,1}$, there exists some $\tau_1 > 0$ such that $|p(t)| > \frac12$ for any $t \in [0,\tau_1]$ and all solutions $(x(\cdot),p(\cdot))$ of $(**)$ with $x_0 \in \partial K$. Thus, $H = \mathscr{H}_F$ close to $(x(t),p(t))$. Now Proposition A.32 can be reformulated as

$$\begin{aligned} \text{Graph } N_{\vartheta_F(t,K)}(\cdot) \subset \big\{ (x(t),\ \lambda\ p(t)) \,\big|\, & (x(\cdot),\, p(\cdot)) \text{ solves system } (**), \\ & x_0 \in \partial K,\ \lambda \ge 0 \big\}, \end{aligned}$$

for all $t \in [0,\tau_1]$. Lemma A.35 yields $\tau \in]0,\tau_1[$ and $\lambda_M > 0$ such that

$$M_t^{\mapsto}(\partial K) := \big\{ (x(t),\, p(t)) \,\big|\, (x(\cdot),\, p(\cdot)) \text{ solves system } (**),\ x_0 \in \partial K \big\}$$

is the graph of a λ_M-Lipschitz continuous function for each $t \in [0,\tau]$.

Then for every point $z \in \partial\vartheta_{\widetilde{F}}(t,K)$, the limiting normal cone $N_{\vartheta_{\widetilde{F}}(t,K)}(z)$ contains exactly one direction and, its unit vector depends on z in a Lipschitz continuous way. (The Lipschitz constant is uniformly bounded by a constant depending on λ_M because the choice of τ_1 ensures $|p(\cdot)| > \frac12$ on $[0,\tau_1]$ for each solution of $(**)$.) Hence, the compact set $\vartheta_{\widetilde{F}}(t,K)$ is N-dimensional $C^{1,1}$ submanifold of $\mathbb{R}^N$ with boundary for all $t \in [0,\tau]$ and, its radius of curvature has a uniform lower bound.

□

A.5.4 How to Guarantee Reversibility of Reachable Sets in Time

The Hamilton condition has led to a necessary condition on boundary points $x \in \partial\, \vartheta_{\widetilde{F}}(t,K)$ and their limiting normal cones in Proposition A.32 (on page 457). If each set $\vartheta_{\widetilde{F}}(t,K)$ $(0 \le t \le T)$ has positive reach $\ge \rho$, then standard hypothesis $(\widetilde{\mathscr{H}})$ turns adjoint arcs into sufficient conditions and, we conclude that the evolution of reachable sets is reversible with respect to time — in the following sense:

Proposition 36. *Suppose standard hypothesis* $(\widetilde{\mathscr{H}})$ *for* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. *Assume for* $K_0 \in \mathscr{K}(\mathbb{R}^N)$ *and* $\rho > 0$ *that every compact reachable set* $K_t := \vartheta_{\widetilde{F}}(t,K_0)$ $(0 \le t \le T)$ *has positive reach* $\ge \rho$ *(in the sense of Definition A.26).*
Then for every $0 \le s \le t < T$, $\quad K_s \;=\; \mathbb{R}^N \setminus \vartheta_{-\widetilde{F}(t-\cdot,\cdot)}(t-s,\, \mathbb{R}^N \setminus K_t)$.

Remark 37. 1. $\mathscr{K}(\mathbb{R}^N) \rightsquigarrow \mathbb{R}^N,\; K_0 \longmapsto \mathbb{R}^N \setminus \vartheta_{-\widetilde{F}(t-\cdot,\cdot)}(t,\, \mathbb{R}^N \setminus \vartheta_{\widetilde{F}}(t,K_0))$ generalizes the morphological operation of closing (of sets in $\mathscr{K}(\mathbb{R}^N)$) that was introduced by Minkowski and is usually defined as

$$\mathscr{P}(X) \rightsquigarrow X, \qquad K \longmapsto (K - tB) \ominus (-tB) \overset{\text{Def.}}{=} \{y \in X \,|\, y - tB \subset K - tB\}$$

for a vector space X and fixed $B \subset X$, $t > 0$ (see e.g. [10, Definition 3.3.1]).

2. In [25], viscosity solutions to the Hamilton-Jacobi-Bellman equation $\partial_t u + H(t,x,Du) = 0$ are investigated and in a word, the continuous differentiability of u is concluded from the reversibility in time:
If $u \in C^0([0,T] \times \mathbb{R}^N, \mathbb{R})$ is a viscosity solution of $\partial_t u + H(t,\cdot,Du) = 0$
and $v(t,x) := u(T-t,x)$ is a viscosity solution of $\partial_t v - H(T-t,\cdot,Dv) = 0$
then adequate assumptions about H ensure $u \in C^1(]0,T[\times \mathbb{R}^N)$.
Referring to the relation between reachable sets and level sets of viscosity solutions, we draw an inverse conclusion since we assume smoothness and obtain reversibility in time.

3. The reversibility in time (in the sense of Proposition A.36) can also be regarded as recovering the initial data. Further results about this problem have already been published by Rzeżuchowski in [164, 165], for example, but they usually assume other conditions. Either the initial set consists of only one point or the Hamiltonian function $\mathscr{H}_F$ is of class C^2.

In Proposition 36, we even suppose a uniform radius ρ of positive reach for $K_t \overset{\text{Def.}}{=} \vartheta_{\widetilde{F}}(t,K_0)$. The essential advantage for the proof is the relation between the boundaries of $K_t \subset \mathbb{R}^N$ and Graph $(t \longmapsto K_t) \subset \mathbb{R} \times \mathbb{R}^N$ stated in the next lemma:

Lemma 38. *Suppose for* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $K \in \mathscr{K}(\mathbb{R}^N)$ *and* $\rho > 0$ *that the map* $[0,T] \rightsquigarrow \mathbb{R}^N$, $t \mapsto \vartheta_{\widetilde{F}}(t,K)$ *is* λ*-Lipschitz continuous (with respect to* $d\!l$*) and each set* $\vartheta_{\widetilde{F}}(t,K)$ $(0 \le t \le T)$ *has positive reach of radius* ρ*.*

Then the topological boundary of Graph $\vartheta_{\widetilde{F}}(\cdot,K)|_{[0,T]}$ *in* $\mathbb{R} \times \mathbb{R}^N$ *is*

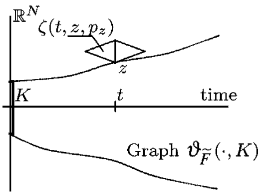
$$\big(\{0\} \times K\big) \;\cup \bigcup_{0<t<T} \big(\{t\} \times \partial\,\vartheta_{\widetilde{F}}(t,K)\big) \;\cup\; \big(\{T\} \times \vartheta_{\widetilde{F}}(T,K)\big).$$

Proof (of Lemma 38). The inclusion

$$\big(\{0\} \times K\big) \;\cup \bigcup_{0<t<T} \big(\{t\} \times \partial\,\vartheta_{\widetilde{F}}(t,K)\big) \;\cup\; \big(\{T\} \times \vartheta_{\widetilde{F}}(T,K)\big) \;\subset\; \partial\,\text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K)$$

is obvious. Due to the Lipschitz continuity of $\vartheta_{\widetilde{F}}(\cdot,K)$, we only have to show

$$\partial\,\text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K) \;\cap\; (]0,T[\, \times \mathbb{R}^N) \;\subset \bigcup_{0<t<T} \big(\{t\} \times \partial\,\vartheta_{\widetilde{F}}(t,K)\big).$$



Every $z \in \partial\,\vartheta_{\widetilde{F}}(t,K)$ $(0 \le t \le T)$ and any unit vector $p_z \in N^P_{\vartheta_{\widetilde{F}}(t,K)}(z) = N_{\vartheta_{\widetilde{F}}(t,K)}(z)$ satisfy

$$\overset{\circ}{\mathbb{B}}_\rho\,(z + \rho\, p_z) \cap \vartheta_{\widetilde{F}}(t,K) \;=\; \emptyset$$

and thus,

$$\big(\{t\} \times \overset{\circ}{\mathbb{B}}_\rho\,(z + \rho\, p_z)\big) \;\cap\; \text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K) \;=\; \emptyset.$$

The λ-Lipschitz continuity of $\vartheta_{\widetilde{F}}(\cdot,K)$ implies

$$\zeta(t,z,p_z) \cap \text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K) \;=\; \emptyset$$

for the open set $\zeta(t,z,p_z) := \big\{ (s,y) \in \mathbb{R}^{1+N} \,\big|\, |z + \rho\, p_z - y| < \rho - \lambda\,|s-t| \big\}$.

Now choose $(t,x) \in \partial\,\text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K)$ with $0 < t < T$ arbitrarily. The continuity of $\vartheta_{\widetilde{F}}(\cdot,K)$ guarantees that Graph $\vartheta_{\widetilde{F}}(\cdot,K)$ is closed and thus, it contains (t,x). Moreover there are sequences $(t_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ in $]0,T[$, $\mathbb{R}^N$ respectively with

$$\begin{aligned} &(t_n,x_n) \;\notin\; \text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K) && \text{for every } n \in \mathbb{N}, \\ &(t_n,x_n) \;\longrightarrow\; (t,x) && \text{for } n \longrightarrow \infty. \end{aligned}$$

For each $n \in \mathbb{N}$, let z_n be an element of the projection $\Pi_{\vartheta_{\widetilde{F}}(t_n,K)}(x_n) \subset \partial\,\vartheta_{\widetilde{F}}(t_n,K)$.

Then, $\quad 0 < |x_n - z_n| \;=\; \text{dist}(x_n,\, \vartheta_{\widetilde{F}}(t_n,K)) \;\le\; |x_n - x| + \text{dist}(x,\, \vartheta_{\widetilde{F}}(t_n,K)) \;\longrightarrow\; 0$

and $\quad p_n := \frac{x_n - z_n}{|x_n - z_n|} \;\in\; N^P_{\vartheta_{\widetilde{F}}(t_n,K)}(z_n) \cap \partial\mathbb{B}_1.$

As mentioned before, we obtain $\zeta(t_n,z_n,p_n) \cap \text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K) \;=\; \emptyset$ for each $n \in \mathbb{N}$. Adequate subsequences (again denoted by) $(t_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$ lead to the additional convergence $p_n \longrightarrow p \in \partial\mathbb{B}_1$ $(n \longrightarrow \infty)$. Finally,

$$\zeta(t,x,p) \cap \text{Graph}\ \vartheta_{\widetilde{F}}(\cdot,K) \;=\; \emptyset.$$

In particular, $\overset{\circ}{\mathbb{B}}_\rho\,(x + \rho\, p) \cap \vartheta_{\widetilde{F}}(t,K) \;=\; \emptyset$ implies $x \in \partial\,\vartheta_{\widetilde{F}}(t,K)$.

$\square$

Proof (of Proposition A.36). $\vartheta_{\widetilde{F}}(s, K_0) \subset \mathbb{R}^N \setminus \vartheta_{-\widetilde{F}(t-\cdot,\cdot)}(t-s, \mathbb{R}^N \setminus K_t)$ is an easy indirect consequence of definitions since it is equivalent to

$$\vartheta_{\widetilde{F}}(s, K_0) \cap \vartheta_{-\widetilde{F}(t-\cdot,\cdot)}(t-s, \mathbb{R}^N \setminus K_t) = \emptyset.$$

For proving the inverse inclusion indirectly at time $s = 0$ (w.l.o.g.), we assume the existence of $t \in [0, T[$ and $y_0 \in \mathbb{R}^N$ with $y_0 \notin K_0 \cup \vartheta_{-\widetilde{F}(t-\cdot,\cdot)}(t, \mathbb{R}^N \setminus K_t)$.
As an immediate consequence of $y_0 \notin \vartheta_{-\widetilde{F}(t-\cdot,\cdot)}(t, \mathbb{R}^N \setminus K_t)$, the reachable set $\vartheta_{\widetilde{F}}(t, y_0)$ is contained in $K_t \overset{\text{Def.}}{=} \vartheta_{\widetilde{F}}(t, K_0)$. Now set

$$\tau := \inf \big\{ s \in [0,t] \;\big|\; \vartheta_{\widetilde{F}}(s, y_0) \subset \vartheta_{\widetilde{F}}(s, K_0) \big\}.$$

In particular, $\tau > 0$ due to $y_0 \notin K_0$.
and $\vartheta_{\widetilde{F}}(\tau, y_0) \subset \vartheta_{\widetilde{F}}(\tau, K_0)$ due to the continuity of the reachable sets.
There are sequences $\tau_n \nearrow \tau$ and $(x_n(\cdot))_{n \in \mathbb{N}}$ in $W^{1,1}([0,T], \mathbb{R}^N)$ satisfying

$$x_n'(\cdot) \in \widetilde{F}(\cdot, x_n(\cdot)) \quad \mathscr{L}^1\text{-a.e.}, \qquad x_n(0) = y_0, \qquad x_n(\tau_n) \notin \vartheta_{\widetilde{F}}(\tau_n, K_0).$$

Standard hypothesis $(\widetilde{\mathscr{H}})$ and the compactness of solutions (as formulated in [180, Theorem 2.5.3]) lead to subsequences (again denoted by) $(\tau_n)_{n \in \mathbb{N}}$, $(x_n(\cdot))_{n \in \mathbb{N}}$ and a solution $x(\cdot) \in W^{1,1}([0,T], \mathbb{R}^N)$ of $x'(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ ($\mathscr{L}^1$-almost everywhere) with

$$x_n(\cdot) \longrightarrow x(\cdot) \text{ uniformly in } [0,T], \qquad x_n'(\cdot) \longrightarrow x'(\cdot) \text{ weakly in } L^1([0,T], \mathbb{R}^N).$$

In particular, $(\tau, x(\tau))$ has to be in the boundary of Graph $\vartheta_{\widetilde{F}}(\cdot, K_0)$. Lemma A.38 and $0 < \tau \le t < T$ ensure $x_\tau := x(\tau) \in \partial K_\tau \overset{\text{Def.}}{=} \partial\, \vartheta_{\widetilde{F}}(\tau, K_0)$.

Moreover, $K_\tau \overset{\text{Def.}}{=} \vartheta_{\widetilde{F}}(\tau, K_0)$ is supposed to have positive reach. Its limiting and proximal normal cone coincide at each boundary point due to Corollary A.29. Thus,

$$\emptyset \neq N_{\vartheta_{\widetilde{F}}(\tau, K_0)}(x_\tau) = N^P_{\vartheta_{\widetilde{F}}(\tau, K_0)}(x_\tau) \subset N^P_{\vartheta_{\widetilde{F}}(\tau, y_0)}(x_\tau).$$

For every unit normal vector $\nu \in N_{\vartheta_{\widetilde{F}}(\tau, K_0)}(x_\tau)$, Proposition A.32 provides a solution $z(\cdot) \in W^{1,1}([0,\tau], \mathbb{R}^N)$ and its adjoint arc $q(\cdot) \in W^{1,1}([0,\tau], \mathbb{R}^N)$ satisfying the corresponding Hamiltonian system and $z(0) \in K_0$, $z(\tau) = x_\tau$, $q(\tau) = \nu$.
The same Cauchy problem is solved by $x(\cdot)$ and its adjoint arc as well. Standard hypothesis $(\widetilde{\mathscr{H}})$ implies the uniqueness of solutions and, its consequence $z(0) = x(0) = y_0 \notin K_0$ leads to a contradiction. □

A.5.5 *How to Make Points Evolve into Convex Sets of Positive Erosion*

Our aim consists in sufficient assumptions for the interior ball condition on $\vartheta_F(t, K)$ — without any regularity assumptions about the initial set $K \in \mathscr{K}(\mathbb{R}^N)$. In particular, we focus on K consisting just of a single point. For this purpose, we are willing to tolerate stronger assumptions about the set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ than standard hypothesis $(\widetilde{\mathscr{H}})$ (specified in Definition A.33 on page 458).

Definition 39. For any $\rho > 0$, a set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ satisfies the so-called *standard hypothesis* $(\widetilde{\mathscr{H}_\circ^\rho})$ if it has the following properties:

1. $\widetilde{F}$ is measurable and, all its values are nonempty convex compact subsets of positive erosion of radius ρ,
2. for every $t \in [0,T]$, $\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot) \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,
3. for every $R > 1$, there exists $\lambda_R(\cdot) \in L^1([0,T])$ such that the derivative of $\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)$ restricted to $\mathbb{B}_R \times (\mathbb{B}_R \setminus \overset{\circ}{\mathbb{B}}_{\frac{1}{R}})$ is $\lambda_R(t)$-Lipschitz continuous for Lebesgue-almost every $t \in [0,T]$,
4. there is $k_{\widetilde{F}} \in L^1([0,T])$ such that for a.e. $t \in [0,T]$ and all $x, p \in \mathbb{R}^N$ $(|p| \geq 1)$,
$$\left\| \partial_{(x,p)} \mathscr{H}_{\widetilde{F}}(t,x,p) \right\|_{\mathrm{Lin}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})} \leq k_{\widetilde{F}}(t) \cdot (1 + |x| + |p|).$$

Remark 40. Standard hypothesis $(\widetilde{\mathscr{H}_\circ^\rho})$ differs from its counterpart $(\widetilde{\mathscr{H}})$ in two respects: The values of $\widetilde{F}$ have uniform positive erosion (additionally) and, its Hamiltonian $\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)$ is even twice continuously differentiable in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$. This second restriction has the advantage that we can apply the tools of matrix Riccati equation (mentioned in subsequent Lemmas A.43 and A.44).

Proposition 41. *In addition to standard hypothesis $(\widetilde{\mathscr{H}_\circ^\rho})$, assume for the set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ that some $\lambda(\cdot) \in L^1([0,T])$ satisfies*
$$\begin{aligned} \|\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)\|_{C^{1,1}(\mathbb{R}^N \times \partial\mathbb{B}_1)} &\overset{\text{Def.}}{=} \|\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} + \mathrm{Lip}\, \partial \mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)|_{\mathbb{R}^N \times \partial\mathbb{B}_1} \\ &< \lambda(t) \end{aligned}$$
at $\mathscr{L}^1$-almost every time $t \in [0,T]$. Choose $K \in \mathscr{K}(\mathbb{R}^N)$ arbitrarily.

Then there exist $\sigma > 0$ and a time $\widehat{\tau} \in \,]0,T]$ (depending only on $\|\lambda\|_{L^1}, \rho, K$) such that the reachable set $\vartheta_{\widetilde{F}}(t,x_0)$ is convex and has positive erosion of radius σt for any $t \in \,]0,\widehat{\tau}[$, $x_0 \in K$.

As a direct consequence, the reachable set $\vartheta_{\widetilde{F}}(t,K_1)$ is the closed (σt)-neighborhood of a compact set for all $t \in \,]0,\widehat{\tau}[$ and each nonempty compact subset $K_1 \subset K$.

The proof of this proposition uses matrix Riccati equations for Hamiltonian systems, but these tools of subsequent Lemma A.43 consider initial values induced by a Lipschitz function ψ. First we specify how to exchange the two components $(x(\cdot), p(\cdot))$ (of a solution and its adjoint arc) for preserving the Hamiltonian structure of their differential equations:

Lemma 42. *Assume the Hamiltonian system for $x(\cdot),\, p(\cdot) \in W^{1,1}([0,T], \mathbb{R}^N)$*
$$\begin{cases} x'(t) = \frac{\partial}{\partial p} H_1(t, x(t), p(t)) \\ p'(t) = -\frac{\partial}{\partial x} H_1(t, x(t), p(t)) \end{cases} \quad \text{a.e. in } [0,T]$$
with sufficiently smooth $H_1 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$. Moreover set
$$y(t) := -p(t), \qquad q(t) := x(t), \qquad H_2(t,\xi,\zeta) := H_1(t,\zeta,-\xi).$$

Then the absolutely continuous functions $(y(\cdot), q(\cdot))$ *satisfy the Hamiltonian system*

$$\begin{cases} y'(t) = \frac{\partial}{\partial q} H_2(t, y(t), q(t)) \\ q'(t) = -\frac{\partial}{\partial y} H_2(t, y(t), q(t)) \end{cases} \qquad a.e. \text{ in } [0,T].$$

□

Lemma 43.
In addition to the assumptions (2.)–(4.) *of Lemma A.35* (*on page 458*)*, suppose for* $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *and the Hamiltonian system*

$$\begin{cases} y'(t) = \frac{\partial}{\partial q} H(t, y(t), q(t)), & y(0) = y_0 \\ q'(t) = -\frac{\partial}{\partial y} H(t, y(t), q(t)), & q(0) = \psi(y_0) \end{cases} \qquad (*)$$

1'. $H(t,\cdot,\cdot)$ *is* twice *continuously differentiable for every* $t \in [0,T]$.

Then for every initial set $K \in \mathscr{K}(\mathbb{R}^N)$, *the following statements are equivalent:*

(i) *For all* $t \in [0,T]$, $M_t^{\mapsto}(K) := \{ (y(t), q(t)) \mid (y(\cdot), q(\cdot)) \text{ solves } (*),\ y_0 \in K \}$ *is the graph of a locally Lipschitz continuous function,*

(ii) *For any solution* $(y(\cdot), q(\cdot)) : [0,T] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ *to initial value problem* $(*)$ *and each cluster point* $Q_0 \in \operatorname{Limsup}_{z \to y_0} \{\nabla \psi(z)\} \subset \mathbb{R}^{N \times N}$, *the following matrix Riccati equation has a solution* $Q(\cdot)$ *on* $[0,T]$

$$\begin{cases} \partial_t Q + \frac{\partial^2 H}{\partial p \, \partial x}(t, y(t), q(t))\, Q + Q\, \frac{\partial^2 H}{\partial x \, \partial p}(t, y(t), q(t)) \\ \qquad + Q\, \frac{\partial^2 H}{\partial p^2}(t, y(t), q(t))\, Q + \frac{\partial^2 H}{\partial x^2}(t, y(t), q(t)) = 0, \\ \qquad\qquad Q(0) = Q_0. \end{cases}$$

If one of these equivalent properties is satisfied and if ψ *is* (*continuously*) *differentiable, then* $M_t^{\mapsto}(K)$ *is even the graph of a* (*continuously*) *differentiable function.*

Proof is given in [80, Theorem 5.3], for the same Hamiltonian system but with $y(T) = y_T$, $q(T) = q_T$ given. Hence, this lemma is a direct consequence considering $-H(T - \cdot, \cdot, \cdot)$ and $(y(T - \cdot), q(T - \cdot))$. □

For preventing singularities of $Q(\cdot)$, the following comparison principle provides a bridge to a *scalar* Riccati equation.

Lemma 44 (Comparison theorem for the matrix Riccati equation, [163, Th.2]**).**
Let $A_j, B_j, C_j : [0,T[\longrightarrow \mathbb{R}^{N \times N}$ $(j = 0,1,2)$ *be bounded continuous matrix-valued functions such that each* $M_j(t) := \begin{pmatrix} A_j(t) & B_j(t) \\ B_j(t)^T & C_j(t) \end{pmatrix}$ *is symmetric.*

Assume that $U_0, U_2 : [0,T[\longrightarrow \mathbb{R}^{N \times N}$ *are solutions to the matrix Riccati equation*

$$\frac{d}{dt} U_j = A_j + B_j U_j + U_j B_j^T + U_j C_j U_j$$

with $M_2(\cdot) \geq M_0(\cdot)$ (*i.e.* $M_2(t) - M_0(t)$ *is positive semi-definite for every* t).

For symmetric $U_1(0) \in \mathbb{R}^{N \times N}$ *with* $U_2(0) \geq U_1(0) \geq U_0(0)$, $M_2(\cdot) \geq M_1(\cdot) \geq M_0(\cdot)$, *given, there exists a solution* $U_1 : [0,T[\longrightarrow \mathbb{R}^{N \times N}$ *to the Riccati equation with matrix* $M_1(\cdot)$. *Moreover,* $U_2(t) \geq U_1(t) \geq U_0(t)$ *for all* $t \in [0,T[$.

Proof (of Proposition A.41).
The integrable bound of $t \mapsto \|\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)\|_{C^{1,1}(\mathbb{R}^N \times \partial\mathbb{B}_1)}$ and Gronwall's Lemma lead to a radius $R = R(\|\lambda\|_{L^1}, K) > 1$ and a time $\widehat{T} = \widehat{T}(\|\lambda\|_{L^1}, K) \in \,]0,T]$ such that

1. $\vartheta_{\widetilde{F}}(t,K) \subset \mathbb{B}_R$ for all $t \in [0,T]$,
2. for every solution $x(\cdot)$ of $x'(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ starting in K and each adjoint $p(\cdot)$ with $\frac{1}{2} \le |p(0)| \le 2$ fulfills $\frac{1}{R} < |p(\cdot)| < R$, $|p(\cdot) - p(0)| < \frac{1}{4R}$ on $[0,\widehat{T}]$.

A smooth cut-off function provides a map $H_1 : [0,\widehat{T}] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ that fulfills the assumptions of Lemma A.43 and

$$H_1 \;=\; \mathscr{H}_{\widetilde{F}} \qquad \text{in } [0,\widehat{T}] \times \mathbb{R}^N \times (\mathbb{R}^N \setminus \mathbb{B}_{\frac{1}{2R}}).$$

Using the transformation of the preceding Lemma A.42, the auxiliary function

$$H_2 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}, \qquad (t,\xi,\zeta) \longmapsto H_1(t,\zeta,-\xi)$$

is still holding the conditions of Lemma A.43. As a consequence, we obtain for any initial point $x_0 \in K$ and time $\tau \in \,]0,\widehat{T}]$ that the following statements are equivalent:

(i) For all $t \in [0,\tau]$, the set M_t^1 of all points $(p(t), x(t))$ with solutions $(x(\cdot), p(\cdot)) \in W^{1,1}([0,t], \mathbb{R}^N \times \mathbb{R}^N)$ of

$$\begin{cases} x'(s) = \frac{\partial}{\partial p} H_1(s, x(s), p(s)), & x(0) = x_0 \\ p'(s) = -\frac{\partial}{\partial x} H_1(s, x(s), p(s)), & p(0) \in \mathbb{B}_2 \setminus \overset{\circ}{\mathbb{B}}_{\frac{1}{2}} \end{cases}$$

is the graph of a continuously differentiable function f_t.

(ii) For all $t \in [0,\tau]$, the set M_t^2 of all points $(y(t), q(t))$ with solutions $(y(\cdot), q(\cdot)) \in W^{1,1}([0,t], \mathbb{R}^N \times \mathbb{R}^N)$ of

$$\begin{cases} y'(s) = \frac{\partial}{\partial q} H_2(s, y(s), q(s)), & y(0) \in \mathbb{B}_2 \setminus \overset{\circ}{\mathbb{B}}_{\frac{1}{2}} \\ q'(s) = -\frac{\partial}{\partial y} H_2(s, y(s), q(s)), & q(0) = x_0 \end{cases}$$

is the graph of a C^1 function g_t (and $g_t(\xi) = f_t(-\xi)$).

(iii) For any solution $(y,q) : [0,t] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ to the initial value problem (ii) $(t \le \tau)$, there is a solution $Q : [0,t] \longrightarrow \mathbb{R}^{N \times N}$ to the Riccati equation

$$\begin{cases} Q' + \frac{\partial^2 H_2}{\partial q\, \partial y}(s, y(s), q(s))\, Q + Q\, \frac{\partial^2 H_2}{\partial y\, \partial q}(s, y(s), q(s)) \\ \quad + Q\, \frac{\partial^2 H_2}{\partial q^2}(t, y(s), q(s))\, Q + \frac{\partial^2 H_2}{\partial y^2}(s, y(s), q(s)) = 0, \\ \hfill Q(0) = 0. \end{cases}$$

(iv) For any solution $(x,p) : [0,t] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ to the initial value problem (i) $(t \le \tau)$, there is a solution $Q : [0,t] \longrightarrow \mathbb{R}^{N \times N}$ to the Riccati equation

$$\begin{cases} Q' - \frac{\partial^2 H_1}{\partial x\, \partial p}(s, x(s), p(s))\, Q - Q\, \frac{\partial^2 H_1}{\partial p\, \partial x}(s, x(s), p(s)) \\ \quad + Q\, \frac{\partial^2 H_1}{\partial x^2}(s, x(s), p(s))\, Q + \frac{\partial^2 H_1}{\partial p^2}(s, x(s), p(s)) = 0, \\ \hfill Q(0) = 0. \end{cases}$$

Now we give a criterion for the choice of $\widehat{\tau} \in \,]0, \widehat{T}]$. Setting

$$\mu(t) \;:=\; \sup_{\substack{|x| \,\le\, R \\ \frac{1}{R} \,\le\, |p| \,\le\, R}} \left\| \begin{pmatrix} \frac{\partial^2}{\partial p^2} \; \mathscr{H}_{\widetilde{F}}(t,x,p) & - \; \frac{\partial^2}{\partial x \, \partial p} \; \mathscr{H}_{\widetilde{F}}(t,x,p) \\ - \; \frac{\partial^2}{\partial p \, \partial x} \; \mathscr{H}_{\widetilde{F}}(t,x,p) & \frac{\partial^2}{\partial x^2} \; \mathscr{H}_{\widetilde{F}}(t,x,p) \end{pmatrix} \right\|_{\mathrm{Lin}(\mathbb{R}^{2N},\mathbb{R}^{2N})}$$

the comparison theorem for matrix Riccati equations (Lemma A.44 extended to integrable coefficients via Lusin's Theorem and approximation, see also [80, § 5.2]) guarantees existence and uniqueness of such a solution $Q \in W^{1,1}([0,t],\, \mathbb{R}^{N \times N})$ for every $t < \min\{T,\, \frac{\pi}{2\,\|\mu\|_{L^1}}\}$. Indeed, for $a(\cdot) = \pm\mu(\cdot) \in L^1([0,T])$, the scalar Riccati equation

$$\tfrac{d}{dt}\, u(t) \;=\; a(t) + a(t)\, u(t)^2, \qquad u(0) = 0$$

has the solution $u(t) \;=\; \tan\bigl(\int_0^t a(s)\; ds\bigr)$ in $[0, \frac{\pi}{2\,\|a\|_{L^1}}[$. Furthermore we obtain the upper bound $\|Q(t)\| \;\le\; \tan\; \|\, \mu|_{[0,t]}\, \|_{L_1}$.

All values of $\widetilde{F}$ are compact convex sets with positive erosion of radius ρ due to standard hypothesis $(\widetilde{\mathscr{H}_\circ^\rho})$. It implies a constant $\widehat{\sigma} = \widehat{\sigma}(\rho, K, R) > 0$ with

$$\xi \cdot \tfrac{\partial^2}{\partial p^2}\; \mathscr{H}_{\widetilde{F}}(t,x,p)\;\; \xi \;\;\ge\;\; 9\, \widehat{\sigma}\, \left|\, \xi \;-\; \tfrac{\xi \cdot p}{|p|^2}\; p \,\right|^2$$

for all $t \in [0,T]$, $|x| \le R$, $\frac{1}{R} \le |p| \le R$, ξ. Using the matrix abbreviation

$$\begin{aligned} D(t,x,p) \;:=\; & -\; \tfrac{\partial^2 \mathscr{H}_{\widetilde{F}}}{\partial x\, \partial p}\,(t,x,p)\;\, Q(t) \;\;-\;\; Q(t)\;\, \tfrac{\partial^2 \mathscr{H}_{\widetilde{F}}}{\partial p\, \partial x}\,(t,x,p) \\ & +\; Q(t)\;\, \tfrac{\partial^2 \mathscr{H}_{\widetilde{F}}}{\partial x^2}\,(t,x,p)\;\, Q(t), \end{aligned}$$

choose $\widehat{\tau} = \widehat{\tau}(\lambda, \rho, K) > 0$ small enough such that

$$\begin{cases} \qquad\qquad \widehat{\tau} \;<\; \min\{\widehat{T},\; \frac{\pi}{2\,\|\mu\|_{L^1}}\}, \\ \displaystyle\int_0^{\widehat{\tau}} \lambda(t)\;\, dt \;<\; 1, \\ \|D(t,x,p)\| \;\le\; \widehat{\sigma} \qquad \text{for every } t \in [0,\widehat{\tau}],\; |x| \le R,\; \frac{1}{R} \le |p| \le R. \end{cases}$$

As a next step, we conclude that the solution $Q(t)$ of (iv) (restricted to $[0,\widehat{\tau}]$) satisfies $Q(t) \le -\,\widehat{\sigma}\, t \cdot \mathbb{I}\mathrm{d}$ in the $(N-1)$-dimensional subspace of $\mathbb{R}^N$ perpendicular to $p(t)$. Indeed, let $(x(\cdot),\, p(\cdot)) \in W^{1,1}([0,\widehat{\tau}],\, \mathbb{R}^N \times \mathbb{R}^N)$ be a solution to the Hamiltonian system (i) and choose an arbitrary unit vector $\xi \in \mathbb{R}^N$ with $|\xi \cdot p(0)| < \frac{1}{4R}$. Then the auxiliary function

$$\varphi:\; [0,\widehat{\tau}] \;\longrightarrow\; \mathbb{R}^N,\;\; t \;\longmapsto\;\; \xi \cdot Q(t)\, \xi \;\;+\;\; \widehat{\sigma}\, t\, \left|\, \xi \;-\; \tfrac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \,\right|^2$$

satisfies $\varphi(0) = 0$ and is absolutely continuous with $\varphi(\cdot) \le 0$. Indeed,

$$\begin{aligned} \varphi'(t) &= \xi \cdot Q'(t)\; \xi + \widehat{\sigma}\, \left|\, \xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right|^2 - 2\, \widehat{\sigma}\, t\, \left(\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\; p(t)\right) \cdot\; \tfrac{d}{dt}\left(\tfrac{\xi \cdot p(t)}{|p(t)|^2}\; p(t)\right) \\ &= \xi \cdot Q'(t)\; \xi + \widehat{\sigma}\, \left|\, \xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right|^2 - 2\, \widehat{\sigma}\, t\, \left(\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\; p(t)\right) \cdot \quad \tfrac{\xi \cdot p(t)}{|p(t)|^2} \quad p'(t) \end{aligned}$$

because $\xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t)$ is perpendicular to $p(t)$.

Now $|p(t)-p(0)| < \frac{1}{4R}$, $\frac{1}{R} \le |p(t)| \le R$ and $|\xi \cdot p(0)| < \frac{1}{4R}$ imply $\left|\frac{\xi \cdot p(t)}{|p(t)|}\right| < \frac{1}{2}$ and $\frac{1}{2}\,|\xi| = 1 - \frac{1}{2} \le \left|\xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right| \le 1 + \frac{1}{2}$. Thus,

$$\begin{aligned}
\varphi'(t) &\le (-9+4+1)\,\widehat{\sigma}\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right|^2 + 2\,\widehat{\sigma}\,t\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right| \tfrac{|\xi|\;|p(t)|}{|p(t)|^2}\,|p'(t)| \\
&\le -4\,\widehat{\sigma}\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right|^2 + 2\,\widehat{\sigma}\,t\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right| \quad \lambda(t) \\
&\le 2\,\widehat{\sigma}\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right| \cdot \Big(-2\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right| + \lambda(t)\,t\Big) \\
&\le 2\,\widehat{\sigma}\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right| \cdot \Big(-2\,\Big(1 - \left|\tfrac{\xi \cdot p(t)}{|p(t)|}\right|\Big) + \lambda(t)\,t\Big) \\
&\le 2\,\widehat{\sigma}\,\left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\,p(t)\right| \cdot \Big(-2\,\Big(1 - \tfrac{1}{2}\Big) + \lambda(t)\,\widehat{\tau}\Big) \\
&\le \widehat{\sigma} \cdot 3 \cdot \big(-1 + \lambda(t)\,\widehat{\tau}\big).
\end{aligned}$$

Now we obtain $\varphi(t) \le 0$ for all $t \in [0, \widehat{\tau}]$ and as a consequence, $Q(t) \le -\widehat{\sigma}\,t \cdot \mathbb{I}\mathrm{d}$ is fulfilled in the subspace of $\mathbb{R}^N$ perpendicular to $p(t)$.

Finally we need the geometric interpretation for concluding convexity and positive erosion of $\vartheta_{\widetilde{F}}(t,x_0)$ (of radius $\widehat{\sigma}\,t$) for each $t \in \,]0,\widehat{\tau}[$ and $x_0 \in K$.
As mentioned before, the existence of the solution $Q(\cdot)$ on $[0,\widehat{\tau}[$ implies for all $t \in [0,\widehat{\tau}[$ that the set M_t^1 is the graph of a C^1 function f_t. Moreover Proposition A.32 (on page 457) guarantees

$$\begin{aligned}
\text{Graph}\, N_{\vartheta_{\widetilde{F}}(t,x_0)} &\subset \big\{(x(t),\, \lambda\, p(t)) \,\big|\, (x(\cdot),p(\cdot)) \text{ solves } (i),\ \lambda \ge 0\big\} \\
&\overset{\text{Def.}}{=} \bigcup_{\lambda \ge 0} \text{Graph}\ (\lambda\, f_t^{-1}).
\end{aligned}$$

Now we obtain at every time $t \in \,]0,\widehat{\tau}[$ that each $p \in \mathbb{R}^N \setminus \{0\}$ belongs to the limiting normal cone of a unique boundary point $z \in \partial\,\vartheta_{\widetilde{F}}(t,x_0)$ and, $z = z(p)$ is continuously differentiable.
In particular, every supporting hyperplane of the closed convex hull $\overline{\text{co}}\ \vartheta_{\widetilde{F}}(t,x_0)$ may have at most one point in common with the compact reachable set $\vartheta_{\widetilde{F}}(t,x_0)$. Thus, $\overline{\text{co}}\ \vartheta_{\widetilde{F}}(t,x_0) \subset \mathbb{R}^N$ is even *strictly* convex and coincides with $\vartheta_{\widetilde{F}}(t,x_0)$ at each time $t \in \,]0,\widehat{\tau}[$. It is sufficient to consider the limiting normal cones of $\vartheta_{\widetilde{F}}(t,x_0)$ *locally* at every boundary point.

Well-known properties of variational equations (see e.g. [80]) and the uniqueness of solutions to the matrix Riccati equation (iv) imply that $-Q(s)$ is the derivative of the C^1 function f_s for $0 < s \le t < \widehat{\tau}$. Indeed, for each solution $(x(\cdot),p(\cdot))$ to the Hamiltonian system (i), set $(y(\cdot),q(\cdot)) := (-p(\cdot),x(\cdot))$ again and let $(U(\cdot),V(\cdot)) : [0,t] \longrightarrow \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N}$ denote the solution to the linearized system

$$\begin{cases}
U'(s) = \frac{\partial^2}{\partial y\,\partial q} H_2(s,\,y(s),\,q(s))\ U(s) + \frac{\partial^2}{\partial q^2} H_2(s,\,y(s),\,q(s))\ V(s), \\
V'(s) = -\frac{\partial^2}{\partial y^2} H_2(s,\,y(s),\,q(s))\ U(s) - \frac{\partial^2}{\partial q\,\partial y} H_2(s,\,y(s),\,q(s))\ V(s), \\
U(0) = \mathbb{I}\mathrm{d}_{\mathbb{R}^{N\times N}}, \qquad V(0) = 0.
\end{cases}$$

Then for any $s \in]0,t]$ and initial direction $u_0 \in \mathbb{R}^N \setminus \{0\}$, $(U(s)\,u_0,\ V(s)\,u_0)$ belongs to the contingent cone of $M_s^2 \subset \mathbb{R}^N \times \mathbb{R}^N$ at $(y(s), q(s))$ (due to the variational equations, see e.g. [80]).
Since M_s^2 is the graph of a continuously differentiable function g_s, we conclude that firstly, this cone $T_{M_s^2}(y(s), q(s))$ is a N-dimensional subspace of $\mathbb{R}^N \times \mathbb{R}^N$ and secondly, $|V(s)\,u_0| \leq \text{const} \cdot \lambda(s) \cdot |U(s)\,u_0|$ (due to Remark A.31 on page 457).
The latter property and the uniqueness of the linearized system ensure $U(s)\,u_0 \neq 0$ for all $u_0 \neq 0$ and thus, $U(s)$ is invertible. Comparing the dimensions leads to

$$T_{M_s^2}(y(s), q(s)) \;=\; (U(s),\, V(s))\ \mathbb{R}^N$$

and $V(s)\,U(s)^{-1}$ is the derivative of g_s at $y(s)$.
Hence, $-V(s)\,U(s)^{-1}$ is the derivative of $f_s = g_s(-\,\cdot)$ at $p(s) = -y(s)$.
Moreover it is easy to check that $V(s)\ U(s)^{-1}$ satisfies the matrix Riccati equation (iii) and thus, its uniqueness implies $V(s)\ U(s)^{-1} = Q(s)$ for $0 < s \leq t < \widehat{\tau}$.

Thus for every time $t \in]0, \widehat{\tau}[$, the derivative of f_t at $p(t)$ is bounded by $\widehat{\sigma}\,t$ from below in a $(N-1)$-dimensional subspace of $\mathbb{R}^N$.
Since $\vartheta_{\widetilde{F}}(t, x_0)$ is convex, it implies that $\vartheta_{\widetilde{F}}(t, x_0)$ has positive erosion of radius increasing (at least) linearly in time. □

A.5.6 Reachable Sets of Balls and Their Complements

In this section, we investigate the proximal radius of boundary points while sets are evolving along differential inclusions. Compact balls and their complements exemplify the key features for short times (as stated in subsequent Proposition A.46). They lead to the main results about proximal radii in both forward and backward time direction as a corollary.
The proofs are based on the Hamiltonian system and its regularity — in the same way as in § A.5.5.

Definition 45. For $\Lambda > 0$ fixed, the set $\mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$ consists of all set-valued maps $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ satisfying

1. $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has nonempty compact convex values,
2. $\mathscr{H}_F(x,p) := \sup\limits_{v \in F(x)} p \cdot v$ is twice continuously differentiable in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$,
3. $\|\mathscr{H}_F\|_{C^2(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} < \Lambda$.

Proposition 46. *Let F be any set-valued map of $\mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$ and $B := \mathbb{B}_r(x_0) \subset \mathbb{R}^N$ a compact ball of positive radius r.*
Then there exists a time $\tau = \tau(r, \Lambda) > 0$ such that for all times $t \in [0, \tau(r, \Lambda)[$,

1.) $\vartheta_F(t, B)$ is convex and has radius of curvature $\geq r - 9\,\Lambda\,(1+r)^2\,t$,
2.) $\vartheta_F(t, \mathbb{R}^N \setminus B)$ is concave and has radius of curvature $\geq r - 9\,\Lambda\,(1+r)^2\,t$.

Restricting ourselves to $0 < r \leq 2$, the time $\tau(r,\Lambda) > 0$ can be chosen as an increasing function of r. The claim of Proposition A.46 does not include, however, that $r - 9\,\Lambda\,(1+r)^2\,t \geq 0$ for all $t \in [0, \tau(r,\Lambda)[$ (because then it is not immediately clear how to choose $\tau(r,\Lambda) > 0$ as increasing with respect to all $r \in]0,2]$).

As an equivalent formulation of statement (1.), the convex set $\vartheta_F(t,B)$ has *positive erosion* of radius $\rho(t) \geq r - 9\,\Lambda\,(1+r)^2\,t$, i.e. there is some $K_t \subset \mathbb{R}^N$ with $\vartheta_F(t,B) = \mathbb{B}_{\rho(t)}(K_t)$.
Strictly speaking, statement (2.) is of more interest here: $\vartheta_F(t, \mathbb{R}^N \setminus B) \subset \mathbb{R}^N$ has *positive reach* $\geq \rho(t) \geq r - 9\,\Lambda\,(1+r)^2\,t$ (in the sense of Federer, see Def. A.26). Roughly speaking, the proofs of these two statements just differ in a sign and thus, both of them are mentioned here.

Applying Proposition A.46 to adequate proximal balls, the inclusion principle of reachable sets and Proposition A.32 (on page 457) have the immediate consequence:

Corollary 47. *For every map* $F \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$ *and radius* $r_0 \in]0,2]$, *there exists some* $\tau = \tau(r_0,\Lambda) > 0$ *such that for any* $K \in \mathscr{K}(\mathbb{R}^N)$, $r \in [r_0, 2]$ *and* $t \in [0,\tau[$,

1. *each* $x_1 \in \partial\,\vartheta_F(t,K)$ *and* $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ *with proximal radius* r *are linked to some* $x_0 \in \partial K$ *and* $\nu_0 \in N^P_K(x_0)$ *with proximal radius* $\geq r - 81\,\Lambda\,t$ *by a solution to* $x'(\cdot) \in F(x(\cdot))$ *and its adjoint arc, respectively.*
2. *each* $x_0 \in \partial K$ *and* $\nu_0 \in N^P_K(x_0)$ *with proximal radius* r *are linked to some* $x_1 \in \partial\,\vartheta_F(t,K)$ *and* $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ *with proximal radius* $\geq r - 81\,\Lambda\,t$ *by a solution to* $x'(\cdot) \in F(x(\cdot))$ *and its adjoint arc, respectively.* □

For describing the time-dependent limiting normals, we use adjoint arcs and benefit from the Hamiltonian system they are satisfying together with the solutions (as formulated in preceding Proposition A.32 on page 457).
In short, the graph of normal cones at time t, $\text{Graph}\ N_{\vartheta_F(t,K)}(\cdot)|_{\partial\,\vartheta_F(t,K)}$, can be traced back to the beginning by means of the Hamiltonian system with $\mathscr{H}_F$.

As in § A.5.5, we take the next order into consideration and, the matrix Riccati equation provides an analytical access to geometric properties like curvature. In particular, Lemma A.43 (on page 465) motivates the assumption that $\mathscr{H}_F$ is twice continuously differentiable in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$ for all maps $F \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$.

For preventing singularities of the matrix solution $Q(\cdot)$ to the Riccati equation, the comparison principle in Lemma A.44 (on page 465) provides a connection with solutions to a *scalar* Riccati equation again.

Proof (of Proposition A.46). Similarly to Proposition A.41 (on page 464), statement (1.) is based on applying Lemma A.43 (on page 465) to the boundary $K := \partial\,\mathbb{B}_r(0)$ and its exterior unit normals, i.e. $\psi(x) := \frac{x}{r}$, after assuming $B = \mathbb{B}_r(0)$ without loss of generality. Obviously, ψ can be extended to $\psi \in C^1(\mathbb{R}^N,\mathbb{R}^N)$.
(Statement (2.) of Proposition 46 is shown in the same way – just with inverse signs, i.e. $\widehat{\psi}(x) := -\frac{x}{r}$ instead. Hence, we do not formulate this part in detail.)

For every point $y_0 \in \partial\,\mathbb{B}_r$, there exist a solution $y(\cdot) \in C^1([0,\infty[,\mathbb{R}^N)$ and its adjoint $q(\cdot) \in C^1([0,\infty[,\mathbb{R}^N)$ satisfying

$$\begin{cases} y'(t) \;=\; \frac{\partial}{\partial q}\ \mathscr{H}_F(y(t),q(t)) \in F(y(t)), & y(0) \;=\; y_0, \\ q'(t) \;=\; -\frac{\partial}{\partial y}\ \mathscr{H}_F(y(t),q(t)), & q(0) \;=\; \psi(y_0) \end{cases} \qquad (*)$$

and, $F \in \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N,\mathbb{R}^N)$ implies the a priori bounds

$$\begin{array}{rcl} & |y(t)-y_0| & \le\ \Lambda\,t, \\ e^{-\Lambda t}\ \le & |q(t)| & \le\ e^{\Lambda t}. \end{array}$$

After restricting to the finite time interval $I_r = [0,t_r[$ (specified explicitly later), a simple cut-off function provides a twice continuously differentiable extension $\mathscr{H} : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ of $\mathscr{H}_F|_{\mathbb{R}^N\times(\mathbb{R}^N\setminus\mathbb{B}^{\circ}_{\exp(-\Lambda t_r)}(0))}$ and finally, Lemma A.43 can be applied to $\partial\mathbb{B}_r$, ψ and $\mathscr{H}_F$.
Furthermore $\mathscr{H}_F(x,p) \stackrel{\text{Def.}}{=} \sup_{v\in F(x)}\ p\cdot v$ is positively homogeneous with respect to p and thus, the second derivatives of $\mathscr{H}_F$ are bounded by $9\,\Lambda\,R^2$ on $\mathbb{R}^N \times (\mathbb{B}_R \setminus \overset{\circ}{\mathbb{B}}_{\frac{1}{R}})$ (according to Lemma 4.33 on page 366). Together with the preceding a priori bounds, we obtain

$$\big\|\,D^2\,\mathscr{H}_F(y(t),q(t))\,\big\|_{\mathrm{Lin}(\mathbb{R}^{2N},\mathbb{R}^{2N})}\ \le\ 9\,\Lambda\,e^{2\Lambda t}.$$

Let $Q(\cdot)$ denote the solution to the matrix Riccati equation

$$\begin{cases} \partial_t\,Q + \frac{\partial^2\mathscr{H}_F}{\partial p\,\partial x}(y(t),q(t))\ Q + Q\ \frac{\partial^2\mathscr{H}_F}{\partial x\,\partial p}(y(t),q(t)) \\ \qquad + Q\ \frac{\partial^2\mathscr{H}_F}{\partial p^2}(y(t),q(t))\ Q + \frac{\partial^2\mathscr{H}_F}{\partial x^2}(y(t),q(t)) = 0, \\ \qquad\qquad Q(0) \;=\; \nabla\psi(y_0) = \frac{1}{r}\cdot \mathbb{I}\mathrm{d}_{\mathbb{R}^N}. \end{cases}$$

Due to the comparison principle in Lemma A.44 (on page 465), $Q(\cdot)$ exists (at least) as long as the two scalar Riccati equations

$$\partial_t\,u_\pm \;=\; \pm 9\,\Lambda\,e^{2\Lambda t} \pm 9\,\Lambda\,e^{2\Lambda t}\,u_\pm^2, \qquad u_\pm(0) = \tfrac{1}{r}$$

have finite solutions and within this period, they fulfill

$$u_-(t)\cdot\mathbb{I}\mathrm{d}_{\mathbb{R}^N}\ \le\ Q(t)\ \le\ u_+(t)\cdot\mathbb{I}\mathrm{d}_{\mathbb{R}^N}.$$

In fact, we get the explicit solutions in $I_r := \big[0,\ \frac{1}{2\Lambda}\cdot\log\big(1+\frac{\pi}{9}-\frac{2}{9}\cdot\arctan\frac{1}{r}\big)\big[$, namely $\quad u_\pm(t) \;=\; \tan\big(\pm\,\frac{9}{2}\,(e^{2\Lambda t}-1) + \arctan\frac{1}{r}\big),$

Hence, $Q(t)$ is positive definite with eigenvalues $\geq u_-(t)$ at every time t of the (maybe smaller) interval $I'_r := I_r \cap [0, \frac{1}{2\Lambda} \cdot \log\big(1 + \frac{2}{9} \cdot \arctan \frac{1}{r}\big)\big[$.

Now we focus on the geometric interpretation of $Q(\cdot)$.
Due to Lemma A.43 (on page 465),

$$M_t^{\mapsto}(\partial\,\mathbb{B}_r) := \big\{ (y(t), q(t)) \,\big|\, (y(\cdot), q(\cdot)) \text{ solves system } (*),\ |y_0| = r \big\}$$

is graph of a continuously differentiable function and, $Q(t)$ is related to its derivative at $y(t)$ as we clarified in the proof of Proposition A.41 (on page 466 ff.). Furthermore the Hamilton condition of Proposition A.32 (on page 457) ensures

$$\text{Graph } N_{\vartheta_F(t,\mathbb{B}_r)}(\cdot) \subset \Big\{ (y(t), \lambda\, q(t)) \,\Big|\, (y(\cdot), q(\cdot)) \text{ solves } (*),\ |y_0| = r,\ \lambda \geq 0 \Big\}$$

and thus, the graph property of $M_t^{\mapsto}(\partial\,\mathbb{B}_r)$ implies that each $q(t)$ is a normal vector to the smooth reachable set $\vartheta_F(t, \mathbb{B}_r)$ at $y(t)$.
As $q(t) \neq 0$ might not have norm 1, the eigenvalues of $Q(t)$ are not always identical to the principal curvatures $(\kappa_j)_{j=1\ldots N}$ of $\vartheta_F(t, \mathbb{B}_r)$ at $y(t)$, but they provide bounds:

$$e^{-\Lambda t} \cdot u_-(t) \;\leq\; \kappa_j \;\leq\; e^{\Lambda t} \cdot u_+(t)$$

due to $e^{-\Lambda t} \leq |q(t)| \leq e^{\Lambda t}$. Thus, $\vartheta_F(t, \mathbb{B}_r)$ is convex for all times $t \in I'_r$ and, the *local* properties of principal curvatures have the *nonlocal* consequence that $\vartheta_F(t, \mathbb{B}_r) \subset \mathbb{R}^N$ has positive erosion of radius

$$\rho(t) \;\geq\; \frac{1}{e^{\Lambda t} \cdot u_+(t)} \;\geq\; r - 9\,\Lambda\,(1+r)^2\,t \qquad \text{for all } t \in I'_r.$$

Indeed, the linear estimate at the end is shown by means of the auxiliary function

$$t \;\longmapsto\; \frac{1}{e^{\Lambda t} \cdot u_+(t)} - r + 9\,\Lambda\,(1+r)^2\,t$$

that is 0 at $t = 0$, has positive derivative at $t = 0$ and is convex (due to nonnegative second derivative in I'_r).

The time $\tau(r, \Lambda) > 0$ is chosen as minimum of $\frac{1}{2\Lambda} \cdot \log\big(1 + \frac{\pi}{9} - \frac{2}{9} \cdot \arctan \frac{1}{r}\big)$, $\frac{1}{2\Lambda} \cdot \log\big(1 + \frac{2}{9} \cdot \arctan \frac{1}{r}\big)$. The linear estimate does not have to be positive in $[0, \tau(r, \Lambda)[$ though. $\square$

A.5.7 The (Uniform) Tusk Condition for Graphs of Reachable Sets

The so-called exterior tusk condition is an essential tool for verifying the boundary regularity of solutions to parabolic differential equations of second order. Indeed, its role is comparable to the exterior cone condition for elliptic differential equations of second order. Effros and Kazdan investigated it in connection with the heat equation in [75] and, Lieberman extended it to more general parabolic equations in [115].

Definition 48 ([114, § 3], [115]). A nonempty subset $M \subset \mathbb{R} \times \mathbb{R}^N$ is called *tusk* in $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N$ if there exist constants $R, \tau > 0$ and a point $x_1 \in \mathbb{R}^N$ with

$$M = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^N \;\Big|\; t_0 - \tau < t < t_0, \; \big|(x - x_0) - \sqrt{t_0 - t} \cdot x_1\big| < R\,\sqrt{t_0 - t}\Big\}.$$

A nonempty subset $\Omega \subset \mathbb{R} \times \mathbb{R}^N$ satisfies the so-called *exterior tusk condition* if for every point $(t,x) \subset \partial\Omega$ belonging to the parabolic boundary of Ω (i.e.

$$\Omega \cap \Big\{(s,y) \in \mathbb{R} \times \mathbb{R}^N \;\Big|\; |x - y| \le \varepsilon,\; t - \varepsilon < s < t\Big\} \neq \emptyset \qquad \text{for any } \varepsilon > 0),$$

there exists a tusk $M \subset \mathbb{R} \times \mathbb{R}^N$ in (t,x) with $\overline{M} \cap \overline{\Omega} = \{(t,x)\}$.

A nonempty subset $\Omega \subset \mathbb{R} \times \mathbb{R}^N$ is said to fulfill the *uniform exterior tusk condition* if it satisfies the exterior tusk conditions and if the scalar geometric parameters $R, \tau > 0$ of the tusks can be chosen independently of the respective points (t,x) of the parabolic boundary of Ω.

Now we focus on the exterior tusk condition for graphs of reachable sets.
In particular, its uniform version can be verified for parts of the complement if the differential inclusion makes every point evolve into convex sets with positive erosion of increasing radius for short times. Thus, Proposition A.41 (on page 464) provides sufficient conditions on the nonautonomous differential inclusion — independently of the compact initial set.

Proposition 49. *For $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ suppose standard hypothesis $(\widetilde{\mathscr{H}})$ with uniform linear growth of $\partial_{(x,p)} \mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)$ (i.e. $k_{\widetilde{F}} \in L^\infty([0,T])$ in Definition A.33) and the following property:*
For every set $\widetilde{K} \in \mathscr{K}([0,T] \times \mathbb{R}^N)$, there exist $\widehat{\tau} \in]0,T]$ and some nondecreasing $\sigma : [0, \widehat{\tau}] \longrightarrow [0, \infty[$ such that the reachable set $\vartheta_{\widetilde{F}(t_0 + \cdot, \cdot)}(s, x_0) \subset \mathbb{R}^N$ is convex and has positive erosion of radius $\sigma(s) > 0$ for any $s \in]0, \widehat{\tau}]$, $(t_0, x_0) \in \widetilde{K}$ with $t_0 + s \le T$.

Then for every initial set $K_0 \in \mathscr{K}(\mathbb{R}^N)$ and any time parameter $\tau_{\min} \in]0,T[$, the complement of the graph of $[0,T] \rightsquigarrow \mathbb{R}^N$, $t \mapsto \vartheta_{\widetilde{F}}(t, K_0)$ (as a subset of $\mathbb{R} \times \mathbb{R}^N$) satisfies the uniform exterior tusk condition in all boundary points in $]\tau_{\min}, T[\times \mathbb{R}^N$.

Corollary 50. *In addition to standard hypothesis* ($\widetilde{\mathscr{H}_\circ^\rho}$) *(on page 464), assume for the set-valued map* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *that some* $\lambda(\cdot) \in L^\infty([0,T])$ *satisfies*

$$\begin{aligned}\|\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)\|_{C^{1,1}(\mathbb{R}^N \times \partial\mathbb{B}_1)} &\stackrel{\text{Def.}}{=} \|\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} + \operatorname{Lip} \partial\mathscr{H}_{\widetilde{F}}(t,\cdot,\cdot)|_{\mathbb{R}^N \times \partial\mathbb{B}_1} \\ &< \lambda(t)\end{aligned}$$

at $\mathscr{L}^1$*-almost every time* $t \in [0,T]$.

Then for every initial set $K_0 \in \mathscr{K}(\mathbb{R}^N)$ *and any time parameter* $\tau_{\min} \in]0,T[$, *the complement of the graph of* $[0,T] \rightsquigarrow \mathbb{R}^N,\ t \mapsto \vartheta_{\widetilde{F}}(t,K_0)$ *(as a subset of* $\mathbb{R} \times \mathbb{R}^N$*) satisfies the uniform exterior tusk condition in all boundary points in* $]\tau_{\min}, T[\, \times \mathbb{R}^N$. □

For proving Proposition A.49, we conclude the exterior tusk condition from a similar property about truncated cones (alias conical frustums). In particular, the possibility of choosing geometric parameters *uniformly* does not depend on the shape of a tusk or a conical frustum. The latter condition, however, is easier to verify for graphs of reachable sets by means of boundary solutions and their adjoints (in the sense of Proposition A.32 on page 457).

Lemma 51 (Conical frustum provides suitable tusk).
Let $\Omega \subset \mathbb{R} \times \mathbb{R}^N$ *be nonempty. Assume* $(t_0, x_0) \in \partial\Omega$ *and* $x_1 \in \mathbb{R}^N$, $h, \lambda > 0$ *to satisfy* $\lambda\, h < |x_0 - x_1|$ *and*

$$\overline{\Omega} \cap \big\{(s,y) \in \mathbb{R} \times \mathbb{R}^N \,\big|\, t_0 - h \le s \le t_0,\ |y - x_1| \le |x_0 - x_1| - \lambda\,(t_0 - s)\big\} = \big\{(t_0, x_0)\big\}.$$

Then there exists a tusk in (t_0,x_0) *whose closure has only* (t_0,x_0) *in common with* $\overline{\Omega}$. *Furthermore the scalar geometric parameters of this tusk depend merely on* h, λ.

Lemma 52 (Graphs of reachable sets have interior conical frustums).
Under the assumptions of Proposition A.49, every accumulation point (t_0,x_0) *of* $\partial\,\big(\text{Graph } \vartheta_{\widetilde{F}}(\cdot,K_0)\big|_{[0,T]}\big) \cap \big(]0,T[\, \times \mathbb{R}^N\big)$ *with* $t_0 > 0$ *has an open conical frustum*

$$\big\{(s,y) \in \mathbb{R} \times \mathbb{R}^N \,\big|\, t_0 - h < s < t_0,\ |y - x_1| < |x_0 - x_1| - \lambda\,(t_0 - s)\big\}$$

(with suitable parameters $h, \lambda > 0$ *and* $x_1 \in \mathbb{R}^N$*) whose closure has only* (t_0,x_0) *in common with the closed complement of* $\text{Graph } \vartheta_{\widetilde{F}}(\cdot,K_0)\big|_{[0,T]} \subset \mathbb{R} \times \mathbb{R}^N$.

If $t_0 > \tau_{\min}$ *with an arbitrarily fixed parameter* $\tau_{\min}$ *in addition, the parameters* $h, \lambda > 0$ *can be chosen independently of* (t_0,x_0), *but just depending on* K_0, $\widetilde{F}, T, \tau_{\min}$.

Proof (of Lemma A.51). Consider the following tusk with $R := \frac{|x_0 - x_1| - \lambda\,h}{\sqrt{h}} > 0$

$$M := \Big\{(s,y) \in \mathbb{R} \times \mathbb{R}^N \,\Big|\, t_0 - h < s < t_0,\ \Big|(y - x_0) - \sqrt{t_0 - s}\cdot \tfrac{x_1 - x_0}{\sqrt{h}}\Big| < R\,\sqrt{t_0 - s}\Big\}.$$

As a simple consequence of the triangle inequality in $\mathbb{R}^N$, M is contained in the given conical frustum and thus, $\overline{\Omega} \cap \overline{M} = \{(t_0, x_0)\}$. □

Proof (of Lemma A.52). As an accumulation point, $(t_0, x_0) \in]0, T] \times \mathbb{R}^N$ can be approximated by a sequence of points in $\partial \left(\text{Graph } \vartheta_{\widetilde{F}}(\cdot, K_0)\big|_{[0,T]}\right) \cap \left(]0, T[\, \times \mathbb{R}^N\right)$. Applying preceding Proposition A.32 (on page 457) to each of these boundary points, an appropriate subsequence provides a solution $x(\cdot) \in W^{1,1}([0,t_0], \mathbb{R}^N)$ and its adjoint $p(\cdot) \in W^{1,1}([0,t_0], \mathbb{R}^N)$ satisfying

$$\begin{cases} x'(t) = \ \ \frac{\partial}{\partial p} \mathscr{H}_{\widetilde{F}}(t, x(t), p(t)) \in \widetilde{F}(t, x(t)), & x(t_0) = x_0, \\ p'(t) = -\frac{\partial}{\partial x} \mathscr{H}_{\widetilde{F}}(t, x(t), p(t)), & |p(t_0)| = 1 \end{cases}$$

and the additional properties for every $s \in [0, t_0[$

$$\begin{cases} x(s) \in \partial \vartheta_{\widetilde{F}}(s, K_0) \\ p(s) \in N_{\vartheta_{\widetilde{F}}(s,K_0)}\big(x(s)\big) \setminus \{0\} \end{cases}$$

due to regularity and uniqueness of the Hamiltonian initial value problem.

Choose any compact neighborhood $\widetilde{C}$ of the graph of $\vartheta_{\widetilde{F}}(\cdot, K_0) : [0, T] \rightsquigarrow \mathbb{R}^N$ in $[0, T] \times \mathbb{R}^N$. Due to the assumption of Proposition A.49, there exist $\widehat{\tau} \in]0, T]$ and a nondecreasing function $\sigma : [0, \widehat{\tau}] \longrightarrow [0, \infty[$ such that $\vartheta_{\widetilde{F}(t+\cdot,\cdot)}(s, y) \subset \mathbb{R}^N$ is convex and has positive erosion of radius $\sigma(s)$ for any $s \in]0, \widehat{\tau}]$, $(t, y) \in \widetilde{C}$ with $t + s \le T$. (If some $\tau_{\min} > 0$ with $\tau_{\min} \le t_0$ is fixed additionally, replace $\widehat{\tau}$ by $\min\{\widehat{\tau}, \tau_{\min}\} > 0$.) Without loss of generality, we assume $\widehat{\tau} < t_0$, $(t_0 - \widehat{\tau}, x(t_0 - \widehat{\tau})) \in \widetilde{C}$.

Set $t_1 := t_0 - \widehat{\tau} > 0$ and $t_2 := t_0 - \frac{\widehat{\tau}}{2} \in]t_1, t_0[$.
At every time $s \in [t_2, t_0[$, the point $x(s)$ belongs to the topological boundary of the convex set $\vartheta_{\widetilde{F}(t_1+\cdot,\cdot)}\big(s - t_1,\, x(t_1)\big)$ with positive erosion of radius $\ge \sigma(\frac{\widehat{\tau}}{2}) =: \rho_{\widehat{\tau}}$. Furthermore the inclusion $\vartheta_{\widetilde{F}(t_1+\cdot,\cdot)}\big(s - t_1,\, x(t_1)\big) \subset \vartheta_{\widetilde{F}}(s, K_0)$ and the convexity of the reachable set $\vartheta_{\widetilde{F}(t_1+\cdot,\cdot)}\big(s - t_1,\, x(t_1)\big)$ imply

$$p(s) \in N_{\vartheta_{\widetilde{F}}(s,K_0)}\big(x(s)\big) \setminus \{0\} \subset N^P_{\vartheta_{\widetilde{F}(t_1+\cdot,\cdot)}(s-t_1,\, x(t_1))}\big(x(s)\big).$$

Now the aspects of (uniform) positive erosion and continuity ensure

$$\mathbb{B}_{\rho_{\widehat{\tau}}}\big(x(s) - \rho_{\widehat{\tau}} \tfrac{p(s)}{|p(s)|}\big) \subset \vartheta_{\widetilde{F}(t_1+\cdot,\cdot)}\big(s - t_1,\, x(t_1)\big) \subset \vartheta_{\widetilde{F}}(s, K_0)$$

for every $s \in [t_2, t_0]$. Moreover, due to the uniform linear growth of $\partial_{(x,p)} \mathscr{H}_{\widetilde{F}}(t, \cdot, \cdot)$, the set-valued map $[t_2, t_0] \rightsquigarrow \mathbb{R}^N$, $s \mapsto \mathbb{B}_{\rho_{\widehat{\tau}}}\big(x(s) - \rho_{\widehat{\tau}} \frac{p(s)}{|p(s)|}\big)$ is Lipschitz continuous with convex values and, its Lipschitz constant Λ depends only on $\widetilde{C}, \widetilde{F}, T, \widehat{\tau}$.

Finally comparing graphs of Lipschitz set-valued maps implies for any $\gamma > \Lambda$ that the truncated cone

$$C_\gamma := \left\{ (s, y) \in \mathbb{R}^{1+N} \,\middle|\, t_0 - \tfrac{\rho_{\widehat{\tau}}}{\gamma} \le s < t_0,\ \left|x_0 - \rho_{\widehat{\tau}} \tfrac{p(t_0)}{|p(t_0)|} - y\right| < \rho_{\widehat{\tau}} - \gamma \cdot (t_0 - s) \right\}$$

is a subset of $\bigcup_{s \in [t_2, t_0]} \Big(\{s\} \times \mathbb{B}_{\rho_{\widehat{\tau}}}\big(x(s) - \rho_{\widehat{\tau}} \frac{p(s)}{|p(s)|}\big)\Big) \subset \mathbb{R} \times \mathbb{R}^N$.
Obviously the modified truncated cone $C_{2\gamma}$ is contained in the interior of its counterpart C_γ and thus, $C_{2\gamma}$ belongs to the interior of Graph $\vartheta_{\widetilde{F}}(\cdot, K_0)|_{[0,T]} \subset \mathbb{R} \times \mathbb{R}^N$. □

A.6 Reynolds Transport Theorem for Differential Inclusions with Carathéodory Maps

Reynolds Transport Theorem concerns the time derivative of a Lebesgue integral whose domain is deformed due to a sufficiently smooth vector field (e.g. [55, § 8.3]):

Theorem 53 (Reynolds Transport Theorem). *Suppose* $w \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. *For a nonempty compact set* $K_0 \subset \mathbb{R}^N$, *let* $K(t) \subset \mathbb{R}^N$ *contain all points* $x(t)$ *of solutions* $x(\cdot) \in C^1([0,t], \mathbb{R}^N)$ *of* $x' = w(x)$, $x(0) \in K_0$.
Then for every $\Psi \in C^1(\mathbb{R} \times \mathbb{R}^N)$, *the function* $\mathbb{I}_w : t \longmapsto \int_{K(t)} \Psi(t,x)\, dx$ *fulfills*

$$\tfrac{d^+}{dt^+}\, \mathbb{I}_w(0) \overset{\text{Def.}}{=} \lim_{t \downarrow 0} \tfrac{\mathbb{I}_w(t) - \mathbb{I}_w(0)}{t} = \int_{K_0} \Big(\partial_t \Psi(0,x) + \operatorname{div}\, (\Psi(0,x)\, w(x)) \Big)\, dx.$$

If, in addition, K_0 *satisfies the assumptions of Gauss' Integral Theorem then*

$$\tfrac{d^+}{dt^+}\, \mathbb{I}_w(0) = \int_{K_0} \partial_t \Psi(0,x)\, dx + \int_{\partial K_0} \Psi(0,x)\, w(x) \cdot \nu_{K_0}(x)\, d\sigma_x$$

with the exterior unit normal ν_{K_0} *to* K_0.

Although the name of Osborne Reynolds (1842 – 1912) is used mainly in continuum mechanics this theorem has broad applications, e.g. in shape optimization and free boundary problems.
Now we focus on the integrals over compact reachable sets of differential inclusions, i.e. for a given function $\psi \in L^1_{\text{loc}}(\mathbb{R}^N)$, we consider

$$\mathbb{I}_{\widetilde{F}} : [0,T] \longrightarrow \mathbb{R}, \qquad t \longmapsto \int_{\vartheta_{\widetilde{F}}(t,K_0)} \psi(x)\, dx.$$

As a key point, a priori assumptions about the regularity of ∂K_0 are avoided completely. However, $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has to fill the gap concerning sufficient conditions. In particular, any generalization of Theorem A.53 (with a boundary integral) has to exclude the example that a nonrectifiable set $K_0 \subset \mathbb{R}^N$ is simply translated. For this reason, $\widetilde{F}$ is supposed to have a continuous selection of its interior $\widetilde{F}(\cdot,\cdot)^\circ$ and, the main result of this section is

Theorem 54. *Assume* $N \geq 2$. *Let* $\rho_{\widetilde{F}}, \mu_{\widetilde{F}} > 0$, $v_{\widetilde{F}} \in C^0([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$ *and* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *be a Carathéodory map with compact convex values and*

$$\mathbb{B}_{\rho_{\widetilde{F}}}(v_{\widetilde{F}}(t,x)) \subset \widetilde{F}(t,x) \subset \mu_{\widetilde{F}}\,(1+|x|) \cdot \mathbb{B}$$

for every $(t,x) \in [0,T] \times \mathbb{R}^N$. *Furthermore assume* $K_0 \in \mathscr{K}(\mathbb{R}^N)$, $\psi \in C^0(\mathbb{R}^N)$.

Then $\mathbb{I}_{\widetilde{F}} : [0,T] \longrightarrow \mathbb{R}$ *is absolutely continuous and has the weak derivative*

$$\tfrac{d}{dt}\, \mathbb{I}_{\widetilde{F}}(t) = \int_{\partial\, \vartheta_{\widetilde{F}}(t,K_0)} \psi(x)\, \sup \Big(\widetilde{F}(t,x) \cdot {}^{\flat}N^B_{\vartheta_{\widetilde{F}}(t,K_0)}(x) \Big)\, d\mathscr{H}^{N-1} x.$$

Here ${}^{\flat}N^B_K(x)$ *denotes the set of Bouligand normal vectors in the unit ball* $\mathbb{B}$, *i.e.*

$${}^{\flat}N^B_K(x) \overset{\text{Def.}}{=} \big\{ v \in \mathbb{B}_1(0) \;\big|\; v \cdot w \leq 0 \;\; \text{for all } w \in T_K(x) \big\} \subset \mathbb{R}^N.$$

Corollary 55 (Reynolds Transport Theorem for differential inclusions).
Let $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ satisfy the assumptions of Theorem 54. Moreover suppose $K_0 \in \mathscr{K}(\mathbb{R}^N)$ and $\Psi \in C^1([0,T] \times \mathbb{R}^N)$.
Then the function $[0,T] \longrightarrow \mathbb{R},\ t \longmapsto \int_{\vartheta_{\widetilde{F}}(t,K_0)} \Psi(t,x)\, dx$ is absolutely continuous and has the weak derivative

$$\int_{\partial\, \vartheta_{\widetilde{F}}(t,K_0)} \Psi(t,x)\ \sup\Big(\widetilde{F}(t,x) \cdot {}^\flat N^B_{\vartheta_{\widetilde{F}}(t,K_0)}(x)\Big)\ d\mathscr{H}^{N-1}x \,+\, \int_{\vartheta_{\widetilde{F}}(t,K_0)} \tfrac{\partial}{\partial t}\Psi(t,x)\ dx.$$

Corollary 56 (for autonomous differential inclusions, [127, Corollary 3.4]**).**
The absolute continuity in the preceding Corollary A.55 also holds for an autonomous Lipschitz continuous map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with nonempty compact convex values if for each $x \in \mathbb{R}^N$, either $0 \in F(x)^\circ$ or $F(x) = \{0\}$.

Sketch of the Proof for the Special Case of Strictly Expanding Sets ($v_{\widetilde{F}} \equiv 0$)

In the special case $v_{\widetilde{F}} \equiv 0$, the vector 0 belongs to the interior of each value of $\widetilde{F}$. Then the reachable sets represent strict expansions in the sense that for every $s < t$,

$$\vartheta_{\widetilde{F}}(s,K_0) \ \subset\ \big(\vartheta_{\widetilde{F}}(t,K_0)\big)^\circ.$$

Due to this observation, we can describe both the reachable sets and their topological boundaries easily via the so-called *minimal time function* $\tau_{\widetilde{F}} : \mathbb{R}^N \longrightarrow [0,\infty]$,

$$\begin{aligned}\tau_{\widetilde{F}}(x) &:= \inf\{\, t \in [0,T] \mid x \in \vartheta_{\widetilde{F}}(t,K_0)\,\},\\ &= \inf\{\, t \in [0,T] \mid K_0 \cap \vartheta_{-\widetilde{F}(t-\cdot,\cdot)}(t,x) \neq \emptyset\,\}.\end{aligned}$$

In many papers about minimal time functions (e.g. [23, 34, 35, 82, 186]), the condition on admitted solutions usually concerns their final points, i.e.

$$x \longmapsto \inf\big\{\, t \in [0,T] \;\big|\; \exists \text{ solution } z(\cdot) :\ z(0) = x,\ z(t) \in K_0 \big\}.$$

Here we consider a state constraint for the initial point instead : $z(0) \in K_0,\ z(\underset{\smile}{t}) = x$. These two definitions can be regarded as equivalent *only* if the function $\widetilde{F}$ does not depend on time explicitly. For an autonomous Lipschitz map $G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with compact convex values, the properties of $\tau_G(\cdot)$ have already been investigated extensively. In particular, $\tau_G(\cdot)$ is the viscosity solution of the Eikonal equation

$$\begin{cases} \sup\big(G(x) \cdot \nabla \tau_G(x)\big) = 1 & \text{in } \vartheta_G([0,T],K_0)^\circ,\\ \qquad\qquad\qquad \tau_G = 0 & \text{in } K_0.\end{cases}$$

In [127], the detailed proof of Theorem A.54 has a rather geometric character and verifies the subsequent properties of $\tau_{\widetilde{F}}(\cdot)$ (only). In particular, no results about viscosity solutions are used there. As we consider just the points of differentiability for a locally Lipschitz continuous function, we do not need stronger regularity assumptions about $\widetilde{F}$. Further characterizations of reachable sets by means of normals can be found in [45, 65, 68].

Lemma 57. *Let $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ satisfy the assumptions of Theorem A.54 with $v_{\widetilde{F}} \equiv 0$, i.e. for some $\mu_{\widetilde{F}}, \rho_{\widetilde{F}} > 0$, $\widetilde{F}$ is a Carathéodory set-valued map with compact convex values and $\rho_{\widetilde{F}} \mathbb{B} \subset \widetilde{F}(t,x) \subset \mu_{\widetilde{F}} (1+|x|) \mathbb{B}$ for every (t,x), and assume $K_0 \in \mathscr{K}(\mathbb{R}^N)$.*
Then the corresponding minimum time function $\tau_{\widetilde{F}} : \vartheta_{\widetilde{F}}(T,K_0) \longrightarrow [0,\infty[$ has the properties for every $t \in [0,T]$:

1. *$\tau_{\widetilde{F}}$ is Lipschitz continuous in $\vartheta_{\widetilde{F}}(T,K_0)$,*
2. *$\vartheta_{\widetilde{F}}(t,K_0) = \tau_{\widetilde{F}}^{-1}([0,t])$,*
3. *the topological boundary $\partial\, \vartheta_{\widetilde{F}}(t,K_0)$ is contained in the level set $\tau_{\widetilde{F}}^{-1}(t)$.*
4. *Let $D_{\widetilde{F}}$ consist of all points in $\vartheta_{\widetilde{F}}(T,K_0)^\circ \setminus K_0$ at which $\tau_{\widetilde{F}}$ is differentiable.*
   Then for every $x \in D_{\widetilde{F}}$, $\quad |\nabla \tau_{\widetilde{F}}(x)| \geq \frac{1}{\mu_{\widetilde{F}} \cdot (1+|x|)}$,
   $\quad N^B_{\vartheta_{\widetilde{F}}(\tau_{\widetilde{F}}(x),K_0)}(x) = [0,\infty[\cdot \nabla \tau_{\widetilde{F}}(x)$
   and for every $t \in]0,T[$, $\quad D_{\widetilde{F}} \cap \tau_{\widetilde{F}}^{-1}(\{t\}) \subset \partial\, \vartheta_{\widetilde{F}}(t,K_0)$.
5. *$|\nabla \tau_{\widetilde{F}}(x)| \cdot \sup \Big(\widetilde{F}(\tau_{\widetilde{F}}(x), x) \cdot {}^\flat N^B_{\vartheta_{\widetilde{F}}(\tau_{\widetilde{F}}(x),K_0)}(x)\Big) = 1$ for $\mathscr{L}^N$-a.e. $x \in D_{\widetilde{F}}$.*

The ball assumption about the values of $\widetilde{F}$ is to guarantee the properties (1.), (4.). Then the other statements result from the strict expansion property of $\vartheta_{\widetilde{F}}(\cdot, K_0)$.
Now Theorem A.54 is a consequence of the so-called co-area formula, which is an important tool in (geometric) measure theory. For any Lipschitz continuous map $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $(m > n)$, let $C_n f(x)$ denote its *n-dimensional co-area factor* if f is differentiable at x :

$$C_n f(x) \stackrel{\text{Def.}}{=} \sqrt{\det(Df(x) \cdot Df(x)^T)}.$$

For the minimum time function $\tau_{\widetilde{F}}(\cdot)$ in particular, the dimension $n = 1$ implies $C_1 \tau_{\widetilde{F}}(x) = |\nabla \tau_{\widetilde{F}}(x)|$ for every $x \in D_F$.

Proposition 58 (Co-area formula, [77, § 3.4.3, Theorem 2], [78, Theorem 3.2.12]**).** *If $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is Lipschitz continuous and $m > n$, then*

$$\int_{\mathbb{R}^m} g(x)\ C_n f(x)\ d\mathscr{L}^m x = \int_{\mathbb{R}^n} \int_{f^{-1}(\{y\})} g(x)\ d\mathscr{H}^{m-n} x\ d\mathscr{L}^n y$$

for every (Lebesgue) $\mathscr{L}^m$ integrable function $g : \mathbb{R}^m \longrightarrow [-\infty,\infty]$. Here $\mathscr{H}^{m-n}$ denotes the $(m-n)$-dimensional Hausdorff measure of nonempty subsets in $\mathbb{R}^m$.

Indeed, from a merely formal point of view, this formula applied to

$$g(x) := \psi(x) \cdot \sup \Big(\widetilde{F}(\tau_{\widetilde{F}}(x), x) \cdot {}^\flat N^B_{\vartheta_{\widetilde{F}}(\tau_{\widetilde{F}}(x),K_0)}(x)\Big)$$

leads to

$$\begin{aligned}\int_{\vartheta_{\widetilde{F}}(t,K_0)\setminus K_0} \psi\ dx &= \int_0^t \int_{\tau_{\widetilde{F}}^{-1}(s)} \psi(y) \cdot \sup \Big(\widetilde{F}(s,y) \cdot {}^\flat N^B_{\vartheta_{\widetilde{F}}(s,K_0)}(y)\Big)\ d\mathscr{H}^{N-1} y\ ds \\ &= \int_0^t \int_{\partial\vartheta_{\widetilde{F}}(s,K_0)} \psi(y) \cdot \sup \Big(\widetilde{F}(s,y) \cdot {}^\flat N^B_{\vartheta_{\widetilde{F}}(s,K_0)}(y)\Big)\ d\mathscr{H}^{N-1} y\ ds.\end{aligned}$$

Sketch of the Proof for the General Case Via Coordinate Transformation

Restricting to sufficiently short time intervals in $[0,T]$, the continuous selection $v_{\widetilde{F}}(\cdot,\cdot)$ can be approximated locally by autonomous functions $w_{\widetilde{F}} \in C_0^\infty(\mathbb{R}^N,\mathbb{R}^N)$. The flow $T_t(x) := \vartheta_{-w_{\widetilde{F}}}(t,x)$ induced by the velocity field $-w_{\widetilde{F}}(\cdot)$ is a diffeomorphism and maps the reachable set $\vartheta_F(t,K_0)$ to $T_t(\vartheta_F(t,K_0)) = \vartheta_G(t,K_0)$ with the set-valued map

$$\widetilde{G}(t,y) := -w_{\widetilde{F}}(y) + DT_t\big(T_t^{-1}(y)\big) \cdot F\left(t, T_t^{-1}(y)\right) \subset \mathbb{R}^N.$$

Moreover this map $\widetilde{G}(\cdot,\cdot)$ satisfies the assumptions of Theorem A.54 at any point of an (initially fixed) compact neighborhood of $K_0 \subset \mathbb{R}^N$ and for $t \in [0,T]$ sufficiently small. Additionally a fixed ball with center at 0 is contained in each value of $\widetilde{G}$ and so, the preceding special case can be applied to integrals over $\vartheta_{\widetilde{G}}(t,K_0)$.
The remaining challenge is now to verify that the coordinate transformation via T_t preserves the structure of the integral representation. In particular, all Lebesgue *and* Hausdorff integrals have to be well-defined and finite (almost everywhere). This requires changes of variables for Hausdorff integrals: the so-called area formula.

Proposition 59 (Generalized area formula, [5, Theorem 2.91], [78, Cor. 3.2.20]**).** *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a Lipschitz continuous function and $E \subset \mathbb{R}^m$ a countably $\mathscr{H}^k$-rectifiable set $(k \le n)$. Then, the multiplicity function*

$$\mathbb{R}^n \longrightarrow \mathbb{R}, \quad y \longmapsto \mathscr{H}^0(E \cap f^{-1}(y))$$

is $\mathscr{H}^k$-measurable in $\mathbb{R}^n$ and for every Borel function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\int_{\mathbb{R}^n} g(y) \cdot \mathscr{H}^0\big(E \cap f^{-1}(y)\big)\, d\mathscr{H}^k y \;=\; \int_E g(f(x)) \cdot J_k\, d^E f_x \; d\mathscr{H}^k x.$$

Here $d^E f_x$ denotes the approximate tangential differential of f at x and, J_k abbreviates the k-dimensional Jacobian, i.e. here $J_k\, d^E f_x \overset{\text{Def.}}{=} \sqrt{\det\big(d^E f_x^T \cdot d^E f_x\big)}$.

The generalized Gauss-Green Theorem is a further tool used in [127, § 6] for investigating the level sets of $\tau_{\widetilde{F}}$, $\tau_{\widetilde{G}}$ and their boundaries. It involves the so-called *measure theoretic boundary* $\partial_* M$ of a nonempty $\mathscr{L}^n$-measurable set $M \subset \mathbb{R}^n$, i.e.

$$\partial_* M := \left\{ x \in \mathbb{R}^n \;\middle|\; \limsup_{r\to 0} \tfrac{\mathscr{L}^n(\mathbb{B}_r(x)\cap M)}{r^N} > 0, \quad \limsup_{r\to 0} \tfrac{\mathscr{L}^n(\mathbb{B}_r(x)\setminus M)}{r^N} > 0 \right\} \subset \partial M.$$

Proposition 60 (Generalized Gauss-Green Theorem, [77, § 5.8, Theorem 1]**).** *Let $M \subset \mathbb{R}^n$ have locally finite perimeter. Then for $\mathscr{H}^{n-1}$-a.e. $x \in \partial_* M$, there is a unique measure theoretic unit outer normal $\nu_M(x)$ such that for all $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$*

$$\int_M \operatorname{div} \varphi \; dx \;=\; \int_{\partial_* M} \varphi \cdot \nu_M \; d\mathscr{H}^{n-1}$$

Indeed the reachable sets $\vartheta_{\widetilde{F}}(t,K_0)$, $\vartheta_{\widetilde{G}}(t,K_0)$ have locally finite perimeter at $\mathscr{L}^1$-almost every time t due to their finite $\mathscr{H}^{N-1}$ measures and [77, § 5.11, Theorem 1]. Moreover, every point $x \in \partial\vartheta_{\widetilde{G}}(t,K_0)$ at which $\tau_{\widetilde{G}}$ is differentiable proves to belong even to the measure theoretic boundary $\partial_* \vartheta_{\widetilde{G}}(t,K_0)$ and thus we can use the Eikonal equation in Lemma A.57 (5.).

A.7 Differential Inclusions with One-Sided Lipschitz Continuous Maps

In [69], Donchev and Farkhi prove the existence of solutions to another type of differential inclusions – with a stability estimate as in Filippov's Theorem A.6 (on page 443) included. Their essential aspect is to replace the classical Lipschitz condition with respect to space by a weakened form (called one-sided Lipschitz condition) in combination with upper semicontinuity and convex values:

Definition 61 ([69, Definition 2.1]). A set-valued map $\widetilde{F} : [0,T] \times \mathbb{R}^N \leadsto \mathbb{R}^N$, $(t,x) \mapsto F(t,x)$ is called *one-sided Lipschitz continuous* with respect to x if there is a function $L(\cdot) \in L^1([0,T])$ such that for every $x, y \in \mathbb{R}^N$, $t \in [0,T]$ and $v \in \widetilde{F}(t,x)$, there exists an element $w \in \widetilde{F}(t,y)$ satisfying

$$\langle x-y,\ v-w \rangle \ \le\ L(t)\ |x-y|^2.$$

Remark 62. 1. As Donchev has already pointed out in several of his papers, $\widetilde{F} : [0,T] \times \mathbb{R}^N \leadsto \mathbb{R}^N$ is one-sided Lipschitz continuous with respect to x if and only if some $L(\cdot) \in L^1([0,T])$ satisfies

$$\mathscr{H}_{\widetilde{F}}\big(x-y,\ \widetilde{F}(t,x)\big)\ -\ \mathscr{H}_{\widetilde{F}}\big(x-y,\ \widetilde{F}(t,y)\big)\ \le\ L(t)\ |x-y|^2$$

for every $x, y \in \mathbb{R}^N$ and $t \in [0,T]$.

2. Obviously, every Lipschitz continuous map is also one-sided Lipschitz continuous, but not vice versa in general. In particular, one-sided Lipschitz continuous maps do not have to be upper or lower semicontinuous.

3. The function $L(\cdot) \in L^1([0,T])$ is assumed to be real-valued, but we do not restrict our considerations to $L(\cdot) \ge 0$. The special case of strictly negative $L(\cdot)$ admits interesting conclusions about asymptotic features which usually do not have counterparts of the (classically) Lipschitz continuous maps.

Theorem 63 (Filippov-like existence for one-sided Lipschitz maps [69, Th. 3.2]). *Let* $\widetilde{F} : [0,T] \times \mathbb{R}^N \leadsto \mathbb{R}^N,\ (t,x) \mapsto \widetilde{F}(t,x)$ *be a nonautonomous Marchaud map (in the sense of Definition A.11 on page 447) being one-sided Lipschitz continuous with respect to x. For* $y(\cdot) \in W^{1,1}([0,T], \mathbb{R}^N)$ *and* $g(\cdot) \in L^1([0,T])$ *suppose*

$$\operatorname{dist}\big(y'(t),\ \widetilde{F}(t,\ y(t))\big)\ \le\ g(t)$$

at Lebesgue-almost every time $t \in [0,T]$.

Then for every initial point $x_0 \in \mathbb{R}^N$*, there exists a solution* $x(\cdot) \in W^{1,1}([0,T], \mathbb{R}^N)$ *of* $x'(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ *a.e. satisfying* $x(0) = x_0$ *and for every* $t \in [0,T]$

$$\big|x(t) - y(t)\big|\ \le\ |x_0 - y(0)|\ e^{\int_0^t L(r)\,dr}\ +\ \int_0^t e^{\int_s^t L(r)\,dr}\, g(s)\ ds\,.$$

Remark 64. The existence results of Theorem A.63 and Filippov's Theorem A.6 differ from each other in an essential aspect:
Under the assumptions of Theorem A.63, not every point $x_0 \in \mathbb{R}^N$ and vector $v_0 \in \widetilde{F}(0,x_0)$ has to be related to a solution $x(\cdot) \in W^{1,1}([0,T],\mathbb{R}^N)$ of $x'(\cdot) \in \widetilde{F}(\cdot,x(\cdot))$ satisfying $x(0) = x_0$ and

$$\lim_{h \downarrow 0} \tfrac{1}{h} \cdot \big(x(h) - x(0)\big) \;=\; v_0.$$

An example is given by the following map $\widetilde{F}$ and the initial data $x_0 := 0 \in \mathbb{R}$, $v_0 := \frac{1}{2}$

$$\widetilde{F}:\ [0,1] \times \mathbb{R} \rightsquigarrow \mathbb{R}, \quad (t,x) \mapsto \begin{cases} -1 & \text{for } x > 0 \\ [-1,1] & \text{for } x = 0 \\ 1 & \text{for } x < 0 \end{cases}$$

Proposition 65. *As in Theorem A.63, let* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $(t,x) \mapsto \widetilde{F}(t,x)$ *be a nonautonomous Marchaud map (in the sense of Definition A.11 on page 447) being one-sided Lipschitz continuous with respect to x.*
In addition suppose $\widetilde{F}(\cdot,\cdot)$ *to be lower semicontinuous at each* $(t,x) \in \{0\} \times \mathbb{R}^N$.

Then for any $x_0 \in \mathbb{R}^N$ *and* $v_0 \in \widetilde{F}(0,x_0)$, *there is a solution* $x(\cdot) \in W^{1,1}([0,T],\mathbb{R}^N)$ *of* $x'(\cdot) \in \widetilde{F}(\cdot,x(\cdot))$ *a.e. satisfying* $x(0) = x_0$ *and*

$$\lim_{h \downarrow 0} \tfrac{1}{h} \cdot \big(x(t) - x_0\big) \;=\; v_0.$$

Proof. Theorem A.63 applied to $y(t) := x_0 + t\ v_0$ provides a solution $x(\cdot) \in W^{1,1}([0,T],\mathbb{R}^N)$ of $x'(\cdot) \in \widetilde{F}(\cdot,x(\cdot))$ a.e. satisfying $x(0) = x_0$ and

$$\begin{aligned} \big|x(h) - x_0 - h\, v_0\big| \;&\le\; \int_0^h e^{\int_s^h L(r)\,dr} \ \ \mathrm{dist}\big(v_0,\ \widetilde{F}(s,\, x_0 + s\, v_0)\big)\ ds \\ &\le\; e^{\|L\|_{L^1([0,T])}} \int_0^h \mathrm{dist}\big(v_0,\ \widetilde{F}(s,\, x_0 + s\, v_0)\big)\ ds\,. \end{aligned}$$

In particular, the lower semicontinuity of $\widetilde{F}$ in $(0,x_0)$ implies

$$\mathrm{dist}\big(v_0,\ \widetilde{F}(s,\, x_0 + s\, v_0)\big) \longrightarrow 0 \qquad \text{for } s \searrow 0$$

and thus,
$$\limsup_{h \downarrow 0} \tfrac{1}{h} \cdot \big|x(h) - x_0 - h\, v_0\big| \;\le\; 0. \qquad \square$$

A.8 Stochastic Differential Inclusions in $\mathbb{R}^N$

A.8.1 Filippov-Like Theorem of Da Prato and Frankowska

Now the focus of interest is an existence theorem for *stochastic* differential inclusions in the Euclidean space $\mathbb{R}^N$. In regard to the mutational framework, we need a priori estimates that compare a given curve with a solution to the inclusion and, they should have a form similar to Filippov's Theorem A.6 (on page 443 f.).
In 1994, Da Prato and Frankowska presented such an existence result for stochastic differential inclusions with globally Lipschitz continuous drift and diffusion terms [53]. Their main statements even concern Itô integral inclusions with a strongly continuous semigroup on a separable Hilbert space. Just one year later, Motyl published independent existence and uniqueness results about stochastic differential inclusions in the Euclidean space under some assumptions of dissipative type [143].
Now we consider only the finite-dimensional case and prove such a Filippov-like theorem essentially by means of the arguments of Aubin, Da Prato and Frankowska in [18, Theorem 4.1]. The comparative estimate, however, is slightly modified so that we can use it more easily in the example of § 3.7 (on page 242 ff.) and thus, the proof is presented completely here.

General assumptions in § A.8

(i) $(\Omega, \mathscr{A}, P)$ is a complete probability space.

(ii) $(\mathscr{A}_t)_{t\geq 0}$ denotes a filtration with the usual conditions, i.e. $(\mathscr{A}_t)_{t\geq 0}$ is a right continuous and increasing family of sub-σ-algebras of $\mathscr{A}$ and, $\mathscr{A}_0$ contains all P-null sets.

(iii) $W = (W_t)_{t\geq 0}$ is an m-dimensional Wiener process.

(iv) For finite $T > 0$ fixed, define the class $\mathscr{L}^2_{\mathscr{A}}([0,T],\mathbb{R}^N)$ of functions $f : [0,T] \times \Omega \longrightarrow \mathbb{R}^N$ with

  (1.) f is jointly $\mathscr{L}^1 \times \mathscr{A}$-measurable,

  (2.) $\displaystyle\int_{[0,T]} \mathbb{E}\big(|f(t,\cdot)|^2\big)\, dt < \infty,$

  (3.) for every $t \in [0,T]$, $\mathbb{E}\big(|f(t,\cdot)|^2\big) < \infty$ and

  (4.) for every $t \in [0,T]$, $f(t,\cdot) : \Omega \longrightarrow \mathbb{R}^N$ is $\mathscr{A}_t$-measurable.

(v) Let $\mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N)$ consist of all linear functions $\mathbb{R}^m \longrightarrow \mathbb{R}^N$.

(vi) $\mathfrak{I}_0(X_0,\gamma,\sigma)$ denotes the Itô process associated with
the initial state $X_0 \in L^2(\Omega,\ \mathscr{A}_0, P;\ \mathbb{R}^N)$,
the drift $\gamma \in \mathscr{L}^2_{\mathscr{A}}([0,T],\ \mathbb{R}^N)$ and
the diffusion $\sigma \in \mathscr{L}^2_{\mathscr{A}}([0,T],\ \mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N))$, i.e. for $t \in [0,T]$,

$$\mathfrak{I}_0(X_0,\gamma,\sigma)\,(t) := X_0 + \int_0^t \gamma(s)\, ds + \int_0^t \sigma(s)\, dW_s,$$

$$\big\|\mathfrak{I}_0(X_0,\gamma,\sigma)\big\|_{\mathfrak{I},[0,t]} := \sqrt{\mathbb{E}(|X_0|^2) + \mathbb{E}\Big(\int_0^t |\gamma|^2\, ds\Big) + \mathbb{E}\Big(\int_0^t |\sigma|^2\, ds\Big)}.$$

Remark 66. The general estimate for every Itô process

$$\begin{aligned}\mathbb{E}\big(\big|\mathfrak{I}_0(X_0,\gamma,\sigma)(t)\big|^2\big) &\le 9\,(1+t)\ \ \big\|\mathfrak{I}_0(X_0,\gamma,\sigma)\big\|^2_{\mathfrak{I},[0,t]}\\ &\le 9\ e^t\qquad \big\|\mathfrak{I}_0(X_0,\gamma,\sigma)\big\|^2_{\mathfrak{I},[0,t]}\end{aligned}$$

results from the Hölder inequality, the Itô isometry (quoted for one dimension in Proposition 3.50 (d) on page 233 f.) and the simple inequality $(r+s)^2 \le 3\,(r^2+s^2)$ for any $r,s \in \mathbb{R}$.

Theorem 67 (Da Prato-Frankowska for stochastic differential inclusions). *Suppose for the set-valued map $\widetilde{F} = (\widetilde{F}_1,\widetilde{F}_2) : [0,T]\times\Omega\times\mathbb{R}^N \rightsquigarrow \mathbb{R}^N \times \mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N)$:*

(i) *$\widetilde{F}$ has nonempty compact values,*
(ii) *for every $x\in\mathbb{R}^N$, $\widetilde{F}(\cdot,\cdot,x)$ is measurable,*
(iii) *there is $\Lambda > 0$ such that for each $t\in[0,T]$, $\omega\in\Omega$, $\widetilde{F}(t,\omega,\cdot)$ is Λ-Lipschitz,*
(iv) *there is $\gamma > 0$ such that $|\widetilde{F}(t,\omega,0)|_\infty \le \gamma$ holds for all $t\in[0,T]$, $\omega\in\Omega$.*

Furthermore let $Y := \mathfrak{I}_0(Y_0,\gamma_0,\sigma_0)$ be any Itô process with drift $\gamma_0 \in \mathscr{L}^2_{\mathscr{A}}([0,T],\,\mathbb{R}^N)$ and diffusion $\sigma_0 \in \mathscr{L}^2_{\mathscr{A}}([0,T],\,\mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N))$.

For every initial random variable $X_0 \in L^2(\Omega,\mathscr{A}_0,P;\ \mathbb{R}^N)$, there exist a drift $\gamma\in\mathscr{L}^2_{\mathscr{A}}([0,T],\,\mathbb{R}^N)$ and a diffusion $\sigma\in\mathscr{L}^2_{\mathscr{A}}([0,T],\,\mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N))$ such that the related Itô process $X := \mathfrak{I}_0(X_0,\gamma,\sigma)$ satisfies both

$$\begin{cases}\gamma(t,\omega) \in \widetilde{F}_1\big(t,\,\omega,\,X_t(\omega)\big)\\ \sigma(t,\omega) \in \widetilde{F}_2\big(t,\,\omega,\,X_t(\omega)\big)\end{cases}$$

for $(\mathscr{L}^1\times P)$-almost all $(t,\omega)\in[0,T]\times\Omega$ and for each $t\in[0,T]$

$$\|X - Y\|^2_{\mathfrak{I},[0,t]} \le C\cdot\Big(\mathbb{E}\big(|X_0-Y_0|^2\big) + \int_0^t \mathbb{E}\Big(\mathrm{dist}\big((\gamma_0(s),\sigma_0(s)),\ \widetilde{F}(s,Y_s)\big)^2\Big)\ ds\Big)\cdot e^{C\cdot(1+t)\ t}$$

with a constant $C > 0$ depending merely on Λ.

The proof is based on essentially the same iterative construction of approximate solutions as [18, Theorem 4.1].
We use the following lemma, which is easy to verify by means of partial integration:

Lemma 68 ([10, Lemma 1.4.3], [18, Lemma 4.2])**.** *Every Lebesgue-integrable function $g : [0,T] \longrightarrow \mathbb{R}$ fulfills for each $n\in\mathbb{N}$*

$$\int_0^T \int_0^t g(s)\ \frac{(t-s)^{n-1}}{(n-1)!}\ ds\ dt = \int_0^T g(s)\ \frac{(T-s)^n}{n!}\ ds$$

Proof (of Theorem A.67).
Proposition A.80 about measurable marginal maps (on page 490) and Selection Theorem A.74 (of Kuratowski and Ryll-Nardzewski on page 489) guarantee the existence of $\gamma_1 \in \mathscr{L}^2_{\mathscr{A}}([0,T],\ \mathbb{R}^N)$ and $\sigma_1 \in \mathscr{L}^2_{\mathscr{A}}([0,T],\ \mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N))$ satisfying for every $(t,\omega) \in [0,T] \times \Omega$

$$\begin{cases} \gamma_1(t,\omega) \in \widetilde{F}_1\big(t,\omega,Y_t(\omega)\big), & \big|\,\gamma_0|_{(t,\omega)} - \gamma_1|_{(t,\omega)}\big| = \mathrm{dist}\big(\gamma_0|_{(t,\omega)},\ \widetilde{F}_1|_{(t,\omega,Y_t(\omega))}\big) \\ \sigma_1(t,\omega) \in \widetilde{F}_2\big(t,\omega,Y_t(\omega)\big), & \big|\sigma_0|_{(t,\omega)} - \sigma_1|_{(t,\omega)}\big| = \mathrm{dist}\big(\sigma_0|_{(t,\omega)},\ \widetilde{F}_2|_{(t,\omega,Y_t(\omega))}\big). \end{cases}$$

Then the Itô process $X^1 := \mathfrak{I}_0(X_0,\gamma_1,\sigma_1)$ fulfills

$$\begin{aligned} \big\|X^1 - Y\big\|^2_{\mathfrak{I},[0,t]} &= \mathbb{E}\big(|X_0 - Y_0|^2\big) + \mathbb{E}\Big(\int_0^t |\gamma_1 - \gamma_0|^2\,ds\Big) + \mathbb{E}\Big(\int_0^t |\sigma_1 - \sigma_0|^2\,ds\Big) \\ &= \mathbb{E}\big(|X_0 - Y_0|^2\big) + \mathbb{E}\Big(\int_0^t \mathrm{dist}\big((\gamma_0,\sigma_0),\ \widetilde{F}(s,\cdot,Y_s)\big)^2\ ds\Big) \end{aligned}$$

at every time $t \in [0,T]$. Now we iterate this construction and obtain three sequences $(\gamma_n)_{n\in\mathbb{N}}$, $(\sigma_n)_{n\in\mathbb{N}}$, $(X^n)_{n\in\mathbb{N}}$ in $\mathscr{L}^2_{\mathscr{A}}([0,T],\ \mathbb{R}^N)$, $\mathscr{L}^2_{\mathscr{A}}([0,T],\ \mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N))$ and $\mathscr{L}^2_{\mathscr{A}}([0,T],\ \mathbb{R}^N)$ respectively with

$$\begin{cases} \gamma_{n+1}|_{(t,\omega)} \in \widetilde{F}_1|_{(t,\omega,X^n_t(\omega))}, & \big|\,\gamma_n|_{(t,\omega)} - \gamma_{n+1}|_{(t,\omega)}\big| = \mathrm{dist}\big(\gamma_n|_{(t,\omega)},\ \widetilde{F}_1|_{(t,\omega,X^n_t(\omega))}\big) \\ \sigma_{n+1}|_{(t,\omega)} \in \widetilde{F}_2|_{(t,X^n_t(\omega))}, & \big|\sigma_n|_{(t,\omega)} - \sigma_{n+1}|_{(t,\omega)}\big| = \mathrm{dist}\big(\sigma_n|_{(t,\omega)},\ \widetilde{F}_2|_{(t,\omega,X^n_t(\omega))}\big) \\ \quad X^{n+1} = \mathfrak{I}_0\big(X_0,\ \gamma_{n+1},\ \sigma_{n+1}\big) \end{cases}$$

for all $(t,\omega) \in [0,T] \times \Omega$. Then the uniform Λ-Lipschitz continuity of $F(t,\omega,\cdot)$ and Remark A.66 imply

$$\begin{aligned} \big\|X^{n+1} - X^n\big\|^2_{\mathfrak{I},[0,t]} &= \mathbb{E}\Big(\int_0^t |\gamma_{n+1} - \gamma_n|^2\,ds\Big) + \mathbb{E}\Big(\int_0^t |\sigma_{n+1} - \sigma_n|^2\,ds\Big) \\ &= \mathbb{E}\Big(\int_0^t \mathrm{dist}\Big(\big(\gamma_n(s,\cdot),\ \sigma_n(s,\cdot)\big),\ \ \widetilde{F}(s,\cdot,X^n_s)\Big)^2\ ds\Big) \\ &\le \mathbb{E}\Big(\int_0^t d\!l\big(\widetilde{F}(s,\cdot,X^{n-1}_s),\ \widetilde{F}(s,\cdot,X^n_s)\big)^2\ ds\Big) \\ &\le \Lambda^2 \cdot \mathbb{E}\Big(\int_0^t \big|X^{n-1}_s - X^n_s\big|^2\,ds\Big) \\ &\le \Lambda^2 \cdot 9\,(1+t) \cdot \int_0^t \big\|X^n - X^{n-1}\big\|^2_{\mathfrak{I},[0,s]}\ ds. \end{aligned}$$

By means of induction with respect to n, we obtain for every $n \in \mathbb{N}$ and $t \in [0,T]$

$$\begin{aligned} &\big\|X^{n+1} - X^n\big\|^2_{\mathfrak{I},[0,t]} \\ \le\ &\big(9\,\Lambda^2\,(1+t)\big)^n \cdot \int_0^t ds_n \int_0^{s_n} ds_{n-1} \ldots \int_0^{s_2} \big\|X^1 - Y\big\|^2_{\mathfrak{I},[0,s_1]}\ ds_1 \\ \le\ &\big(9\,\Lambda^2\,(1+t)\big)^n \cdot \int_0^t \big\|X^1 - Y\big\|^2_{\mathfrak{I},[0,s_n]}\ \tfrac{(t-s_n)^{n-1}}{(n-1)!}\ ds_n \\ \le\ &\tfrac{1}{n!}\,\big(9\,\Lambda^2\,(1+t)\,t\big)^n \cdot \big\|X^1 - Y\big\|^2_{\mathfrak{I},[0,t]} \\ \le\ &\tfrac{1}{n!}\,\big(3\,\Lambda\ \ (1+t)\big)^{2n} \cdot \big\|X^1 - Y\big\|^2_{\mathfrak{I},[0,t]} \end{aligned}$$

due to Lemma A.68.

The series

$$c(t) := \sum_{n=0}^{\infty} \tfrac{1}{\sqrt{n!}}\ (3\,\Lambda\ (1+t))^n$$

is absolutely convergent for every $t \in \mathbb{R}$ as d'Alembert's ratio test reveals. Hence, $(X^n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathfrak{I},[0,t]}$ for each $t \in [0,T]$.

Then there exist limits $\gamma \in \mathscr{L}^2_{\mathscr{A}}([0,T],\,\mathbb{R}^N)$, $\sigma \in \mathscr{L}^2_{\mathscr{A}}([0,T],\,\mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^N))$ and $X \in \mathscr{L}^2_{\mathscr{A}}([0,T],\,\mathbb{R}^N)$ of $(\gamma_n)_{n\in\mathbb{N}}$, $(\sigma_n)_{n\in\mathbb{N}}$, $(X^n)_{n\in\mathbb{N}}$ respectively with

$$X = \mathfrak{I}_0\big(X_0,\gamma,\sigma\big).$$

Furthermore, we conclude

$$\gamma(t,\omega) \in \widetilde{F}_1\big(t,\,\omega,X_t(\omega)\big), \qquad \sigma(t,\omega) \in \widetilde{F}_2\big(t,\,\omega,X_t(\omega)\big)$$

for $(\mathscr{L}^1\times P)$-*almost* all $(t,\omega) \in [0,T]\times\Omega$ from the facts that some subsequences of $(\gamma_n)_{n\in\mathbb{N}}$, $(\sigma_n)_{n\in\mathbb{N}}$, $(X^n)_{n\in\mathbb{N}}$ converge to their respective limits pointwise almost everywhere in $[0,T]\times\Omega$ and that $\widetilde{F}(t,\omega,\cdot)$ is continuous by assumption (iii).

Finally, X satisfies at each time $t \in [0,T]$

$$\begin{aligned}
\big\|X-Y\big\|_{\mathfrak{I},[0,t]} &\le \sum_{n=1}^{\infty} \big\|X^{n+1}-X^n\big\|_{\mathfrak{I},[0,t]} && + \big\|X^1-Y\big\|_{\mathfrak{I},[0,t]}\\
&\le \sum_{n=1}^{\infty} \tfrac{(3\,\Lambda\,(1+t))^n}{\sqrt{n!}}\ \big\|X^1-Y\big\|_{\mathfrak{I},[0,t]} && + \big\|X^1-Y\big\|_{\mathfrak{I},[0,t]}\\
&= \sum_{n=0}^{\infty} \tfrac{(3\,\Lambda\,(1+t))^n}{\sqrt{n!}}\ \big\|X^1-Y\big\|_{\mathfrak{I},[0,t]},
\end{aligned}$$

i.e.
$$\begin{aligned}
\big\|X-Y\big\|^2_{\mathfrak{I},[0,t]} &\le c(t)^2\ \big\|X^1-Y\big\|^2_{\mathfrak{I},[0,t]}\\
&= c(t)^2\ \Big(\mathbb{E}\big(|X_0-Y_0|^2\big)+\mathbb{E}\Big(\int_0^t \mathrm{dist}\big((\gamma_0,\sigma_0),\ \widetilde{F}(s,Y_s)\big)^2\,ds\Big)\Big).
\end{aligned}$$

In regard to the claimed estimate, we have to verify $c(t)^2 \le \mathrm{const}\cdot e^{\,\mathrm{const}\cdot(1+t)\ t}$. Due to absolute convergence, $c(\cdot)>0$ is analytic in $[0,\infty[$ and,

$$\begin{aligned}
0 \le \tfrac{d}{dt}\,c(t) &= \sum_{n=1}^{\infty} \tfrac{n}{\sqrt{n!}}\ (3\,\Lambda)^n\ (1+t)^{n-1}\\
&= 3\,\Lambda + \sum_{n=2}^{\infty} \sqrt{\tfrac{n}{(n-1)!}}\ (3\,\Lambda)^n\ (1+t)^{n-1}\\
&\le 3\,\Lambda + \sum_{n=2}^{\infty} \sqrt{\tfrac{2}{(n-2)!}}\ (3\,\Lambda)^n\ (1+t)^{n-1}\\
&= 3\,\Lambda + \sqrt{2}\ (3\,\Lambda)^2\,(1+t)\cdot \sum_{m=0}^{\infty} \tfrac{1}{\sqrt{m!}}\ (3\,\Lambda)^m\ (1+t)^m\\
&= 3\,\Lambda + \sqrt{2}\ (3\,\Lambda)^2\,(1+t)\cdot c(t)
\end{aligned}$$

implies

$$c(t) \le \big(c(0) + 3\,\Lambda\,t\big)\cdot e^{18\,\Lambda^2\,(1+t)\,t} \le \big(c(0)+1\big)\cdot e^{3\,\Lambda\,(1+6\,\Lambda)\,(1+t)\,t}$$

for all $t \ge 0$ by means of Gronwall's inequality. (This upper bound is quite simple to prove, but obviously not optimal.) □

A.8.2 A Sufficient Condition on Invariant Subsets

In the field of stochastic differential inclusions, the aspects of invariance and viability have been investigated thoroughly by several authors like Aubin, Da Prato and Frankowska [18], Michta [139, 140], Motyl [142], Truong-Van & Truong [178]. A broad survey is presented in [104].

For the sake of completeness, we introduce the required notions of contingent cone briefly and then just quote the sufficient result, which can be regarded as the stochastic counterpart of Proposition A.8 (on page 445).

Definition 69 (Stochastic contingent set [17, Definition 1.1], [18]**).**
Let $K : \Omega \rightsquigarrow \mathbb{R}^N, \omega \mapsto K_\omega$ be a random closed set, i.e. here: K is an $\mathscr{A}_0$-measurable set-valued map with nonempty closed values. For some $t \geq 0$, consider a $\mathscr{A}_t$-measurable selection $x : \Omega \longrightarrow \mathbb{R}^N$ of K.
The *stochastic contingent set* $T_K^S(t,x)$ to K at x with respect to $\mathscr{A}_t$ is defined as the set of pairs (η, v) of $\mathscr{A}_t$-random variables satisfying the following property: There exist sequences $(h_n)_{n\in\mathbb{N}}$ in $]0,\infty[$ and $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ of $\mathscr{A}_{t+h_n}$-random variables such that

$$\begin{cases} h_n \longrightarrow 0 & \text{for } n \longrightarrow \infty, \\ \mathbb{E}\big(|a_n|^2\big) \longrightarrow 0 & \text{for } n \longrightarrow \infty, \\ \mathbb{E}\big(|b_n|^2\big) \longrightarrow 0 & \text{for } n \longrightarrow \infty, \\ \mathbb{E}(b_n) = 0, & \\ b_n \text{ is independent of } \mathscr{A}_t & \end{cases}$$

and for each $n \in \mathbb{N}$, the sum $x + v\,\big(W_{t+h_n} - W_t\big) + h_n\,\eta + h_n\,a_n + \sqrt{h_n}\,b_n$ is a square integrable selection of K.

Proposition 70 (Sufficient condition for invariance, [18, Theorem 5.1]**).**
Let $F = (F_1, F_2) : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N \times \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^N)$ be a Lipschitz continuous set-valued map with nonempty compact values. Suppose $K : \Omega \rightsquigarrow \mathbb{R}^N$ to be an $\mathscr{A}_0$-measurable set-valued map with nonempty closed values satisfying for all $t \geq 0$ and each $\mathscr{A}_t$-measurable selection $x : \Omega \longrightarrow \mathbb{R}^N$ of K

$$F(x) \subset T_K^S(t,x) \qquad \textit{almost everywhere in } \Omega.$$

Then the random closed set K is invariant *under F in the following sense: Every Itô process $X := \mathfrak{I}_0(X_0, \gamma, \sigma)$ starting in K and solving the stochastic diff. inclusion*

$$dX_t \,\in\, F_1(X_t)\,dt \,+\, F_2(X_t)\,dW_t$$

i.e. $X_0 \in L^2(\Omega, \mathscr{A}_0, P;\, \mathbb{R}^N)$, $\gamma \in \mathscr{L}^2_{\mathscr{A}}([0,T],\, \mathbb{R}^N)$ and $\sigma \in \mathscr{L}^2_{\mathscr{A}}([0,T],\, \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^N))$ with

$$\begin{cases} X_0 \in K \quad a.e., \\ \gamma(t,\omega) \in F_1\big(X_t(\omega)\big) \\ \sigma(t,\omega) \in F_2\big(X_t(\omega)\big) \end{cases}$$

for $(\mathscr{L}^1 \times P)$-almost all $(t,\omega) \in [0,T] \times \Omega$, satisfies $X_t \in K$ almost everywhere in Ω for each $t \geq 0$.

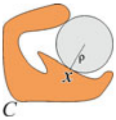
A.9 Proximal Normals of Set Sequences in $\mathbb{R}^N$

Comparing the proximal normals of a converging sequence $(K_n)_{n\in\mathbb{N}}$ in $(\mathscr{K}(\mathbb{R}^N), dl)$ with the normals of its limit $K \in \mathscr{K}(\mathbb{R}^N)$, the following inclusion is not difficult to prove by means of exterior balls and, it has already been quoted in Proposition 4.23 (on page 360)

$$\text{Graph } N_K^P \ \subset\ \text{Limsup}_{n\to\infty} \text{ Graph } N_{K_n}^P$$

(see e.g. [50, Lemma 4.1]). Of course, the equality here is not fulfilled in general. A key advantage of the subset $N_{K,\rho}^P$ $(\rho > 0)$ specified equivalently in Definition 4.40 (on page 373) is that an inverse inclusion is satisfied.
The following proposition provides the inclusions in both directions and their proofs.

Definition 71. Let $C \subset \mathbb{R}^N$ be a nonempty closed set.



For any $\rho > 0$, the set $N_{C,\rho}^P(x) \subset \mathbb{R}^N$ consists of all proximal normal vectors $\eta \in N_C^P(x) \setminus \{0\}$ with the proximal radius $\geq \rho$ (and thus might be empty).
Furthermore define ${}^\flat N_{C,\rho}^P(x) := N_{C,\rho}^P(x) \cap \mathbb{B}$.

Proposition 72. *Let $(K_n)_{n\in\mathbb{N}}$ be a converging sequence in $\mathscr{K}(\mathbb{R}^N)$ and K its limit. Π_{K_n}, $\Pi_K : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ denote the projections on K_n, K $(n \in \mathbb{N})$ respectively, i.e.,*

$$\Pi_K : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N, \quad x \mapsto \big\{y \in K \,\big|\, |y-x| = \text{dist}(x,K)\big\} \subset \mathbb{R}^N.$$

Then,

(1.) $\text{Limsup}_{n\to\infty} \text{ Graph } {}^\flat N_{K_n,\rho}^P \subset \text{Graph } {}^\flat N_{K,\rho}^P$ *for any* $\rho > 0$,

(2.) $\underset{\substack{y\to x\\ n\to\infty}}{\text{Limsup}}\ \Pi_{K_n}(y) \subset \Pi_K(x)$ *for any* $x \in \mathbb{R}^N$,

(3.) $\text{Graph } {}^\flat N_{K,\rho}^P \subset \text{Liminf}_{n\to\infty} \text{ Graph } {}^\flat N_{K_n,r}^P$ *for any* $0 < r < \rho$.

Proof.
(1.) Choose any converging sequence $\big((x_{n_j}, p_{n_j})\big)_{j\in\mathbb{N}}$ with $p_{n_j} \in N_{K_{n_j},\rho}^P(x_{n_j}) \cap \partial\mathbb{B}$ and set $x := \lim_{j\to\infty} x_{n_j} \in K$, $p := \lim_{j\to\infty} p_{n_j} \in \partial\mathbb{B}$. According to Definition A.23 (on page 454), each K_{n_j} is contained in the complement of the open ball with center $x_{n_j} + \rho\, p_{n_j}$ and radius ρ,

$$K_{n_j} \ \subset\ \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_\rho\big(x_{n_j} + \rho\, p_{n_j}\big).$$

As an indirect consequence, $j \longrightarrow \infty$ leads to

$$K \ \subset\ \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_\rho(x + \rho\, p),$$

i.e.

$$p \in N_{K,\rho}^P(x).$$

(2.) Let $r > 0$ and $n \in \mathbb{N}$ be arbitrary. For $y \in \mathbb{B}_r(x)$ given, choose any $z \in \Pi_{K_n}(y)$ and $\xi \in \Pi_K(z)$. Then,

$$|\xi - z| \le d\!l(K_n, K)$$

and

$$\begin{aligned} |x-\xi| &\le |x-y| + |y-z| + |z-\xi| \\ &\le |x-y| + \operatorname{dist}(y,K) + d\!l(K,K_n) + |z-\xi| \\ &\le |x-y| + |y-x| + \operatorname{dist}(x,K) + d\!l(K,K_n) + d\!l(K_n,K) \\ &\le 2r + \operatorname{dist}(x,K) + 2\, d\!l(K_n,K). \end{aligned}$$

Thus, $\Pi_{K_n}(y) \subset \mathbb{B}_{d\!l(K_n,K)}\Big(K \cap \mathbb{B}_{2r + \operatorname{dist}(x,K) + 2\, d\!l(K_n,K)}(x)\Big)$ for any $y \in \mathbb{B}_r(x)$.

The set-valued map $[0,\infty[\rightsquigarrow \mathbb{R}^N,\ r \mapsto K \cap \mathbb{B}_r(x)$ is upper semicontinuous (due to [19, Corollary 1.4.10]) and in the closed interval $[\operatorname{dist}(x,K),\infty[$, it has nonempty compact values. For every $\eta > 0$, there exists $\rho = \rho(x,\eta) \in]0,\eta[$ such that

$$K \cap \mathbb{B}_{r'}(x) \subset \mathbb{B}_\eta\big(\Pi_K(x)\big)$$

for all $r' \in \big[\operatorname{dist}(x,K),\ \operatorname{dist}(x,K) + 2\rho\big]$. Due to $d\!l(K_n,K) \longrightarrow 0\ (n \longrightarrow \infty)$, there is an index $m \in \mathbb{N}$ with $d\!l(K_n,K) \le \frac{\rho}{4}$ for all $n \ge m$. Thus we obtain for every point $y \in \mathbb{B}_{\rho/4}(x) \cap \mathbb{B}_r(x)$ and index $n \ge m$

$$\begin{aligned} \Pi_{K_n}(y) &\subset \mathbb{B}_{\frac{\rho}{4}}\big(K \cap \mathbb{B}_{2\frac{\rho}{4} + \operatorname{dist}(x,K) + 2\frac{\rho}{4}}(x)\big) &&= \mathbb{B}_{\frac{\rho}{4}}\big(K \cap \mathbb{B}_{\operatorname{dist}(x,K)+\rho}(x)\big) \\ &\subset \mathbb{B}_{\frac{\rho}{4}}\big(\mathbb{B}_\eta(\Pi_K(x))\big) &&\subset \mathbb{B}_{2\eta}\big(\Pi_K(x)\big), \end{aligned}$$

i.e. $\operatorname{Limsup}_{\substack{y \to x \\ n \to \infty}} \Pi_{K_n}(y) \subset \Pi_K(x)$.

(3.) Choose any $x \in \partial K$ and $p \in N^P_{K,\rho}(x) \ne \emptyset$ with $|p| = 1$.

Then x is the unique projection of $x + \delta\, p$ on the set K for every $\delta \in]0,\rho[$. Considering now a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \Pi_{K_n}(x + \delta\, p) \subset K_n$, the preceding statement (2.) implies $x_n \longrightarrow x$ and, the definition of proximal normal ensures

$$p_n := \frac{x + \delta\, p - x_n}{|x + \delta\, p - x_n|} \in {}^\flat N^P_{K_n}(x_n)$$

converging to p for $n \longrightarrow \infty$.
Finally the proximal radius of p_n is $\ge |x + \delta\, p - x_n| \ge \delta - |x - x_n|$, and thus,

$$(x,p) \in \operatorname{Liminf}_{n\to\infty} \operatorname{Graph}\, {}^\flat N^P_{K_n,r} \quad \text{for every } 0 < r < \delta < \rho.$$

□

A.10 Tools for Set-Valued Maps

A.10.1 Measurable Set-Valued Maps

In this section we summarize some useful results about set-valued maps in regard to measurability. The monograph of Castaing and Valadier [40] is usually regarded as a standard reference providing many of the well-known results. Here we quote the corresponding theorems from the monograph of Aubin and Frankowska [19].

Definition 73 ([19, Definition 8.1.1]). Consider a measurable space $(\Omega, \mathscr{A})$, a complete separable metric space E and a set-valued map $F : \Omega \leadsto E$ with closed images.
F is called *measurable* if the inverse image of each open set is a measurable set, i.e., for every open set $O \subset E$,

$$F^{-1}(O) \stackrel{\text{Def.}}{=} \left\{\omega \in \Omega \;\middle|\; F(\omega) \cap O \neq \emptyset\right\} \in \mathscr{A}.$$

Theorem 74 (Kuratowski and Ryll-Nardzewski [109], [19, Theorem 8.1.3]). *Let E be a complete separable metric space, $(\Omega, \mathscr{A})$ a measurable space, $F : \Omega \leadsto E$ a measurable set-valued map with nonempty closed values.*
Then there exists a measurable selection of F, i.e., a measurable single-valued function $f : \Omega \longrightarrow E$ satisfying $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$.

Theorem 75 (Characterization Theorem [19, Theorem 8.1.4]). *Let $(\Omega, \mathscr{A}, \mu)$ be a complete σ-finite measure space, E a complete separable metric space and $F : \Omega \leadsto E$ a set-valued map with nonempty closed values.*
Then the following properties are equivalent:

(i) *F is measurable.*
(ii) *The graph of F belongs to $\mathscr{A} \otimes \mathscr{B}$.*
(iii) *$F^{-1}(C) \in \mathscr{A}$ for every closed set $C \subset E$.*
(iv) *$F^{-1}(B) \in \mathscr{A}$ for every Borel set $B \subset E$.*
(v) *For each element $x \in E$, the function $\operatorname{dist}(x, F(\cdot)) : \Omega \longrightarrow [0, \infty[$ is measurable.*
(vi) *There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable selections of F such that*

$$F(\omega) = \overline{\bigcup_{n \in \mathbb{N}} f_n(\omega)} \qquad \text{for every } \omega \in \Omega.$$

Corollary 76 (Upper and lower semicontinuous maps [19, Proposition 8.2.1]). *Consider a metric space Ω and a complete σ-finite measure space $(\Omega, \mathscr{A}, \mu)$ such that $\mathscr{A}$ contains all open subsets of Ω. Let E be a complete separable metric space and $F : \Omega \leadsto E$ a set-valued map with nonempty closed images.*

If F is upper semicontinuous, then F is measurable.
If F is lower semicontinuous, then F is measurable.

Proposition 77 (Closed union and intersection [19, Theorem 8.2.4]**).**
Let $(\Omega, \mathscr{A}, \mu)$ be a complete σ-finite measure space, E a complete separable metric space and $F_n : \Omega \rightsquigarrow E$ $(n \in \mathbb{N})$ set-valued maps with nonempty closed values.

Then the following set-valued maps are measurable:

$$\Omega \rightsquigarrow E, \quad \omega \mapsto \overline{\bigcup_{n\in\mathbb{N}} F_n(\omega)},$$
$$\Omega \rightsquigarrow E, \quad \omega \mapsto \bigcap_{n\in\mathbb{N}} F_n(\omega).$$

Proposition 78 (Direct image [19, Theorem 8.2.8]**).**
Let $(\Omega, \mathscr{A}, \mu)$ be a complete σ-finite measure space, E_1, E_2 complete separable metric spaces and $F : \Omega \rightsquigarrow E_1$ a measurable set-valued map with nonempty closed values. Consider a Carathéodory set-valued map $G : \Omega \times E_1 \rightsquigarrow E_2$, i.e.,
for every $x \in E_1$, the map $G(\,\cdot\,, x) : \Omega \rightsquigarrow E_2$ is measurable and
for every $\omega \in \Omega$, the map $G(\omega, \cdot) : E_1 \rightsquigarrow E_2$ is continuous.

Then the set-valued map $\Omega \rightsquigarrow E_2$, $\omega \mapsto \overline{G(\omega, F(\omega))}$ is measurable.

Proposition 79 (Inverse image, Filippov selection [19, Theorems 8.2.9, 8.2.10]**).**
Consider a complete σ-finite measure space $(\Omega, \mathscr{A}, \mu)$, complete separable metric spaces E_1, E_2 and measurable set-valued maps $F : \Omega \rightsquigarrow E_1$, $G : \Omega \rightsquigarrow E_2$ with nonempty closed values. Let $g : \Omega \times E_1 \longrightarrow E_2$ be a Carathéodory function.

Then the set-valued map
$$\Omega \rightsquigarrow E_1, \quad \omega \mapsto \big\{x \in F(\omega) \,\big|\, g(\omega, x) \in G(\omega)\big\} \subset E_1$$
is measurable.

Consequently, if $g(\omega, F(\omega)) \cap G(\omega)$ is nonempty for every $\omega \in \Omega$, then there exists a measurable selection $f : \Omega \longrightarrow E_1$ of F such that for every $\omega \in \Omega$, the element $g(\omega, f(\omega))$ belongs to $G(\omega)$.

In particular, for every measurable function $h : \Omega \longrightarrow E_2$ with $h(\omega) \in g(\omega, F(\omega))$ for almost all $\omega \in \Omega$, there exists a measurable selection $f : \Omega \longrightarrow E_1$ of F with $h = g(\cdot, f(\cdot))$ almost everywhere in Ω.

Proposition 80 (Marginal map [19, Theorem 8.2.11]**).**
Consider a complete σ-finite measure space $(\Omega, \mathscr{A}, \mu)$, a complete separable metric space E, a measurable set-valued map $F : \Omega \rightsquigarrow E$ with nonempty closed values and a real-valued Carathéodory function $f : \Omega \times E \longrightarrow \mathbb{R}$.

Then the so-called marginal function
$$\Omega \longrightarrow \mathbb{R} \cup \{-\infty\}, \quad \omega \longmapsto \inf_{x\in F(\omega)} f(\omega, x)$$
is measurable. Furthermore the so-called marginal map
$$\Omega \rightsquigarrow E, \quad \omega \mapsto \left\{x \in F(x) \,\middle|\, f(\omega, x) = \inf_{y\in F(\omega)} f(\omega, y)\right\} \subset E$$
is measurable.

A.10.2 Parameterization of Set-Valued Maps

Proposition 81 **(**[19, Theorem 9.7.2]**).**
Consider a metric space X and a set-valued map $G : [a,b] \times X \rightsquigarrow \mathbb{R}^N$ *satisfying*

1. *G has nonempty compact convex values,*
2. $G(\cdot,x) : [a,b] \rightsquigarrow \mathbb{R}^N$ *is measurable for every* $x \in X$,
3. *there exists* $k(\cdot) \in L^1([a,b])$ *such that for every* $t \in [a,b]$, *the set-valued map* $G(t,\cdot) : X \rightsquigarrow \mathbb{R}^N$ *is* $k(t)$*-Lipschitz continuous.*

Then there exists a single-valued function $g : [a,b] \times X \times \mathbb{B}_1 \longrightarrow \mathbb{R}^N$ (*with the closed unit ball* $\mathbb{B}_1 \subset \mathbb{R}^N$) *fulfilling for all* $t \in [a,b]$, $x \in X$, $u,v \in \mathbb{B}_1$ *respectively*

1. $G(t,x) = \bigcup_{w \in \mathbb{B}_1} g(t,x,w)$,
2. $g(\cdot,x,u) : [a,b] \longrightarrow \mathbb{R}^N$ *is measurable,*
3. $g(t,\cdot,u) : X \longrightarrow \mathbb{R}^N$ *is* $c \cdot k(t)$*-Lipschitz continuous*
4. $|g(t,x,u) - g(t,x,v)| \leq c \, \|G(t,x)\|_\infty \, |u-v|$

with a constant $c > 0$ *independent of G.*

A.11 Compactness of Continuous Functions Between Metric Spaces

The essential compactness result about continuous functions between metric spaces is the Arzelà-Ascoli Theorem. We use it in the following version of Green and Valentine:

Theorem 82 (Arzelà-Ascoli in metric spaces [88]**).**
Let (E_1,d_1), (E_2,d_2) *be two precompact metric spaces, i.e. for any* $\varepsilon > 0$, *each set* E_i $(i = 1,2)$ *can be covered by finitely many* ε*-balls with respect to metric* d_i.
Moreover, suppose the sequence $(f_n)_{n\in\mathbb{N}}$ *of functions* $E_1 \longrightarrow E_2$ *to be uniformly equi-continuous* (*i.e. with a common modulus of continuity in* E_1).
Then there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ *being Cauchy sequence with respect to uniform convergence. If* (E_2,d_2) *is complete in addition, then* $(f_{n_j})_{j\in\mathbb{N}}$ *converges uniformly to a continuous function* $E_1 \longrightarrow E_2$.

Kisielewicz characterized *weakly* compact sets in the space of Banach-valued continuous functions. His result can be interpreted as a “weak counterpart” of the Arzelà-Ascoli Theorem.

Proposition 83 (Kisielewicz [102, Theorem 4]**).**
Let S be a compact Hausdorff space and X a Banach space.
A subset $W \subset C^0(S,X)$ *is weakly compact in* $\big(C^0(S,X), \|\cdot\|_{\sup}\big)$ *if it is bounded, equi-continuous and if for every* $s \in S$, *the set* $\{f(s) \mid s \in S\}$ *is relatively weakly compact in* X.

A.12 Bochner Integrals and Weak Compactness in L^1

The so-called Bochner integral extends the familiar concept of integration from real-valued functions to Banach-valued functions on the basis of "simple" functions.

Definition 84 ([63]). Let (Ω,Σ,μ) be a finite measure space and X a Banach space. A function $f:\Omega \longrightarrow X$ is called *simple* if there exist $x_1,x_2\ldots x_n \in X$ and $E_1,E_2\ldots E_n \in \Sigma$ such that $f=\sum_{j=1}^n x_j\,\chi_{E_j}$ with $\chi_{E_j}:\Omega \longrightarrow \{0,1\}$ denoting the characteristic function of $E_j \subset \Omega$.
A function $f:\Omega \longrightarrow X$ is called μ*-measurable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions $\Omega \longrightarrow X$ with $\|f-f_n\|_X \longrightarrow 0$ μ-almost everywhere for $n\to\infty$.
A μ-measurable function $f:\Omega \longrightarrow X$ is called *Bochner integrable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions $\Omega \longrightarrow X$ such that

$$\lim_{n\to\infty} \int_\Omega \|f-f_n\|_X \, d\mu \;=\; 0.$$

Then, the *Bochner integral* of f over $E\in\Sigma$ is defined by $\displaystyle\int_E f\,d\mu := \lim_{n\to\infty}\int_E f_n\,d\mu$.

Let $L^1(\mu,X)$ denote the Banach space of Bochner integrable functions $\Omega \longrightarrow X$ equipped with its usual L^1 norm.

In the nineties, Ülger proved that restricting the values of Bochner integrable functions to a weakly compact subset of X implies the relative weak compactness of these functions in $L^1(\mu,X)$. For real-valued Lebesgue integrable functions, this is closely related with Alaoglu's Theorem and a compact embedding.

Proposition 85 ([179, Proposition 7]). *Let (Ω,Σ,μ) be a probabilistic space, X an arbitrary Banach space. For any weakly compact subset $W\subset X$, the set*

$$\big\{\,h\in L^1(\mu,X) \;\big|\; h(\omega)\in W \text{ for } \mu\text{-almost every } \omega\in\Omega\big\}$$

is relatively weakly compact.

An earlier version of this result is presented in [61] and, [62] considers weak compactness of Bochner integrable functions with values in an arbitrary Banach space under weaker assumptions (see also [22]). The next proposition of Ülger provides a "weakly pointwise" characterization of weakly convergent sequences in $L^1(\mu,X)$.

Proposition 86 ([179, Corollary 5]). *Let (Ω,Σ,μ) be a probabilistic space and X an arbitrary Banach space as in preceding Proposition A.85.*
Set $W := \big\{g\in L^1(\mu,X) \,\big|\, |g(\omega)|\le 1 \text{ for } \mu\text{-almost every } \omega\in\Omega\big\}$.

A sequence $\big(g_n(\cdot)\big)_{n\in\mathbb{N}}$ in $W\subset L^1(\mu,X)$ converges weakly to $g\in L^1(\mu,X)$ if and only if for any subsequence $\big(g_{n_k}(\cdot)\big)_{k\in\mathbb{N}}$ given, there exists a sequence $\big(h_k(\cdot)\big)_{k\in\mathbb{N}}$ with $h_k\in co\big\{g_{n_k},\, g_{n_{k+1}}\,\ldots\big\}$ such that for μ-almost every $\omega\in\Omega$,

$$h_k(\omega) \longrightarrow g(\omega) \qquad (k\longrightarrow\infty) \qquad \text{weakly in } X.$$

Appendix B
Bibliographical Notes

Chapter 1

This chapter reflects the theory of mutational equations as it was introduced by Jean-Pierre Aubin in the 1990s [10, 12, 13]. It extends earlier results about integral funnel equations – for describing set evolutions with feedback. Similar concepts have been introduced by Russian mathematicians in the 1980s and 1990s. Among the more popular examples for metric spaces are the so-called *quasidifferential equations* of Panasyuk (see [150, 153] and references there). Further approaches to generalized differential equations in metric spaces are suggested in [31, 108, 113, 146] later.
Both the structure and the proofs in Chapter 1 are adapted to the generalizations in subsequent chapters so that the new aspects there are easier to identify.

§ 1.9.3 provides new results in comparison with Aubin's monograph [10]: The link between morphological primitives and reachable sets of nonautonomous differential inclusions. The analytical tools are summarized in Appendix A.3.
The examples of morphological primitives in § 1.9.4 are motivated by several questions of Robert Baier during our joint research stay at the Hausdorff Research Institute for Mathematics (HIM) in Bonn in spring 2008.
§ 1.9.5 is mostly based on earlier results of Anne Gorre quoted in Aubin's monograph [13]. Proposition 69 provides a partial answers to an open question that Jean-Pierre Aubin posed the author in November 2007. The closely related conclusions are drawn in Corollary 78.

§ 1.10 presents a set-valued approach to image segmentation that was published by the author in [131] in 2001.

§ 1.11 was developed during the stay at HIM in Bonn after the author had learned more about one-sided Lipschitz maps in the survey lectures of Tzanko Donchev.

Chapter 2

This chapter provides the first extensions of the mutational framework in comparison with Aubin's monograph [10]. They are based on the key notion that the parameters of transitions are just locally uniform.
Continuity parameters *with linear growth* were introduced in the first version of preprint [129] about transport equations for Radon measures in 2005. Later the linear growth condition was weakened to locally uniform bounds as in this chapter. These details were presented in the preprint [126] for the first time and used in [91]. The results about existence with delay and under state constraints in § 2.3.5 and § 2.3.6 respectively have been developed in the initial version of this monograph, i.e. habilitation thesis [117].

The example in § 2.4 dealing with semilinear evolution equations (and the weak topology) in the mutational framework has already been suggested in the author's Ph.D. thesis [130] in 2004.

The Cauchy problem of nonlinear transport equations for Radon measures on $\mathbb{R}^N$ was discussed in the preprint [126] with the same kind of transitions, but another metric and restricted to positive Radon measures with compact support. Hence the results of § 2.5 using the $W^{1,\infty}$ dual metric and solutions in the mutational framework are new in the initial version [117] of this book.

The nonlinear structured population model in § 2.6 provides the main conclusions of [91], which was jointly elaborated with Piotr Gwiazda (Warsaw) and Anna Marciniak-Czochra (Heidelberg).

In § 2.7, morphological equations are modified in a very "natural" way as transitions on $\mathscr{K}(\mathbb{R}^N)$ are now induced by reachable sets of differential inclusions *with linear growth*. In particular, this opens the door to applying the mutational framework to reachable sets of *linear* differential inclusions.

Chapter 3

It provides three substantial contributions of this monograph to mutational analysis:

1. Continuity conditions on distances make the triangle inequality dispensable,
2. sequential continuity of transitions with respect to state and time are handled by separate families of distances,
3. ω-contractivity of transitions (in the sense that the initial distance between states may grow at most exponentially while evolving along one and the same transition) proves to be dispensable under additional assumptions.

Currently the author is not aware of any other approach similar to mutational or quasidifferential equations beyond metric *spaces.*

Nonlocal stochastic differential equations as discussed in § 3.6 were introduced in the initial version [117] of this monograph (to the best of our knowledge). A relevant extension to nonadditive noise is sketched in [119] and generalized in [105].

In comparison with the thesis [117], the statements about time-dependent random closed sets in § 3.7 and nonlocal parabolic equations in cylindrical domains in § 3.9 belong to the new contributions here.

The only further suggestion (for constructing stochastic birth-and-growth processes via random closed sets in continuous time) which the author has found in the literature so far was made by Aletti, Bongiorno and Capasso in the preprint [1] in 2008. Their proposal is based on random closed sets in a reflexive Banach space and takes the aspect of predictability (via filtration) into consideration explicitly. Due to the Aumann integral as an essential ingredient, however, it is restricted to both convex-valued and expanding growth processes [1, Theorem 1.3.2]. Moreover, it assumes the nucleation process to be expanding and so, the final birth-and-growth processes are expanding P-almost surely in Ω [1, Theorem 1.4.9]. The set-valued approach in § 3.7 here has been developed independently (as a part of a DFG project, the author applied for in 2007). From our current point of view, this concept can be adapted to random closed sets in a separable Banach space Y if $\mathscr{RC}^2(\Omega, Y)$ proves to be complete with respect to $d\!\!l_{\mathscr{RC}}$ and if a counterpart of the Da Prato-Frankowska Theorem A.67 holds (see e.g. the original article [53]).

With regard to § 3.8, nonlinear continuity equations with coefficients of bounded variation were investigated as examples of mutational equations in the preprint [129] after attending the lectures of Prof. Ambrosio in a C.I.M.E. summer school in 2005.

The conclusions about semilinear evolution equations in § 3.10 and about parabolic differential equations in noncylindrical domains in § 3.11 respectively are also developed originally in the author's thesis [117] and published here now.

During the Czech-German-French Conference on Optimization in Heidelberg in September 2007 and a workshop at HIM Bonn in March 2008, José Alberto Murillo Hernández (Cartagena, Spain) reported about the heat equation in a domain governed by a morphological equation — similarly to § 3.11.5.
His conclusions were based on the results [116] of Límaco, Medeiros and Zuazua and thus, the noncylindrical domain had to obey bi-Lipschitz transformations to a reference domain. As a consequence, the morphological transitions were restricted to bounded Lipschitz continuous vector fields (instead of the set-valued maps in $\mathrm{LIP}(\mathbb{R}^N, \mathbb{R}^N)$).

Chapters 4 and 5

The author suggested the notion of distribution-like solutions in his Ph.D. thesis [130], but still for tuples with non-symmetric distance functions which fulfill the timed triangle inequality. The example in § 4.4 was also presented in [124, 130]. The second geometric example here in § 4.5 was introduced in [120] in 2008.

In regard to mutational inclusions, the existence results of § 5.1 have been developed in connection with the thesis [117] recently and, they are published in [118]. § 5.2 about the viability theorem for morphological inclusions was prepared in [122] and published in its full generality in [121].
The corresponding approach to control problems (in § 5.3) has its origin in preprint [125] and was motivated by conversations with Zvi Artstein at Weizmann Institute of Science in Rehovot (Israel) in summer 2007.

Appendix A

The generalizations of Gronwall's inequality are essentially new. In particular, Proposition A.2 has less restrictive assumptions than all the other versions which the author found in the literature. It lays the foundations for concluding global estimates from local properties (Lebesgue-almost everywhere). Proposition A.4 has already been presented in Ph.D. thesis [130].

Section A.3 provides the tools for the link between morphological primitives and reachable sets: the integral funnel equation in Proposition A.13. Following a strategy close to the one of Frankowska, Plaskacz and Rzeżuchowski in [83], the author has proved this connection in 2006 and reused these arguments in [121, Corollary 3.14] and [122] later. He developed these proofs independently from earlier results of Tolstonogov [177], which the author found while writing his thesis [117] since 2008.

Most of the results in section A.5 were introduced and proved in [120, 124, 130]. In particular, they were developed by the author independently from the article [33] of Cannarsa and Frankowska (about the interior sphere property of reachable sets of control equations). The consequences of the uniform tusk condition in A.5.7 are presented in [117] and published here for the first time.
Originally Reynolds Transport Theorem in § A.6 was extended to differential inclusions for applications in image segmentation [131]. The author published its complete proof in [127].

Sections A.2, A.4, A.7, A.8 and A.10 – A.12 summarize standard results which are mostly quoted and prove to be useful in this monograph.

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