Appendix A
Simplicial Methods

A.1 Chain Complexes

For any commutative ring R, a *chain complex* K of R-modules is a family $\{K_n,\ d_n\}$ of R-modules K_n and R-homomorphisms $d_n : K_n \to K_{n-1}$, defined for all integers n such that $d_n d_{n+1} = 0$. An *n-cycle* of K is an element of the submodule $C_n(K) = \ker d_n$, and an *n-boundary* is an element of $d_{n+1}(K_{n+1})$. The *homology* of K, denoted $H(K)$, is the family of modules

$$H_n(K) = \ker d_n / \operatorname{im} d_{n+1}.$$

If K and K' are complexes, a *chain transformation* $f : K \to K'$ is a family of module homomorphisms $f_n : K_n \to K'_n$, such that for all n

$$d'_n f_n = f_{n-1} d_n.$$

Denote by $\mathcal{C}h(R)$ the category of chain complexes of R-modules, i.e., the category whose class $Ob(\mathcal{C}h(R))$ of objects consists of the chain complexes K, and the set $\operatorname{Hom}_{\mathcal{C}h(R)}(K,\ K')$ of morphisms between two objects $K,\ K'$ is the set of all chain transformations $f : K \to K'$. Every chain transformation $f \in \operatorname{Hom}_{\mathcal{C}h(R)}(K,\ K')$ induces a family of homomorphisms

$$H_n(f) : H_n(K) \to H_n(K')$$

defined by

$$H_n(f)(c + dK_{n+1}) = f(c) + dK'_{n+1},\ \ c \in \ker d_n.$$

A *chain homotopy* s between two chain transformations $f,\ g \in \operatorname{Hom}_{\mathcal{C}h(R)}$ $(K,\ K')$, denoted $s : f \simeq g$, is a family of module homomorphisms

$$s_n : K_n \to K'_{n+1}$$

such that

$$d'_{n+1}s_n + s_{n-1}d_n = f_n - g_n.$$

Theorem A.1 *If* $s : f \simeq g : K \to K'$, *then*

$$H_n(f) = H_n(g) : H_n(K) \to H_n(K'),\ -\infty < n < \infty.$$

A chain transformation $f \in \mathrm{Hom}_{\mathcal{C}h(R)}(K, K')$ is said to be a *chain equivalence* if there exists a chain transformation $h \in \mathrm{Hom}_{\mathcal{C}h(R)}(K', K)$ and homotopies $hf \simeq 1_K,\ fh \simeq 1_{K'}$.

Corollary A.2 *If* $f \in \mathrm{Hom}_{\mathcal{C}h(R)}(K, K')$ *is a chain equivalence, then the induced map* $H_n(f) : H_n(K) \to H_n(K')$ *is an isomorphism for each* n.

A.2 Simplicial Objects

Let $\mathcal{C}$ be a category. A simplicial object X_* in $\mathcal{C}$ is a family $\{X_i\}_{i\geq 0},\ X_i \in Ob(\mathcal{C})$ together with two families of morphisms

$$d_i \in \mathrm{Hom}_{\mathcal{C}}(X_q, X_{q-1}),\ s_i \in \mathrm{Hom}_{\mathcal{C}}(X_q, X_{q+1}),\ 0 \leq i \leq q,$$

called the *face* and the *degeneracy* maps respectively, which satisfy the following identities:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i,\ i < j, \\ s_i s_j &= s_{j+1} s_i,\ i \leq j, \\ d_i s_j &= s_{j-1} d_i,\ i < j, \\ d_j s_j &= d_{j+1} s_j = id, \\ d_i s_j &= s_j d_{i-1},\ i > j+1. \end{aligned} \tag{A.1}$$

A simplicial morphism $f : X_* \to Y_*$ is a family $f_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i),\ i \geq 0$, of morphisms compatible with the face and the degeneracy maps. The category of simplicial objects in $\mathcal{C}$ will be denoted by $\mathcal{SC}$.

The *simplicial category* (also called *ordinal number category*) Δ consists of the objects

$$Ob(\Delta) = \{[n] := \{0, 1, \ldots, n\}\}$$

and order preserving maps $\{f : [n] \to [m]\}$ as elements of $\mathrm{Hom}_{\Delta}([n], [m])$. In particular, there are the following morphisms, called the face and degeneracy maps, in this category:

$$
\begin{aligned}
&\delta_i : [n-1] \to [n],\ 0 \le i \le n, \\
&\sigma_i : [n+1] \to [n],\ 0 \le i \le n, \\
&\delta_i : \{0\,,1\,,\ldots\,,n-1\} \to \{0\,,1\,,\ldots\,,i-1\,,i+1\,,\ldots\,,n\}, \\
&\sigma_i : \{0\,,1\,,\ldots\,,n+1\} \to \{0\,,1\,,\ldots\,,i\,,i\,,\ldots\,,n\}.
\end{aligned}
$$

It is easy to check that the maps $\delta_i,\ \sigma_i$ satisfy the following *cosimplicial relations*:

$$
\begin{aligned}
\delta_j\delta_i &= \delta_i\delta_{j-1},\ i<j, \\
\sigma_j\sigma_i &= \sigma_i\sigma_{j+1},\ i \le j, \\
\sigma_j\delta_i &= \delta_i\sigma_{j-1},\ i<j, \\
\sigma_j\delta_j &= \sigma_j\delta_{j+1} = id, \\
\sigma_j\delta_i &= \delta_{i-1}\sigma_j,\ i > j+1.
\end{aligned} \tag{A.2}
$$

Furthermore, all elements of $\mathrm{Hom}_\Delta(-\,,-)$ can be written as compositions of these face and degeneracy maps. It thus turns out that a simplicial object in a category $\mathcal{C}$ is simply a contravariant functor from the simplicial category Δ to $\mathcal{C}$, i.e.

$$\mathcal{SC} = \{\Delta^{op} \to \mathcal{C}\},$$

where Δ^{op} denotes the opposite category of the category Δ. By a simplicial group (resp. ring, abelian group, topological space, etc.) we shall mean a simplicial object in the category of groups (resp. the corresponding category).

Example A.3

For a given topological space X, the *total singular complex* $S(X)$ of X is the simplicial set defined as follows.
For $n \ge 0$, let

$$\Delta_n = \{(x_0\,,\ldots\,,x_n) \in \mathbb{R}^{n+1} \mid 0 \le x_i \le 1,\ \sum_{i=0}^{n} x_i = 1\}.$$

Define

$$
\begin{aligned}
&e_i : \Delta_{n-1} \to \Delta_n,\ 0 \le i \le n, \\
&f_j : \Delta_{n+1} \to \Delta_n,\ 0 \le i \le n,
\end{aligned}
$$

by

$$
\begin{aligned}
&e_i : (x_0,\ \ldots\ ,\ x_n) \mapsto (x_0,\ \ldots\ ,\ x_{i-1},\ 0,\ x_i,\ \ldots\ ,\ x_n), \\
&f_j : (x_0,\ \ldots\ ,\ x_{n+1}) \mapsto (x_0,\ \ldots\ ,\ x_{j-1}, x_j + x_{j+1},\ x_{j+2},\ \ldots\ ,\ x_{n+1}).
\end{aligned}
$$

A *singular n-simplex* of X is a continuous map $\sigma : \Delta_n \to X$. The family $\{S(X)_n\}_{n\ge 0}$ of sets is a simplicial set with the face and degeneracy maps given by

$$d_i : S(X)_n \to S(X)_{n-1},\ 0 \le i \le n,$$
$$s_j : S(X)_n \to S(X)_{n+1},\ 0 \le j \le n,$$

defined by

$$d_i(\sigma) = \sigma \circ e_i,\ s_j(\sigma) = \sigma \circ f_j,\ \sigma \in S(X)_n.$$

Let R be a simplicial ring. Then a simplicial abelian group M is called a left *simplicial R-module* (or, simply an R-module), if there exists a simplicial map $f : R \times M \to M$ such that, for each i, f_i defines an R_i-module structure on M_i. Similarly, if G is a simplicial group, we can define *simplicial G-set* (resp. *simplicial G-space*) to be a simplicial set X (resp. topological space) with a simplicial map $G \times X \to X$.

A.3 Geometric Realization Functor

Let X be a simplicial set. The *geometric realization* $|X|$ of X is the topological space obtained from the disjoint union

$$\bigcup_n (X_n \times \Delta_n),$$

where the set X_n is viewed as a topological space with discrete topology, by making the following identifications:

$$(d_i x,\, p) \sim (x,\, e_i p),\ (x,\, p) \in X_n \times \Delta_{n-1},$$
$$(s_i x,\, p) \sim (x,\, f_i p),\ (x,\, p) \in X_{n-1} \times \Delta_n.$$

This construction defines the *geometric realization functor*

$$|\ | : \mathcal{S}\mathsf{Set} \to \mathsf{Top}$$

from the category $\mathcal{S}\mathsf{Set}$ of simplicial objects in the category Set of sets to the category Top of topological spaces.

The geometric realization can also be described as the coequalizer

$$\bigsqcup_{\phi:[n]\to[m]} (X_m \times \Delta_n) \rightrightarrows \bigsqcup_n (X_n \times \Delta_n) \longrightarrow |X|.$$

A.4 Skeleton and Coskeleton Functors

Let Δ_k be the full sub-category of the category Δ consisting of sets of cardinality at most $k+1,\ k \ge 0$. Then any element from

$$\mathcal{S}_k\mathcal{C} := \{\Delta_k^{op} \to \mathcal{C}\}$$

is called a *k-truncated simplicial object in $\mathcal{C}$*. Clearly, for any $k \geq 0$, we have a functor

$$\mathrm{Tr}^k : \mathcal{S}\mathcal{C} \to \mathcal{S}_k\mathcal{C},$$

which "truncates" the simplicial object at level k, i.e., forgets the part of simplicial object which appears in dimensions greater than k. It is known that, in case $\mathcal{C}$ has finite colimits, the functor Tr^k has a left adjoint functor sk^k, called the *k-skeleton* functor. Similarly, if $\mathcal{C}$ has finite projective limits, then Tr^k admits a right adjoint functor $cosk^k$, called the *k-coskeleton* functor. The k-skeleton functor can be constructed precisely by iterating the process of taking the so-called simplicial cokernels. For the detailed description of this construction, see [Dus75].

Example A.4

Let F be a free group with generators $\{x_i\}_{i\in I}$ and R its normal subgroup generated, as a normal subgroup, by the set $\{r_j\}_{j\in J}$. Consider the free product $F_1 = F * F_R$, where F_R is the free group with basis $\{y_j\}_{j\in J}$. Then we have the following three homomorphisms between free groups:

$$\begin{aligned}
&d_0 : F_1 \to F,\ x_i \mapsto x_i,\ i \in I,\ y_j \mapsto 1,\ j \in J,\\
&d_1 : F_1 \to F,\ x_i \mapsto x_i,\ i \in I,\ y_j \mapsto r_j,\ j \in J,\\
&s_0 : F \to F_1,\ x_i \mapsto x_i,\ i \in I.
\end{aligned}$$

It easy to see that the simplicial identities are satisfied for these maps and we have the 1-truncated simplicial group

$$S(X\,,\mathcal{R}) = F_1 \;\begin{smallmatrix}\xrightarrow{d_0}\\ \xrightarrow{d_1}\\ \xleftarrow{s_0}\end{smallmatrix}\; F.$$

We describe the 1-skeleton of this simplicial group. We have

$$\begin{aligned}
&\mathrm{sk}^1\, S(X\,,\mathcal{R})_0 = F = F(x_i,\ i \in I),\\
&\mathrm{sk}^1\, S(X\,,\mathcal{R})_1 = F_1 = F(s_0(x_i)\,,r_j,\ i \in I,\ j \in J),\\
&\mathrm{sk}^1\, S(X\,,\mathcal{R})_2 = F(s_1s_0(x_i)\,,s_0(r_j)\,,s_1(r_j),\ i \in I,\ j \in J),\\
&\mathrm{sk}^1\, S(X\,,\mathcal{R})_3 = F(s_2s_1s_0(x_i)\,,s_1s_0(r_j)\,,s_2s_1(r_j)\,,s_2s_0(r_j),\ i \in I,\ j \in J),\\
&\ldots
\end{aligned}$$

where, for a set X, by $F(X)$ we mean the free group generated by X. One can easily write the simplicial maps in $\mathrm{sk}^1(S(X,\mathcal{R}))$ in a natural way.

In a similar way, we can define the simplicial Lie algebra $\mathrm{sk}^1\, S(X,\mathcal{R})$ for the case of a free Lie algebra F generated the the set $\{x_i\}_{i\in I}$ and its ideal R, which is a smallest ideal containing the set of elements $\{r_j \in F\},\ j \in J$. Then we get the 1-truncated simplicial Lie algebra $S(X,\mathcal{R})$ and its 1-skeleton, viewed as a simplicial Lie algebra.

A.5 Moore Complex and Homotopy Groups

Let $\mathcal{C}$ be one of the following categories:

Gr:= the category of groups,
Lie:= the category of Lie algebras,
${}_R\mathsf{Mod}$:= the category of R-modules for some commutative ring R with identity.

For a given simplicial object $X \in \mathcal{SC}$, define a complex $(N_*(X),\, \bar{d}_*)$, called the *Moore complex* of X, by setting

$$N_n(X) = \bigcap_{0\le i<n} \ker(d_i : X_n \to X_{n-1}), \tag{A.3}$$

and the homomorphism $\bar{d}_n$ to be the restriction of $d_n : X_n \to X_{n-1}$ on $N_n(X)$.

The *homotopy groups* $\pi_i(X),\ i \ge 0$, of a given simplicial object $X_* \in \mathcal{SC}$ are defined as the homologies of its Moore complex:

$$\pi_i(X) := H_i(N_*(X), \bar{d}_*),\ i \ge 0. \tag{A.4}$$

It is easy to show that, for any $X \in \mathcal{SC}$, $\pi_i(X)$ is an abelian group for $i \ge 1$.

For a given simplicial group G, denote by $Z_n(G)$ the nth chain subgroup of G_n, i.e.,

$$Z_n(G) = \ker(\bar{d}_n) = \bigcap_{0\le i\le n} \ker(d_i),$$

and by $B_n(G)$ the nth boundary subgroup of G_n, i.e.,

$$B_n = \operatorname{im}(\bar{d}_{n+1}).$$

Thus we have, by definition,

$$\pi_n(G) = Z_n(G)/B_n(G),\ n \ge 0.$$

In the case of an abelian simplicial group G, there is an equivalent way to compute the homotopy groups. Consider the chain complex $\{G_n,\ d_n\}$, where

$$d_n = \sum_{i=0}^{n} (-1)^i d_i : G_n \to G_{n-1}.$$

It can be checked directly that $d_n \circ d_{n+1} = 0$, and

$$\pi_i(G) = H_i(G_n,\, d_n).$$

The following Proposition follows directly from the definition of homotopy groups of simplicial groups (resp. R-modules).

Proposition A.5 *Let $1 \to H \to G \to K \to 1$ be a short exact sequence of simplicial groups (resp. simplicial R-modules). Then there exists an induced long exact sequence of homotopy groups:*

$$\ldots \to \pi_{i+1}(H) \to \pi_{i+1}(G) \to \pi_{i+1}(K) \to \pi_i(H) \to \ldots$$

It is easy to see that, for any simplicial group G, the π_0-functor coincides with the coequilizer functor:

$$\pi_0(G) = \mathrm{coeq}(G_1 \overset{d_0,\, d_1}{\rightrightarrows} G_0).$$

A similar formula holds for simplicial R-modules.

For a given element $f \in G_0$, there is a simplicial automorphism $F_f : G \to G$ defined by

$$F_f : x \mapsto (s_0^n f)^{-1} x s_0^n f,\ x \in G_n. \tag{A.5}$$

Let $f \in B_0(G)$, that is $f = d_1 f_1$, where $d_0 f_1 = 1$. Then

$$s_0^n d_1 f_1 = s_0^{n-1} d_2 s_0 f_1 = \ \ldots \ = d_{n+1} s_0^n f_1,\ n \geq 1$$

and

$$d_i s_0^n f_1 = s_0 d_{i-1} s_0^{n-1} f_1 = \ \ldots \ = s_0^n d_0 f_1 = 1,\ 0 \leq i \leq n;$$

hence

$$s_0^n f = s_0^n d_1 f_1 \in B_n(G).$$

Therefore, the map F_f defines an action of the group $\pi_0(G)$ on the abelian group $\pi_n(G),\ n \geq 1$, i.e., $\pi_n(G)$ *can be viewed as a* $\mathbb{Z}[\pi_0(G)]$*-module.*

The computation of $\pi_1(G)$, even for the case of quite simple simplicial groups G, can turn out to be nontrivial. We present the computation for the case of simplicial groups which generalizes Example A.4.

Example A.6

(Brown-Loday [Bro87]) Let G be a simplicial group, such that G_2 is generated by degeneracy elements, i.e.,

$$G_2 = \langle s_0(G_1),\, s_1(G_1) \rangle. \tag{A.6}$$

Then

$$\mathrm{im}(\overline{d}_2) = [\ker(d_1),\, \ker(d_2)]. \tag{A.7}$$

Hence, we have

$$\pi_1(G) = \frac{\ker(d_0) \cap \ker(d_1)}{[\ker(d_0),\, \ker(d_1)]}. \tag{A.8}$$

In Example A.4, the condition (A.6) clearly holds for the 1-skeleton $\operatorname{sk}^1 S(X, \mathcal{R})$. Therefore, the formula (A.8) holds. Clearly, we have

$$\ker(d_0) = \langle y_j,\ j \in J\rangle^{F_1},$$
$$\ker(d_1) = \langle y_j r_j^{-1},\ j \in J\rangle^{F_1},$$

and

$$\pi_1(\operatorname{sk}^1 S(X, \mathcal{R})) = \frac{\langle y_j,\ j \in J\rangle^{F_1} \cap \langle y_j r_j^{-1},\ j \in J\rangle^{F_1}}{[\langle y_j,\ j \in J\rangle^{F_1}, \langle y_j r_j^{-1},\ j \in J\rangle^{F_1}]}. \qquad \text{(A.9)}$$

The action of $\pi_0(G)$ on $\pi_1(\operatorname{sk}^1 S(X, \mathcal{R}))$ is given by conjugation:

$$fR \circ x[\ker(d_0), \ker(d_0)] = f^{-1}xf[\ker(d_0), \ker(d_1)],$$
$$x \in \ker(d_0) \cap \ker(d_1),\ x \in F(X).$$

For the case of Lie algebras one can get an analogous result. First note, that for a simplicial Lie algebra G in which G_2 is generated by the degeneracy elements, the relation (A.7) again holds (where the bracket $[.,.]$ denotes the product in a Lie algebra) [Akc02]. Therefore, for a free Lie algebra F with generating set $\{x_i\}_{i\in I}$ and a subset $\{r_j \in F\}_{j\in J}$, one has

$$\pi_1(\operatorname{sk}^1 S(X, \mathcal{R})) = \frac{(y_j,\ j \in J)F \cap (y_j - r_j,\ j \in J)F}{[(y_j,\ j \in J)F, (y_j - r_j,\ j \in J)F]}. \qquad \text{(A.10)}$$

We say that the Moore complex $N_*(X)$ of a simplicial object X in a category $\mathcal{C}$ is of length $\leq k$ if $N_n(X) = 0$ for all $n \geq k+1$. The simplicial objects with Moore complex of length $\leq n$ form a category; we denote this category by $\mathcal{SC}(n)$.

Proposition A.7 *If R is a principal ideal domain and X is a projective simplicial R-module, then $N(X)$ is a complex of projective R-modules.*

A.6 Dold-Kan Correspondence

The following result is the key to the construction of derived functors.

Theorem A.8 *The functor $N : \mathcal{S}_R\mathsf{Mod} \to \mathcal{C}h(R)$ is an equivalence of categories.*

To prove this result it is clearly enough to construct an inverse map

$$N^{-1} : \mathcal{C}h(R) \to \mathcal{S}_R\mathsf{Mod},$$

which is constructed by setting

$$(N^{-1}C)_n = \bigoplus_{f:[n]\twoheadrightarrow[m]} f^*(C_m),\ C \in \mathcal{C}h(R).$$

For example, for a given chain complex C of R-modules, the first few terms of $N^{-1}C$ can be written as

$$\begin{aligned}
(N^{-1}C)_0 &= C_0,\\
(N^{-1}C)_1 &= C_1 \oplus s_0(C_0),\\
(N^{-1}C)_2 &= C_2 \oplus s_0(C_1) \oplus s_1(C_1) \oplus s_0 s_0(C_0).
\end{aligned}$$

A.7 Eilenberg-Zilber Equivalence

For (A, ∂_1), $(B, \partial_2) \in Ch(R)$, the *tensor product* $(A \otimes_R B, \partial) \in Ch(R)$ is defined as follows:

$$\begin{aligned}
(A \otimes_R B)_n &= \oplus_{p+q=n} A_p \otimes_R B_q,\\
\partial(a \otimes b) &= \partial_1 a \otimes b + (-1)^{dim(a)} a \otimes \partial_2 b.
\end{aligned}$$

For $X, Y \in \mathcal{S}_R\mathsf{Mod}$, the *tensor product* $X \otimes_R Y \in \mathcal{S}_R\mathsf{Mod}$ is defined as follows:

$$\begin{aligned}
(X \otimes_R Y)_n &= X_n \otimes_R Y_n,\\
\partial_i(x \otimes y) &= \partial_i x \otimes \partial_i y,\ 0 \le i \le n,\\
s_i(x \otimes y) &= s_i x \otimes s_i y,\ 0 \le i \le n.
\end{aligned}$$

For $x \in X_n$, $y \in Y_n$, the map

$$f : x \otimes y \mapsto \sum_{p+q=n} \partial_{n-p+1} \ldots \partial_{n-1}\partial_n x \otimes \partial_0^q y,$$

called the *Alexander-Whitney map*, induces the homomorphism of the normalized complexes

$$\bar{f} : N(X \otimes_R Y) \to N(X) \otimes_R N(Y). \tag{A.11}$$

The converse map

$$\bar{\nabla} : N(X) \otimes_R N(Y) \to N(X \otimes_R Y) \tag{A.12}$$

is induced by the 'shuffle-map', defined as follows. For $p, q \ge 1$, let

$$(a; b) = (a_1, \ldots, a_p; b_1, \ldots, b_q)$$

be a permutation of $(0, \ldots, p+q-1)$, such that $a_1 < \ldots < a_p$, $b_1 < \ldots < b_q$. We will refer to such $(a; b)$ as a $(p; q)$-shuffle. Denote $sign(a; b)$ to be the sign of the permutation $(a; b)$. For $x \in X_p$, $y \in Y_q$, define

$$\nabla : x \otimes y \mapsto \sum_{(p;q)-shuffles\ (a;b)} (-1)^{sign(a;b)} s_{a_p} \ldots s_{a_1} x \otimes s_{b_q} \ldots s_{b_1} y.$$

The maps (A.11) and (A.12) define the isomorphism of the chain complexes:

$$N(X \otimes_R Y) \simeq N(X) \otimes_R N(Y), \tag{A.13}$$

called the *Eilenberg-Zilber equivalence.*

In the case when R is a principal ideal domain, and X is a free R-simplicial module, Künneth formula implies that there exists the following split exact sequence of R-modules

$$\begin{aligned} 0 \to \oplus_i H_n(N(X)) \otimes_R H_{n-i}(N(Y)) \to H_n(N(X) \otimes_R N(Y)) \to \\ \oplus_i \operatorname{Tor}_1^R(H_i(N(X)), H_{n-i-1}(N(Y))) \to 0, \end{aligned} \tag{A.14}$$

which can be written as

$$\begin{aligned} 0 \to \oplus_i \pi_n(X) \otimes_R \pi_{n-i}(Y) \to \pi_n(X \otimes_R Y) \to \\ \oplus_i \operatorname{Tor}_1^R(\pi_i(X), \pi_{n-i-1}(Y)) \to 0, \end{aligned} \tag{A.15}$$

due to the Eilenberg-Zilber equivalence (A.13).

A.8 Classifying Functor $\overline{W}$ and Homology

Let G be a simplicial group. Define the simplicial set WG by setting:

$$WG_n = G_n \times G_{n-1} \times \ldots \times G_0, \ n \geq 0 \tag{A.16}$$

with face and degeneracy maps

$$\begin{aligned} d_i(g_n, \ldots, g_0) &= (d_i g_n, d_{i-1} g_{n-1}, \ldots, (d_0 g_{n-i}) g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \ i<n, \\ d_n(g_n, \ldots, g_0) &= (d_n g_n, d_{n-1} g_{n-1}, \ldots, d_1 g_1), \\ s_i(g_n, \ldots, g_0) &= (s_i g_n, s_{i-1} g_{n-1}, \ldots, s_0 g_{n-1}, 1, g_{n-i-1}, \ldots, g_0). \end{aligned}$$

The simplicial set WG has a natural structure as a G-set, namely the one where the left G-action is given by

$$g \circ (g_n, \ldots, g_0) \mapsto (g g_n, g_{n-1}, \ldots, g_0), \ g \in G_n.$$

The space $\overline{W}G$ is the quotient of WG by the left G-action. Let $q : WG \to \overline{W}G$ be the quotient map.

A *reduced simplicial set* is a simplicial set having only one vertex i.e., a simplicial set X in which X_0 is a singleton. Denote the sub-category of the reduced simplicial sets by $\mathsf{r}\mathcal{S}\mathsf{Set}$. Then the construction $\overline{W}G$ defines a functor

$$\overline{W} : \mathsf{Gr} \to \mathsf{r}\mathcal{S}\mathsf{Set}, \tag{A.17}$$

called a *classifying space functor* on the category Gr of groups. Clearly, the components of $\overline{W}G$ can be written as

$$\overline{W}G_0 = 1,\ \overline{W}G_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0,\ n > 0.$$

The face and degeneracy maps are defined as

$$\begin{aligned}
&d_0(g) = 1,\ d_1(g) = 1,\ g \in \overline{W}G_1,\ \ s_0(1) = 1,\\
&d_0(g_n, \ldots, g_0) = (g_{n-1}, \ldots, g_0),\\
&d_{i+1}(g_n, \ldots, g_0) = (d_i g_n, \ldots, d_1 g_{n-i+1},\ g_{n-i-1} d_0 g_{n-i},\ g_{n-i-2}, \ldots, g_0);\\
&s_0(g_{n-1}, \ldots, g_0) = (1, g_{n-1}, \ldots, g_0);\\
&s_{i+1}(g_{n-1}, \ldots, g_0) = (s_i g_n, \ldots, s_0 g_{n-i}, 1, g_{n-i-1}, \ldots, g_0).
\end{aligned}$$

Let M be a simplicial G-module. Following Quillen, define a graded abelian group, called *homology of G with coefficients in M* by setting

$$H_*(G, M) := \pi_*(\mathbb{Z}[WG] \otimes_{\mathbb{Z}[G]} M),$$

where $\mathbb{Z}[WG]$ and $\mathbb{Z}[G]$ are free abelian simplicial groups obtained by applying the group ring functor to WG and G respectively.

For a group G, consider the constant simplicial group with $G_i = G$ and all face and degeneracy maps equal to the identity map. Let M be a G-module, then, clearly, $H_*(G, M)$ is the same as ordinary group homology of G with coefficients in M. Clearly, $\mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[WG] = \mathbb{Z}[\overline{W}G]$, where $\mathbb{Z}$ is viewed as a constant simplicial G-module. Hence,

$$H_*(G, \mathbb{Z}) = \pi_*(\mathbb{Z}[WG]).$$

A.9 Bisimplicial Groups

For a given category $\mathcal{C}$, a *bisimplicial object* in $\mathcal{C}$ is a functor

$$\Delta^{op} \times \Delta^{op} \to \mathcal{C}.$$

Clearly, in analogy with simplicial objects, any bisimplicial object can be viewed as a set of objects $X_{m,n} \in \mathcal{C}$, connected by certain maps. In this way,

fixing the first index m, one gets the simplicial object $X_{m,*}$ and fixing the second index n, the simplicial object $X_{*,n}$.

Let G be a bisimplicial group, i.e., a bisimplicial object in the category Gr. It can be viewed as the data

$$G = \{G_{m,n},\ d_j^h,\ s_j^h,\ d_j^v,\ s_j^v\},$$

where each $G_{m,n}$ $(m,\ n \geq 0)$ is a group and

$$\begin{aligned}
d_j^h &: G_{m,n} \to G_{m-1,n},\ 0 \leq j \leq m,\\
s_j^h &: G_{m,n} \to G_{m+1,n},\ 0 \leq j \leq m,\\
d_j^v &: G_{m,n} \to G_{m,n-1},\ 0 \leq j \leq n,\\
s_j^v &: G_{m,n} \to G_{m,n+1},\ 0 \leq j \leq n.
\end{aligned}$$

are homomorphisms satisfying appropriate relations. Define the *diagonal* $\mathcal{D}G$ of the bisimplicial group G to be the simplicial group given by setting

$$(\mathcal{D}G)_n = G_{n,n},\ d_j = d_j^h \circ d_j^v,\ s_j = s_j^h \circ s_j^v.$$

Theorem A.9 (Quillen [Qui66]). *For a bisimplicial group G there are two spectral sequences:*

$$\begin{aligned}
E^2_{p,q} &= \pi_p^h \pi_q^v(G) \Longrightarrow \pi_{p+q}(\mathcal{D}G),\\
E^2_{p,q} &= \pi_p^v \pi_q^h(G) \Longrightarrow \pi_{p+q}(\mathcal{D}G),
\end{aligned}$$

where $\pi_p^h\pi_q^v(G)$ (resp. $\pi_p^v\pi_q^h(G)$) is the p-th homotopy group of the "horizontal" (resp. "vertical") simplicial group obtained by taking the q-th homotopy group of each of the "vertical" (resp. "horizontal") simplicial groups.

For a simplicial group $X_* \in \mathcal{S}\mathsf{Gr}$, there is a natural first quadrant spectral sequence

$$E^1_{p,q} = H_q(X_p, R) \Longrightarrow H_{p+q}(\overline{W}X_*, R), \tag{A.18}$$

$$d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}. \tag{A.19}$$

A.10 Certain Simplicial Constructions

Kan's Construction

Let us recall the Kan's *loop group construction.* This is the functor

$$G : \mathsf{r}\mathcal{S}\mathsf{Set} \to \mathcal{S}\mathsf{Gr}, \tag{A.20}$$

with the properties:

$$(i) \quad \pi_{n-1}(GX) \simeq \pi_n(X), \; X \in \mathcal{S}\mathsf{Gr},$$
$$(ii) \quad (GX)_n \text{ is a free group}, \; n \geq 0.$$

For a given $X \in \mathsf{r}\mathcal{S}\mathsf{Set}$, define the loop group $GX \in \mathcal{S}\mathsf{Gr}$ as follows:

Let $(GX)_n$ be the free group on the elements of X_{n+1} modulo the n-degenerate elements, i.e., elements of the form $s_0x, \; x \in X_n$; thus $(GX)_n$ can be presented as

$$(GX)_n = F(X_{n+1})/F(s_0(X_n)), \; n \geq 0.$$

The face and degeneracy maps are defined by

$$\tau(d_0x)d_0\tau(x) = \tau(d_1x),$$
$$d_i\tau(x) = \tau(d_{i+1}x), \; i > 0,$$
$$s_i\tau(x) = \tau(s_{i+1}x), \; i \geq 0,$$

where $\tau(x)$ denotes the class of the element $x \in X_{n+1}$ in $(GX)_n$. Obviously, for any $X \in \mathsf{r}\mathcal{S}\mathsf{Set}$, $(GX)_n$ is the free group on the set $X_{n+1}\backslash s_0X_n$.

The loop functor G is a left adjoint functor to the classifying space functor $\overline{W}$. These functors have the following main property:

For any reduced simplicial set X and a simplicial group Γ, the canonical maps

$$G(\overline{W}\Gamma) \mapsto \Gamma, \quad X \mapsto \overline{W}(GX)$$

are weak homotopy equivalences (see, for example, [Goe99, Section V]).

A simplicial group $X \in \mathcal{S}\mathsf{Gr}$ is called *free* if, for all $n \geq 0$, the components X_n are free groups, and there are sets of free generators $B_n \subset X_n, \; n \geq 0$, such that $s_i(B_n) \subseteq B_{n+1}$ for all $n \geq 0, \; 0 \leq i \leq n$.

It is easy to see that free simplicial groups and CW-complexes have a lot of similar properties. To handle these similarities, Kan introduced the concept of *CW-basis* for a given free simplicial group [Kan59].

Let F be a free simplicial group. A family $\{\mathfrak{B}_n\}_{n\geq 0}, \; \mathfrak{B}_n \subset F_n$, is called a *CW-basis* of F if

(i) $\mathfrak{B}_n$ is a basis of F_n;
(ii) $s_i(x) \in \mathfrak{B}_{n+1}$ for all $x \in \mathfrak{B}_n, \; n \geq 0, \; 0 \leq i \leq n$;
(iii) if $x \in \mathfrak{B}_n$ is non-degenerate, then $d_i(x) = 1, \; n \geq 1, \; 0 \leq i \leq n-1$.

The non-degenerate elements of a CW-basis of F are called *generators* of F. For a given generator $x \in \mathfrak{B}_n$, the element $d_n(x) \in \mathfrak{B}_{n-1}$ is called the *attaching element* of x.

It is shown in [Kan58b] that *any free simplicial group has a CW-basis.* It follows directly from the definitions that a free simplicial group is determined

by the generators of its CW-basis together with their attaching elements. Similar is the case for CW-complexes: every CW-complex is determined by its cells and attaching maps.

Let X be a reduced simplicial set, Γ a simplicial group. Then a *twisting function* $\tau : X \to \Gamma$ is a function which lowers dimension by one and is such that, for every $n > 0$ and $\sigma \in X_n$, the following conditions are satisfied:

$$d_i(\tau(\sigma)) = \tau(d_i(\sigma)),\ 0 \le i < n-1,$$

$$d_{n-1}(\tau(\sigma)) = \tau(d_{n-1}(\sigma))\tau(d_n(\sigma))^{-1},$$

$$s_i(\tau(\sigma)) = \tau(s_i(\sigma)),\ 0 \le i < n,$$

$$\tau(s_n(\sigma)) = 1.$$

The twisting functions have a strong relation with the functor G (A.20) discussed above [Kan59]. Namely, for any reduced simplicial set X and a twisting function $\tau : X \to \Gamma$, there exists a simplicial homomorphism $g : GX \to \Gamma$, such that $g(\bar{\sigma}) = \tau(\sigma),\ \sigma \in X$, where $\bar{\sigma}$ is the image of σ in GX. Therefore, there is a one to one correspondence between twisting functions $X \to \Gamma$ and simplicial homomorphisms $GX \to \Gamma$.

Let K be a CW-complex with a single 0-cell. The total singular complex $S(K)$ contains *the first Eilenberg sub-complex* $S^1(X)$, which consists of n-simplices $\sigma : \Delta_n \to K$ of $S(K)$, such that $\sigma(A_i)$ is the 0-cell in K for $0 \le i \le n$, where $A_i,\ 0 \le i \le n$ are vertices of Δ_n. The Kan's construction is a simplicial map

$$\tau : S^1(K) \to B_K,$$

where B_K is a free simplicial group, τ a twisting function, which satisfies the following conditions:

(i) the elements $\tau(\sigma_c)$ are distinct and form the generators of a CW-basis of B_K, where c runs through the cells of K of dimension at least one;

(ii) for every sub-complex $L \subset K$, $\tau(S(L)) \subset B(L)$, where $B(L) \subset B_K$ denotes the simplicial subgroup of B_K, generated by elements $\tau(\sigma_c),\ \sigma_c \in L$.

A twisting functor satisfying the above conditions defines a combinatorial homotopical model for based loops ΩK over the complex K. Namely, it defines a weak homotopical equivalence

$$\Omega K \simeq |B_K|.$$

In the reverse direction, for any free simplicial group B, there exists a reduced CW-complex K_B, together with a twisting function $\tau : S(K_B) \to B$, which satisfies the above conditions (i)-(ii) and is such that there are weak homotopy equivalences $K_{B_K} \simeq K$ and $B_{K_B} \simeq B$.

Example A.10

Let F be a free group with generators $\{x_i\}_{i\in I}$ and R a normal subgroup of F generated, as a normal subgroup, by the set $\{r_j\}_{j\in J}$. Form the standard two-dimensional complex K corresponding to the group presentation

$$\langle x_i,\ i\in I \mid r_j,\ j\in J\rangle.$$

The complex K is reduced, it has a single 0-cell, 1-cells are in one-to-one correspondence with the basis elements $\{x_i\}_{i\in I}$, 2-cells are in one-to-one correspondence with the elements $\{r_j\}_{j\in J}$. Then Kan's construction is the 1-skeleton of the truncation, described above:

$$B_K = \mathrm{sk}^1(S(X,\mathcal{R})).$$

Hence, the second homotopy module of K can be described as

$$\pi_2(K) \simeq \frac{\langle y_j,\ j\in J\rangle^{F_1} \cap \langle y_j r_j^{-1},\ j\in J\rangle^{F_1}}{[\langle y_j,\ j\in J\rangle^{F_1}, \langle y_j r_j^{-1},\ j\in J\rangle^{F_1}]},$$

with the action of $\pi_1(K) = \pi_0(\mathrm{sk}^1\, S(X,\mathcal{R}))$, defined via the degeneracy map s_0.

Milnor's F[K]-construction

For a given pointed simplicial set K, the $F[K]$-construction [Mil56] is the simplicial group with $F[K]_n = F(K_n \setminus *)$, where $F(-)$ is the free group functor. There is the following weak homotopy equivalence:

$$|F[K]| \simeq \Omega\Sigma|K|.$$

Consider the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$:

$$S^1_0 = \{*\},\ S^1_1 = \{*,\sigma\},\ S^1_2 = \{*, s_0\sigma, s_1\sigma\},\ \dots, S^1_n = \{*, x_0, \dots, x_n\},$$

where $x_i = s_n \dots \hat{s}_i \dots s_0\sigma$. The $F[S^1]$-construction then clearly has the following terms:

$$\begin{aligned}
&F[S^1]_0 = 0,\\
&F[S^1]_1 = F(\sigma),\ \text{free abelian group generated by } \sigma,\\
&F[S^1]_2 = F(s_0\sigma, s_1\sigma),\\
&F[S^1]_3 = F(s_i s_j \sigma \mid 0 \le j \le i \le 2),\\
&\dots
\end{aligned}$$

The face and degeneracy maps are determined naturally (with respect to the standard simplicial identities) for these simplicial groups. For example, the first nontrivial maps are defined as follows:

$$\begin{aligned}
&\partial_i : F[S^1]_2 \to F[S^1]_1,\ i = 0, 1, 2,\\
&\partial_0 : s_0\sigma \mapsto \sigma,\ s_1\sigma \mapsto 1,\\
&\partial_1 : s_0\sigma \mapsto \sigma,\ s_1\sigma \mapsto \sigma,\\
&\partial_2 : s_0\sigma \mapsto 1,\ s_1\sigma \mapsto \sigma.
\end{aligned}$$

The above construction gives a possibility to define the homotopy groups $\pi_n(S^2)$ combinatorially, in terms of free groups. Since the geometrical realization of $F[S^1]$ is weakly homotopically equivalent to the loop space ΩS^2, the homotopy groups $\pi_n(S^2)$ are naturally isomorphic to the homotopy groups of the Moore complex of $F[S^1]$: $\pi_{n+1}(S^2) \simeq Z_n(F[S^1])/B_n(F[S^1])$. Here Z_n and B_n denote the cycles and the boundaries of the Moore complex of the corresponding simplicial group.

The explicit structure of the cycles and boundaries for $F[S^1]$ can be given in terms of certain normal subgroups in $F[S^1]$. This was realized by Jie Wu [Wu,01]. In fact, we have

$$\pi_{n+1}(S^2) \cong \frac{\langle y_{-1}\rangle^F \cap \langle y_0\rangle^F \cap \ldots \cap \langle y_{n-1}\rangle^F}{[[y_{-1}, y_0, \ldots, y_{n-1}]]}, \tag{A.21}$$

where F is a free group with generators $y_0, \ldots, y_{n-1}$, $y_{-1} = (y_0 \ldots y_{n-1})^{-1}$, the group $[[y_{-1}, y_0, \ldots, y_{n-1}]]$ is the normal closure in F of the set of left-ordered commutators

$$[z_1^{\varepsilon_1}, \ldots, z_t^{\varepsilon_t}] \tag{A.22}$$

with the properties that $\varepsilon_i = \pm 1$, $z_i \in \{y_{-1}, \ldots, y_{n-1}\}$ and all elements in $\{y_{-1}, \ldots, y_{n-1}\}$ appear at least once in the sequence of elements z_i in (A.22).

Hurewicz Theorem

The following simplicial analog of Hurewicz Theorem was obtained by Kan (see [Kan58a, Theorems 17.5, 17.6]):

> Let F be a simplicial free group with $\pi_i(F) = 0,\ i \le n\ (n \ge 0)$, then $\pi_i(F/[F, F]) = 0,\ i \le n$, and the natural homomorphism $\pi_{n+1}(F) \to \pi_{n+1}(F/[F, F])$ is an isomorphism.

A.11 Free Simplicial Resolutions

For a given category $\mathcal{C}$, an *augmented simplicial object* in $\mathcal{C}$ is a pair

$$(X, X_{-1}),\ X \in \mathcal{SC},\ X_{-1} \in \mathcal{C}$$

together with a morphism $d_0 \in \mathrm{Hom}_{\mathcal{C}}(X_0, X_{-1})$, such that

$$d_0 d_0 = d_0 d_1 : X_1 \to X_{-1}.$$

The augmented simplicial objects in $\mathcal{C}$ naturally form a category, which we denote by $\mathsf{a}\mathcal{SC}$; the m-truncated augmented simplicial category can also be defined in the obvious way; we denote it by $\mathsf{a}\mathcal{SC}_m$.

An augmented simplicial group (X, X_{-1}) is called a *resolution* of X_{-1} if $\pi_n(X) = 0,\ n > 0$ and $\pi_0(X) = X_{-1}$. The resolution (X, X_{-1}) of a group X_{-1} is called *free simplicial resolution* of X_{-1} if X is free.

A method of construction of free simplicial resolutions in the category of commutative algebras was described first by Andre [And70]. We start with the description of general construction of free simplicial resolutions from [Keu].

For a given $(X, X_{-1}) \in \mathsf{a}\mathcal{SC}$, define the nth *simplicial kernel* to be

$$Z_n(X, X_{-1}) = \{(x_0, \dots , x_{n+1}) \in X_n^{n+2} \mid d_i x_j = d_{j-1} x_i,\ i < j\}.$$

Analogically the group $Z_n(X, X_{-1})$ can be defined for any $(X, X_{-1}) \in \mathsf{a}\mathcal{SC}_m$ for $m \geq n$. For such an object, there are $n + 2$ natural morphisms

$$\begin{aligned} &p_i : Z_n(X, X_{-1}) \to X_n, \\ &p_i : (x_1, \dots , x_{n+1}) \mapsto x_i,\ i = 0, \dots , n + 1 \end{aligned}$$

and $n + 1$ morphisms

$$\begin{aligned} &q_i : X_n \to Z_n(X, X_{-1}) \\ &q_i : x \to (s_{i-1} d_0 x, \dots , s_{i-1} d_{i-1} x, x, x, s_i d_{i+1} x, \dots , s_i d_n x),\ x \in X_n. \end{aligned}$$

The free simplicial resolution of a given object $X_{-1} \in \mathcal{C}$ can be constructed inductively. Firstly we choose a set E of generators of X_{-1}. Define X_0 to be a free object in $\mathcal{C}$ with basis E, which we denote by $F(E_0)$, and define $d_0 : F(E_0) \to X_{-1}$ to be the natural surjection. Now suppose we have defined an object $(X, X_{-1}) \in \mathsf{a}\mathcal{SC}_n$, where X_n is a free object in $\mathcal{C}$ over some set E_n. Complete the set $\bigcup_{i=0}^n q_i(E_n)$ to the set Y_n of generators of $Z_n(X, X_{-1})$. Then put $E_{n+1} = \{e_n \mid e_n \in Y_n\}$, and define X_{n+1} to be a free object $F(E_{n+1})$ in $\mathcal{C}$ on the generating set E_{n+1}. The maps d_i, s_i are defined by setting

$$d_i = p_i,\ s_i = q_i, \quad i = 0, \dots , n + 1,$$

where we use the identification of generators of $Z_n(X, X_{-1})$ with elements in X_{n+1}.

Similarly one can construct a free simplicial resolution starting with a free simplicial group X which is aspherical up to a fixeddimension, say n.

The above method gives an algorithm for converting X into a free simplicial resolution without changing $X_i,\ i \leq n$.

The free simplicial resolution of a given group can be constructed as an inductive limit of skeleton filtration, also constructed by step-by-step procedure [Mut99]. Let G be a group, given as a quotient $G = F/R$, where F is a free group with basis $\{x_i\}_{i\in I}$ and R its normal subgroup generated, as a normal subgroup, by the set $\{r_j\}_{j\in J}$. We use the notation from Example A.4. The first step in the construction of a free simplicial resolution of G is the 1-skeleton $F^1 = \mathrm{sk}^1\, S(X, \mathcal{R})$. Clearly,

$$\pi_0(F^1) = G.$$

Now, in general, $\pi_1(F^1)$, which is generated by cosets

$$a_i[\ker(d_0),\, \ker(d_1)],\ \ i \in T,$$

is nontrivial. We can then define a new simplicial group F^2:

$$F_0 = F_0,\ F_1 = F_1,\ F_2 = \mathrm{sk}^1(F^1) * F(a_i \mid i \in T),\ \ldots$$

with obvious face and degeneracy maps, where all terms in dimensions ≥ 3 are free groups generated by degeneracy elements. Clearly,

$$\pi_0(F^2) = G,\ \ \pi_1(F^2) = 0.$$

Continuing this process by induction, "killing" homotopy groups at each dimension, we get the required free simplicial resolution of a given group.

A functorial construction for a free simplicial resolution of a given group can be given by the composition of the classifying space functor and Kan's loop group construction. Every group Γ can be viewed as a simplicial group X with $X_n = \Gamma$ and all face and degeneracy maps an identity. The composition of the functors G and $\overline{W}$ defines the free loop construction over a group Γ:

$$G\overline{W} : \mathsf{Gr} \to \mathcal{S}\mathsf{Gr},$$

which has the property:

$$\pi_0(G\overline{W}\Gamma) = \Gamma,\ \ \pi_i(G\overline{W}\Gamma) = 0,\ \ i > 0.$$

Clearly, $G\overline{W}\Gamma$ is a free simplicial group; thus it can be viewed as a free simplicial resolution of Γ.

Theorem A.11 (Comparison theorem [Keu]). *Let (X, X_{-1}) be a free simplicial resolution of X_{-1}, (Y, Y_{-1}) a resolution of Y_{-1} and $\alpha : X_{-1} \to Y_{-1}$ a group homomorphism. Then*

(i) there exists a simplicial map $\gamma : X \to Y$, such that $\pi_0(\gamma) = \alpha$;

(ii) for any simplicial map $\gamma' : X \to Y$ with $\pi_0(\gamma') = \alpha$ there exists a simplicial homotopy $h : \gamma \to \gamma'$.

Proof. (i) Let $\{B_n \subset X_n\}_{n\geq 0}$ be the sets of free generators with $s_i(B_n) \subset B_{n+1},\ n \geq 0,\ 0 \leq i \leq n$. We construct γ by induction. Let

$$\gamma_{-1} : X_{-1} \to Y_{-1}$$

be the map α. Now, since $\pi_0(Y) = Y_{-1}$, the map $d_0 : Y_0 \to Y_1$ is an epimorphism. Hence, for any $x_0 \in B_0$, there exists an element $y \in Y_0$, such that

$$d_0 y = \gamma_{-1} d_0 x. \tag{A.23}$$

Define the map

$$\gamma_0 : X_0 \to Y_0$$

by setting $\gamma_0 : x \mapsto y,\ x \in B_0$ where y is defined by (A.23). Now suppose we have defined maps $\gamma_i : X_i \to Y_i,\ -1 \leq i \leq n-1,\ n \geq 1$, which satisfy the standard simplicial identities. We proceed to define $\gamma_n : X_n \to Y_n$. Clearly, for $x = s_i(x_0) \in B_n,\ x_0 \in B_{n-1},\ 0 \leq i \leq n-1$, we can define $\gamma_n(x) = s_i\gamma_{n-1}(x_0)$. Now let $x \in B_n \setminus \bigcup_{i=0}^{n-1} s_i(B_{n-1})$. Then

$$(\gamma_{n-1} d_0 x, \ldots, \gamma_{n-1}\gamma_n x) \in Z_{n-1}X.$$

The asphericity of X implies that there exists an element $y \in Y_n$, such that

$$d_i y = \gamma_{n-1} d_i y,\ 0 \leq i \leq n;$$

define $\gamma_n(x) = y$. This completes the construction of γ_n.

(ii) The inductive construction of the simplicial homotopy $h : \gamma \to \gamma'$ is standard; one can find all details in [Keu]. □

A.12 Functorial Properties

Let R be a ring and n a non-negative integer. Define the *strong truncation functor*

$$\mathrm{str}_n : \mathcal{C}h(R) \to \mathcal{C}h(R)$$

in the category of chain complexes over the ring R by setting

$$\begin{aligned}
\mathrm{str}_n(K)_i &= K_i,\ i < n,\\
\mathrm{str}_n(K)_n &= K_n/\ker(d_n),\\
\mathrm{str}_n(C)_i &= 0,\ i \geq n+1,
\end{aligned}$$

for $K \in \mathcal{C}h(R)$ with obvious boundary maps.

The following results, which are due to Gruenenfelder, generalize the results from [Dol58a]. We follow the treatment in [Gru80].

Theorem A.12 *Let R be a principal ideal domain and X a projective simplicial R-module. Let $X' \in \mathcal{S}_R\mathsf{Mod}$ and $f_i : \pi_i(X) \to \pi_i(X'),\ i \geq 0$. Then there exists a map $f : X \to X'$, which induces the maps f_i.*

Proof. We construct the required map as $N^{-1}g : X \to X'$, where $g : N(X) \to N(X')$ is the map between Moore complexes which induces the maps $f_i : H_i(N(X)) \to H_i(N(X'))$.

Since X is a projective simplicial R-module, $N(X)_i$ are projective R-modules for $i \geq 0$ by Proposition A.7. The component $N(X)_i$ can be presented as $N(X)_i = \ker(\bar{d}_i) \oplus \operatorname{im}(\bar{d}_i)$. Therefore, the R-modules $\ker(\bar{d}_i)$ and $\operatorname{im}(\bar{d}_i)$, $i \geq 0$, are projective. Thus, there exist maps g'_i, g''_i such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(\bar{d}_{i+1}) \arrow[r] \arrow[d, "g''_i"'] & \ker(\bar{d}_i) \arrow[r] \arrow[d, "g'_i"'] & H_i(N(X)) \arrow[r] \arrow[d, "f_i"'] & 0 \\
0 \arrow[r] & \operatorname{im}(\bar{d}'_{i+1}) \arrow[r] & \ker(\bar{d}'_i) \arrow[r] & H_i(N(X')) \arrow[r] & 0,
\end{tikzcd}$$

and we have the required map

$$g''_i \oplus g'_{i+1} : \operatorname{im}(\bar{d}_{i+1}) \oplus \ker(\bar{d}_{i+1}) \to \operatorname{im}(\bar{d}'_{i+1}) \oplus \ker(\bar{d}'_{i+1}). \ \square$$

Theorem A.13 *Let S be a ring, R a principal ideal domain and $X \in \mathcal{S}_R\mathsf{Mod}$, $X' \in \mathcal{S}_S\mathsf{Mod}$ projective simplicial modules. Let $F :\ {}_R\mathsf{Mod} \to\ {}_S\mathsf{Mod}$ be a covariant functor. Then*

(i) Any sequence of homomorphisms $\{f_i : \pi_i(X) \to \pi_i(X')\}$, $i \leq n$ induces homomorphisms $\{f_{F,\,i} : \pi_i(F(X)) \to \pi_i(F(X'))\}$, $i \leq n$.

(ii) If f_i is bijective for $i < n$ and surjective for $i = n$, then the same is true for $f_{F,\,i}$, $i \leq n$.

Proof. The homomorphisms f_i can be extended to homomorphisms between homotopy groups of strong truncations of the given simplicial modules:

$$f_i : \pi_i(\mathrm{str}_{n+1}(X)) \to \pi_i(\mathrm{str}_{n+1}(X')),\ i \geq 0.$$

By Theorem A.12, we can assume that the sequence $f_i,\ i \geq 0$, is induced by a map

$$f : \mathrm{str}_n(X) \to \mathrm{str}_n(X').$$

Since $\mathrm{str}_{n+1}(X)$ is a direct summand of X, such that $\mathrm{str}_{n+1}(X)_i = X_i,\ i \leq n$, it follows that $F(\mathrm{sk}_{n+1}(X))$ is a direct summand of $F(X)$ with

$$F(\mathrm{str}_{n+1}(X))_i = F(X)_i,\ i \geq n.$$

Thus

$$\pi_i(F(X)) = \pi_i(F(\mathrm{str}_{n+1}(X))),\ i \leq n.$$

The sequence of maps $\{f_{F,i}\},\ i \leq n$, can be defined as

$$f_{F,i} : \pi_i(F(X)) = \pi_i(F(\mathrm{str}_{n+1}(X))) \to \pi_i(F(\mathrm{str}_{n+1}(X'))) = \pi_i(F(X')).$$

(ii) Consider the following decomposition of the functor $N \circ f : N(\mathrm{str}_n(X)) \to N(\mathrm{str}_n(X')) :$

$$\begin{array}{ccccccccc}
N(\mathrm{str}_n(X)): & \mathrm{im}(d_{n+1}) & \longrightarrow & N(X)_n & \longrightarrow & N(X)_{n-1} & \longrightarrow & \dots \\
\big\downarrow & \big\downarrow & & \big\| & & \big\| & & \\
C: & P & \longrightarrow & N(X)_n & \longrightarrow & N(X)_{n-1} & \longrightarrow & \dots \\
\beta \big\downarrow & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
N(\mathrm{str}_n(X')): & \mathrm{im}(d'_{n+1}) & \longrightarrow & N(X')_n & \longrightarrow & N(X')_{n-1} & \longrightarrow & \dots,
\end{array}$$

where the left hand square is a pullback. First note, that the natural pullback properties imply that the map β induces isomorphisms

$$H_i(\beta) : H_i(C) \to H_i(N(\mathrm{str}_n(X')))$$

for all $i \geq 0$. Since P is projective R-module, the map

$$N^{-1} \circ \beta \circ N : N^{-1}(C) \to \mathrm{str}_n(X')$$

is a homotopy equivalence of *strongly* truncated simplicial groups. Obviously, the functor F preserves homotopy; hence

$$F(N^{-1} \circ \beta \circ N) : F(N^{-1}(C)) \to F(\mathrm{str}_n(X'))$$

is a homotopy equivalence. We have $C_i = N(\mathrm{str}_n(X))_i,\ i \leq n$, and a monomorphism $C_{n+1} \to N(\mathrm{str}_n(X))_{n+1}$. Hence $F(C)_i = F(N(\mathrm{str}_n(X'))_i$, $i \leq n$, therefore

$$\pi_i(F(N^{-1} \circ \beta \circ N)) : \pi_i(F(N^{-1}(C))) \to \pi_i(F(\mathrm{str}_n(X')))$$

is an isomorphism for $i < n$. The fact that the simplicial R-modules under consideration are projective implies that $\pi_n(F(N^{-1} \circ \beta \circ N))$ is an epimorphism. Therefore the map

$$\pi_i(F(X)) \simeq \pi_i(F(\mathrm{str}_n(X)) \to \pi_i(F(N^{-1}(C)) \to \pi_i(F(\mathrm{str}_n(X')) \simeq \pi_i(X')$$

is an isomorphism for $i < n$, and an epimorphism for $i = n$. □

A.13 Derived Functors

Derived functors play a fundamental role in different areas of mathematics. The reader can find in [Ina98] an exposition of the general technique of derived functors. Here we mention only some examples of derived functors on the category of groups. However, it may be noted that the same construction is possible for other *algebraic* categories, like Lie algebras, crossed modules, cat^1-groups etc.

Let $T : \mathsf{Gr} \to \mathsf{Gr}$ be a functor on the category Gr of groups, $T(1) = 1$. Theorem A.11 clearly implies the following

Proposition A.14 *Let (X, X_{-1}) and (Y, Y_{-1}) be two free simplicial resolutions of $X_{-1} = Y_{-1}$. Then, for all $n \geq 0$,*

$$\pi_n(T(X)) = \pi_n(T(Y)).$$

From the above Proposition the definition of *left derived functors* of the functor T, denoted $\mathcal{L}_i T,\ i \geq 0$, follows naturally:

$$\mathcal{L}_0 T : \mathsf{Gr} \to \mathsf{Gr},\ \mathcal{L}_i T : \mathsf{Gr} \to \mathsf{Ab},\ i \geq 1,$$

are defined as

$$\mathcal{L}_i T : \varGamma \mapsto \pi_i(T(F_*)),\ i \geq 0,$$

where $F_* \in \mathcal{S}\mathsf{Gr}$ is a free simplicial resolution of $\varGamma \in \mathsf{Gr}$, and Ab is the category of abelian groups. Proposition A.14 implies that the resulting groups are independent of the choice of the free simplicial resolution F_*.

Example A.15

Let

$$Z_2 : \mathsf{Gr} \to \mathsf{Ab}, \quad \varGamma \mapsto \varGamma/\gamma_2(\varGamma),\ \varGamma \in \mathsf{Gr},$$

be the abelianization functor. Let X be a free simplicial resolution of $\varGamma$. By definition, X_i is free for all $i \geq 0$,

$$\pi_0(X) = \varGamma,\ \pi_i(X) = 0,\ i \geq 1.$$

Form the following bisimplicial group:

$$Y_{i,j} = \mathbb{Z}[(\overline{W} X_i)_j] = \mathbb{Z}[X_i^{\times(j-1)}].$$

Computing first homotopy groups of $Y_{i,j}$ for a fixed j, we get

$$\pi_0(\mathbb{Z}[(\overline{W} X_i)_j]) = \mathbb{Z}[\overline{W}_j(\varGamma)],$$
$$\pi_n(\mathbb{Z}[(\overline{W} X_i)_j]) = 0,\ n \geq 1.$$

On the other hand, computing first homotopy groups with fixed indices i, we get

$$\pi_n(\mathbb{Z}[(\overline{W}X_i)_j]) = H_n(X_i),\ n \geq 0.$$

Since X_i are free, we conclude that the last term is $\mathbb{Z}$ in dimension zero, abelianization $X_i/\gamma_2(X_i)$ in dimension one, and trivial in all other dimensions. Thus, by Theorem A.9,

$$\mathcal{L}_n Z_2(\Gamma) := \pi_n(X/\gamma_2(X)) = \pi_{n+1}(\mathbb{Z}[\overline{W}(\Gamma)]) = H_{n+1}(\Gamma),\ n \geq 0.$$

Example A.16

Let $Z_n : \mathsf{Gr} \to \mathsf{Gr},\ n \geq 2$, be the functor which maps every group Γ to its quotient by the nth term of its lower central series:

$$Z_n : \Gamma \mapsto \Gamma/\gamma_n(\Gamma).$$

Let $\Gamma = F/R$ and X a free simplicial resolution of Γ with first two terms F_0, F_1 as in Example A.4, i.e.,

$$F_0 = F, \quad F_1 = F(y_j \mid j \in J) * F,$$

where y_j's are in one to one correspondence with a normal basis of R. Then, clearly,

$$\mathcal{L}_0 Z_n(\Gamma) = \Gamma/\gamma_n(\Gamma),\ n \geq 2.$$

The short exact sequence of simplicial groups

$$1 \to \gamma_n(X) \to X \to X/\gamma_n(X) \to 1,$$

gives the following long exact sequence:

$$\ldots \to \pi_1(X) \to \pi_1(X/\gamma_n(X)) \to \pi_0(\gamma_n(X)) \to \pi_0(X) \to \pi_0(X/\gamma_n(X)) \to 1.$$

Since X is aspherical and $\mathcal{L}_0 Z_n = Z_n$, we have

$$\mathcal{L}_1 Z_n(\Gamma) = \pi_1(X/\gamma_n(X)) = \ker\{\pi_0(\gamma_n(X)) \to \gamma_n(\Gamma)\}.$$

The group $\pi_0(\gamma_n(X))$ is a coequalizer of the diagram

$$\gamma_n(F_1) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \gamma_n(F),$$

where the maps $d_0,\ d_1$ are restrictions of degeneracy maps in X. Clearly,

$$\ker(d_0) = [\langle F(y_j \mid j \in J)\rangle^{F_1},\ {}_{n-1}F_1]$$

and

$$\mathrm{im}(d_1|_{\ker(d_0)}) = [R,\ {}_{n-1}F].$$

Hence,

$$\pi_0(\gamma_n(X)) = \frac{\gamma_n(F)}{[R,\ {}_{n-1}F]}$$

and therefore,

$$\mathcal{L}_1 Z_n(\varGamma) = \ker\{\frac{\gamma_n(F)}{[R,\ {}_{n-1}F]} \to \frac{\gamma_n(F)}{R \cap \gamma_n(F)}\} = \frac{R \cap \gamma_n(F)}{[R,\ {}_{n-1}F]}.$$

This is the same as the nth Baer invariant of $\varGamma$, which we denote by $M^{(n)}(\varGamma)$ (see Section 1.4); that is, $\mathcal{L}_1 Z_n(\varGamma) = M^{(n)}(\varGamma),\ n \geq 2$.

Example A.17

For a given group F and its normal subgroup R, define the series $\{\delta_n(R,\ F)\}_{n\geq 1}$ by setting:

$$\delta_1(R,\ F) = [R,\ F],\ \delta_{n+1}(R,\ F) = [\delta_n(R,\ F),\ \delta_n(F)].$$

Let $\varGamma$ be a group given by a free presentation $\varGamma = F/R$. Consider the functor

$$\mathfrak{D}_n : \mathsf{Gr} \to \mathsf{Gr}, \quad \varGamma \mapsto \varGamma/\delta_n(\varGamma),\ n \geq 1.$$

Then the same method as in the previous example gives the following:

$$\mathcal{L}_0\mathfrak{D}_n = \mathfrak{D}_n,\ \mathcal{L}_1\mathfrak{D}_n = \frac{R \cap \delta_n(F)}{\delta_n(R,\ F)}.$$

For example, the first derived functor of the metabelianization functor is

$$\mathcal{L}_1\mathfrak{D}_2 = \frac{R \cap [[F,\ F],\ [F,\ F]]}{[[R,\ F],\ [F,\ F]]}.$$

Example A.18

Consider the commutator subgroup functor $T : G \mapsto [G,\ G]$. Then the foregoing arguments easily imply that for a given group $G = F/R$, one has the following:

$$\mathcal{L}_i T(G) = \begin{cases} \frac{[F,\,F]}{[R,\,F]},\ i = 0, \\ H_{i+2}(G),\ i > 0. \end{cases} \tag{A.24}$$

A.14 Quadratic Functors

A functor $F : \mathsf{Ab} \to \mathsf{Ab}$ is quadratic, i.e., has degree ≤ 2, if $F(0) = 0$ and if the cross effect

$$F(A|B) = \ker(F(A \oplus B) \to F(A) \oplus F(B)), \quad A,\ B \in \mathsf{Ab},$$

is biadditive. We then have a binatural isomorphism

$$F(A \oplus B) = F(A) \oplus F(B) \oplus F(A|B)$$

given by $(F_{i_1}; F_{i_2}; i_{12})$, where $i_1 : A \hookrightarrow A \oplus B;\ i_2 : A \hookrightarrow A \oplus B$ and $i_{12} : F(A|B) \hookrightarrow F(A \oplus B)$ are the inclusions. Moreover, for any $A \in \mathsf{Ab}$ one gets the diagram [Bau00]

$$F\{A\} := (F(A) \xrightarrow{H} F(A|A) \xrightarrow{P} F(A)). \tag{A.25}$$

Here $P = F(p_1 + p_2)i_{12} : F(A|A) \hookrightarrow F(A \oplus A) \to F(A)$ is given by the codiagonal $p_1+p_2 : A \oplus A \to A$, where p_1 and p_2 are the projections. Moreover, H is determined by the equation $i_{12}H = F(i_1 + i_2) - F(i_1) - F(i_2)$ where $i_1 + i_2 : A \to A \oplus A$ is the diagonal map.

We recall from [Eil54] the definitions of certain quadratic functors.

Tor (A, C):

For abelian groups A and C, the abelian group $\mathrm{Tor}(A, C)$ ([Eil54], §11, p. 85) has generators

$$(a, m, c),\ a \in A,\ c \in C,\ 0 < m \in \mathbb{Z},\ ma = mc = 0$$

and relations

$$\begin{aligned}
(a_1 + a_2, m, c) &= (a_1, m, c) + (a_2, m, c),\ \text{if } ma_1 = ma_2 = mc = 0\\
(a, m, c_1 + c_2) &= (a, m, c_1) + (a, m, c_2),\ \text{if } ma = mc_1 = mc_2 = 0\\
(a, mn, c) &= (na, m, c),\ \text{if } mna = mc = 0\\
(a, mn, c) &= (a, m, nc),\ \text{if } ma = mnc = 0.
\end{aligned}$$

In particular, we have the functor $A \mapsto \mathrm{Tor}(A, A))$ on the category Ab. We denote the class of the triple (a, m, c) by $\tau_m(a, c)$.

$\mathbf{\Omega}$(A): Let A be an Abelian group. Then the group $\Omega(A)$, defined by Eilenberg-MacLane ([Eil54], p. 93), is the Abelian group generated by symbols $w_n(x),\ 0 < n \in \mathbb{Z}, x \in A,\ nx = 0$ with defining relations

$$\begin{aligned}
&w_{nk}(x) = kw_n(x),\ nx = 0,\\
&kw_{nk}(x) = w_n(kx),\ nkx = 0,\\
&w_n(kx + y) - w_n(kx) - w_n(y) = w_{nk}(x + y) - w_{nk}(x) - w_{nk}(y),\ nkx = ny = 0,\\
&w_n(x+y+z) - w_n(x + y) - w_n(x + z) - w_n(y + z) + w_n(x) + w_n(y) + w_n(z) = 0,\\
&nx = ny = nz = 0.
\end{aligned}$$

We continue to denote the class of the element $w_n(x)$ in $\Omega(A)$ by $w_n(x)$ itself.
Eilenberg-MacLane ([Eil54], p. 94) constructed a map

$$E : \mathrm{Tor}(A,\, A) \to \Omega(A)$$

by setting

$$\tau_n(a,\, c) \mapsto w_n(a+c) - w_n(a) - w_n(c).$$

A natural map

$$T : \Omega(A) \to \mathrm{Tor}(A,\, A) \tag{A.26}$$

can be defined by setting

$$w_n(x) \mapsto \tau_n(x,\, x),\ x \in A,\ nx = 0.$$

Clearly the composite map

$$E \circ T : \Omega(A) \to \Omega(A)$$

is multiplication by 2; for, as a consequence of the defining relations the elements $w_n(x)$ satisfy

$$w_n(mx) = m^2 w_n(x)$$

for all $m \in \mathbb{Z},\ 0 < n \in \mathbb{Z},\ x \in A$.
Whitehead functor $\mathbf{\Gamma_2(A)}$: The homogeneous component $\mathbf{\Gamma}_2(A)$ of the graded functor $\Gamma(A)$ (see (5.36)-(5.39)) of degree 2 can be identified with the Whitehead functor ([Whi50], [Eil54], pp. 92 & 110) $\Gamma_2 : \mathsf{Ab} \to \mathsf{Ab},\ A \mapsto \Gamma_2(A)$, where the group $\Gamma_2(A)$ is defined for $A \in \mathsf{Ab}$ to be the group given by generators $\gamma(a)$, one for each $x \in A$, subject to the defining relations

$$\gamma(-x) = \gamma(x), \tag{A.27}$$

$$\gamma(x+y+z) - \gamma(x+y) - \gamma(x+z) - \gamma(y+z) + \gamma(x) + \gamma(y) + \gamma(z) = 0 \tag{A.28}$$

for all $x,\, y,\, z \in A$.
$\mathbf{R(A)}$: For $A \in \mathsf{Ab}$, let ${}_2A$ denote the subgroup consisting of elements x satisfying $2x = 0$. Define $R(A)$ ([Eil54]. p. 120) to be the quotient group of $\mathrm{Tor}(A,\, A) \oplus \Gamma_2(A)$ by the relations

$$\tau_m(x,\, x) = 0, \quad mx = 0, \tag{A.29}$$

$$\gamma_2(s+t) - \gamma_2(s) - \gamma_2(t) = \tau_2(s,\, t), \quad s,\, t \in_2 A. \tag{A.30}$$

The functors $A \mapsto \Gamma_2(A),\ \mathrm{Tor}(A,\, A),\ \Omega(A),\ R(A)$ are all quadratic functors on the category of Abelian groups, i.e., all these functors have degree ≤ 2. Furthermore, $R(A) = H_5K(A; 2);\ \Omega(A) = H_7K(A; 3)/(\mathbb{Z}/3\mathbb{Z} \otimes A)$, $R(A|B) = \Omega(A|B) = \mathrm{Tor}(A,\, B)$ and $R(\mathbb{Z}) = \Omega(\mathbb{Z}) = 0$ and $R(\mathbb{Z}/n) = \mathbb{Z}/(2,\, n);\ \Omega(\mathbb{Z}/n) = \mathbb{Z}/n$.

The square functor. Let Ab_g denote the category of graded Abelian groups. For $A,\ B \in \mathsf{Ab}$, let $A \otimes B$ and $A * B := \mathrm{Tor}(A,\ B)$ be respectively the tensor product and the torsion product of $A,\ B$. The notion of tensor product of Abelian groups extends naturally to that of *tensor product* $A \otimes B$ *of graded Abelian groups* $A,\ B$ by setting

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j .$$

We also need the *ordered tensor product* $A \overset{>}{\otimes} B$ of graded Abelian groups, which is defined by setting

$$(A \overset{>}{\otimes} B)_n = \bigoplus_{i+j=n,\ i>j} A_i \otimes B_j$$

for $A,\ B \in \mathsf{Ab}_g$. In an analogous manner, we can define, for $A,\ B \in \mathsf{Ab}_g$, *torsion product* $A * B$ and *ordered torsion product* $A \overset{>}{*} B$ as

$$A * B = \bigoplus_{i+j=n} A_i * B_j, \quad (A \overset{>}{*} B)_n = \bigoplus_{i+j=n,\ i>j} A_i * B_j .$$

The tensor product, torsion product and the ordered tensor and torsion product are, in an obvious way, bifunctors on the category Ab_g.

Let $\wedge^2$ be the exterior square functor on the category Ab. The *weak square functor*

$$sq^{\otimes} : \mathsf{Ab}_g \to \mathsf{Ab}_g$$

is defined by

$$sq^{\otimes}(A)_n = \begin{cases} \Gamma_2(A_m),\ \text{if } n = 2m,\ m \text{ odd}, \\ \wedge^2(A_m),\ \text{if } n = 2m,\ m \text{ even}, \\ 0,\ \text{otherwise}. \end{cases}$$

Let $(\mathbb{Z}_2)_{odd}$ be the graded Abelian group which is $\mathbb{Z}_2$ in odd degree ≥ 1 and trivial otherwise; thus $(\mathbb{Z}_2)_{odd}$ is the reduced homology of the classifying space $\mathbb{R}P_\infty = K(\mathbb{Z}_2, 1)$. The *square functor* $Sq^{\otimes} : \mathsf{Ab}_g \to \mathsf{Ab}_g$ is defined as follows:

$$Sq^{\otimes}(A) = A \overset{>}{\otimes} (A \oplus (\mathbb{Z}_2)_{odd}) \oplus sq^{\otimes}(A).$$

Define next the *torsion square functor*

$$Sq^{\star}(A) : \mathsf{Ab}_g \to \mathsf{Ab}_g$$

by setting

$$Sq^{\star}(A) = (A \overset{>}{*} (A \oplus (\mathbb{Z}_2)_{odd})) \oplus sq^{\star}(A),$$

where

$$sq^{\star}(A)_n = \begin{cases} \Omega(A_m), & n = 2m, \ m \text{ even} \\ R(A_m), & n = 2m, \ m \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

Now we are ready to formulate so-called universal coefficient theorem for the functor $\wedge^2$ due to Baues and Pirashvili [Bau00]. Let X be a simplicial group which is free Abelian in each degree. Then there exists [[Bau00], (4.1)] a natural short exact sequence of graded Abelian groups

$$0 \to Sq^{\otimes}(\pi_*(X)) \to \pi_*(\wedge^2 X) \to Sq^{\star}(\pi_*(X))[-1] \to 0 \tag{A.31}$$

where $\pi_*(X)$ and $\pi_*(\wedge^2 X)$ are the graded homotopy groups of X and $\wedge^2 X$ respectively.

The exact sequence (A.31) leads to the description of the derived functors of the second lower central quotient functor. Since for a free group F, there is a natural isomorphism

$$\gamma_2(F)/\gamma_3(F) \simeq \wedge^2(F_{ab}),$$

for every free simplicial resolution $F_* \to G$, we obtain the following natural exact sequences:

$$0 \to H_2(G) \otimes H_1(G) \to \pi_1(\gamma_2(F_*)/\gamma_3(F_*)) \to \Omega(H_1(G)) \to 0$$
$$0 \to H_3(G) \otimes H_1(G) \oplus \mathit{\Gamma}_2(H_2(G)) \to \pi_2(\gamma_2(F_*)/\gamma_3(F_*)) \\ \to \mathrm{Tor}(H_2(G), H_1(G)) \to 0$$

Similar descriptions exist for other quadratic functors (see [Bau00]). Consider the functor

$$\bar{\Gamma} : \mathsf{Gr} \to \mathsf{Ab},$$

which is the composition of the abelianization and the functor Γ_2. For every group G there exists the following exact sequence of groups (see [Bau00]):

$$0 \to H_2(G) \otimes (H_1(G) \oplus \mathbb{Z}_2) \to \mathcal{L}_1\bar{\Gamma}(G) \to R(H_1(G)) \to 0.$$

A.15 Derived Functors in the Sense of Dold and Puppe

Let $T : \mathsf{Ab} \to \mathsf{Ab}$ be a functor with $T(0) = 0$ and A an abelian group. For $n \geq 0$, consider a free simplicial abelian group P_* with the following properties:

(i) $P_i = 0, \ i < n,$

(ii) $\pi_n(P_*) = A,$

(iii) $\pi_i(P_*) = 0, \ i \neq n.$

Define the ith derived functor of T (in the sense of Dold and Puppe [Dol61]) for the pair (A, n) as

$$\mathfrak{L}_i T(A, n) := \pi_i(T(P_*)).$$

Standard arguments, similar to ones given in Proposition A.14, show that this definition is independent of a choice of P_*. As an example of P_*, we can choose the free abelian simplicial group

$$P_* = N^{-1}((A_1 \hookrightarrow A_0)[n]),$$

where N^{-1} is the inverse map to the Dold-Kan map (see A.6), A_1 and A_0 are free abelian and the sequence

$$0 \to A_1 \to A_0 \to A \to 0$$

is exact. For $n = 0$, we will use the notation $\mathfrak{L}_i T(A) := \mathfrak{L}_i T(A, 0)$.

The derived functors in the sense of Dold and Puppe play a fundamental role in topology in view of the following fact. For every abelian group A and $n \geq 1$, there exists a natural spectral sequence

$$E^2_{p,q} = \mathfrak{L}_{p+q} \, \mathrm{SP}^q(A, n) \Rightarrow H_{p+q} K(A, n) \tag{A.32}$$

which converges to the homology of the Eilenberg-MacLane space $K(A, n)$. This sequence degenerates [Bre99] and, therefore, the derived functors $\mathfrak{L}_{p+q} \, \mathrm{SP}^q(A, n)$ define a canonical filtration of $H_{p+q} K(A, n)$.

Clearly, the sequence (A.31) provides a method of computing the derived functors of the exterior square. The universal coefficient theorem for quadratic functors SP^2 and Γ given in [Bau00] imply the following description of derived functors:

$$\mathfrak{L}_i \, \mathrm{SP}^2(A, n) = \begin{cases} \mathrm{SP}^2(A), \ i = 0, \ n = 0, \\ A \overset{\wedge}{*} A, \ i = 1, \ n = 0 \\ \Gamma_2(A), \ i = 2n, \ n \neq 0 \text{ even}, \\ \Lambda^2(A) \oplus \mathrm{Tor}(A, \mathbb{Z}_2), \ i = 2n, \ n \neq 1 \text{ odd}, \\ R(A), \ i = 2n+1, \ n \neq 0 \text{ even}, \\ \Omega(A), \ i = 2n+1, \ n \text{ odd}, \\ A \otimes \mathbb{Z}_2, \ i = n+2, n+4, \dots, n + 2[\frac{n-1}{2}], \\ \mathrm{Tor}(A, \mathbb{Z}_2), \ i = n+3, n+5, \dots, n+2[\frac{n-1}{2}]+1, \ i \neq 2n, \\ 0, \ \text{otherwise}; \end{cases}$$

$$\mathfrak{L}_i\Lambda^2(A,\,n) = \begin{cases} \Lambda^2(A),\; i=0,\; n=0,\\ \Gamma_2(A),\; i=2n,\; n \text{ odd},\\ \Lambda^2(A)\oplus \operatorname{Tor}(A,\,\mathbb{Z}_2),\; i=2n,\; n\neq 0 \text{ even},\\ R_2(A),\; i=2n+1,\; n \text{ odd},\\ \Omega_2(A),\; i=2n+1,\; n \text{ even},\\ A\otimes\mathbb{Z}_2,\; i=n+1,\,\dots,\,n+2[\frac{n-1}{2}]+1,\; i\neq 2n,\\ \operatorname{Tor}(A,\,\mathbb{Z}_2),\; i=n+2,\,n+4,\,\dots,\,n+2[\frac{n-1}{2}],\\ 0,\; \text{otherwise}; \end{cases}$$

$$\mathfrak{L}_i\Gamma_2(A,\,n) = \begin{cases} \Gamma_2(A),\; i=2n,\; n \text{ even},\\ \Lambda^2(A)\oplus \operatorname{Tor}(A,\,\mathbb{Z}_2),\; i=2n,\; n \text{ odd},\\ R_2(A),\; i=2n+1,\; n \text{ even},\\ \Omega_2(A),\; i=2n+1,\; n \text{ odd},\\ A\otimes\mathbb{Z}_2,\; i=n, n+2,\,\dots,\,n+2[\frac{n-1}{2}],\; n>0,\\ \operatorname{Tor}(A,\,\mathbb{Z}_2),\, i{=}n{+}1, n{+}3,\dots, n+2[\frac{n-1}{2}]+1,\, n>0,\; i{\neq}2n,\\ 0,\; \text{otherwise}. \end{cases}$$

For polynomial functors of higher degrees the functorial description of the derived functors is a deep problem. For example, consider the tensor cube. It follows from the work of MacLain [Mac60] that the derived functors can be described as follows.

$$\mathfrak{L}_i \otimes^3 (A) = \begin{cases} \otimes^3(A),\; i=0,\\ (\operatorname{Tor}(A,\,A)\otimes A)^{\oplus 3}/Jac_{\otimes},\; i=1,\\ \operatorname{Tor}(\operatorname{Tor}(A,\,A),\,A),\; i=2 \end{cases}$$

where $Jac_{\otimes}$ is the subgroup in $(\operatorname{Tor}(A,\,A)\otimes A)^{\oplus 3}$, generated by elements

$$(a,\,n,\,b)\otimes c + (c,\,n,\,a)\otimes b + (b,\,n,\,c)\otimes a,\;\; a,\,b,\,c\in A,\;\; na=nb=nc=0.$$

The derived functor $\mathfrak{L}_1 \otimes^3 (A)$ is a part of the following short exact sequence [Mac60]:

$$0 \to \operatorname{Tor}(A,\,A)\otimes A \to \mathfrak{L}_1 \otimes^3 (A) \to \operatorname{Tor}(A\otimes A,\,A) \to 0.$$

For an analogous description of the derived functors of the symmetric and the exterior powers see [Bre99] and [Jea02].

A.16 Derived Limits and Fibration Sequence

Let

$$\dots \xrightarrow{f_n} G_n \xrightarrow{f_{n-1}} G_{n-1} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_1} G_1 \xrightarrow{f_0} G_0 \tag{A.33}$$

be an inverse system of groups in the category Gr. Then the group $\Pi_{\mathsf{Gr}} = \prod_i G_i$, defined as the unrestricted product, acts on the set $\Pi_{\mathsf{Set}} = \prod_i G_i$ by setting

$$(g_0, \dots, g_n, \dots) \circ (x_0, \dots, x_n, \dots) = (g_0 x_0 f_0(g_1^{-1}), \dots, g_n x_n f_n(g_{n+1}^{-1}), \dots),$$

$x_i,\ g_i \in G_i$. Let $\mathbf{1} = (1, 1, \dots, 1, \dots)$ be the identity element in $\Pi_i G_i$. Then, by definition,

$$\varprojlim_i G_i = \{g \in \Pi_{\mathsf{Gr}} G_i \mid g \circ \mathbf{1} = \mathbf{1}\}.$$

The *derived limit* $\varprojlim_i^1 G_i$ of the system (A.33) is defined to be the set of orbits of Π_{Set} under the above action of Π_{Gr}:

$$\varprojlim_i^1 G_i := \Pi_{Set} G_i / \{x \sim g \circ x : \ g \in \Pi_{\mathsf{Gr}},\ x \in \Pi_{Set} G_i\}.$$

In general, $\varprojlim_i^1 G_i$ is a pointed set, but in case the group G_i are all abelian, it has a natural structure of an abelian group. Clearly, we can define the inverse and derived limit in the case of an inverse system of abelian groups by the exact sequence

$$1 \to \varprojlim_i G_i \to \prod_i G_i \xrightarrow{f} \prod_i G_i \to \varprojlim_i^1 G_i \to 1,$$

where the homomorphism $f : \prod_i G_i \to \prod_i G_i$ is defined by

$$(g_0, g_1, \dots, g_n, \dots) \mapsto (g_0 f_0(g_1^{-1}), g_1 d_1(g_2^{-1}), \dots, g_n f_n(g_{n+1}^{-1}), \dots).$$

The following result is well-known.

Proposition A.19 *Let*

$$1 \to \{G'_n\} \to \{G_n\} \to \{G''_n\} \to 1 \tag{A.34}$$

be a short exact sequence of inverse systems of groups. Then there is a sequence

$$\begin{aligned} 1 \to \varprojlim_n G'_n \to \varprojlim_n G_n \to \varprojlim_n G''_n \to \\ \varprojlim_i{}^1 G'_n \to \varprojlim_i{}^1 G_n \to \varprojlim_i{}^1 G''_n \to 1. \end{aligned} \tag{A.35}$$

of groups and pointed spaces which is exact as a sequence of groups at the first three terms, as a sequence pointed sets at the last three terms, and the set map

$$\varprojlim_n G''_n \to \varprojlim_i^1 G'_n$$

extends to a natural action of $\varprojlim_i G''_n$ *on* $\varprojlim_i^1 G'_n$ *such that elements of* $\varprojlim_i^1 G'_n$ *are in the same orbit if and only if they have the same image in* $\varprojlim_i^1 G_n$.

In case (A.34) is a short exact sequence of abelian groups, then the sequence (A.35) is a long exact sequence of abelian groups.

It is of interest to note a characterization for the vanishing of the derived limit of a given inverse system of abelian groups.

An inverse system (A.34) of abelian groups is said to satisfy the *Mittag-Leffler condition* if, for every $m \geq 0$, the chain

$$\mathrm{im}(f_m) \supseteq \mathrm{im}(f_m \circ f_{m+1}) \supseteq \mathrm{im}(f_m \circ f_{m+1} \circ f_{m+2}) \supseteq \dots$$

is stationary.

Proposition A.20 (Gray [Gra66]). *Let* $(A.34)$ *be an inverse system of countable abelian groups. Then* $\varprojlim_i^1 G_i = 0$ *if and only if this inverse system satisfies the Mittag-Leffler condition.*

Thus, in particular, for any inverse system of finite abelian groups, or for any inverse system of epimorphisms of abelian groups, its derived limit vanishes always.

Example A.21

Let p be a prime. Consider the inverse system

$$\dots \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \dots \to \mathbb{Z}$$

of monomorphisms of the additive group of integers, defined by p-multiplication, i.e.,

$$z \mapsto pz, \;\; z \in \mathbb{Z}.$$

Then one has

$$\varprojlim_n^1 \mathbb{Z} = (\varprojlim_n \mathbb{Z}_{p^n})/\mathbb{Z},$$

i.e., p-adic integers modulo the rational integers.

Proposition A.22 (Harlap [Har75]). *Let* $(A.34)$ *be an inverse system of finitely generated abelian groups. Then*
either $(A.34)$ *satisfies the Mittag-Leffler condition and* $\varprojlim_i^1 G_i = 0$,
or $\varprojlim_i^1 G_i$ *is uncountable.*

Theorem A.23 [Bou72]. *Let* $\mathcal{C}$ *be a category with initial object* $* \in Ob(\mathcal{C})$ *and*

$$\dots \to X_n \to X_{n-1} \to \dots \to X_0 \to *$$

a tower of fibrations in $\mathcal{SC}$ *with compatible base points. Then for any* $i \geq 0$, *there exists the following natural exact sequence:*

$$* \to \varprojlim_n^1 \pi_{i+1}(X_n) \to \pi_i(\varprojlim_n X_n) \to \varprojlim_n \pi_i(X_n) \to *.$$

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