Appendix

A The Mass Transportation Problem

The mass transportation problem was first proposed by G. Monge in 1781 [49] as follows: *"Étant donnés dans l'espace, deux volumes egaux entr'eux, & terminés chacune par une ou plusieurs surfaces courbes donnés; trouver dans le second volume le point où doit être transportée chaque molécule du premier, pour que la somme des produits des molecules multipliées chacune par l'espace parcouru soit un minimum"*. In modern language, we can rewrite his question as follows: given two sets A, $B \subseteq \mathbb{R}^3$ with the same volume, we look for a measurable map $T : A \to B$ which describes a way to transport the set A onto B, and so that the cost

$$\int_A |T(x) - x| \, dx \tag{A.1}$$

is minimal. Saying that T "describes a transportation" means that, by moving the mass on A according to T, we completely cover B, that is, for any measurable set $E \subseteq B$ one must have

$$\mathrm{meas}\Big(\Big\{x \in A : T(x) \in E\Big\}\Big) = \mathrm{meas}(E)\,. \tag{A.2}$$

A first generalization that can be done is to consider a density of the material which may be non-constant or even singular; then, instead of two sets, we can consider two positive Borel measures f^+ and f^- over a Polish space X which have the same total mass, that is $\|f^+\| = \|f^-\|$. Usually, one assumes this total mass to be unitary for simplicity. A Borel map $T : X \to X$ is then called a *transport map* if $T_\# f^+ = f^-$, where $T_\#$ stands for the push-forward operator defined in Appendix B.2; equivalently, we can say that T is a transport map if for all Borel sets $E \subseteq X$ one has

$$f^+\Big(\Big\{x \in X : T(x) \in E\Big\}\Big) = f^-(E)\,,$$

which is the exact analogue of (A.2) with $f^+ = \chi_A$ and $f^- = \chi_B$.

Another possible generalization is to consider a cost function $c : X \times X \to \mathbb{R}^+$ such that $c(x,y)$ represents the cost to move a unit mass from x to y. Therefore, the cost associated with the transport map T is simply

$$\int_X c(x, T(x))\, df^+(x)\,, \tag{A.3}$$

which again generalizes (A.1) with $c(x,y) = |y-x|$ when $X = \mathbb{R}^3$ and $f^+ = \chi_A$, $f^- = \chi_B$.

Despite the simplicity of the problem considered, it reveals to be very hard to attack. The main reasons are two: first of all, the set of the admissible maps has no good structure; indeed, even in the simplest case considered by Monge, the admissibility of T relies on the validity of the highly non-linear equation

$$\bigl|\mathrm{Det}\, DT\bigr| = 1\,.$$

A second reason is that also the cost given by (A.1) or (A.3) depends in a quite involved way on the map T so again there is no kind of linearity giving some help.

We can also note that, in general, the problem of finding a minimizer of the cost has no solution, and it can also easily happen that there are no admissible transport maps at all. For instance, if $f^+ = \delta_x$ is a Dirac mass at a point x, then for any map $T : X \to X$ one has $T_\# f^+ = \delta_{T(x)}$; therefore, there are no transport maps at all unless f^- is a Dirac mass itself. In fact, the only obstacle to the existence of tranport maps is the presence of Dirac masses in f^+: indeed, the following theorem is well known (for a proof in the general case of Polish spaces one can refer to [59]).

Theorem A.1. *Let X be a Polish space and f^+, f^- be two positive Borel measures on X with the same total mass. If f^+ is non-atomic (that is, for any $x \in X$ one has $f^+\bigl(\{x\}\bigr) = 0$) then there exist transport maps from f^+ to f^-.*

Even if the set of transport maps is nonempty, the existence of optimal transport maps may fail: consider for instance the case when $X = \mathbb{R}^2$, f^+ is the Hausdorff one-dimensional measure with density 1 on the segment $\{0\} \times [0,1]$, and f^- is the Hausdorff one-dimensional measure with density $1/2$ on the two segments $\{\pm 1\} \times [0,1]$; consider also the easiest cost function $c(x,y) = |y-x|$. In this case, it is immediate to understand that the infimum of the cost of the transport maps cannot be achieved, since otherwise an optimal transport map should split every point in the support of f^+ and move half of it in the left segment of the support of f^- and half in the right one.

The first big step forward in the study of the mass transportation is due to the work by Kantorovich [42, 43]; to explain it, let us start from the consideration that, in both of the above examples, the obstacle was the impossibility to split the masses by a map. The idea of Kantorovich, then, was to allow this possibility; hence, the mass which is initially at x, should not be entirely

transported to a point $T(x)$, rather it may be distributed on the support of f^-. Formally, this means that a new notion of transportation is introduced.

Definition A.2. A *transport plan* is a positive measure $\gamma \in \mathcal{M}^+(X \times X)$ such that the two projections $\pi_{1\#}\gamma$ and $\pi_{2\#}\gamma$ coincide with f^+ and f^- respectively, where π_1 and π_2 are the projections of $X \times X$ on the first and second factor respectively.

It is important to notice the meaning of the above definition: the measure γ corresponds to transporting a mass $\gamma(C \times D)$ from the set C to the set D for any choice of the sets $C, D \subseteq X$. Hence, every transport map T correspond naturally to a transport plan γ_T, namely

$$\gamma_T = \big(\mathrm{Id}, T\big)_{\#} f^+ .$$

It is easy to generalize the cost of a transport map to transport plans: indeed, we will say that the cost of the transport plan γ is

$$\iint_{X \times X} c(x,y)\, d\gamma(x,y) .$$

With this definition, the cost of the transport map T equals the cost of the transport plan γ_T, so that the Kantorovich setting of the problem is indeed a generalization of the one by Monge.

It is immediate to see that both of the above mentioned difficulties for transport maps are immediately solved for transport plans: indeed, the transport plans form a bounded, convex and weakly* closed subset of the measures on $X \times X$, so they have a very good and perfectly known structure; in addition, the cost of the transport plans is linear with respect to γ.

Hence, in this new formulation, the existence of transports and of optimal transports always occurs. Indeed, the set of transport plans is nonempty, since the measure $f^+ \otimes f^-$ is always an admissible transport plan. Moreover, also the existence of optimal transport plans (i.e. those minimizing the cost) becomes very easy: in fact, if φ is a lower semicontinuous function on the Polish space Y and $\mu_n \stackrel{*}{\rightharpoonup} \mu$ are Borel measures on Y then

$$\int_Y \varphi(y)\, d\mu(y) \leq \liminf \int_Y \varphi(y)\, d\mu_n(y) .$$

Therefore, provided that c is lower semicontinuous, the problem of finding an optimal transport plan is trivial: it suffices to take any weak* limit of any minimizing sequence of transport plans.

Even if the Kantorovich formulation of the problem always admits optimal solutions, deducing the existence of transport maps remains a difficult issue. The main idea to show the existence of transport maps was found in the 70's by Sudakov [67]: indeed, in the case $c(x,y) = |y - x|$ with ambient space $X = \mathbb{R}^n$, it is known that any optimal transport plan moves the mass along

non-intersecting segments. More precisely, if γ is an optimal transport plan and (x_1, y_1), (x_2, y_2) belong to the support of γ, then the segments x_1y_1 and x_2y_2 cannot cross except in a common endpoint. Then $\mathbb{R}^n$ can be filled by non-intersecting segments (called *transport rays*) so that all the mass can be moved by the optimal transport plans only along those rays. Since it is possible to determine these segments knowing only f^+ and f^-, the idea of Sudakov was to consider the transport problem on each of these rays and then "glue" all the information found. This argument would reduce the mass transportation to one-dimensional problems, which are easily discussed, so this provides a possible strategy to show the existence of transport maps.

Nevertheless, two decades passed before the Sudakov argument were rigorously completed by means of fine tools of Geometric Measure Theory. One of the technical difficulties was the following: as we saw, the presence of Dirac masses can prevent the existence of optimal transport maps; however, even if f^+ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^n$, in principle it may happen that its "restrictions" to the transport rays have Dirac masses. By the way, also the definition of this restrictions needs to be carefully precised by means of measure disintegration. Anyway, Sudakov's formal idea has given a powerful strategy to attack the problem in the last years, and all the different proofs now available concerning the linear cost (i.e. $c(x, y) = |y - x|$) rely somehow on this idea.

The first proof of the existence of optimal transport maps was given independently by Brenier and Knott–Smith in the '80s [22, 23, 45], but instead of the Monge case of the linear cost, the quadratic one was considered.

Theorem A.3. *Consider the case $X = \mathbb{R}^n$ with the quadratic cost $c(x, y) = |y-x|^2$, and assume that $f^+(S) = 0$ for any set S such that $\mathscr{H}^{n-1}(S) < +\infty$. Then there is a unique optimal transport plan, which in particular corresponds to an optimal transport map.*

In the following years this result has been widely generalized to other cases of strictly convex cost functions such as, for instance, $c(x, y) = |y - x|^p$ with $p > 1$; some references for these results are for instance [60, 40, 71].

The situation in the original Monge case is completely different, mainly due to the fact that the linear cost $c(x, y) = |y-x|$ is convex but not strictly convex. The first existence result for this case was given by Evans and Gangbo in 1999 in [36], and in the following few years their result was generalized in other papers [1, 70, 25]. We can summarize all their results in the following one.

Theorem A.4. *Assume that $X = \mathbb{R}^n$ and that $c(x, y) = |y - x|$, and let f^+ be a measure with compact support and absolutely continuous with respect to the Lebesgue measure. Then there exists an optimal transport map.*

We emphasize the big difference between the results for the linear and the quadratic costs. First of all, with the quadratic cost (or generally, with many strictly convex costs) one has the uniqueness of an optimal transport plan, which is also a map; on the other hand, with the linear cost there are many

optimal transport plans, and many of them (but not all) are in particular transport maps. Moreover, while in the strictly convex case the essential assumption is that f^+ does not charge $(n-1)-$dimensional sets (in the sense of Theorem A.3), in the linear case it is essential that f^+ be absolutely continuous: the fact that this stronger assumption is indeed necessary is shown by a counterexample given in [4].

Also for the linear case many subsequent generalizations have been shown: the case where the ambient space X is a manifold and $c(x,y) = d(x,y)$ is the distance on the manifold is considered in [39], while the case when $X = \mathbb{R}^n$ and $c(x,y) = \|y - x\|$ for a convex norm $\|\cdot\|$ different from the Euclidean one is considered in [25, 4, 3].

For the interested reader, there is a large number of books and wide surveys about the mass transportation problem from many different points of view. A very short list can be given by [71, 72, 60, 1, 4, 57].

B Some Tools from Geometric Measure Theory

In this chapter we present some well known results about the definition of measures and their main properties and about some specific tools from Geometric Measure Theory, such as the Disintegration Theorem or the $\Gamma-$convergence; more precise and detailed results can be found in many complete books on these subjects, for instance in [33, 2].

B.1 Measures as Duals of the Continuous Functions

We start from the following definition.

Definition B.1. A topological space X is called a *Polish space if it is separable and metrizable with a distance making it complete.*

Recall that a space is said to be metrizable if it can endowed by some distance inducing the same topology; note also that the completeness, but not the separability, depends on the distance. In this paper, we always consider each Polish space endowed with its Borel $\sigma-$algebra, denoted by $\mathscr{B}(X)$.

Given a Polish space X, we consider the space $\mathcal{M}^+(X)$ of the finite positive Borel measures on it, which is defined as the set of all countably additive functions from $\mathscr{B}(X)$ to $\mathbb{R}^+$. It is well known that $\mathcal{M}^+(X)$ is a (strongly) closed subset of $\big(C_b(X)\big)'$, the dual space of the space $C_b(X)$ of the continuous and bounded real functions on X, endowed with the norm

$$\|u\|_{C_b(X)} := \sup\big\{|u(x)|,\ x \in X\big\}.$$

We denote by $\langle \cdot, \cdot \rangle$ the duality between $C_b(X)'$ and $C_b(X)$, and hence, in particular, the duality between $\mathcal{M}^+(X)$ and $C_b(X)$. Even though $\mathcal{M}^+(X)$ is not a linear space because we deal only with positive measures, we endow it with the norm induced by the inclusion in $C_b(X)'$, that is

$$\|\mu\|_{\mathcal{M}^+(X)} := \sup \left\{ \langle \mu, u \rangle, \ \|u\|_{C_b(X)} = 1 \right\} = \mu(X).$$

The subspace of $C_b(X)'$ made by the increasing functionals, i.e. all those $\mu \in C_b(X)'$ such that $\langle \mu, u \rangle \geq 0$ whenever $u \geq 0$, is denoted by $C_b(X)'_+$: of course, the inclusion $\mathcal{M}^+(X) \subseteq C_b(X)'_+$ holds.

We can also consider on $\mathcal{M}^+(X)$ the weak* convergence induced by $C_b(X)$: we say that a sequence $\mu_n \in \mathcal{M}^+(X)$ weakly* converge to μ, and we write $\mu_n \stackrel{*}{\rightharpoonup} \mu$, if

$$\langle \mu_n, u \rangle \longrightarrow \langle \mu, u \rangle \qquad \text{for each } u \in C_b(X).$$

Definition B.2. Given a positive measure $\eta \in \mathcal{M}^+(X)$, we define the set $\mathscr{B}_\eta(X)$ of the *measurable sets with respect to* η to be the smallest $\sigma-$algebra containing both the set $\mathscr{B}(X)$ of all Borel sets and the set of all the η-negligible subsets of X. Moreover, we say that a set $B \subseteq X$ is *universally measurable* if $B \in \mathscr{B}_\eta(X)$ for any positive measure η.

Definition B.3. Given any positive measure $\mu \in \mathcal{M}^+(X)$ we define the *support of* μ, written $\operatorname{spt}\mu$, to be the smallest closed set $K \subseteq X$ such that

$$\mu(X \setminus K) = 0.$$

More in general, one says that the measure μ is *concentrated* on Γ if

$$\mu(X \setminus \Gamma) = 0.$$

For any measure μ, the support $\operatorname{spt}\mu$ is a well-defined closed set of full measure; in particular, it is the smallest closed set of full measure; on the other hand, there are in general many sets on which μ is concentrated. However, it is often very useful to select some set of full measure for μ with particular properties, which may also differ from the support. In particular, a measure can be concentrated in a set much smaller than its support: if for example $X = \mathbb{R}$ and $\mu = \sum_i 2^{-i} \delta_{q_i}$ where $\mathbb{Q} = \{q_i\}$ is the set of the rationals, then μ is concentrated in the countable set $\mathbb{Q}$ but its support is the whole $\mathbb{R}$.

A first result on Polish spaces is given by the following

Theorem B.4 (Ulam). *Given a Polish space X and $\mu \in \mathcal{M}^+(X)$, there is a $\sigma-$compact set in which μ is concentrated.*

This result is of primary importance, and can be found for example in [33]; one can refer to the same book for the proof of the following very important theorem.

Theorem B.5 (Prokhorov). *A bounded set $\{\mu_i,\, i \in I\}$ in $\mathcal{M}^+(X)$ is weakly* sequentially relatively compact if and only if for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subseteq X$ such that*

$$\mu_i(X \setminus K_\varepsilon) \leq \varepsilon \qquad \text{for any } i \in I. \tag{B.1}$$

The property (B.1) is called *tightness* of the set $\{\mu_i\}$; hence, Prokhorov Theorem says that a bounded set of measures is weakly* sequentially relatively compact if and only if the tightness property holds.

As we mentioned before, any measure is an element of $C_b(X)'_+$; however, it is fundamental to know when the opposite is also true: in fact, it is very often useful to define a measure μ by specifying the value of $\langle \mu, u \rangle$ for any $u \in C_b(X)$. When doing so, it is sufficient to check that $\langle \mu, \cdot \rangle$ is linear, continuous and increasing to derive that μ belongs to $C_b(X)'_+$, (and this usually follows trivially from the definition); but on the other hand, to establish that μ is in fact a measure is often not straightforward since the inclusion $\mathcal{M}^+(X) \subseteq C_b(X)'_+$ may in general be strict for Polish spaces.

Hence, we collect now some results which give necessary and sufficient conditions for an element of $C_b(X)'_+$ to be in $\mathcal{M}^+(X)$. We will notice that these results cover many important situations, in particular by Corollary B.8 they always apply to transport plans. The most important result to deal with the question whether an element of $C_b(X)'_+$ is a measure is given by Daniell's Theorem.

Theorem B.6 (Daniell). *Let $\mu \in C_b(X)'_+$: then $\mu \in \mathcal{M}^+(X)$ if and only if for every sequence $\{h_n\} \subseteq C_b(X)$ such that $h_n \searrow 0$ (i.e. h_n is pointwise decreasing to 0) one has $\langle \mu, h_n \rangle \longrightarrow 0$.*

Note that, in the hypotheses above, $\langle \mu, h_n \rangle$ is a real positive decreasing sequence, then $\langle \mu, h_n \rangle \longrightarrow l$ for some $l \geq 0$; the Theorem claims that μ is a measure if and only if this limit l is never strictly positive. Once again, one can find the proof of a more general assertion in the book [33]. We remark now some easy consequences of Daniell's Theorem.

Proposition B.7. *Let $\mu \in C_b(X)'_+$: then $\mu \in \mathcal{M}^+(X)$ if and only if for each $\varepsilon > 0$ there is a compact set K such that $\langle \mu, u \rangle \leq \varepsilon \|u\|_{C_b(X)}$ whenever $u \in C_b(X)$, $u \geq 0$ on X, $u = 0$ on K.*

Proof. If $\mu \in \mathcal{M}^+(X)$ then the stated property immediately follows from Ulam's Theorem. On the other hand, assume that the property holds and take a sequence $h_n \searrow 0$ as in Daniell's Theorem: to show that $\mu \in \mathcal{M}^+(X)$ it is sufficient to check that $\langle \mu, h_n \rangle \to 0$. Fix then $\varepsilon > 0$ and consider the corresponding compact set K.

Since K is compact, there is an integer m such that $h_m \leq \varepsilon$ on K; let then U be an open neighborhood of K such that $h_m < 2\varepsilon$ on U. Notice that this is possible since h_m is continuous, and that the open set U depends on ε, on K and on m. Take now a partition of the unity $\{\varphi_1, \varphi_2\}$ associated to

the open sets U and $X \setminus K$, i.e. $\varphi_i : X \to [0,1]$ is a continuous function for $i = 1, 2$, $\varphi_1 + \varphi_2 \equiv 1$, and φ_1 (resp. φ_2) is positive only inside the open set U (resp. $X \setminus K$). We recall that this is always possible, since X is metrizable: for example, if we denote by $d_1(x)$ and $d_2(x)$ the distance of x from $X \setminus U$ and K respectively, it suffices to note that d_1 and d_2 are continuous, that $d_1 + d_2$ is everywhere strictly positive and finally to define

$$\varphi_1(x) := \frac{d_1(x)}{d_1(x) + d_2(x)}, \qquad \varphi_2(x) := \frac{d_2(x)}{d_1(x) + d_2(x)}.$$

Now we write $h_m = g_1 + g_2$, where $g_i = \varphi_i h_m$: then we note that $g_1 \leq h_m$ on U and $g_1 = 0$ outside of U: consequently $\|g_1\|_{C_b(X)} \leq 2\varepsilon$. On the other hand, $g_2 = 0$ on K and, of course, $\|g_2\|_{C_b(X)} \leq \|h_m\| \leq \|h_1\|$, since the sequence $\{h_n\}$ is decreasing. Then, thanks to the hypothesis, we can evaluate

$$\langle \mu, h_m \rangle = \langle \mu, g_1 \rangle + \langle \mu, g_2 \rangle \leq 2\varepsilon \, \|\mu\|_{C_b(X)'_+} + \varepsilon \|h_1\|_{C_b(X)'_+} \leq \varepsilon\big(2\|\mu\| + \|h_1\|\big).$$

Since the sequence $\{h_n\}$ is decreasing and by the generality of ε we deduce $\langle \mu, h_n \rangle \to 0$ and then the thesis follows by Daniell's Theorem. □

We can derive the following very useful corollary.

Corollary B.8. *Assume that $\gamma \in C_b(X \times Y)'_+$ and that $\pi_1\gamma \in \mathcal{M}^+(X)$ and $\pi_2\gamma \in \mathcal{M}^+(Y)$. Then $\gamma \in \mathcal{M}^+(X \times Y)$.*

Proof. By Proposition B.7, given an $\varepsilon > 0$ there are two compact sets $K_1 \subseteq X$ and $K_2 \subseteq Y$ such that $\langle \pi_1\gamma, v \rangle \leq \varepsilon\|v\|$ (resp. $\langle \pi_2\gamma, w \rangle \leq \varepsilon\|w\|$) whenever $v \in C_b(X)$, $v \geq 0$ on X, $v = 0$ on K_1 (resp. $w \in C_b(Y)$, $w \geq 0$ on Y, $w = 0$ on K_2). We consider then the compact set $K = K_1 \times K_2$ in $X \times Y$: given any $u \in C_b(X \times Y)$ such that $u \geq 0$ on X and $u = 0$ on K, and given a $\delta > 0$, we can find an open neighborhood $U_1 \times U_2$ of K such that $0 \leq u \leq \delta$ in $U_1 \times U_2$. Moreover, arguing as in Proposition B.7, we can find two functions $v : X \to [0, \|u\|]$ and $w : Y \to [0, \|u\|]$ such that $v = 0$ on K_1, $w = 0$ on K_2, $v = \|u\|$ out of U_1 and $w = \|u\|$ out of U_2. Since $v(x) + w(y) \geq u(x,y)$ whenever $(x,y) \notin U_1 \times U_2$ and $\langle \gamma, v(x) + w(y) \rangle = \langle \pi_1\gamma, v \rangle + \langle \pi_2\gamma, w \rangle$, arguing as in Proposition B.7 one derives

$$\langle \gamma, u \rangle \leq 2\,\varepsilon\|u\| + \delta\|\gamma\|;$$

by the generality of δ, the thesis follows from Proposition B.7. □

The corollary above is very important in mass transportation problem: in fact, it implies in particular that any continuous and linear functional on $X \times Y$ with marginals f^+ and f^- is in fact a measure, and hence it is a transport plan.

Let us now briefly discuss the problem whether or not the inclusion $\mathcal{M}^+(X) \subseteq C_b(X)'_+$ is strict: first of all, we have an immediate result.

Lemma B.9. *If X is compact, then $\mathcal{M}^+(X) = C_b(X)'_+$.*

Proof. This trivially follows from Daniell's Theorem, since a sequence of continuous functions pointwise decreasing to 0 in a compact space is easily seen to be converging uniformly. □

However, the result above cannot be extended, as the following example shows.

Example B.10. Take $X = \mathbb{R}$, define $u_0 \equiv 1$ and for any integer $n \geq 1$ let $u_n : \mathbb{R} \to [0,1]$ be a continuous function such that $u_n \equiv 1$ on $[-n,n]$ and $u_n \equiv 0$ outside of $(-(n+1), n+1)$. Define D to be the span of $\{u_n,\ n \geq 0\}$, that is a subspace of $C_b(X)$. Let moreover $\tilde{\mu}$ be the linear functional on D given by $\langle \tilde{\mu}, u_0 \rangle = 1$ and $\langle \tilde{\mu}, u_n \rangle = 0$ for any $n \geq 1$, which is easily checked to be continuous and to have unit norm. By Hahn-Banach Theorem, there is an element μ of $C_b(X)'_+$ extending $\tilde{\mu}$. Since the functions $h_n = u_0 - u_n$ are as in the claim of Daniell's Theorem but $\langle \mu, h_n \rangle = 1$ for any n, μ is not a measure, then $\mu \in C_b(X)'_+ \setminus \mathcal{M}^+(X)$.

The example above can be clearly extended to cover all the non-compact Polish spaces, therefore in Lemma B.9 the opposite implication is valid too, so that $\mathcal{M}^+(X) = C_b(X)'_+$ if and only if X is compact.

Remark B.11. Recall that Riesz Theorem claims that $\mathcal{M}^+(\mathbb{R})$ is exactly the set of all increasing linear functionals on $C_c(\mathbb{R})$. In general, using $C_c(X)$ in place of $C_b(X)$ to define a duality on $\mathcal{M}^+(X)$ has the advantage that Riesz Theorem, which states that $\mathcal{M}^+(X) = C_c(X)'_+$, is true in a wide generality: for example, making use of Daniell's Theorem, one can quite easily show that this is true whenever X is a countable union of open sets $\{U_n\}$ with compact closures $\{K_n\}$ such that $K_n \subseteq U_{n+1}$ for any integer n; in particular, Riesz Theorem is true whenever X is a locally compact Polish space. Notice in particular that, since $C_c(X) \subseteq C_b(X)$, whenever Riesz Theorem holds for X one has that for any $\mu \in C_b(X)'_+$ there is a $\tilde{\mu} \in \mathcal{M}^+(X)$ for which one has $\langle \mu, u \rangle = \langle \tilde{\mu}, u \rangle$ for every $u \in C_c(X)$.

However, it is often preferable to use the duality with $C_b(X)$ whenever one wants to deal with non compactly supported measures. In this book, the two choices are equivalent since we assume our ambient space to be a compact subset of $\mathbb{R}^n$.

We conclude this section by stating two other important results about Polish spaces, which can be found for example in [28] (in a more general version).

Theorem B.12 (Measurable selection). *Let X and Y be two Polish spaces, and let η be a Borel measure on X. If $\Delta \subseteq X \times Y$ is $\mathscr{B}_\eta(X) \otimes \mathscr{B}(Y)$-measurable and the projection of Δ on X is a set of full $\eta-$measure, then there exists a measurable selection of Δ, i.e. an $\eta-$measurable function $\sigma : X \to Y$ such that for $\eta-$a.e. $x \in X$ one has $(x, \sigma(x)) \in \Delta$.*

Theorem B.13 (Projection). *Let X and Y be two Polish spaces, and let $\Delta \subseteq X \times Y$ be a Borel set; then the projection of Δ on X is universally measurable.*

B.2 Push-forward and Tensor Product of Measures

Given a measurable function $\varphi : X \to Y$, we may define a linear and mass-preserving operator $\varphi_\# : \mathcal{M}^+(X) \to \mathcal{M}^+(Y)$ (by mass-preserving we mean that $\|\mu\|_{\mathcal{M}^+(X)} = \|\varphi_\#\mu\|_{\mathcal{M}^+(Y)}$), according to the following formula:

$$\varphi_\#\mu(B) := \mu\big(\varphi^{-1}(B)\big) \qquad \forall\, A \in \mathscr{B}\,(Y).$$

It is easily noticed that $\varphi_\#\mu \in \mathcal{M}^+(Y)$, and in particular for any $\alpha \in C_b(Y)$

$$\langle \varphi_\#\mu, \alpha\rangle = \int_Y \alpha(y)\, d\varphi_\#\mu(y) = \int_X \alpha\big(\varphi(x)\big)\, d\mu(x).$$

An immediate but very useful consequence of the definition is that for any $\varphi : X \to Y$ and $\psi : Y \to Z$ it is

$$(\psi \circ \varphi)_\#\mu = \psi_\#(\varphi_\#\mu). \tag{B.2}$$

Given two measures $\mu \in \mathcal{M}^+(X)$ and $\nu \in \mathcal{M}^+(Y)$, we define the *tensor product* of μ and ν, $\mu \otimes \nu \in \mathcal{M}^+(X \times Y)$, as the unique measure on $X \times Y$ satisfying

$$\mu \otimes \nu(A \times B) := \mu(A) \cdot \nu(B) \qquad \forall\, A \in \mathscr{B}\,(X),\; B \in \mathscr{B}\,(Y).$$

The fact that there is a measure satisfying the above property follows from Fubini-Tonelli Theorem, while the uniqueness occurs because the smallest $\sigma-$algebra containing all the sets $A \times B$ with $A \in \mathscr{B}\,(X)$, $B \in \mathscr{B}\,(Y)$ is the whole $\mathscr{B}\,(X \times Y)$. In particular, the marginals of $\mu \otimes \nu$ are

$$\pi_{1\#}(\mu \otimes \nu) = \nu(Y)\mu, \qquad\qquad \pi_{2\#}(\nu \otimes \mu) = \mu(X)\nu\,.$$

An interesting particular case is when μ and ν are probability measures; so, the projections of $\mu \otimes \nu$ are precisely μ and ν: this is particularly useful in the study of the mass transportation, since if $\|f^+\| = \|f^-\| = 1$ then $f^+ \otimes f^-$ is always a transport plan between f^+ and f^-.

B.3 Measure Valued Maps and Disintegration Theorem

Here we briefly introduce a couple of tools concerning the measure-valued maps and then present the Disintegration Theorem.

Definition B.14. A map $\tau : X \to \mathcal{M}^+(Y)$ is called Borel measure-valued map (resp. $\mu-$measurable measure valued map, where $\mu \in \mathcal{M}^+(X)$) if for any Borel set $B \subseteq Y$ the function $x \to \tau(x)(B)$ is Borel (resp. $\mu-$measurable). Equivalently, τ is said to be Borel (resp. $\mu-$measurable) if for any bounded Borel $\varphi : X \times Y \to \mathbb{R}$ the function

$$x \to \int_Y \varphi(x,y)\, d\tau(x)(y)$$

is Borel (resp. $\mu-$measurable).

Thanks to the notion above, we can generalize the idea of a tensor product between two measures: consider a measure $\nu \in \mathcal{M}^+(Y)$ and a $\nu-$measurable measure valued map $y \mapsto \gamma_y$ with $\gamma_y \in \mathcal{M}^+(X)$. We define the *tensor product* between $\{\gamma_y\}$ and ν to be the measure $\gamma \in \mathcal{M}^+(X)$ given by the formula

$$\langle \gamma, \varphi \rangle := \int_Y \langle \gamma_y, \varphi \rangle \, d\nu(y)$$

for any $\varphi \in C_b(X)$. We always denote this measure by

$$\gamma := \gamma_x \otimes \nu.$$

We can now present the Disintegration Theorem: this result allows to decompose a measure γ over the space X with respect to a Borel function $\alpha : X \to Y$, where this “decomposition” is intended as a tensor product between suitable probability measures and the push-forward of γ. The proof of this Theorem can be found, for instance, in [2] or [33].

Theorem B.15 (Disintegration). *Let $\alpha : X \to Y$ be a given Borel map and $\gamma \in \mathcal{M}^+(X)$ is a given measure, and define $\mu \in \mathcal{M}^+(Y)$ by setting $\mu := \alpha_\# \gamma$. Then there exists a $\mu-$measurable measure valued function $y \mapsto \gamma_y$ such that γ_y is a probability measure on X for any y and*

(i) $\gamma = \gamma_y \otimes \mu$;
(ii) γ_y *is concentrated on* $\{x : \alpha(x) = y\}$ *for* $\mu-$*a.e.* $y \in Y$.

Moreover, the measures γ_y are uniquely determined by (i) and (ii) for $\mu-$a.e. $y \in Y$.

We state and prove now a useful consequence of the above theorem.

Lemma B.16. *The operations of disintegration and of composition commute, i.e. if*

$$\gamma = \gamma_y \otimes \alpha_\# \gamma$$

is the disintegration of $\gamma \in \mathcal{M}^+(X)$ with respect to some $\alpha : X \to Y$ and a function $\beta : X \to Z$ is given, then

$$\beta_\# \gamma = \beta_\# \gamma_y \otimes \alpha_\# \gamma. \tag{B.3}$$

In particular, if $\alpha = \delta \circ \beta$ for some $\delta : Z \to Y$ and

$$\beta_{\#}\gamma = \mu_y \otimes \delta_{\#}\big(\beta_{\#}\gamma\big) \tag{B.4}$$

is the disintegration of $\beta_{\#}\gamma$ with respect of δ, for a.e. $y \in Y$ it is

$$\mu_y = \beta_{\#}\gamma_y. \tag{B.5}$$

Proof. The first part is easy: for any $\varphi \in C_c(Z)$, recalling the properties of the push-forward one has

$$\begin{aligned}\langle \beta_{\#}\gamma, \varphi\rangle &= \int_X \varphi(\beta(x))\, d\gamma(x) = \int_Y \Big(\int_X \varphi(\beta(x))\, d\gamma_y(x)\Big)\, d\nu(y)\\ &= \int_Y \Big(\int_Z \varphi(z)\, d\beta_{\#}\gamma_y(z)\Big)\, d\nu(y) = \langle \beta_{\#}\gamma_y \otimes \nu, \varphi\rangle,\end{aligned}$$

thus the claim follows.

Concerning the second part, by the properties of disintegration (B.3) becomes

$$\beta_{\#}\gamma = \beta_{\#}\gamma_y \otimes \big(\delta_{\#}\beta_{\#}\gamma\big).$$

Recall now the disintegration (B.4): according to Theorem B.15, for a.e. $y \in Y$ the measure μ_y is concentrated on the set $\{z \in Z :\ \delta(z) = y\}$; analogously, γ_y is concentrated on the set $\{x \in X :\ \alpha(x) = y\} = \{x \in X :\ \delta(\beta(x)) = y\}$. But then $\beta_{\#}\gamma_y$ is concentrated on

$$\beta\Big(\big\{x \in X :\ \delta(\beta(x)) = y\big\}\Big) \subseteq \big\{z \in Z :\ \delta(z) = y\big\};$$

moreover, $\alpha_{\#}\gamma = \delta_{\#}(\beta_{\#}\gamma)$ by (B.2). Then, by the uniqueness part of Theorem B.15, we infer the validity of (B.5) and hence also the second claim is achieved. □

B.4 Γ−convergence

In this section we briefly recall the definition and the main properties of the $\Gamma-$convergence; for a more complete and precise reference we address the reader to the books [29, 12].

The notion of $\Gamma-$convergence, first proposed by De Giorgi in [31, 32], is the following: let X be a metric space, and assume that we are given a sequence of functionals $g_n : X \to \overline{\mathbb{R}}$ and a functional $g : X \to \overline{\mathbb{R}}$. We say that g_n $\Gamma-$converges to g, or $g_n \xrightarrow{\Gamma} g$, if the following hold:

$$\textbf{(i)} \qquad \forall\, x,\ \ \forall\, \{x_n\} \to x, \quad g(x) \le \liminf_{n\to\infty} g_n(x_n)\,;$$

$$\textbf{(ii)} \qquad \forall\, x,\ \ \exists\, \{x_n\} \to x : \quad g(x) \ge \limsup_{n\to\infty} g_n(x_n)\,.$$

The first property is usually called the *liminf inequality* and the second one *limsup inequality*. Note that thanks to **(i)**, one could simply write in **(ii)** $g(x) = \lim g_n(x_n)$ instead of $g(x) \geq \limsup g_n(x_n)$. Moreover, given an $x \in X$, any sequence $\{x_n\} \to x$ for which the property **(ii)** is fulfilled is called *recovery sequence*. The first fundamental property that one can immediately notice is the following.

Proposition B.17. *If $g_n \xrightarrow{\Gamma} g$ and there exists a compact set $K \subseteq X$ so that for any $n \in \mathbb{N}$ one has $\inf_X g_n = \inf_K g_n$, then g admits a minimum and $\inf g_n \to \min g$. Moreover, for any sequence x_n such that $g_n(x_n) - \inf g_n \to 0$ and that $x_n \to \bar{x}$, one has that $\bar{x}$ is a minimum point for g.*

Proof. It suffices to take $x_n \in K$ so that $g_n(x_n) \leq \inf g_n + 1/n$; by the compactness of K we know the existence of a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that $x_{n_i} \to \bar{x}$ for a certain $\bar{x} \in K$: moreover, we can also assume that

$$g_{n_i}(x_{n_i}) \xrightarrow[i\to\infty]{} \liminf_{n\to\infty} \inf_X g_n \,.$$

By the liminf property we know then that

$$g(\bar{x}) \leq \liminf_{i\to\infty} g_{n_i}(x_{n_i}) = \liminf_{n\to\infty} \inf_X g_n \,. \tag{B.6}$$

On the other hand, take any $\tilde{x} \in X$: by the limsup property we know the existence of a sequence $x_n \in X$ for which $x_n \to \tilde{x}$ and

$$g(\tilde{x}) \geq \limsup_{n\to\infty} g_n(x_n) \geq \limsup_{n\to\infty} \inf_X g_n \,. \tag{B.7}$$

From (B.6) and (B.7) we deduce that $\bar{x}$ is a minimum point for g, as well as that $\inf_X g_n$ converges, for $n \to \infty$, to $\min_X g$. The thesis then immediately follows. $\square$

We claim now the second property, which is also very important, namely a compactness result for $\Gamma-$convergence.

Theorem B.18. *Assume that X is separable. Then, for any sequence of functions $g_n : X \to \overline{\mathbb{R}}$, there exists a subsequence g_{n_i} which admits a $\Gamma-$limit.*

The above result is very strong: indeed, given any sequence of functions, it allows us to assume, up to a subsequence, that they $\Gamma-$converge to some limit.

We present now the definition of the $\Gamma - \liminf$ and $\Gamma - \limsup$ of a sequence of functions. Given a sequence $\{g_n\}$, we define

$$\begin{aligned} \Gamma - \liminf_{n\to\infty} g_n(x) &:= \inf \Big\{ \liminf_{n\to\infty} g_n(x_n) :\ x_n \to x \Big\}\,; \\ \Gamma - \limsup_{n\to\infty} g_n(x) &:= \inf \Big\{ \limsup_{n\to\infty} g_n(x_n) :\ x_n \to x \Big\}\,. \end{aligned} \tag{B.8}$$

It is clear from the definitions that one has always

$$\Gamma - \liminf g_n \le \Gamma - \limsup g_n \,,$$

and that the sequence $\Gamma-$converges (to a function g) if and only if

$$\Gamma - \liminf g_n = \Gamma - \limsup g_n \quad (= g)\,.$$

Proposition B.19. *One has*

$$\Gamma - \liminf_{n\to\infty} g_n = \sup_{n\in\mathbb{N}} \Big(\mathrm{env}\big(\inf_{m\ge n} g_m\big)\Big)\,, \tag{B.9}$$

where env(φ) *denotes the lower semicontinuous envelope of any function* $\varphi : X \to \overline{\mathbb{R}}$. *In particular,* $\Gamma - \liminf g_n$ *is always lower semicontinuous in* X.

Proof. The equivalence (B.9) is verified directly from the definition (B.8). The lower semicontinuity of the $\Gamma - \liminf$ follows from (B.9) once one reminds that the supremum of lower semicontinuous functions is still lower semicontinuous. □

Finally, one can show an important property of the $\Gamma - \liminf$ of the sequence g_n: it is the infimum of all the possible $\Gamma-$limits of subsequences of $\{g_n\}$; analagously, the $\Gamma - \limsup$ is the supremum of all the possible $\Gamma-$limits of subsequences of $\{g_n\}$.

We conclude this section pointing out an useful consequence of (B.9) in the setting of the weak* convergence of measures.

Lemma B.20. *Let* X *be a Polish space,* $\{\nu_n\} \in \mathcal{M}^+(X)$ *a sequence of measures weakly* converging to* ν, *and* $\{g_n\} : X \to \mathbb{R}$ *a sequence of l.s.c. functions. Then*

$$\int_X \Gamma - \liminf_{n\to\infty} g_n \, d\nu \le \liminf_{n\to\infty} \int_X g_n \, d\nu_n \,.$$

Proof. Defining for simplicity

$$\tau_n := \mathrm{env}\big(\inf_{m\ge n} g_m\big)\,,$$

we fix $j \in \mathbb{N}$ and evaluate

$$\begin{aligned}\liminf_{n\to\infty} \int_X g_n \, d\nu_n &\ge \liminf_{n\to\infty} \int_X \big(\inf_{m\ge j} g_m\big)\, d\nu_n \\ &\ge \liminf_{n\to\infty} \int_X \mathrm{env}\big(\inf_{m\ge j} g_m\big)\, d\nu_n \\ &= \liminf_{n\to\infty} \int_X \tau_j \, d\nu_n \ge \int_X \tau_j \, d\nu.\end{aligned}$$

Since this is true for any $j \in \mathbb{N}$, by the Lebesgue monotone convergence theorem and (B.9) the thesis follows. □

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