Appendix A
Entropy Formula of Pesin Type for One-sided Stationary Random Maps

In this appendix we consider random dynamical systems (abbreviated as RDS's) generated by compositions of one-sided stationary random endomorphisms of class C^2 of a compact manifold following the line of [52]. We will first introduce the notions of entropy and Lyapunov exponents for such RDS's and then prove that the entropy formula of Pesin type holds if the sample measures of an invariant measure are absolutely continuous with respect to the Lebesgue measure on the manifold. This result covers those obtained by Ledrappier and Young [44] and Liu [48] for i.i.d. (independent and identically distributed) random diffeomorphisms or (noninvertible) endomorphisms and that obtained by [48] for two-sided stationary random endomorphisms. As far as the phase spaces are compact and finite dimensional manifolds without boundary, this result may be considered as the almost final form of Pesin entropy formula for RDS's with absolutely continuous invariant or sample measures.

A.1 Basic Notions

A.1.1 Set-up

In this appendix M will always be a compact Riemannian manifold without boundary. Let $C^r(M,M)(r \geq 1$ integer) be the space of all C^r endomorphisms on M endowed with the C^r topology and the Borel σ-algebra. Put $\Omega = C^r(M,M)^{\mathbb{Z}^+}$ and let it have the product σ-algebra $\mathscr{A}$. Assume that P is a probability on $(\Omega, \mathscr{A})$ which is invariant under the left shift operator θ on Ω. For each $\omega \in \Omega$ we write $\omega = (f_0(\omega), f_1(\omega), \cdots)$ and define

$$f_\omega^n = \begin{cases} \mathrm{id} & \text{if } n = 0 \\ f_{n-1}(\omega) \circ \cdots \circ f_0(\omega) & \text{if } n > 0. \end{cases} \tag{A.1}$$

We are concerned with the asymptotic behavior of these composed maps for P-a.e. ω. This set-up will be referred to as $\mathscr{X}^+(M,\mathrm{P})$ in the rest of this appendix and it falls into the general framework of the theory of RDS's, for which we refer the reader to Kifer [35], Arnold [3] and Liu [49].

A.1.2 Invariant Measures and Sample Measures

Let $\mathscr{X}^+(M,\mathrm{P})$ be given. In what follows we will use $\mathscr{B}(X)$ to denote the Borel σ-algebra of a topological space X.

Definition A.1.1 An *invariant measure* of $\mathscr{X}^+(M,\mathrm{P})$ is defined as a probability μ on $(\Omega \times M, \mathscr{A} \times \mathscr{B}(M))$ which has marginal P on Ω and which is invariant under the skew-product transformation

$$\Theta : \Omega \times M \to \Omega \times M, \qquad (\omega,x) \mapsto (\theta\omega, f_0(\omega)x)$$

associated with $\mathscr{X}^+(M,\mathrm{P})$.

Such invariant measures always exist (see Arnold [3]), and let now μ be such an invariant measure. Since $C^r(M,M)$ is a Polish space (see Hirsch [24]), $(\Omega \times M, \mu)$ with the μ-completion of $\mathscr{A} \times \mathscr{B}(M)$ constitutes a Lebesgue space. According to Rokhlin [74], one can speak of the conditional measure μ_ω of μ on $\{\omega\} \times M$ (identified with M) for P-a.e. ω. $\{\mu_\omega\}_{\omega\in\Omega}$ is P-mod 0 uniquely defined and is called the family of *sample measures* of μ. It is easy to see that, for any $n \geq 1$, the sample measures have the invariance property

$$\int_{\theta^{-n}\{\omega\}} f^n_{\omega'} \mu_{\omega'} d\mathrm{P}^{(n)}_\omega(\omega') = \mu_\omega, \quad \mathrm{P}-\text{a.e. } \omega \tag{A.2}$$

which will play an important role in the treatment of this appendix, where $\{\mathrm{P}^{(n)}_\omega\}_{\omega\in\Omega}$ is a canonical system of conditional measures of P associated with the partition $\{\theta^{-n}\{\omega\} : \omega \in \Omega\}$ of Ω.

A.1.3 Entropy

Given an invariant measure μ, there are some quantities to describe the complexity of the dynamical behavior of $\mathscr{X}^+(M,\mathrm{P})$ with respect to μ. One is the entropy $h_\mu(\mathscr{X}^+(M,\mathrm{P}))$, among the others are the Lyapunov exponents $\lambda_i(\omega,x)$, $1 \leq i \leq r(\omega,x)$ which will be introduced in the next subsection. The entropy $h_\mu(\mathscr{X}^+(M,\mathrm{P}))$ is defined as follows and it describes the average (on ω) information creation rate of the time evolution of the system $(\mathscr{X}^+(M,\mathrm{P}),\mu)$.

Proposition A.1.1 *For any finite partition ξ of M the limit*

$$h_\mu(\mathscr{X}^+(M,\mathrm{P}),\xi) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty}\frac{1}{n}\int H_{\mu_\omega}\left(\bigvee_{k=0}^{n-1}(f_\omega^k)^{-1}\xi\right)d\mathrm{P}(\omega) \tag{A.3}$$

exists.

Proof. Define $k(x) = \begin{cases} x\log x & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$, write $\xi_\omega^n = \bigvee_{k=0}^{n-1}(f_\omega^k)^{-1}\xi$ and put $a_n = \int H_{\mu_\omega}(\xi_\omega^n)d\mathrm{P}(\omega)$. Then for all $n,m\in\mathbb{Z}^+$

$$\begin{aligned}
a_{n+m} &= \int H_{\mu_\omega}(\xi_\omega^{n+m})d\mathrm{P}(\omega) \\
&\le \int H_{\mu_\omega}(\xi_\omega^n)d\mathrm{P}(\omega) + \int H_{\mu_\omega}((f_\omega^n)^{-1}\xi_{\theta^n\omega}^m)d\mathrm{P}(\omega) \\
&= a_n + \int H_{f_\omega^n\mu_\omega}(\xi_{\theta^n\omega}^m)d\mathrm{P}(\omega) \\
&= a_n - \int_\Omega\int_{\theta^{-n}\{\omega'\}}\sum_{C\in\xi_{\omega'}^m} k((f_\omega^n\mu_\omega)(C))d\mathrm{P}_{\omega'}^{(n)}(\omega)d\mathrm{P}(\omega') \\
&\le a_n - \int_\Omega\sum_{C\in\xi_{\omega'}^m} k\left(\int_{\theta^{-n}\{\omega'\}}(f_\omega^n\mu_\omega)(C)d\mathrm{P}_{\omega'}^{(n)}(\omega)\right)d\mathrm{P}(\omega') \\
&\qquad \text{(by the convexity of } k(x) \text{ on } [0,+\infty)) \\
&= a_n + \int_\Omega H_{\mu_{\omega'}}(\xi_{\omega'}^m)d\mathrm{P}(\omega') \quad (\text{by } (A.2)) \\
&= a_n + a_m.
\end{aligned}$$

The limit $\lim_{n\to+\infty}\frac{1}{n}a_n$ thus exists and equals $\inf_{n\ge1}\frac{1}{n}a_n$.

Definition A.1.2 The number $h_\mu(\mathscr{X}^+(M,\mathrm{P})) \stackrel{\mathrm{def}}{=} \sup_\xi h_\mu(\mathscr{X}^+(M,\mathrm{P}),\xi)$ is called the *entropy* of $(\mathscr{X}^+(M,\mathrm{P}),\mu)$, where the supremum is taken over the set of all finite partitions ξ of M.

By the same argument as Bogenschütz [9, Theorem 3.1] one has

$$h_\mu(\mathscr{X}^+(M,\mathrm{P})) = h_\mu^{\mathscr{F}}(\Theta), \tag{A.4}$$

where $h_\mu^{\mathscr{F}}(\Theta)$ is the conditional entropy of $\Theta : (\Omega\times M,\mu) \hookleftarrow$ with respect to $\mathscr{F} \stackrel{\mathrm{def}}{=} \{A\times M : A\in\mathscr{A}\}$.

A.1.4 Lyapunov Exponents

Let $\mathscr{X}^+(M,\mathrm{P})$ be of class C^1 (i.e., $r=1$) and let μ be an invariant measure of $\mathscr{X}^+(M,\mathrm{P})$. If $\int\log^+|T_xf_0(\omega)|d\mu(\omega,x) < +\infty$, the Oseledec multiplicative ergodic

theorem applied to $\Theta : (\Omega \times M, \mu) \hookleftarrow$ yields that, for μ-a.e. $(\omega, x) \in \Omega \times M$, there are measurable (in (ω, x)) numbers

$$+\infty > \lambda_1(\omega,x) > \lambda_2(\omega,x) > \cdots > \lambda_{r(\omega,x)}(\omega,x) \geq -\infty$$

and an associated sequence of subspaces of T_xM

$$V^{(0)}(\omega,x) = T_xM \supset V^{(1)}(\omega,x) \supset \cdots \supset V^{(r(\omega,x))}(\omega,x) = \{0\}$$

(all measurable in (ω, x)) such that

$$\lim_{n\to+\infty} \frac{1}{n} \log |T_x f_\omega^n \xi| = \lambda_i(\omega,x)$$

for all $\xi \in V^{(i-1)}(\omega,x) \backslash V^{(i)}(\omega,x)$, $1 \leq i \leq r(\omega,x)$. The numbers $\lambda_i(\omega,x)$, $1 \leq i \leq r(\omega,x)$ are called the *Lyapunov exponents* of $\mathscr{X}^+(M,\mathrm{P})$ at (ω,x), and $m_i(\omega,x) \stackrel{\text{def}}{=} \dim V^{(i-1)}(\omega,x) - \dim V^{(i)}(\omega,x)$ is called the *multiplicity* of $\lambda_i(\omega,x)$.

A.2 Statement of the Main Result

A.2.1 Ruelle Inequality

Let $\mathscr{X}^+(M,\mathrm{P})$ be given and let μ be an $\mathscr{X}^+(M,\mathrm{P})$-invariant measure. Roughly speaking, the entropy and the Lyapunov exponents provide two different ways of measuring the complexity of the dynamical behavior of $(\mathscr{X}^+(M,\mathrm{P}),\mu)$. The entropy does it from the point-view of information, and the positive exponents measure geometrically how fast nearby orbits diverge (via the corresponding unstable manifolds theory formulated on the inverse limit space of $\Theta : (\Omega \times M, \mu) \hookleftarrow$). Concerning the relationship between these two kinds of quantities there is first the following result.

Proposition A.2.1 ***(Ruelle Inequality)*** *Assume $\mathscr{X}^+(M,\mathrm{P})$ is of class C^1 (i.e., $r = 1$) and $\log^+ |f_0(\omega)|_{C^1} \in L^1(\Omega,\mathrm{P})$, where $|f|_{C^1} \stackrel{\text{def}}{=} \sup_{x\in M} |T_x f|$ for $f \in C^1(M,M)$. Then for any $\mathscr{X}^+(M,\mathrm{P})$-invariant measure μ one has*

$$h_\mu(\mathscr{X}^+(M,\mathrm{P})) \leq \int \sum_i \lambda_i(\omega,x)^+ m_i(\omega,x) d\mu. \tag{A.5}$$

This was first proved by Ruelle [77] (also by an unpublished work of Margulis) for a single C^1 map and a similar result was proved by Bahnmüller and Bogeuschütz [4] for RDS's over ergodic and invertible measure-preserving "noise" systems (see that paper for previous works by others). To prove the present result, let $\Omega^* = C^r(M,M)^{\mathbb{Z}}$ (with the product σ-algebra $\mathscr{A}^*$) and define

$$\Theta^* : \Omega^* \times M \to \Omega^* \times M, \quad (\omega^*, x) \mapsto (\theta^* \omega^*, f_0(\omega^*)x)$$

where θ^* is the left shift operator on Ω^* and $(\cdots, f_{-1}(\omega^*), f_0(\omega^*), f_1(\omega^*), \cdots)$ is the sequence of maps corresponding to ω^*. It is easy to show that there is a unique probability μ^* on $(\Omega^* \times M, \mathscr{A}^* \times \mathscr{B}(M))$ such that $\Theta^* \mu^* = \mu^*$ and $\Pi \mu^* = \mu$, where $\Pi : \Omega^* \times M \to \Omega \times M$, $(\omega^*, x) \mapsto ((f_0(\omega^*), f_1(\omega^*), \cdots), x)$ is the natural projection. By arguments similar to Liu [48, Prop.2.2] one has

$$h_\mu^{\mathscr{F}}(\Theta) = h_{\mu^*}^{\mathscr{F}^*}(\Theta^*) \tag{A.6}$$

where $\mathscr{F}^* = \{A^* \times M : A^* \in \mathscr{A}^*\}$. By [4],

$$h_{\mu^*}^{\mathscr{F}^*}(\Theta^*) \le \int \sum_i \lambda_i(\omega^*, x)^+ m_i(\omega^*, x) d\mu^*$$

if $\log^+ |f_0(\omega^*)|_{C^1} \in L^1(\Omega^*, \mathrm{P}^*)$, where P^* is the marginal of μ^* on Ω^* ($(\Omega^*, \theta^*, \mathrm{P}^*)$ is clearly the natural extension of $(\Omega, \theta, \mathrm{P})$) and $\{(\lambda_i(\omega^*, x), m_i(\omega^*, x)) : i = 1, \cdots, r(\omega^*, x)\}$ is the Lyapunov spectrum of $\Theta^* : (\Omega^* \times M, \mu^*) \hookleftarrow$ at μ^*-a.e. (ω^*, x). This together with (A.6) proves Proposition A.2.1 since

$$\int \sum_i \lambda_i(\omega^*, x)^+ m_i(\omega^*, x) d\mu^* = \int \sum_i \lambda_i(\omega, x)^+ m_i(\omega, x) d\mu.$$

A.2.2 Pesin (Entropy) Formula

Our main result of this appendix is the following theorem, where $|f|_{C^2}$ is the C^2-norm of $f \in C^2(M, M)$ (see [51] for the definition) and $D(f) \overset{\text{def}}{=} \inf_{x \in M} |\det T_x f|$.

Theorem A.2.2 ***(Pesin Formula)*** *Let $\mathscr{X}^+(M, \mathrm{P})$ be of class C^2 and assume $\log^+ |f_0(\omega)|_{C^2} + \log D(f_0(\omega)) \in L^1(\Omega, \mathrm{P})$. Let μ be an $\mathscr{X}^+(M, \mathrm{P})$-invariant measure. If $\mu_\omega \ll Leb$ for P-a.e. ω, then there holds the equality*

$$h_\mu(\mathscr{X}^+(M, \mathrm{P})) = \int \sum_i \lambda_i(\omega, x)^+ m_i(\omega, x) d\mu. \tag{A.7}$$

The proof of this theorem will be given in Section 3.

Some results in this direction have been obtained previously and let us indicate how they can be covered by Theorem A.2.2. The formula (A.7) was first proved by Pesin [63] for a single diffeomorphism and later by Thieullen [90] for a single noninvertible map (along a line different from Pesin's), and these results correspond to the situation of P being supported by a single point $(f, f, \cdots) \in \Omega$ for some $f \in C^2(M, M)$ (we remark that [63] and [90] made a weaker smoothness

assumption, i.e., a $C^{1+\alpha}$ one on the map under consideration, and the map considered in [90] is allowed to have singularities; see Remark A.1 about the $C^{1+\alpha}$ assumption in the random case). Ledrappier and Young [44] considered the case of $\mathscr{X}^+(M,\mathrm{P})$ where $f_i(\omega), i=0,1,2,\cdots$ are assumed to be independent and identically distributed random diffeomorphisms (i.e., $\mathrm{P}=\nu^{\mathbb{Z}^+}$ for some probability ν on $\mathrm{Diff}^2(M)$) and proved (A.7) for a probability ρ on M which is stationary for the Markov process induced by $\mathscr{X}^+(M,\mathrm{P})$ on M (or equivalently $\int f\rho d\nu(f)=\rho$) and which satisfies $\rho \ll Leb$, and this result was extended by Liu [48, Sect.2.2] to i.i.d. random endomorphisms case. Since in these cases $\nu^{\mathbb{Z}^+}\times\rho$ constitutes an invariant measure of $\mathscr{X}^+(M,\mathrm{P})$, these results are covered by Theorem A.2.2. Getting rid of the i.i.d. assumption, Liu [48, Sect.2.1] considered the two-sided model $\mathscr{X}(M,\mathrm{P}^*)$ defined similarly to $\mathscr{X}^+(M,\mathrm{P})$ but over $\Omega^*=C^2(M,M)^{\mathbb{Z}}$ and proved a result similar to Theorem A.2.2. This result can also be easily deduced from Theorem A.2.2 (apart from some slight changes concerning integrability conditions) by projecting $(\mathscr{X}(M,\mathrm{P}^*),\mu^*)$ (μ^* an $\mathscr{X}(M,P^*)$-invariant measure) to a one-sided RDS $(\mathscr{X}^+(M,\mathrm{P}),\mu)$ with $\mu=\Pi\mu^*$, since the projecting procedure preserves the smoothness of the sample measures. Note however that extending a one-sided RDS $(\mathscr{X}^+(M,\mathrm{P}),\mu)$ to a two-sided one $(\mathscr{X}(M,\mathrm{P}^*),\mu^*)$ will generally destroy the smoothness of the sample measures (but can only guarantee the SRB property of them, see Liu [49, Remark 2.9] and Corollary A.2.2.1 below), hence Theorem A.2.2, especially the "i.i.d." results in [44] and [48, Sect.2.2], can not be covered by the "two-sided" result in [48, Sect.2.1].

Theorem A.2.2 together with the main result of Bahnmüller and Liu [5] (see also [44] for the i.i.d. case) yields the following corollary. See [5] for the definition of SRB measures of RDS's.

Corollary A.2.2.1 *Assume $\mathscr{X}^+(M,\mathrm{P})$ is given such that $f_0(\omega)\in\mathrm{Diff}^2(M)$ for* P*-a.e. ω and $\log^+|f_0(\omega)_{C^2}+\log^+|f_0(\omega)^{-1}|_{C^2}\in L^1(\Omega,\mathrm{P})$. Let μ be an $\mathscr{X}^+(M,\mathrm{P})$-invariant measure and let $(\Omega^*\times M,\Theta^*,\mu^*)$ be the natural extension of $(\Omega\times M,\Theta,\mu)$. If $\mu_\omega \ll Leb$ for* P*-a.e. ω, then μ^* is an SRB measure.*

Remark A.1. Since we need to work with Lebesgue spaces, we consider here C^2 rather than $C^{1+\alpha}(0<\alpha\le 1)$ endomorphisms for the reason that $C^2(M,M)$ is Polish whereas $C^{1+\alpha}(M,M)$ is in general not separable (see [38]). It is not clear to the authors if the C^2 assumption could be reduced to a $C^{1+\alpha}$ one by a suitable trick for the purpose of this appendix (note however that a C^1 assumption is usually not sufficient, see Pugh [68] for a counterexample concerning the absolute continuity property of the stable manifolds).

Remark A.2. One may consider the following slightly more general framework of RDS's. Let W be a Polish space, $\tilde{P}$ a probability on $(W,\mathscr{B}(W))$ and $\tau:(W,\mathscr{B}(W),\tilde{P})\hookleftarrow$ a measure-preserving transformation (take the classical Winner space as an example). Assume that

$$\mathscr{G}:W\to C^r(M,M)$$

($r \geq 0$ integer) is a measurable map and one considers for $\tilde{\mathrm{P}}$-a.e. w the composed maps

$$g_w^n \stackrel{\text{def}}{=} \begin{cases} id & \text{for } n = 0 \\ g(\tau^{n-1}w) \circ \cdots \circ g(w) & \text{for } n > 0, \end{cases}$$

where $g(w) \stackrel{\text{def}}{=} \mathscr{G}(w)$. This defines an RDS and we will denote it still by the notation $\mathscr{G}$. Various notions defined for $\mathscr{X}^+(M,\mathrm{P})$ can be adapted verbatim to the case of $\mathscr{G}$ by considering the skew-product transformation $G : W \times M \to W \times M,\ (w,x) \mapsto (\tau w, g(w)x)$. The map $\Sigma : W \to \Omega,\ w \mapsto (g(w), g(\tau w), \cdots)$ projects $\tilde{P}$ to a probability P on Ω and this gives rise to an RDS $\mathscr{X}^+(M,\mathrm{P})$. Clearly any invariant measure $\tilde{\mu}$ of $\mathscr{G}$ can deduce an invariant measure μ of $\mathscr{X}^+(M,\mathrm{P})$ via the map $\tilde{\Sigma} : W \times M \to \Omega \times M,\ (w,x) \mapsto (\Sigma w, x)$. By arguments similar to the proof of Proposition A.1.1 one can see that the entropy $h_{\tilde{\mu}}(\mathscr{G})$ of $(\mathscr{G}, \tilde{\mu})$ is equal to $h_\mu(\mathscr{X}^+(M,\mathrm{P}))$ and then has

Theorem 2.2′ *Let the RDS $\mathscr{G}$ be as introduced above and let it be of class C^2. Assume* $\log^+ |g(w)|_{C^2} + \log D(g(w)) \in L^1(W, \tilde{P})$ *and let $\tilde{\mu}$ be an invariant measure of $\mathscr{G}$. If $\tilde{\mu}_w \ll Leb$ for $\tilde{P}$-a.e. w, or more generally, if $\int_{\Sigma^{-1}\omega} \tilde{\mu}_w d\tilde{P}_\omega(w) \ll Leb$ where $\tilde{P}_\omega$ is the conditional measure of $\tilde{P}$ on $\Sigma^{-1}\omega$, then*

$$h_{\tilde{\mu}}(\mathscr{G}) = \int \sum_i \lambda_i(w,x)^+ m_i(w,x) d\tilde{\mu}.$$

Using the natural extension of $(W, \tau, \tilde{P})$, one can have a result similar to Corollary A.2.2.1.

A.2.3 Pesin Formula for Some Particular RDS's

Expanding in average RDS's. Let $\mathscr{X}^+(M,\mathrm{P})$ be of class C^r ($r \geq 1$ integer) and assume that $\log|f_0(\omega)|^-_{C^1} \in L^1(\Omega, \mathrm{P})$, where $|f|^-_{C^1} \stackrel{\text{def}}{=} \inf_{\xi \in TM, |\xi|=1} |Tf\xi|$ for $f \in C^r(M,M)$. If P-a.e.

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log|f_k(\omega)|^-_{C^1} =: a(\omega) > 0$$

(the limit exists for P-a.e. ω by Birkhoff theorem), $\mathscr{X}^+(M,\mathrm{P})$ is then said to be *expanding in average*. The two-sided version of this model was introduced by Khanin and Kifer [34] and thermodynamic formalism was developed there for this two-sided version. Particularly, existence of invariant measures whose sample measures are absolutely continuous with respect to Lebesgue is assured there under suitable smoothness and integrability conditions on the random maps. Using the extension technique in Subsection 2.1, this implies that for a C^2 expanding in average RDS

$\mathscr{X}^+(M,\mathrm{P})$ with $\log^+|f_0(\omega)|_{C^2} \in L^1(\Omega,P)$ there is an (in fact unique) invariant measure μ which satisfies $\mu_\omega \ll Leb$ for P-a.e. ω. By Theorem A.2.2, this μ satisfies

$$\begin{aligned} h_\mu(\mathscr{X}^+(M,\mathrm{P})) &= \int \sum_i \lambda_i(\omega,x)(\omega,x)^+ m_i(\omega,x)d\mu \\ &= \int \sum_i \lambda_i(\omega,x) m_i(\omega,x)d\mu = \int \log|\det T_x f_0(\omega)|d\mu. \end{aligned}$$

This formula can also follow from the equilibrium state arguments in [34] together with the extension technique in Subsection 2.1, but now we obtain it directly. This can be regarded as a particular result of thermodynamic formalism of one-sided expanding in average RDS's, a fairly full study of which is still lacking (see Baladi [6] for related results).

Markov RDS's. Let $\Omega = \{f_1,\cdots,f_N\}^{\mathbb{Z}^+}$, where $f_1,\cdots,f_N$ are a finite number of $C^r(r \geq 0)$ maps on M, and let P be a probability on Ω. If the coordinate process $f_n(\omega)$, $n = 0,1,\cdots$ on (Ω,P) constitutes a time homogeneous stationary Markov process, namely, if $\mathrm{P}(f_n(\omega) = f_{i_n}|f_0(\omega) = f_{i_0},\cdots,f_{n-1}(\omega) = f_{i_{n-1}}) = p_{i_{n-1}i_n}, \forall n \geq 1$, $\forall i_0,i_1,\cdots,i_n \in \{1,\cdots,N\}$ and $\mathrm{P}(f_0(\omega) = f_i) = \pi_i, 1 \leq i \leq N$ for some transition matrix (p_{ij}) and some probability vector $(\pi_1,\cdots,\pi_N)$ with $\sum_{i=1}^N \pi_i p_{ij} = \pi_j$, $1 \leq j \leq N$, $\mathscr{X}^+(M,\mathrm{P})$ is then called a *finite-stated one-sided Markov RDS*. Let now $\mathscr{X}^+(M,\mathrm{P})$ be such an RDS and assume $\pi_i > 0$ for all i. In this case there is a special kind of $\mathscr{X}^+(M,\mathrm{P})$-invariant measures μ such that at P-a.e. ω, μ_ω depends only on $f_0(\omega)$, i.e., there are probabilities $\mu_1,\cdots,\mu_N$ on M such that $\mu_\omega = \mu_i$ whenever $f_0(\omega) = f_i$ for P almost all ω. In fact, for any $\mu_1,\cdots,\mu_N \in$ Prob(M) (Prob(X) is the space of all Borel probabilities on a topological space X) there is clearly $\mu \in \mathrm{Prob}(\Omega \times M)$ with marginal P on Ω such that $\mu_\omega = \mu_i$ whenever $f_0(\omega) = f_i$, and the $\mathscr{X}^+(M,\mathrm{P})$-invariance of μ is equivalent to the equations $\mu_i = \sum_{j=1}^N \frac{\pi_j p_{ji}}{\pi_i} f_j \mu_j$. It is easy to see that the map $L : \mathrm{Prob}(M)^N \to \mathrm{Prob}(M)^N$, $(\mu_1,\cdots,\mu_N) \mapsto (\sum_{j=1}^N \frac{\pi_j p_{j1}}{\pi_1} f_j \mu_j,\cdots,\sum_{j=1}^N \frac{\pi_j p_{jN}}{\pi_N} f_j \mu_j)$ has fixed points. This proves the existence of the special invariant measures μ. By Theorem A.2.2, to obtain Pesin formula for $\mathscr{X}^+(M,\mathrm{P})$ with such an invariant measure μ, it is sufficient to check $\mu_i \ll Leb$ for $i = 1,\cdots,N$ if f_i's are C^2 and have no singularities.

A.3 Proof of Theorem A.2.2

By Proposition A.2.1, it remains to prove

$$h_\mu(\mathscr{X}^+(M,\mathrm{P})) \geq \int \sum_i \lambda_i(\omega,x)^+ m_i(\omega,x)d\mu. \tag{A.8}$$

We will follow essentially the line of Ledrappier and Young [44] and especially that of Liu [48]. We first introduce the stable manifolds of $(\mathscr{X}^+(M,\mathrm{P}),\mu)$ and the proof of (A.8) consists in analysis along these manifolds.

Let $\Gamma \subset \Omega \times M$ be a μ-full set such that $\Theta\Gamma \subset \Gamma$ and each point in Γ is regular in the sense of Oseledec as described in Subsection 1.4. Set

$$I = \{(\omega,x) \in \Gamma : \lambda_i(\omega,x) \geq 0, 1 \leq i \leq r(\omega,x)\}$$

and $\Delta = \Gamma \backslash I$. It is clear that $\Theta I \subset I$ and $\Theta\Delta \subset \Delta$.

For $(\omega,x) \in \Delta$, put $E^s(\omega,x) = \bigcup_{\lambda_i(\omega,x)<0} V^{(i)}(\omega,x)$ and define the (global) stable manifold of $\mathscr{X}^+(M,\mathrm{P})$ at point (ω,x) by

$$W^s(\omega,x) \stackrel{\text{def}}{=} \{y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f_\omega^n x, f_\omega^n y) < 0\}.$$

Then for μ-a.e. $(\omega,x) \in \Delta$ there exists a sequence of $C^{1,1}$ embedded $\dim E^s(\omega,x)$-dimensional discs $\{W_n(\omega,x)\}_{n=0}^{+\infty}$ such that $f_n(\omega)W_n(\omega,x) \subset W_{n+1}(\omega,x)$ for all $n \geq 0$ and

$$W^s(\omega,x) = \bigcup_{n=0}^{+\infty} (f_\omega^n)^{-1} W_n(\omega,x)$$

(see [51, Chapter III] for a proof). The integrability condition $\log D(f_0(\omega)) \in L^1(\Omega,\mathrm{P})$ implies that for P-a.e. ω the maps f_ω^n have no singularities for all $n \geq 0$ and hence $W^s(\omega,x)$ is a $C^{1,1}$ immersed submanifold of M for μ-a.e. $(\omega,x) \in \Delta$. We define $W^s(\omega,x) = \{x\}$ for any $(\omega,x) \in I$.

Since $\mu_\omega \ll Leb$ for P-a.e. ω, there exists a measurable partition η of $\Omega \times M$ which has the following properties:

i) $\Theta^{-1}\eta \leq \eta$, $\sigma \stackrel{\text{def}}{=} \{\{\omega\} \times M : \omega \in \Omega\} \leq \eta$;
ii) η is subordinate to W^s-manifolds of $(\mathscr{X}^+(M,\mathrm{P}),\mu)$, i.e., for μ-a.e. (ω,x) $\eta_\omega(x) \stackrel{\text{def}}{=} \{y : (\omega,y) \in \eta(\omega,x)\} \subset W^s(\omega,x)$ and it contains an open neighborhood of x in $W^s(\omega,x)$, this neighborhood being taken in the submanifold topology of $W^s(\omega,x)$;
iii) For every $B \in \mathscr{B}(\Omega \times M)$ the function $(\omega,x) \mapsto \lambda^s_{(\omega,x)}(\eta_\omega(x) \bigcap B_\omega)$ is measurable and μ-a.e. finite, where $B_\omega = \{y : (\omega,y) \in B\}$ and $\lambda^s_{(\omega,x)}$ is the Lebesgue measure on $W^s(\omega,x)$ induced by its inherited Riemannian structure as a submanifold of M ($\lambda^s_{(\omega,x)} = \delta_x$ if $W^s(\omega,x) = \{x\}$);
iv) $(\mu_\omega)_x^{\eta_\omega} \ll \lambda^s_{(\omega,x)}$ for μ-a.e. (ω,x), where $(\mu_\omega)_x^{\eta_\omega}$ is the conditional measure of μ_ω on $\eta_\omega(x)$.

A similar proof of the existence of such a partition η can be found in [51, IV.2] and [48]. It is worth mentioning that Lemma IV.2.2. in [51] remains true in the present case, i.e., $\mathscr{B}^I \subset \mathscr{B}^s$ μ-mod(0), where $\mathscr{B}^I = \{B \in \mathscr{B}_\mu(\Omega \times M) : \Theta^{-1}B = B\}$ ($\mathscr{B}_\mu(\Omega,M)$ the completion of $\mathscr{B}(\Omega \times M)$ with respect to μ) and $\mathscr{B}^s = \{B \in \mathscr{B}_\mu(\Omega \times M) : B = \bigcup_{(\omega,x)\in B}\{\omega\} \times W^s(\omega,x)\}$ (see [5] for a proof).

Let η be as given above. By a computation similar to [44, (4.8)] one has $\lim_{n\to+\infty} \frac{1}{n} H_\mu(\eta|\Theta^{-n}\eta \bigvee \sigma) \leq h_\mu(\mathscr{X}^+(M,\mathrm{P}))$ if

$$H_\mu(\eta|\Theta^{-n}\eta \vee \sigma) < +\infty \tag{A.9}$$

for all $n \geq 1$. Hence, in order to prove (A.8), it is sufficient to prove that for every $n \geq 1$ there holds (A.9) and

$$\frac{1}{n} H_\mu(\eta|\Theta^{-n}\eta \vee \sigma) \geq \int \sum_i \lambda_i(\omega,x)^+ m_i(\omega,x) d\mu. \tag{A.10}$$

Now we fix $n \geq 1$ arbitrarily. By the definition of conditional entropies one has

$$\begin{aligned} H_\mu(\eta|\Theta^{-n}\eta \vee \sigma) &= -\int \log \mu_{(\omega,x)}^{\Theta^{-n}\eta\vee\sigma}(\eta(\omega,x)) d\mu(\omega,x) \\ &= -\int_\Omega \int_M \log(\mu_\omega)_x^{(f_\omega^n)^{-1}\eta_{\theta^n\omega}}(\eta_\omega(x)) d\mu_\omega(x) d\mathrm{P}(\omega) \end{aligned}$$

where $\{\nu_z^\xi\}_{z\in X}$ denotes a canonical system of conditional measures of ν associated with a measurable partition ξ of a Lebesgue space $(X, \mathscr{A}, \nu)$.

In what follows we write $\lambda = Leb$. Since $\mu \ll \lambda \times P$ we can define

$$\varphi = \frac{d\mu}{d(\lambda \times \mathrm{P})}$$

which implies $\varphi_\omega(\cdot) \stackrel{\text{def}}{=} \varphi(\omega, \cdot) = \frac{d\mu_\omega}{d\lambda}(\cdot)$ for P-a.e. ω. Put $\Lambda = \{(\omega,x) : \varphi(\omega,x) > 0\}$. By assumption, $f_\omega^n : M \to M$ has no singularities for P-a.e. ω. From this together with the assumption $\mu_\omega \ll \lambda$, P-a.e. ω it follows easily that P-a.e. $f_\omega^n \mu_\omega \ll \mu_{\theta^n\omega}$. We now fix arbitrarily an ω with these above properties.

Choose a countable Borel partition $\{A_{\omega,i}^n\}_{i=1}^{+\infty}$ of M such that $f_{\omega,i}^n \stackrel{\text{def}}{=} f_\omega^n|_{A_{\omega,i}^n}$ is injective for each i, and define Borel Probabilities $\rho_{\omega,n}$ and $\nu_{\omega,n}$ on M such that for any Borel set $B \subset A_{\omega,i}^n$

$$\rho_{\omega,n}(B) = (f_\omega^n \mu_\omega)(f_{\omega,i}^n B), \qquad \nu_{\omega,n}(B) = \mu_{\theta^n\omega}(f_{\omega,i}^n B).$$

We now state some preliminary facts 1)-4) as follows

1) For any Borel set $B \subset \Lambda_{\theta^n\omega}$,

$$\begin{aligned} (f_\omega^n \mu_\omega)(B) = \mu_\omega((f_\omega^n)^{-1}B) &= \int_{(f_\omega^n)^{-1}B} \varphi_\omega(x) d\lambda(x) \\ = \int_B (\mathscr{L}_{f_\omega^n} \varphi_\omega)(x) d\lambda(x) &= \int_B \frac{(\mathscr{L}_{f_\omega^n}\varphi_\omega)(x)}{\varphi_{\theta^n\omega}(x)} d\mu_{\theta^n\omega}(x), \end{aligned}$$

where $\mathscr{L}_f$ is the Ruelle transfer operator of $f \in C^1(M,M)$ defined by $(\mathscr{L}_f l)(x) = \sum_{y\in f^{-1}\{x\}} \frac{l(y)}{|\det T_y f|}$ for measurable function $l : M \to \mathbb{R}$. Hence

$$\frac{df_\omega^n \mu_\omega}{d\mu_{\theta^n\omega}}(x) = \frac{(\mathscr{L}_{f_\omega^n}\varphi_\omega)(x)}{\varphi_{\theta^n\omega}(x)} =: \Psi_n(\omega,x), \ \ \mu_{\theta^n\omega}-\text{a.e. } x.$$

Furthermore, for any Borel set $B \subset M$ one has

$$\begin{aligned}\int_B \int_{\theta^{-n}\{\omega\}} (\mathscr{L}_{f^n_{\omega'}} \varphi_{\omega'})(y) dP^{(n)}_\omega(\omega') d\lambda(y) &= \int_{\theta^{-n}\{\omega\}} \int_B (\mathscr{L}_{f^n_{\omega'}} \varphi_{\omega'})(y) d\lambda(y) dP^{(n)}_\omega(\omega') \\ &= \int_{\theta^{-n}\{\omega\}} \int_{(f^n_{\omega'})^{-1}B} \varphi_{\omega'}(y) d\lambda(y) dP^{(n)}_\omega(\omega') \\ &= \int_{\theta^{-n}\{\omega\}} \mu_{\omega'}((f^n_{\omega'})^{-1}B) dP^{(n)}_\omega(\omega') \\ &= \int_{\theta^{-n}\{\omega\}} (f^n_{\omega'}\mu_{\omega'})(B) dP^{(n)}_\omega(\omega') \\ &= \mu_\omega(B) = \int_B \varphi_\omega(y) d\lambda(y).\end{aligned}$$

Hence

$$\int_{\theta^{-n}\{\omega\}} (\mathscr{L}_{f^n_{\omega'}} \varphi_{\omega'})(y) dP^{(n)}_\omega(\omega') = \varphi_\omega(y) \tag{A.11}$$

holds for λ-a.e. y and then holds for μ_ω-a.e. y.

2) For any Borel set $B \subset (f^n_\omega)^{-1}\Lambda_{\theta^n\omega}$, putting $B_i = B \bigcap A^n_{\omega,i}$ one has

$$\begin{aligned}\rho_{\omega,n}(B) &= \sum_i f^n_\omega \mu_\omega(f^n_{\omega,i} B_i) = \sum_i \int_{f^n_{\omega,i}B_i} \frac{(\mathscr{L}_{f^n_\omega}\varphi_\omega)(x)}{\varphi_{\theta^n\omega}(x)} d\mu_{\theta^n\omega} \\ &= \sum_i \int_{B_i} \frac{(\mathscr{L}_{f^n_\omega}\varphi_\omega)(f^n_\omega x)}{\varphi_{\theta^n\omega}(f^n_\omega x)} d\nu_{\omega,n} = \int_B \frac{(\mathscr{L}_{f^n_\omega}\varphi_\omega)(f^n_\omega x)}{\varphi_{\theta^n\omega}(f^n_\omega x)} d\nu_{\omega,n}\end{aligned}$$

which yields

$$\frac{d\rho_{\omega,n}}{d\nu_{\omega,n}}(x) = \frac{(\mathscr{L}_{f^n_\omega}\varphi_\omega)(f^n_\omega x)}{\varphi_{\theta^n\omega}(f^n_\omega x)} = \Psi_n(\omega, f^n_\omega x) \tag{A.12}$$

for $\nu_{\omega,n}$-a.e. $x \in (f^n_\omega)^{-1}\Lambda_{\theta^n\omega}$.

3) For any Borel set $B \subset (f^n_\omega)^{-1}\Lambda_{\theta^n\omega}$ one has

$$\begin{aligned}\mu_\omega(B) &= \sum_i \int_{B_i} \varphi_\omega(x) d\lambda = \sum_i \int_{f^n_{\omega,i}B_i} \varphi_\omega((f^n_{\omega,i})^{-1}y) |\det T_y (f^n_{\omega,i})^{-1}| d\lambda \\ &= \sum_i \int_{f^n_{\omega,i}B_i} \varphi(\omega, (f^n_{\omega,i})^{-1}y) |\det T_{(f^n_{\omega,i})^{-1}y} f^n_\omega|^{-1} \frac{1}{\varphi(\theta^n\omega, y)} d\mu_{\theta^n\omega} \\ &= \sum_i \int_{B_i} \frac{\varphi(\omega,x)}{\varphi(\theta^n\omega, f^n_{\omega,i}x)} |\det T_x f^n_\omega|^{-1} d\nu_{\omega,n} \\ &= \int_B \frac{\varphi(\omega,x)}{\varphi\circ\Theta^n(\omega,x)} |\det T_x f^n_\omega|^{-1} d\nu_{\omega,n}\end{aligned}$$

which yields

$$\frac{d\mu_\omega}{d\nu_{\omega,n}}(x) = \frac{\varphi(\omega,x)}{\varphi\circ\Theta^n(\omega,x)} |\det T_x f^n_\omega|^{-1} =: \Phi_n(\omega,x)$$

for $\nu_{\omega,n}$-a.e. $x \in (f^n_\omega)^{-1}\Lambda_{\theta^n\omega}$.

It is easy to see that μ_ω is equivalent to $\nu_{\omega,n}$ on $\Lambda_\omega\bigcap(f_\omega^n)^{-1}\Lambda_{\theta^n\omega}$ and $\frac{d\nu_{\omega,n}}{d\mu_\omega}=\frac{1}{\Phi_n(\omega,x)}$ for μ_ω-a.e. $x\in\Lambda_\omega\bigcap(f_\omega^n)^{-1}\Lambda_{\theta^n\omega}$. This together with (A.12) yields $\frac{d\rho_{\omega,n}}{d\mu_\omega}(x)=\frac{\Psi_n(\omega,f_\omega^n x)}{\Phi_n(\omega,x)}$ for μ_ω-a.e. $x\in\Lambda_\omega\bigcap(f_\omega^n)^{-1}\Lambda_{\theta^n\omega}$. Hence

$$f_\omega^n:(\Lambda_\omega\bigcap(f_\omega^n)^{-1}\Lambda_{\theta^n\omega},\mu_\omega)\to(f_\omega^n\Lambda_\omega\bigcap\Lambda_{\theta^n\omega},f_\omega^n\mu_\omega)$$

has the Jacobian

$$J(f_\omega^n)(x)=\frac{\Psi_n(\omega,f_\omega^n x)}{\Phi_n(\omega,x)}$$

(see Parry [61] or see [48, 3.1.1] for the definition). Thus, if ξ is a measurable partition of $f_\omega^n\Lambda_\omega\bigcap\Lambda_{\theta^n\omega}$, for μ_ω-a.e. $x\in\Lambda_\omega\bigcap(f_\omega^n)^{-1}\Lambda_{\theta^n\omega}$ one has

$$(\mu_\omega)_x^{(f_\omega^n)^{-1}\xi}(B)=\int_{f_\omega^n B}\frac{1}{J(f_\omega^n)\circ(f_{\omega,i}^n)^{-1}}d(f_\omega^n\mu_\omega)_{f_\omega^n x}^{\xi} \tag{A.13}$$

for any Borel set $B\subset((f_\omega^n)^{-1}\xi)(x)\bigcap A_{\omega,i}^n$ (see [48, Lemma 3.1]). Since $\mu_\omega(\Lambda_\omega\bigcap(f_\omega^n)^{-1}\Lambda_{\theta^n\omega})=1$ and $(f_\omega^n\mu_\omega)(f_\omega^n\Lambda_\omega\bigcap\Lambda_{\theta^n\omega})=1$ for any measurable partition ξ of M (A.13) holds for μ_ω-a.e. $x\in M$ and for any Borel set $B\subset((f_\omega^n)^{-1}\xi)(x)\bigcap A_{\omega,i}^n$.

4) One can define a Borel measure λ^* on $\Omega\times M$ by

$$\lambda^*(B)=\int\lambda_{(\omega,x)}^s(\eta_\omega(x)\bigcap B_\omega)d\mu(\omega,x)\text{ for Borel set }B\subset\Omega\times M.$$

Recalling that $\mu(B)=\int(\mu_\omega)_x^{\eta_\omega}(\eta_\omega(x)\bigcap B_\omega)d\mu(\omega,x)$ and $(\mu_\omega)_x^{\eta_\omega}\ll\lambda_{(\omega,x)}^s$, we have $\mu\ll\lambda^*$. Put $g(\omega,x)=\frac{d\mu}{d\lambda^*}(\omega,x)$. We then have for μ-a.e. (ω,x)

$$g(\omega,y)=\frac{d(\mu_\omega)_x^{\eta_\omega}}{d\lambda_{(\omega,x)}^s}(y),\quad \lambda_{(\omega,x)}^s\text{--a.e. }y\in\eta_\omega(x) \tag{A.14}$$

(see [51, Proposition IV.2.2]).

Put for μ-a.e. $(\omega,x)\in\Omega\times M$

$$\begin{aligned}
W_n(\omega,x)&=(\mu_\omega)_x^{(f_\omega^n)^{-1}\eta_{\theta^n\omega}}(\eta_\omega(x)),\\
X_n(\omega,x)&=\frac{\varphi(\omega,x)}{\varphi\circ\Theta^n(\omega,x)}\frac{g\circ\Theta^n(\omega,x)}{g(\omega,x)},\\
Y_n(\omega,x)&=\begin{cases}\frac{|\det T_x f_\omega^n|_{E^s}|}{|\det T_x f_\omega^n|} & \text{if}(\omega,x)\in\Delta,\\ \frac{1}{|\det T_x f_\omega^n|} & \text{if}(\omega,x)\in I,\end{cases}\\
Z_n(\omega,x)&=\int_{\eta_{\theta^n\omega}(f_\omega^n x)}\Psi_n(\omega,y)d(\mu_{\theta^n\omega})_{f_\omega^n x}^{\eta_{\theta^n\omega}}(y).
\end{aligned}$$

It is easy to see that W_n, X_n, Y_n, Z_n are all measurable and μ-a.e. finite. We now present several claims, whose proofs will be given a bit later.

Claim 1. $W_n = \frac{X_n Y_n}{Z_n}$ μ-a.e. on $\Omega \times M$;
Claim 2. $\log Y_n \in L^1(\Omega \times M, \mu)$ and $-\int \frac{1}{n} \log Y_n d\mu = \int \sum_i \lambda_i(\omega,x)^+ m_i(\omega,x) d\mu$.
Claim 3. $\log Z_n \in L^1(\Omega \times M, \mu)$ and $\int \log Z_n d\mu \geq 0$.
Claim 4. $\log X_n \in L^1(\Omega \times M, \mu)$ and $\int \log X_n d\mu = 0$.

By these claims one can easily see that $\log W_n \in L^1(\Omega \times M, \mu)$ and

$$-\frac{1}{n}\int \log W_n d\mu \geq \int \sum_i \lambda_i(\omega,x)^+ m_i(\omega,x) d\mu.$$

This proves (A.9) and (A.10) and thus proves Theorem A.2.2. □

Proof of Claim 1. Clearly one has

$$(\mu_\omega)_x^{(f_\omega^n)^{-1}\varepsilon}(\{x\}) = \frac{1}{J(f_\omega^n)(x)}, \quad \mu\text{−a.e. } (\omega,x) \tag{A.15}$$

where ε is the partition of M into single points. Notice that if $(\omega,x) \in I$, then the left side of (A.15) is $W_n(\omega,x)$ and $Z_n(\omega,x) = \Psi_n(\omega, f_\omega^n x)$. Thus $W_n = \frac{X_n Y_n}{Z_n}$ holds μ-a.e. on I. Now we consider μ-a.e. $(\omega,x) \in \Delta$. By fact 1), one has

$$\frac{d(f_\omega^n \mu_\omega)_{f_\omega^n x}^{\eta_{\theta^n\omega}}}{d(\mu_{\theta^n\omega})_{f_\omega^n x}^{\eta_{\theta^n\omega}}}(\cdot) = \frac{\Psi_n(\omega,\cdot)}{Z_n(\omega,x)}. \tag{A.16}$$

Then, for any Borel set $B \subset \eta_\omega(x)$

$$\begin{aligned}
&(\mu_\omega)_x^{\eta_\omega}(B) = \frac{(\mu_\omega)_x^{(f_\omega^n)^{-1}\eta_{\theta^n\omega}}(B)}{W_n(\omega,x)} \\
&= \frac{1}{W_n(\omega,x)} \sum_i \int_{f_{\omega,i}^n B_i} \frac{1}{J(f_\omega^n)\circ (f_{\omega,i}^n)^{-1}(y)} d(f_\omega^n \mu_\omega)_{f_\omega^n x}^{\eta_{\theta^n\omega}}(y) \qquad \text{(by (A.13))} \\
&= \frac{1}{W_n(\omega,x)} \sum_i \int_{f_{\omega,i}^n B_i} \frac{1}{J(f_\omega^n)\circ (f_{\omega,i}^n)^{-1}(y)} \frac{\Psi_n(\omega,y)}{Z_n(\omega,x)} d(\mu_{\theta^n\omega})_{f_\omega^n x}^{\eta_{\theta^n\omega}}(y) \qquad \text{(by (A.16))} \\
&= \frac{1}{W_n(\omega,x)Z_n(\omega,x)} \sum_i \int_{f_{\omega,i}^n B_i} \frac{\Psi_n(\omega,y)}{J(f_\omega^n)\circ (f_{\omega,i}^n)^{-1}(y)} g(\theta^n\omega,y) d\lambda_{\Theta^n(\omega,x)}^s(y) \\
&= \frac{1}{W_n(\omega,x)Z_n(\omega,x)} \sum_i \int_{B_i} \frac{\Psi_n(\omega,f_\omega^n y)}{J(f_\omega^n)(y)} g(\theta^n\omega, f_\omega^n y) |\det(T_y f_\omega^n|_{E^s(\omega,y)})| d\lambda_{(\omega,x)}^s(y) \\
&= \frac{1}{W_n(\omega,x)Z_n(\omega,x)} \int_B \frac{\Psi_n(\omega,f_\omega^n y)}{J(f_\omega^n)(y)} g\circ\Theta^n(\omega,y) |\det(T_y f_\omega^n|_{E^s(\omega,y)})| d\lambda_{(\omega,x)}^s(y).
\end{aligned}$$

Since B is arbitrary, this together with (A.14) yields

$$\frac{1}{W_n(\omega,x)Z_n(\omega,x)}\frac{\Psi_n(\omega,f_\omega^n y)}{J(f_\omega^n)(y)}g\circ\Theta^n(\omega,y)|\det T_y f_\omega^n|_{E^s(\omega,y)}|=g(\omega,y)$$

for $\lambda^s_{(\omega,x)}$-a.e. $y\in\eta_\omega(x)$, and hence

$$W_n(\omega,y)=\frac{X_n(\omega,y)Y_n(\omega,y)}{Z_n(\omega,y)}\text{ for }(\mu_\omega)_x^{\eta_\omega}\text{-a.e. }y\in\eta_\omega(x)$$

since $(\mu_\omega)_x^{\eta_\omega}\ll\lambda^s_{(\omega,x)}$ and $W_n(\omega,y)=W_n(\omega,x)$ and $Z_n(\omega,y)=Z_n(\omega,x)$ for any $y\in\eta_\omega(x)$. This shows that $W_n=\frac{X_nY_n}{Z_n}$ holds for μ-a.e. (ω,x) on Δ. □

The proof of Claim 2 is almost the same as that of [48, Claim 4.2] and is omitted here.

Proof of Claim 3 and Claim 4. For $f\in C^1(M,M)$ with no singularities, denote by $\deg(f)$ the number of elements of $\{y: f(y)=x\}$ (it is finite and independent of $x\in M$). Since $\deg(f_\omega^1)\le\lambda(M)|f_\omega^1|_{C^1}^{\dim M}$ (see [48]), from $|f|_{C^1}\le|f|_{C^2}$ and $\log^+|f_0(\omega)|_{C^2}\in L^1(\Omega,P)$ it follows that $\log\deg(f_0(\omega))\in L^1(\Omega,\mathrm{P})$ (recall that $f_0(\omega)$ has no singularities by assumption).

Then one has

$$\begin{aligned}
\infty &> \int_\Omega \log\deg(f_\omega^n)d\mathrm{P}\\
&\ge \int_\Omega H_{\mu_\omega}(\varepsilon|(f_\omega^n)^{-1}\varepsilon)d\mathrm{P}\\
&= -\int_\Omega\int_M \log(\mu_\omega)_x^{(f_\omega^n)^{-1}\varepsilon}(\{x\})d\mu_\omega d\mathrm{P}\\
&= \int_{\Omega\times M}\log J(f_\omega^n)(x)d\mu\\
&= \int_{\Omega\times M}(\log\Psi_n(\omega,f_\omega^n x)+\log\frac{\varphi\circ\Theta^n(\omega,x)}{\varphi(\omega,x)}+\log|\det T_x f_\omega^n|)d\mu
\end{aligned}$$

Noting that $k(x)\ge e^{-1}\log e^{-1}$, $x\in[0,+\infty)$ ($k(x)$ is as introduced in the proof of Proposition A.1.1), hence

$$\begin{aligned}
&\int_\Omega\int_M \log^-\Psi_n(\omega,f_\omega^n x)d\mu_\omega(x)d\mathrm{P}(\omega)\\
&= \int_\Omega\int_M \log^-\Psi_n(\omega,y)df_\omega^n\mu_\omega(y)d\mathrm{P}(\omega)\\
&= \int_\Omega\int_M k^-(\Psi_n(\omega,y))d\mu_{\theta^n\omega}(y)d\mathrm{P}(\omega)\\
&\ge \int_\Omega\int_M e^{-1}\log e^{-1}d\mu_{\theta^n\omega}(y)d\mathrm{P}(\omega)\\
&= e^{-1}\log e^{-1}>-\infty
\end{aligned}$$

one has $\log^-\Psi_n(\omega,f_\omega^n x)$ is μ-integrable.

Since

$$\log^{+} \frac{\varphi \circ \Theta^n(\omega,x)}{\varphi(\omega,x)} \le -\log(\mu_\omega)_x^{(f_\omega^n)^{-1}\varepsilon}(\{x\}) - \log^{-} \Psi_n(\omega, f_\omega^n x) - \log^{-} |\det T_x f_\omega^n|,$$

we know that $\log^{+} \frac{\varphi\circ\Theta^n}{\varphi}$ is μ-integrable. Then, by [41, Prop.2.2], $\log \frac{\varphi\circ\Theta^n}{\varphi}$ is integrable and $\int_{\Omega\times M} \log \frac{\varphi\circ\Theta^n}{\varphi} d\mu = 0$. On the other hand,

$$\log^{+} \Psi_n(\omega, f_\omega^n x) \le -\log(\mu_\omega)_x^{(f_\omega^n)^{-1}\varepsilon} - \log^{-} \frac{\varphi \circ \Theta^n(\omega,x)}{\varphi(\omega,x)} - \log^{-} |\det T_x f_\omega^n|$$

and hence $\log^{+} \Psi_n(\omega, f_\omega^n x)$ is μ-integrable. In what follows, given a probability space $(X, \mathscr{A}, m)$ and a sub-σ-algebra $\mathscr{H}$ of $\mathscr{A}$, we use $E_m(\cdot|\mathscr{H})$ to denote the corresponding conditional expectation operator. By $\mathscr{B}_{n,\omega}$ we will denote the σ-algebra generated by the partition $\eta_{\theta^n\omega}$ of M. Then

$$\begin{aligned}
&\int_\Omega \int_M \log^{+} Z_n(\omega,x) d\mu_\omega d\mathrm{P} \\
&= \int_\Omega \int_M \log^{+}\Big(\int_{\eta_{\theta^n\omega}(f_\omega^n x)} \Psi_n(\omega,z) d(\mu_{\theta^n\omega})_{f_\omega^n x}^{\eta_{\theta^n\omega}}\Big) d\mu_\omega(x) d\mathrm{P} \\
&= \int_\Omega \int_M \log^{+}\Big(\int_{\eta_{\theta^n\omega}(y)} \Psi_n(\omega,z) d(\mu_{\theta^n\omega})_y^{\eta_{\theta^n\omega}}\Big) d f_\omega^n \mu_\omega(y) d\mathrm{P} \\
&= \int_\Omega \int_M \Psi_n(\omega,y) \log^{+}\Big(\int_{\eta_{\theta^n\omega}(y)} \Psi_n(\omega,z) d(\mu_{\theta^n\omega})_y^{\eta_{\theta^n\omega}}\Big) d\mu_{\theta^n\omega}(y) d\mathrm{P} \\
&= \int_\Omega \int_M k^{+}(V_{\omega,n}(y)) d\mu_{\theta^n\omega}(y) d\mathrm{P} \\
&\quad \Big(\text{where } V_{\omega,n}(y) = \int_{\eta_{\theta^n\omega}(y)} \Psi_n(\omega,z) d(\mu_{\theta^n\omega})_y^{\eta_{\theta^n\omega}}\Big) \\
&= \int_\Omega \int_M k^{+}(E_{\mu_{\theta^n\omega}}(\Psi_n(\omega,\cdot)|\mathscr{B}_{n,\omega})(y)) d\mu_{\theta^n\omega}(y) d\mathrm{P} \\
&\le \int_\Omega \int_M E_{\mu_{\theta^n\omega}}(k^{+}(\Psi_n(\omega,\cdot))|\mathscr{B}_{n,\omega})(y) d\mu_{\theta^n\omega}(y) d\mathrm{P} \\
&= \int_\Omega \int_M k^{+}(\Psi_n(\omega,y)) d\mu_{\theta^n\omega}(y) d\mathrm{P} \\
&= \int_\Omega \int_M \log^{+} \Psi_n(\omega,y) d f_\omega^n \mu_\omega(y) d\mathrm{P} \\
&= \int_\Omega \int_M \log^{+} \Psi_n(\omega, f_\omega^n x) d\mu_\omega(y) d\mathrm{P} < \infty.
\end{aligned}$$

This shows that $\log^{+} Z_n \in L^1(\Omega\times M, \mu)$ which together with Claim 1 and Claim 2 yields $\log^{+} X_n \in L^1(\Omega\times M, \mu)$ since $\log W_n \le 0$. Hence $\int \log X_n d\mu = 0$ and then

$\log^- Z_n \in L^1(\Omega \times M, \mu)$ since $\log^- Z_n \geq \log^- X_n + \log^- Y_n$ μ-a.e.. We have thus proved Claim 4 and $\log Z_n \in L^1(\Omega \times M, M)$. Finally,

$$\begin{aligned}
&\int \log Z_n d\mu = \int_\Omega \int_M k(V_{\omega,n}(y)) d\mu_{\theta^n\omega} d\mathrm{P} \geq k(\int_\Omega \int_M V_{\omega,n}(y) d\mu_{\theta^n\omega} d\mathrm{P}) \\
&= k(\int_\Omega \int_M \Psi_n(\omega, y) d\mu_{\theta^n\omega} d\mathrm{P}) \\
&= k(\int_\Omega \int_M \int_{\{\omega' : \theta^n\omega' = \theta^n\omega\}} \frac{(\mathscr{L}_{f^n_\omega}\varphi_\omega)(y)}{\varphi_{\theta^n\omega}(y)} d\mathrm{P}^{(n)}_{\theta^n\omega}(\omega') d\mu_{\theta^n\omega}(y) d\mathrm{P}(\omega)) \\
&= k(\int_\Omega \mu_{\theta^n\omega}(M) d\mathrm{P}(\omega)) \ \ (\text{by} \ \ (A.11)) \\
&= 0. \square
\end{aligned}$$

Appendix B
Large Deviations in Axiom A Endomorphisms

Here we present some large deviation estimates for Axiom A endomorphisms by applying a general large deviation theorem in Kifer [37] and Ruelle's Smale space technique in [78]. This is the work of Liu et al [53].

B.1 Introduction and Statement of Main Results

Consider a discrete time dynamical system generated by a measurable self-map $f : X \hookleftarrow$ of some measurable space $(X, \mathscr{B})$. Let P be a reference probability measure on $(X, \mathscr{B})$ and let $\psi : X \to \mathbb{R}$ be an observable. If $\frac{1}{n} \sum_{k=0}^{n-1} \psi \circ f^k$ converges to some constant ψ^* P-a.e. as $n \to \infty$, then, for given $\varepsilon > 0$,

$$Q_n(\varepsilon) := \{x \in X : |\frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k x) - \psi^*| > \varepsilon\}$$

satisfies $\mathrm{P}(Q_n(\varepsilon)) \to 0$ as $n \to +\infty$. Large deviation theory in this set-up deals with estimates of the exponential speed of this last convergence to zero. More precisely and more generally, large deviation questions concern estimates of the following form:

$$\limsup_{n \to +\infty} \frac{1}{n} \log \mathrm{P}\{x \in X : \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k x) \in K\} \leq -\inf_{z \in K} I(z) \tag{B.1}$$

for any closed set $K \subset \mathbb{R}$ and

$$\liminf_{n \to +\infty} \frac{1}{n} \log \mathrm{P}\{x \in X : \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k x) \in G\} \geq -\inf_{z \in G} I(z) \tag{B.2}$$

for any open set $G \subset \mathbb{R}$, where $I : \mathbb{R} \to [0, +\infty)$ is a lower semi-continuous function and is called a *rate function*. Such questions have been well studied by Orey and Pelikan [59] for Anosov diffeomorphisms and by Young [96], among other things,

for Axiom A attractors. Developing ideas of [21, 89, 18, 1], Kifer [37] presents a unified approach to large deviations of dynamical systems and stochastic processes based on the existence of a pressure function and on the uniqueness of equilibrium states for certain potentials, and this approach enables one to generalize results from [59] and [96] and to recover the large deviation estimates in Donsker and Varadhan [15]. In this appendix we apply Kifer's results in [37], together with Ruelle's Smale space technique in [78], to give some large deviation estimates for Axiom A endomorphisms.

Our set-up and main results are as follows. Let M be a Riemannian manifold without boundary, O an open subset of M with compact closure and $f : O \to M$ a $C^r (r \geq 1)$ map. Let $\Lambda = f(\Lambda) \subset O$ be a compact invariant set of f and let

$$\Lambda^f := \{\tilde{x} = (x_i)_{-\infty}^{+\infty} : x_i \in \Lambda,\ f(x_i) = x_{i+1}, i \in \mathbb{Z}\}$$

be the orbit space of (Λ, f) with $\theta : \Lambda^f \to \Lambda^f$ denoting the left shift operator on Λ^f. Write $E = p^* T_\Lambda M$ for the pull-back bundle of $T_\Lambda M$ via the natural projection $p : \Lambda^f \to \Lambda, \tilde{x} \mapsto x_0$, and write $E_{\tilde{x}} = p^*_{\tilde{x}} T_{x_0} M \underset{p^*_{\tilde{x}}}{\overset{p_*}{\rightleftharpoons}} T_{x_0} M$ for the natural isomorphisms between the fibres $E_{\tilde{x}}$ and $T_{x_0} M$. A fibre-preserving map on E which covers θ can be defined by $p^*_{\theta\tilde{x}} \circ Tf \circ p_* : E_{\tilde{x}} \to E_{\theta\tilde{x}}$ for all $\tilde{x} \in \Lambda^f$, and for simplicity of notation we will denote it still by Tf.

Definition B.1.1 Λ is called a *hyperbolic set* of f if there is a continuous splitting $E = E^s \oplus E^u$ together with constants $C > 0$ and $0 < \lambda < 1$ such that

$$TfE^s \subset E^s, \qquad TfE^u = E^u$$

and for all $n \geq 0$

$$|Tf^n \xi| \leq C\lambda^n |\xi| \ \text{ for } \ \xi \in E^s, |Tf^n \eta| \geq C^{-1} \lambda^{-n} |\eta| \ \text{ for } \ \eta \in E^u.$$

Via a change of Riemannian metric we may—and will—assume that $C = 1$. Note that there may be points in Λ at which Tf is degenerate, and that the splitting $E_{\tilde{x}} = E^s_{\tilde{x}} \oplus E^u_{\tilde{x}}$ may depend on the past of $\tilde{x}$, i.e., it may happen that $p_* E^u_{\tilde{x}} \neq p_* E^u_{\tilde{y}}$ while $p(\tilde{x}) = p(\tilde{y})$. In what follows we denote by C_f the set of points in O at which Tf is degenerate, and by m the Lebesgue measure on M.

A hyperbolic set Λ is said to be an *Axiom A basic set* if Λ is locally maximal (i.e., there exists a neighborhood U of Λ such that $\bigcap_{n=-\infty}^{+\infty} f^n U = \Lambda$) and f is positively topologically transitive on it (i.e., $(f^n x_0)_{n \geq 0}$ is dense in Λ for some $x_0 \in \Lambda$). (It can be shown that periodic points are dense in an Axiom A basic set.) If an Axiom A basic set Λ has arbitrarily small open neighborhood U such that $f\bar{U} \subset U$ and $\bigcap_{n=0}^{+\infty} f^n U = \Lambda$, it is then called an *Axiom A attractor*, and U is called a *basin of attraction of* Λ. Applying Ruelle's Smale space technique, Qian and Zhang [72] presents an ergodic theory of such an Axiom A basic set Λ. In particular, they proved that (Λ, f) admits a unique equilibrium state μ_ϕ for each Hölder continuous $\phi : \Lambda \to \mathbb{R}$ and, in case of Λ being an attractor of $f \in C^2(O, M)$ with basin of attraction U and $m(C_f) = 0$, Λ supports a unique f-invariant measure ρ, called the *SRB measure*, which is generic with respect to Lebesgue measure in the following sense: for m-a.e. $x \in \bar{U}$ one has

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\psi(f^k x)=\int_\Lambda \psi d\rho \text{ for all } \psi\in C(\bar U).$$

Our main results of this note are as follows, where $\mathscr{P}(X)$ denotes the space of Borel probability measures on a compact metric space X endowed with the topology of weak convergence.

Theorem B.1.1. (1) *Let Λ be an Axiom A basic set of $f\in C^1(O,M)$, let $\phi:\Lambda\to\mathbb{R}$ be Hölder continuous and let μ_ϕ be the unique equilibrium state. Then there hold*

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mu_\phi\{x\in\Lambda:\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}\in K\}\le-\inf\{J(\nu):\nu\in K\} \tag{B.3}$$

for any closed $K\subset\mathscr{P}(\Lambda)$ and

$$\liminf_{n\to+\infty}\frac{1}{n}\log\mu_\phi\{x\in\Lambda:\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}\in G\}\ge-\inf\{J(\nu):\nu\in G\} \tag{B.4}$$

for any open $G\subset\mathscr{P}(\Lambda)$, where

$$J(\nu)=\begin{cases}P_f(\phi)-\int\phi d\nu-h_\nu(f) & \text{if}\quad \nu\in\mathscr{P}_f(\Lambda)\\ +\infty & \text{otherwise,}\end{cases} \tag{B.5}$$

$\mathscr{P}_f(\Lambda)$ is the set of f-invariant measures on Λ, $P_f(\phi)$ is the pressure of f for ϕ and $h_\nu(f)$ is the entropy of (f,ν).

(2) *Let Λ be an Axiom A attractor of $f\in C^2(O,M)$ and let ρ be the SRB measure on Λ. Then (B.3) and (B.4) hold true with μ_ϕ being replaced by ρ and with $J(\cdot)$ being defined by*

$$J(\nu)=\begin{cases}\int\sum_i\lambda_i(x)^+m_i(x)d\nu-h_\nu(f) & \text{if } \nu\in\mathscr{P}_f(\Lambda)\\ +\infty & \text{otherwise}\end{cases} \tag{B.6}$$

where $\lambda_i(x),1\le i\le r(x)$ are the Lyapunov exponents of f at x, $m_i(x)$ is the multiplicity of $\lambda_i(x)$ and $a^+:=\max\{a,0\}$.

(3) *Assume the circumstances of* (2). *Let U be a sufficiently small basin of attraction of Λ and let $\bar m$ be the normalized Lebesgue measure on $\bar U$. Then (B.3) and (B.4) hold true with μ_ϕ and Λ being replaced by $\bar m$ and $\bar U$ respectively and with $J(\cdot)$ being given by (B.6).*

The proof of this theorem will be given in Section 2. From

Theorem B.1.1 and the contraction principle there follows

Corollary B.1.1.1. *Let $\psi:O\to\mathbb{R}$ be a continuous function and let us in the circumstances of Theorem B.1.1 (3). For $J(\cdot)$ given by (B.6) put*

$$I(z)=\inf\{J(\nu):\int\psi d\nu=z\}.$$

Then (B.1) and (B.2) hold true with P and X being taken respectively as $\bar{m}$ and $\bar{U}$. In particular, for $\varepsilon > 0$ there exists $h > 0$ such that

$$\bar{m}\{x \in \bar{U} : |\frac{1}{n}\sum_{k=0}^{n-1}\psi(f^k x) - \int \psi d\rho| \geq \varepsilon\} \leq e^{-hn}$$

when n is sufficiently large.

Similar things hold true in the circumstances of Theorem B.1.1 (1) or (2).

B.2 Proof of Theorem B.1.1

B.2.1 A Large Deviation Theorem from Kifer [37]

Let (X,d) be a compact metric space, $f : (X,d) \hookleftarrow$ a continuous map, and, as before, $\mathscr{P}(X)$ the space of Borel probabilities on X endowed with the weak convergence topology , and $\mathscr{P}_f(X)$ the set of those elements in $\mathscr{P}(X)$ which are f-invariant. Put for $x \in X, \varepsilon > 0$ and $n \in \mathbb{N}$

$$B_f(x,\varepsilon,n) = \{y \in X : d(f^k x, f^k y) \leq \varepsilon, 0 \leq k \leq n-1\}.$$

The following theorem is a special case of the general large deviation results of [37], and we will apply it to Axiom A endomorphisms in this appendix.

Theorem B.2.1. *Suppose that $\mu \in \mathscr{P}(X)$, the support of μ is the whole X, and there is $\phi \in C(X)$ such that for any given small $\varepsilon > 0$ and for all $n \geq 1, x \in X$*

$$A_\varepsilon(n)^{-1} \leq \mu(B_f(x,\varepsilon,n))\exp\left(-\sum_{k=0}^{n-1}\phi(f^k x)\right) \leq A_\varepsilon(n) \tag{B.7}$$

where $A_\varepsilon(n) > 0$ is a constant satisfying $\frac{1}{n}\log A_\varepsilon(n) \to 0$ as $n \to +\infty$. Then for any $\psi \in C(X)$ there holds

$$\lim_{n\to\infty}\frac{1}{n}\log\int\exp\left(\sum_{k=0}^{n-1}\psi(f^k x)\right)d\mu(x) = P_f(\phi+\psi) = P_{f|_Y}(\phi+\psi) \tag{B.8}$$

where $P_f(\cdot)$ denotes the pressure of f and Y is the closure of $\bigcup_{\nu\in\mathscr{P}_f(X)}\mathrm{supp}\nu$. Suppose further that the entropy $h_\nu(f)$ is upper semicontinuous at all $\nu \in \mathscr{P}_f(X)$ and define

$$J(\nu) = \begin{cases} -\int \phi d\nu - h_\nu(f) & \text{if } \nu \in \mathscr{P}_f(X) \\ & +\infty \text{ otherwise.} \end{cases} \tag{B.9}$$

Then the above conclusion implies

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mu\{x:\sum_{k=0}^{n-1}\delta_{f^kx}\in K\}\leq-\inf\{J(\nu):\nu\in K\} \tag{B.10}$$

for any closed set $K\subset\mathscr{P}(X)$. If, moreover, there exist a countable number of functions $\psi_1,\psi_2,\cdots\in C(X)$ such that their span $\Gamma=\{\sum_{i=1}^n\beta_i\psi_i:\beta_i\in\mathbb{R},n\in\mathbb{N}\}$ is dense in $C(X)$ with respect to the supremum norm and that for each $\psi\in\Gamma$ there is a unique $\nu_\psi\in\mathscr{P}(X)$ satisfying

$$P_f(\phi+\psi)=\int\psi d\nu_\psi-J(\nu_\psi), \tag{B.11}$$

then one has for any open $G\subset\mathscr{P}(X)$

$$\liminf_{n\to+\infty}\frac{1}{n}\log\mu\{x:\sum_{k=0}^{n-1}\delta_{f^kx}\in G\}\geq-\inf\{J(\nu):\nu\in G\}. \tag{B.12}$$

B.2.2 Smale Spaces

Here we recall the notion and some properties of Smale spaces from Ruelle [78].

Definition B.2.1 Suppose that (X,d) is a compact metric space and $f:X\to X$ a homeomorphism. (X,d,f) is said to be a *Smale space* if for suitable $\varepsilon>0, 0<\delta<\varepsilon, 0<\lambda<1$ there exists a continuous map $[\cdot,\cdot]:\{(x,y)\in X\times X:d(x,y)<\varepsilon\}\to X$ with the following properties:

(1) $[x,x]=x$ and $[[x,y],z]=[x,z],[x,[y,z]]=[x,z]$ when the two sides of these relations are well defined.

(2) $f[x,y]=[fx,fy]$ when both sides are well defined and

$$d(f^ny,f^nz)\leq\lambda^nd(y,z)\quad\text{for } y,z\in V_x^+(\delta),n>0,$$

$$d(f^{-n}y,f^{-n}z)\leq\lambda^nd(y,z)\quad\text{for } y,z\in V_x^-(\delta),n>0,$$

where $V_x^+(\delta)=\{u:u=[u,x],d(x,u)<\delta\}$ and $V_x^-(\delta)=\{v:v=[x,v],d(x,v)<\delta\}$.

Here are some properties of a Smale space (X,d,f). Define $C^f(X)$ to be the space of functions $\phi\in C(X)$ which satisfy the following conditions: There exist $\delta>0$ and $K\geq0$ such that if $d(f^kx,f^ky)<\delta$ for $k=0,1,\cdots,n$ then

$$\left|\sum_{k=0}^n\phi(f^kx)-\sum_{k=0}^n\phi(f^ky)\right|\leq K \tag{B.13}$$

($\phi\in C^f(X)$ if it is Hölder continuous, see Ruelle [78, pp. 136]). If (X,f) is positively topologically transitive, then it has a unique equilibrium state μ_ϕ for each

$\phi \in C^f(X)$. (X,d,f) is expansive. In the case of (X,d,f) being topologically mixing, it has the specification property, and hence there is the following proposition which follows from Katok and Hasselblatt [32, Lemma 20.3.4 and Theorem 20.3.7]. For the transitive case, the proposition can be reduced to the mixing case by the spectral decomposition of Smale spaces (see also [78]) and by considering $(f^l, \sum_{i=0}^{l-1} \phi \circ f^i)$ restricted to one of the basic sets, say X_0, in the spectral decomposition, where l is the number of the basic sets (note that $\sum_{i=0}^{l-1} \phi \circ f^i \in C^{f^l}(X_0)$ if $\phi \in C^f(X)$; when f is not Hölderian, nor is $\sum_{i=0}^{l-1} \phi \circ f^i$ even if ϕ is).

Proposition B.2.2 *Assume that (X,d,f) is a positively topologically transitive Smale space and $\phi \in C^f(X)$. Let μ_ϕ be the unique equilibrium state of (X,d,f) for ϕ. Then, for small $\varepsilon > 0$ there exist $A_\varepsilon, B_\varepsilon > 0$ such that for $x \in X$ and $n \in \mathbb{N}$ one has*

$$A_\varepsilon \leq \mu_\phi(B_f(x,\varepsilon,n)) \exp\{-\sum_{k=0}^{n-1} \phi(f^k x) + nP_f(\phi)\} \leq B_\varepsilon.$$

B.2.3 Smale Space Property of Locally Maximal Hyperbolic Sets

We first collect in the following proposition some properties of local stable and unstable manifolds of a hyperbolic set, for details see Qian and Zhang [72] which is the first paper to apply the Smale space technique in the study of Axiom A endomorphisms.

Proposition B.2.3 *Let Λ be a hyperbolic set of $f \in C^r(O,M)$ $(r \geq 1)$. Then there exist in M a continuous family of C^r embedded* $\dim E^s$*-dimensional discs $\{W^s_{\mathrm{loc}}(x_0)\}_{x_0 \in \Lambda}$ and a continuous family of C^r embedded* $\dim E^u$*-dimensional discs $\{W^u_{\mathrm{loc}}(\tilde{x})\}_{\tilde{x} \in \Lambda^f}$ with the following properties:*

(1) *For each $\tilde{x} = (x_i)_{i \in \mathbb{Z}} \in \Lambda^f$, both $W^s_{\mathrm{loc}}(x_0)$ and $W^u_{\mathrm{loc}}(\tilde{x})$ contain x_0, and $fW^s_{\mathrm{loc}}(x_0) \subset W^s_{\mathrm{loc}}(fx_0), fW^u_{\mathrm{loc}}(\tilde{x}) \supset W^u_{\mathrm{loc}}(\theta\tilde{x})$.*

(2) *There is $0 < \bar{\lambda} < 1$ such that, for any $\tilde{x} \in \Lambda^f$*

$$d(fy,fz) \leq \bar{\lambda} d(y,z) \text{ if } y,z \in W^s_{\mathrm{loc}}(x_0)$$

and for each $y_0 \in W^u_{\mathrm{loc}}(\tilde{x})$ there is a unique $y_{-1} \in W^u_{\mathrm{loc}}(\theta^{-1}\tilde{x})$ with $fy_{-1} = y_0$, and this y_{-1} and similar z_{-1} for $z_0 \in W^u_{\mathrm{loc}}(\tilde{x})$ satisfy

$$d(y_{-1},z_{-1}) \leq \bar{\lambda} d(y_0,z_0).$$

(3) *There is $\delta > 0$ such that, for any $x_0 \in \Lambda, \tilde{y} \in \Lambda^f$ with $d(x_0,y_0) < \delta$, $W^s_{\mathrm{loc}}(x_0)$ intersects transversely with $W^u_{\mathrm{loc}}(\tilde{y})$ at a unique point $[x_0,\tilde{y}]$ which depends continuously on $(x_0,\tilde{y}) \in \{(u_0,\tilde{v}) \in \Lambda \times \Lambda^f : d(u_0,v_0) < \delta\}$, and, if furthermore Λ is locally maximal, then there is a unique $\tilde{z} \in \Lambda^f$ satisfying $z_0 = [x_0,\tilde{y}]$ and $z_i \in W^u_{\mathrm{loc}}(\theta^{-i}\tilde{y})$.*

We remark that the local stable manifolds $W^s_{\text{loc}}(x_0), x_0 \in \Lambda$ can be constructed by the usual standard argument, but, since Λ may contain degenerate points, the local unstable manifolds $W^u_{\text{loc}}(\tilde{x}), \tilde{x} \in \Lambda^f$ can not be constructed similarly. However, one may construct $W^u_{\text{loc}}(\tilde{x})$ in the following (standard as well) way: Let $\tilde{x} = (x_i)_{i\in\mathbb{Z}} \in \Lambda^f$. For small $r > 0$ let $h_{\tilde{x}} : E^u_{\tilde{x}}(r) \to E^s_{\tilde{x}}(r)$ be a Lipschitz map with $h_{\tilde{x}}(0) = 0$ and $\text{Lip}(h_{\tilde{x}}) \leq 1$, where $E^a_{\tilde{x}}(r) = \{\xi \in E^a_{\tilde{x}} : |\xi| < r\}, a = u, s$. Then one can show that there is a similar map $h_{\theta\tilde{x}} : E^u_{\theta\tilde{x}}(r) \to E^s_{\theta\tilde{x}}(r)$ such that

$$(\exp^{-1}_{x_1} \circ f \circ \exp_{x_0})\text{Graph}(h_{\tilde{x}}) \supset \text{Graph}(h_{\theta\tilde{x}}) \tag{B.14}$$

(see Liu [45, Proposition 2.6] for details). Starting from

$$\begin{aligned} h_{\theta^{-n}\tilde{x}} : E^u_{\theta^{-n}\tilde{x}}(r) &\to E^s_{\theta^{-n}\tilde{x}}(r) \\ \xi &\mapsto 0 \end{aligned}$$

via the relation (B.14) one ends by succession with a C^r function $h^{(n)}_{\tilde{x}} : E^u_{\tilde{x}}(r) \to E^s_{\tilde{x}}(r)$ with $h^{(n)}_{\tilde{x}}(0) = 0$ and $\text{Lip}(h^{(n)}_{\tilde{x}}) \leq 1$. It is easy to show that $h^{(n)}_{\tilde{x}}$ converges as $n \to +\infty$ uniformly to a similar function $h^{(\infty)}_{\tilde{x}} : E^u_{\tilde{x}}(r) \to E^s_{\tilde{x}}(r)$ whose graph gives $W^u_{\text{loc}}(\tilde{x})$ under the exponential map.

Assume in what follows that Λ is locally maximal. Let $0 < \bar{\lambda} < 1$ be as given in Proposition B.2.3 and define a metric on Λ^f by

$$d_f(\tilde{x}, \tilde{y}) = \left(\sum_{i=-\infty}^{+\infty} 2^{-|i|} d(x_i, y_i)^N \right)^{\frac{1}{N}}, \qquad \tilde{x}, \tilde{y} \in \Lambda^f$$

where $N > 0$ is an integer such that $\bar{\lambda}^N < \frac{1}{2}$.

For sufficiently small $\varepsilon > 0$ define

$$[\cdot,\cdot] : \{(\tilde{x}, \tilde{y}) \in \Lambda^f \times \Lambda^f : d_f(\tilde{x}, \tilde{y}) < \varepsilon\} \to \Lambda^f, (\tilde{x}, \tilde{y}) \mapsto \tilde{z}$$

where $\tilde{z}$ is the unique point in Λ^f given in Proposition B.2.3 (3) corresponding to x_0 and $\tilde{y}$. It is then easy to have the following

Proposition B.2.4 *(Λ^f, d_f, θ) is a Smale space.*

Clearly, when (Λ, f) is positively topologically transitive, so is (Λ^f, θ).

B.2.4 Proof of Theorem B.1.1

In what follows we will always endow Λ^f with the metric $d_f(\cdot,\cdot)$.

Proof of Theorem B.1.1 (1). Each Hölder continuous $\phi : \Lambda \to \mathbb{R}$ gives a Hölder continuous $\bar{\phi} = \phi \circ p : \Lambda^f \to \mathbb{R}$ which hence belongs to $C^\theta(\Lambda^f)$. By results in the

last two subsections, there is a unique equilibrium state $\bar{\mu}_{\bar{\phi}}$ of θ for $\bar{\phi}$ and $\mu_\phi = p\bar{\mu}_{\bar{\phi}}$ gives the unique equilibrium state of f for ϕ. Noting that there is a countable set of Hölder continuous functions which is dense in $C(\Lambda^f)$ and the entropy map of (Λ^f, θ) is upper semicontinuous, by Theorem B.2.1 and Proposition B.2.2 one has (B.3) and (B.4) for $(\Lambda^f, \theta, \bar{\mu}_{\bar{\phi}})$ with rate function

$$\bar{J}(\bar{\nu}) = \begin{cases} P_\theta(\bar{\phi}) - \int \bar{\phi}\, d\bar{\nu} - h_{\bar{\nu}}(\theta) & \text{if } \bar{\nu} \in \mathscr{P}_\theta(\Lambda^f) \\ +\infty & \text{otherwise.} \end{cases} \tag{B.15}$$

Since $P_\theta(\bar{\phi}) = P_f(\phi)$ and for any $\nu \in \mathscr{P}_f(\Lambda)$ there is a unique $\bar{\nu} \in \mathscr{P}_\theta(\Lambda^f)$ such that $p\bar{\nu} = \nu$ and this $\bar{\nu}$ satisfies $h_{\bar{\nu}}(\theta) = h_\nu(f)$, one has for any $\nu \in \mathscr{P}(\Lambda)$

$$\inf_{p\bar{\nu}=\nu} \bar{J}(\bar{\nu}) = J(\nu)$$

where $J(\nu)$ is given by (B.5). One obtains then Theorem B.1.1 (1) by the contraction principle. □

Proof of Theorem B.1.1 (2). Define

$$\bar{\phi}^u(\tilde{x}) = -\log|\det(T_{x_0} f|_{E^u_{\tilde{x}}})|, \tilde{x} \in \Lambda^f. \tag{B.16}$$

It is Hölder continuous (see [72]) and the unique equilibrium state $\bar{\mu}_{\bar{\phi}^u}$ of θ for $\bar{\phi}^u$ projects under p to the SRB measure ρ, and in this case

$$\inf_{p\bar{\nu}=\nu} \bar{J}^u(\bar{\nu}) = \begin{cases} \int \sum_i \lambda^i(x)^+ m_i(x) d\nu - h_\nu(f) & \text{if } \nu \in \mathscr{P}_f(\Lambda) \\ +\infty & \text{otherwise} \end{cases}$$

since $P_\theta(\bar{\phi}^u) = 0$, where $\bar{J}^u(\bar{\nu})$ is given by (B.15) corresponding to $\bar{\phi}^u$. This proves Theorem B.1.1 (2). □

In order to prove Theorem B.1.1 (3), we need the following result.

Lemma B.2.5 *Let Λ be a hyperbolic set of $f \in C^2(O, M)$. Then each Hölder continuous $\bar{\phi} : \Lambda^f \to \mathbb{R}$ is homologous to some $\hat{\phi} \in C(\Lambda^f)$ which satisfies $\hat{\phi}(\tilde{x}) = \hat{\phi}(\tilde{y})$ whenever $x_i = y_i$ for $i \le 0$, i.e., there is $\bar{u} \in C(\Lambda^f)$ such that*

$$\bar{\phi} = \hat{\phi} + \bar{u} - \bar{u} \circ \theta.$$

Proof. For each $x_0 \in \Lambda$ pick $(z_{i,x_0})_{i\in\mathbb{Z}} \in \Lambda^f$ with $z_{0,x_0} = x_0$. Define $r : \Lambda^f \to \Lambda^f$ by $r(\tilde{x}) = \tilde{x}^* = (x_i^*)_{i\in\mathbb{Z}}$ where

$$x_i^* = \begin{cases} x_i & \text{for } i \ge 0 \\ z_{i,x_0} & \text{for } i < 0. \end{cases}$$

Let $\bar{u} : \Lambda^f \to \mathbb{R}$ be defined by

$$\bar{u}(\tilde{x}) = \sum_{j=0}^{+\infty} [\bar{\phi}(\theta^j \tilde{x}) - \bar{\phi}(\theta^j r(\tilde{x}))]$$

Then $\hat{\phi} = \bar{\phi} + \bar{u} \circ \theta - \bar{u}$ satisfies the requirements (see Bowen [10, Lemma 1.6] for a similar argument). □

Proof of Theorem B.1.1 (3). Let $\bar{\phi}^u$ be given by (B.16).

Let now Λ be an Axiom A attractor of $f \in C^2(O,M)$. Lemma B.2.5 tells that $\bar{\phi}^u$ is homologous to $\phi^u \circ p$ for some $\phi^u \in C(\Lambda)$. Extend ϕ^u to a continuous function $\Phi^u : V \to \mathbb{R}$ where V is a neighborhood of Λ. Take $\varepsilon_0 > 0$ such that Proposition IV.IV.3.8 holds for $\varepsilon = \varepsilon_0$ and for all small $\delta > 0$. Let now U be a basin of attraction of Λ such that

$$\bar{U} \subset \bigcup_{x_0 \in \Lambda} [W^s_{\text{loc}}(x_0) \bigcap B(x_0, \varepsilon_0)]$$

(this union contains an open neighborhood of Λ, see Chapter IV; see also [72]), $\bar{U} \subset V$ and, moreover, $U_1 \supset \bar{U} \supset U \supset \bar{U}_2$ for two other basins of attraction U_1, U_2 of Λ. Put

$$a_n = \sup\{|\Phi^u(x) - \Phi^u(y)| : x, y \in \bar{U}, d(x,y) \le \varepsilon_0 \bar{\lambda}^n\}.$$

From Proposition IV.IV.3.8 it follows that for any $y_0 \in \bar{U}$, small $\delta > 0$ and all $n \ge 0$ one has

$$A_\delta(n)^{-1} \le m(B_f(y_0, \delta, n)) \exp\left(-\sum_{k=0}^{n-1} \Phi^u(f^k y_0)\right) \le A_\delta(n)$$

where $A_\delta(n) = A_{\varepsilon_0,\delta} \exp(\sum_{k=0}^{n-1} a_k)$ which clearly satisfies $\frac{1}{n} \log A_\delta(n) \to 0$ as $n \to \infty$. Then, by a minor modification of the proof of Kifer [37, Proposition 3.2], one can prove that for any $\psi \in C(\bar{U})$

$$P_{f|_{U_2}}(\Phi^u + \psi) \le \lim_{n\to\infty} \frac{1}{n} \log \int \exp\left(\sum_{k=0}^{n-1} \psi(f^k x)\right) d\bar{m}(x) \le P_{f|_{U_1}}(\Phi^u + \psi)$$

which implies

$$\lim_{n\to+\infty} \frac{1}{n} \int \exp\left(\sum_{k=0}^{n-1} \psi(f^k x)\right) d\bar{m}(x) = P_{f|_\Lambda}(\phi^u + \psi),$$

where $\bar{m}$ is the normalized Lebesgue measure on $\bar{U}$ and, when working on U_1, one may take continuous extensions of Φ^u and ψ. Noting that for each Hölder continuous $\psi : \bar{U} \to \mathbb{R}$ there is a unique equilibrium state of $f|_{\bar{U}}$ for $\Phi^u + \psi$ and the entropy map of $f|_U$ is upper semicontinuous, one obtains Theorem B.1.1 (3) by applying Theorem B.2.1 to $X = \bar{U}, \mu = \bar{m}$ and by

$$\int \Phi^u d\nu = \int \sum_i \lambda_i(x)^+ m_i(x) d\nu(x) \quad \text{for all } \nu \in \mathscr{P}_f(\bar{U}). \ \square$$

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