A

Large Polynomials

In this appendix, we quarantine off some of the larger polynomials which would otherwise disrupt the flow of the text of Chaps. 2 and 3.

A.1 $\mathcal{H}^4$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{\mathcal{H}^4}(X,Y)$, mentioned on p. 36:

$$
\begin{aligned}
&1 - 6X^8Y^5 + 5X^9Y^5 + 4X^8Y^7 - 8X^9Y^7 + 3X^{10}Y^7 + 4X^{16}Y^8 - 8X^{17}Y^8\\
&+ 3X^{18}Y^8 - X^8Y^9 + 3X^9Y^9 - 3X^{10}Y^9 - X^{16}Y^{10} + 5X^{17}Y^{10} - 6X^{18}Y^{10}\\
&+ X^{19}Y^{10} - X^{24}Y^{11} + 3X^{25}Y^{11} - 3X^{26}Y^{11} + 8X^{17}Y^{12} - 10X^{18}Y^{12}\\
&+ 3X^{19}Y^{12} + 8X^{25}Y^{13} - 10X^{26}Y^{13} + 3X^{27}Y^{13} - 5X^{17}Y^{14} + 15X^{18}Y^{14}\\
&- 9X^{19}Y^{14} - 19X^{25}Y^{15} + 43X^{26}Y^{15} - 24X^{27}Y^{15} + 2X^{28}Y^{15} - 3X^{18}Y^{16}\\
&+ 5X^{19}Y^{16} - X^{20}Y^{16} - 5X^{33}Y^{16} + 15X^{34}Y^{16} - 9X^{35}Y^{16} + 8X^{25}Y^{17}\\
&- 30X^{26}Y^{17} + 32X^{27}Y^{17} - 7X^{28}Y^{17} - X^{29}Y^{17} + 8X^{33}Y^{18} - 30X^{34}Y^{18}\\
&+ 32X^{35}Y^{18} - 7X^{36}Y^{18} - X^{37}Y^{18} + 3X^{26}Y^{19} - 9X^{27}Y^{19} + 7X^{28}Y^{19}\\
&- 3X^{42}Y^{19} + 5X^{43}Y^{19} - X^{44}Y^{19} - 3X^{33}Y^{20} + 8X^{34}Y^{20} - 17X^{35}Y^{20}\\
&+ 15X^{36}Y^{20} - 3X^{37}Y^{20} - 3X^{27}Y^{21} + X^{28}Y^{21} + X^{29}Y^{21} + 3X^{42}Y^{21}\\
&- 9X^{43}Y^{21} + 7X^{44}Y^{21} + 4X^{34}Y^{22} - 12X^{35}Y^{22} - 2X^{36}Y^{22} + 13X^{37}Y^{22}\\
&- 5X^{38}Y^{22} + 4X^{42}Y^{23} - 12X^{43}Y^{23} - 2X^{44}Y^{23} + 13X^{45}Y^{23} - 5X^{46}Y^{23}\\
&+ 9X^{35}Y^{24} - 10X^{36}Y^{24} - 10X^{37}Y^{24} + 9X^{38}Y^{24} - 3X^{51}Y^{24} + X^{52}Y^{24}\\
&+ X^{53}Y^{24} - 3X^{42}Y^{25} + 18X^{43}Y^{25} - 16X^{44}Y^{25} - 16X^{45}Y^{25} + 18X^{46}Y^{25}\\
&- 3X^{47}Y^{25} + X^{36}Y^{26} + X^{37}Y^{26} - 3X^{38}Y^{26} + 9X^{51}Y^{26} - 10X^{52}Y^{26}\\
&- 10X^{53}Y^{26} + 9X^{54}Y^{26} - 5X^{43}Y^{27} + 13X^{44}Y^{27} - 2X^{45}Y^{27} - 12X^{46}Y^{27}\\
&+ 4X^{47}Y^{27} - 5X^{51}Y^{28} + 13X^{52}Y^{28} - 2X^{53}Y^{28} - 12X^{54}Y^{28} + 4X^{55}Y^{28}
\end{aligned}
$$

$$
\begin{aligned}
&+ 7X^{45}Y^{29} - 9X^{46}Y^{29} + 3X^{47}Y^{29} + X^{60}Y^{29} + X^{61}Y^{29} - 3X^{62}Y^{29} \\
&- 3X^{52}Y^{30} + 15X^{53}Y^{30} - 17X^{54}Y^{30} + 8X^{55}Y^{30} - 3X^{56}Y^{30} - X^{45}Y^{31} \\
&+ 5X^{46}Y^{31} - 3X^{47}Y^{31} + 7X^{61}Y^{31} - 9X^{62}Y^{31} + 3X^{63}Y^{31} - X^{52}Y^{32} \\
&- 7X^{53}Y^{32} + 32X^{54}Y^{32} - 30X^{55}Y^{32} + 8X^{56}Y^{32} - X^{60}Y^{33} - 7X^{61}Y^{33} \\
&+ 32X^{62}Y^{33} - 30X^{63}Y^{33} + 8X^{64}Y^{33} - 9X^{54}Y^{34} + 15X^{55}Y^{34} - 5X^{56}Y^{34} \\
&- X^{69}Y^{34} + 5X^{70}Y^{34} - 3X^{71}Y^{34} + 2X^{61}Y^{35} - 24X^{62}Y^{35} + 43X^{63}Y^{35} \\
&- 19X^{64}Y^{35} - 9X^{70}Y^{36} + 15X^{71}Y^{36} - 5X^{72}Y^{36} + 3X^{62}Y^{37} - 10X^{63}Y^{37} \\
&+ 8X^{64}Y^{37} + 3X^{70}Y^{38} - 10X^{71}Y^{38} + 8X^{72}Y^{38} - 3X^{63}Y^{39} + 3X^{64}Y^{39} \\
&- X^{65}Y^{39} + X^{70}Y^{40} - 6X^{71}Y^{40} + 5X^{72}Y^{40} - X^{73}Y^{40} - 3X^{79}Y^{41} \\
&+ 3X^{80}Y^{41} - X^{81}Y^{41} + 3X^{71}Y^{42} - 8X^{72}Y^{42} + 4X^{73}Y^{42} + 3X^{79}Y^{43} \\
&- 8X^{80}Y^{43} + 4X^{81}Y^{43} + 5X^{80}Y^{45} - 6X^{81}Y^{45} + X^{89}Y^{50}.
\end{aligned}
$$

A.2 $\mathfrak{g}_{6,4}$, Counting All Subrings

The following polynomial is $W^{\leq}_{\mathfrak{g}_{6,4}}(X,Y)$, mentioned on p. 44:

$$
\begin{aligned}
&1 + X^{4}Y^{2} - X^{5}Y^{3} + X^{6}Y^{3} - X^{6}Y^{4} - X^{7}Y^{4} - X^{9}Y^{4} + X^{10}Y^{4} - X^{9}Y^{5} \\
&- 2X^{10}Y^{5} - 3X^{11}Y^{5} - 2X^{12}Y^{5} - X^{13}Y^{5} + X^{10}Y^{6} + X^{11}Y^{6} + 2X^{12}Y^{6} \\
&+ X^{13}Y^{6} + X^{14}Y^{6} - X^{15}Y^{6} - X^{13}Y^{7} - X^{14}Y^{7} - 2X^{15}Y^{7} - X^{16}Y^{7} \\
&- X^{17}Y^{7} + X^{14}Y^{8} + 2X^{15}Y^{8} + 3X^{16}Y^{8} + 3X^{17}Y^{8} + X^{18}Y^{8} - X^{19}Y^{8} \\
&+ X^{20}Y^{8} - X^{17}Y^{9} + X^{18}Y^{9} + 2X^{19}Y^{9} + 2X^{20}Y^{9} + 2X^{21}Y^{9} + X^{22}Y^{9} \\
&+ X^{22}Y^{10} + X^{23}Y^{10} + X^{24}Y^{10} - X^{21}Y^{11} - X^{22}Y^{11} + X^{26}Y^{11} \\
&+ X^{27}Y^{11} - X^{24}Y^{12} - X^{25}Y^{12} - X^{26}Y^{12} - X^{26}Y^{13} - 2X^{27}Y^{13} \\
&- 2X^{28}Y^{13} - 2X^{29}Y^{13} - X^{30}Y^{13} + X^{31}Y^{13} - X^{28}Y^{14} + X^{29}Y^{14} \\
&- X^{30}Y^{14} - 3X^{31}Y^{14} - 3X^{32}Y^{14} - 2X^{33}Y^{14} - X^{34}Y^{14} + X^{31}Y^{15} \\
&+ X^{32}Y^{15} + 2X^{33}Y^{15} + X^{34}Y^{15} + X^{35}Y^{15} + X^{33}Y^{16} - X^{34}Y^{16} \\
&- X^{35}Y^{16} - 2X^{36}Y^{16} - X^{37}Y^{16} - X^{38}Y^{16} + X^{35}Y^{17} + 2X^{36}Y^{17} \\
&+ 3X^{37}Y^{17} + 2X^{38}Y^{17} + X^{39}Y^{17} - X^{38}Y^{18} + X^{39}Y^{18} + X^{41}Y^{18} \\
&+ X^{42}Y^{18} - X^{42}Y^{19} + X^{43}Y^{19} - X^{44}Y^{20} - X^{48}Y^{22}.
\end{aligned}
$$

A.3 T_4, Counting All Subrings

The following polynomial is $W^{\leq}_{T_4}(X,Y)$, mentioned on p. 45:

$$1 + X^{4}Y^{2} + X^{5}Y^{2} - 2X^{5}Y^{3} - 3X^{6}Y^{3} + X^{6}Y^{4} - X^{8}Y^{4} - 2X^{9}Y^{4} - 2X^{10}Y^{4}$$

$$
\begin{aligned}
&+X^{11}Y^5-2X^{12}Y^5-4X^{13}Y^5-X^{14}Y^5+X^{11}Y^6+X^{12}Y^6+4X^{13}Y^6\\
&+2X^{14}Y^6+2X^{15}Y^6-2X^{16}Y^6-2X^{17}Y^6+X^{14}Y^7+3X^{15}Y^7+2X^{16}Y^7\\
&+4X^{17}Y^7+2X^{18}Y^7-X^{19}Y^7+X^{20}Y^7-X^{15}Y^8+3X^{18}Y^8+6X^{19}Y^8\\
&+4X^{20}Y^8+3X^{21}Y^8-X^{18}Y^9-4X^{19}Y^9-4X^{20}Y^9-2X^{21}Y^9+5X^{22}Y^9\\
&+2X^{23}Y^9+4X^{24}Y^9+2X^{25}Y^9-X^{20}Y^{10}-3X^{22}Y^{10}-8X^{23}Y^{10}\\
&-3X^{24}Y^{10}-X^{25}Y^{10}+2X^{26}Y^{10}+2X^{27}Y^{10}+2X^{23}Y^{11}-2X^{24}Y^{11}\\
&-4X^{25}Y^{11}-8X^{26}Y^{11}-11X^{27}Y^{11}-4X^{28}Y^{11}+X^{29}Y^{11}+3X^{30}Y^{11}\\
&+X^{25}Y^{12}+4X^{26}Y^{12}+3X^{27}Y^{12}+2X^{28}Y^{12}-6X^{29}Y^{12}-11X^{30}Y^{12}\\
&-6X^{31}Y^{12}-4X^{32}Y^{12}+X^{33}Y^{12}+2X^{29}Y^{13}+6X^{30}Y^{13}+5X^{31}Y^{13}\\
&-5X^{33}Y^{13}-6X^{34}Y^{13}-2X^{35}Y^{13}-X^{31}Y^{14}+4X^{32}Y^{14}+6X^{33}Y^{14}\\
&+11X^{34}Y^{14}+6X^{35}Y^{14}-2X^{36}Y^{14}-3X^{37}Y^{14}-4X^{38}Y^{14}-X^{39}Y^{14}\\
&-3X^{34}Y^{15}-X^{35}Y^{15}+4X^{36}Y^{15}+11X^{37}Y^{15}+8X^{38}Y^{15}+4X^{39}Y^{15}\\
&+2X^{40}Y^{15}-2X^{41}Y^{15}-2X^{37}Y^{16}-2X^{38}Y^{16}+X^{39}Y^{16}+3X^{40}Y^{16}\\
&+8X^{41}Y^{16}+3X^{42}Y^{16}+X^{44}Y^{16}-2X^{39}Y^{17}-4X^{40}Y^{17}-2X^{41}Y^{17}\\
&-5X^{42}Y^{17}+2X^{43}Y^{17}+4X^{44}Y^{17}+4X^{45}Y^{17}+X^{46}Y^{17}-3X^{43}Y^{18}\\
&-4X^{44}Y^{18}-6X^{45}Y^{18}-3X^{46}Y^{18}+X^{49}Y^{18}-X^{44}Y^{19}+X^{45}Y^{19}\\
&-2X^{46}Y^{19}-4X^{47}Y^{19}-2X^{48}Y^{19}-3X^{49}Y^{19}-X^{50}Y^{19}+2X^{47}Y^{20}\\
&+2X^{48}Y^{20}-2X^{49}Y^{20}-2X^{50}Y^{20}-4X^{51}Y^{20}-X^{52}Y^{20}-X^{53}Y^{20}\\
&+X^{50}Y^{21}+4X^{51}Y^{21}+2X^{52}Y^{21}-X^{53}Y^{21}+2X^{54}Y^{22}+2X^{55}Y^{22}\\
&+X^{56}Y^{22}-X^{58}Y^{22}+3X^{58}Y^{23}+2X^{59}Y^{23}-X^{59}Y^{24}-X^{60}Y^{24}-X^{64}Y^{26}.
\end{aligned}
$$

A.4 $L_{(3,2,2)}$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{L_{(3,2,2)}}(X,Y)$, mentioned on p. 50:

$$
\begin{aligned}
&1-X^3Y^2+X^4Y^3+X^5Y^3+X^6Y^4-X^6Y^5-X^7Y^5+X^9Y^5-X^{10}Y^7\\
&-X^{11}Y^8-X^{12}Y^8+X^{13}Y^8-X^{12}Y^9+X^{13}Y^9-2X^{14}Y^9-X^{15}Y^9\\
&+X^{14}Y^{10}-X^{16}Y^{10}-X^{17}Y^{10}+X^{15}Y^{11}-2X^{16}Y^{11}-X^{18}Y^{11}+X^{20}Y^{11}\\
&+X^{16}Y^{12}+X^{18}Y^{12}-X^{19}Y^{12}+X^{20}Y^{12}-X^{21}Y^{12}-X^{22}Y^{12}+X^{19}Y^{13}\\
&-X^{20}Y^{13}-2X^{22}Y^{13}-X^{23}Y^{13}+3X^{22}Y^{14}-2X^{23}Y^{14}+X^{24}Y^{14}\\
&-X^{26}Y^{14}+X^{22}Y^{15}+X^{23}Y^{15}+X^{25}Y^{15}+X^{23}Y^{16}+X^{24}Y^{16}-2X^{25}Y^{16}\\
&+2X^{26}Y^{16}-X^{27}Y^{16}-X^{25}Y^{17}+2X^{26}Y^{17}+X^{27}Y^{17}-X^{28}Y^{17}-X^{30}Y^{17}\\
&-X^{26}Y^{18}+X^{27}Y^{18}+2X^{28}Y^{18}+2X^{29}Y^{18}-X^{30}Y^{18}+X^{31}Y^{18}-X^{29}Y^{19}\\
&+2X^{30}Y^{19}+X^{33}Y^{19}-X^{30}Y^{20}+X^{31}Y^{20}-X^{32}Y^{20}+3X^{33}Y^{20}+X^{35}Y^{20}
\end{aligned}
$$

$$- 2X^{33}Y^{21} + 2X^{34}Y^{21} - X^{37}Y^{21} - X^{34}Y^{22} + 3X^{35}Y^{22} - 2X^{36}Y^{22}$$
$$+ 3X^{37}Y^{22} + X^{39}Y^{22} - X^{34}Y^{23} - X^{35}Y^{23} - X^{37}Y^{23} + X^{38}Y^{23} + 3X^{40}Y^{23}$$
$$- X^{41}Y^{23} - X^{38}Y^{24} + X^{39}Y^{24} - X^{40}Y^{24} + X^{41}Y^{24} - X^{42}Y^{24} - X^{38}Y^{25}$$
$$- 2X^{39}Y^{25} - X^{40}Y^{25} - X^{41}Y^{25} + 2X^{42}Y^{25} - 2X^{43}Y^{25} + 2X^{44}Y^{25}$$
$$+ X^{42}Y^{26} - X^{43}Y^{26} + 2X^{45}Y^{26} - 3X^{44}Y^{27} - 2X^{45}Y^{27} + 2X^{46}Y^{27}$$
$$+ X^{48}Y^{27} - X^{45}Y^{28} - 2X^{47}Y^{28} - X^{48}Y^{28} + X^{49}Y^{28} + X^{44}Y^{29} - X^{45}Y^{29}$$
$$- X^{46}Y^{29} - X^{49}Y^{29} - X^{50}Y^{29} - X^{49}Y^{30} + X^{50}Y^{30} - X^{51}Y^{30} - X^{52}Y^{30}$$
$$+ X^{53}Y^{30} + X^{48}Y^{31} - X^{50}Y^{31} - 2X^{51}Y^{31} - 2X^{52}Y^{31} + 2X^{53}Y^{31}$$
$$- X^{54}Y^{31} + 2X^{51}Y^{32} - X^{53}Y^{32} - X^{55}Y^{32} - X^{56}Y^{32} + X^{52}Y^{33} - 3X^{56}Y^{33}$$
$$- X^{54}Y^{34} + 2X^{55}Y^{34} + 2X^{56}Y^{34} + X^{56}Y^{35} - X^{58}Y^{35} - X^{60}Y^{35} + X^{57}Y^{36}$$
$$+ X^{60}Y^{36} - X^{63}Y^{36} + X^{61}Y^{37} + X^{62}Y^{37} + X^{62}Y^{38} + X^{63}Y^{38} + X^{64}Y^{38}$$
$$- X^{65}Y^{38} + X^{66}Y^{38} - X^{65}Y^{40} + X^{66}Y^{40} + X^{67}Y^{40} + X^{68}Y^{40} - X^{69}Y^{40}$$
$$+ X^{69}Y^{41} - X^{69}Y^{42} + X^{72}Y^{42} + X^{71}Y^{43} - X^{72}Y^{43} - X^{73}Y^{45} - X^{74}Y^{45}$$
$$+ X^{75}Y^{45} - X^{78}Y^{47}.$$

A.5 $\mathcal{G}_3 \times \mathfrak{g}_{5,3}$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{\mathcal{G}_3\times\mathfrak{g}_{5,3}}(X,Y)$, mentioned on p. 56:

$$1 + X^{6}Y^{3} - X^{6}Y^{5} - X^{7}Y^{7} - X^{12}Y^{7} - X^{14}Y^{8} - X^{13}Y^{9} - X^{15}Y^{10}$$
$$- X^{20}Y^{10} + X^{13}Y^{11} - X^{14}Y^{11} - X^{15}Y^{11} + X^{14}Y^{12} + X^{15}Y^{12} - X^{16}Y^{12}$$
$$+ X^{20}Y^{12} - X^{21}Y^{12} + X^{19}Y^{14} + 2X^{21}Y^{14} - X^{22}Y^{14} - X^{23}Y^{14} + X^{23}Y^{15}$$
$$+ X^{26}Y^{15} + 2X^{22}Y^{16} + X^{26}Y^{16} + X^{27}Y^{16} - X^{28}Y^{16} - X^{26}Y^{17} + X^{27}Y^{17}$$
$$+ X^{28}Y^{17} + X^{29}Y^{17} + X^{23}Y^{18} + X^{28}Y^{18} - X^{27}Y^{19} + X^{28}Y^{19} + X^{30}Y^{19}$$
$$+ X^{35}Y^{19} + X^{29}Y^{20} - X^{28}Y^{21} + X^{31}Y^{21} - X^{33}Y^{21} + X^{36}Y^{21} - X^{35}Y^{22}$$
$$- X^{29}Y^{23} - X^{34}Y^{23} - X^{36}Y^{23} + X^{37}Y^{23} - X^{36}Y^{24} - X^{41}Y^{24} - X^{35}Y^{25}$$
$$- X^{36}Y^{25} - X^{37}Y^{25} + X^{38}Y^{25} + X^{36}Y^{26} - X^{37}Y^{26} - X^{38}Y^{26} - 2X^{42}Y^{26}$$
$$- X^{38}Y^{27} - X^{41}Y^{27} + X^{41}Y^{28} + X^{42}Y^{28} - 2X^{43}Y^{28} - X^{45}Y^{28} + X^{43}Y^{30}$$
$$- X^{44}Y^{30} + X^{48}Y^{30} - X^{49}Y^{30} - X^{50}Y^{30} + X^{49}Y^{31} + X^{50}Y^{31} - X^{51}Y^{31}$$
$$+ X^{44}Y^{32} + X^{49}Y^{32} + X^{51}Y^{33} + X^{50}Y^{34} + X^{52}Y^{35} + X^{57}Y^{35} + X^{58}Y^{37}$$
$$- X^{58}Y^{39} - X^{64}Y^{42}.$$

A.6 $\mathfrak{g}_{6,12}$, Counting All Subrings

The following polynomial is $W^{\leq}_{\mathfrak{g}_{6,12}}(X,Y)$, mentioned on p. 59:

$$
\begin{aligned}
&1 + X^2Y + 2X^4Y^2 - X^5Y^3 + 2X^6Y^3 - X^6Y^4 - X^7Y^4 + 4X^8Y^4 + X^9Y^4\\
&- 3X^8Y^5 - 5X^9Y^5 + X^{10}Y^5 + X^9Y^6 - X^{10}Y^6 - 3X^{11}Y^6 + 3X^{12}Y^6\\
&- X^{13}Y^6 + X^{11}Y^7 - 5X^{12}Y^7 - 8X^{13}Y^7 - 2X^{14}Y^7 - 3X^{15}Y^7 + 5X^{13}Y^8\\
&+ X^{14}Y^8 - X^{15}Y^8 - 2X^{16}Y^8 - 6X^{17}Y^8 + 2X^{15}Y^9 - 2X^{16}Y^9 + X^{17}Y^9\\
&- 2X^{18}Y^9 - 7X^{19}Y^9 - 2X^{20}Y^9 - X^{21}Y^9 + 4X^{17}Y^{10} + 3X^{18}Y^{10} + 8X^{19}Y^{10}\\
&- X^{20}Y^{10} - 4X^{21}Y^{10} - X^{23}Y^{10} - X^{18}Y^{11} + X^{20}Y^{11} + 11X^{21}Y^{11}\\
&+ X^{22}Y^{11} - 4X^{23}Y^{11} - 4X^{24}Y^{11} - 2X^{25}Y^{11} + 2X^{22}Y^{12} + 13X^{23}Y^{12}\\
&+ 8X^{24}Y^{12} + 8X^{25}Y^{12} + X^{26}Y^{12} - X^{27}Y^{12} - 3X^{23}Y^{13} - 2X^{24}Y^{13}\\
&+ 8X^{25}Y^{13} + 3X^{26}Y^{13} + 5X^{27}Y^{13} + X^{29}Y^{13} - 3X^{25}Y^{14} - 2X^{26}Y^{14}\\
&+ 6X^{27}Y^{14} + 8X^{28}Y^{14} + 13X^{29}Y^{14} + 3X^{30}Y^{14} + 2X^{31}Y^{14} - 5X^{27}Y^{15}\\
&- 5X^{28}Y^{15} - 4X^{29}Y^{15} - 3X^{30}Y^{15} + 9X^{31}Y^{15} + 6X^{32}Y^{15} + 4X^{33}Y^{15}\\
&- 2X^{29}Y^{16} - 3X^{30}Y^{16} - 8X^{31}Y^{16} - 5X^{32}Y^{16} + 6X^{33}Y^{16} + 2X^{34}Y^{16}\\
&+ 6X^{35}Y^{16} + 2X^{36}Y^{16} - 2X^{31}Y^{17} - X^{32}Y^{17} - 11X^{33}Y^{17} - 11X^{34}Y^{17}\\
&- X^{36}Y^{17} + 4X^{37}Y^{17} + X^{38}Y^{17} - 12X^{35}Y^{18} - 11X^{36}Y^{18} - 8X^{37}Y^{18}\\
&- 6X^{38}Y^{18} + 6X^{39}Y^{18} + 2X^{40}Y^{18} + 2X^{35}Y^{19} + 6X^{36}Y^{19} - 6X^{37}Y^{19}\\
&- 8X^{38}Y^{19} - 11X^{39}Y^{19} - 12X^{40}Y^{19} + X^{37}Y^{20} + 4X^{38}Y^{20} - X^{39}Y^{20}\\
&- 11X^{41}Y^{20} - 11X^{42}Y^{20} - X^{43}Y^{20} - 2X^{44}Y^{20} + 2X^{39}Y^{21} + 6X^{40}Y^{21}\\
&+ 2X^{41}Y^{21} + 6X^{42}Y^{21} - 5X^{43}Y^{21} - 8X^{44}Y^{21} - 3X^{45}Y^{21} - 2X^{46}Y^{21}\\
&+ 4X^{42}Y^{22} + 6X^{43}Y^{22} + 9X^{44}Y^{22} - 3X^{45}Y^{22} - 4X^{46}Y^{22} - 5X^{47}Y^{22}\\
&- 5X^{48}Y^{22} + 2X^{44}Y^{23} + 3X^{45}Y^{23} + 13X^{46}Y^{23} + 8X^{47}Y^{23} + 6X^{48}Y^{23}\\
&- 2X^{49}Y^{23} - 3X^{50}Y^{23} + X^{46}Y^{24} + 5X^{48}Y^{24} + 3X^{49}Y^{24} + 8X^{50}Y^{24}\\
&- 2X^{51}Y^{24} - 3X^{52}Y^{24} - X^{48}Y^{25} + X^{49}Y^{25} + 8X^{50}Y^{25} + 8X^{51}Y^{25}\\
&+ 13X^{52}Y^{25} + 2X^{53}Y^{25} - 2X^{50}Y^{26} - 4X^{51}Y^{26} - 4X^{52}Y^{26} + X^{53}Y^{26}\\
&+ 11X^{54}Y^{26} + X^{55}Y^{26} - X^{57}Y^{26} - X^{52}Y^{27} - 4X^{54}Y^{27} - X^{55}Y^{27}\\
&+ 8X^{56}Y^{27} + 3X^{57}Y^{27} + 4X^{58}Y^{27} - X^{54}Y^{28} - 2X^{55}Y^{28} - 7X^{56}Y^{28}\\
&- 2X^{57}Y^{28} + X^{58}Y^{28} - 2X^{59}Y^{28} + 2X^{60}Y^{28} - 6X^{58}Y^{29} - 2X^{59}Y^{29}\\
&- X^{60}Y^{29} + X^{61}Y^{29} + 5X^{62}Y^{29} - 3X^{60}Y^{30} - 2X^{61}Y^{30} - 8X^{62}Y^{30}\\
&- 5X^{63}Y^{30} + X^{64}Y^{30} - X^{62}Y^{31} + 3X^{63}Y^{31} - 3X^{64}Y^{31} - X^{65}Y^{31}\\
&+ X^{66}Y^{31} + X^{65}Y^{32} - 5X^{66}Y^{32} - 3X^{67}Y^{32} + X^{66}Y^{33} + 4X^{67}Y^{33}\\
&- X^{68}Y^{33} - X^{69}Y^{33} + 2X^{69}Y^{34} - X^{70}Y^{34} + 2X^{71}Y^{35} + X^{73}Y^{36} + X^{75}Y^{37}.
\end{aligned}
$$

A.7 $\mathfrak{g}_{1357G}$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{\mathfrak{g}_{1357B}}(X,Y)$, mentioned on p. 67:

$$
\begin{aligned}
&1 + X^3Y^3 - X^3Y^5 - 2X^6Y^7 - 2X^4Y^8 + X^5Y^8 - X^7Y^8 + X^4Y^9 - 2X^5Y^9 \\
&- X^9Y^9 - X^7Y^{10} - X^{10}Y^{10} - X^7Y^{11} - X^8Y^{11} + X^9Y^{11} - X^{10}Y^{11} \\
&+ 3X^7Y^{12} - 3X^8Y^{12} - 2X^9Y^{12} + X^{10}Y^{12} - X^7Y^{13} + 3X^8Y^{13} + X^{10}Y^{13} \\
&- 3X^{11}Y^{13} + X^9Y^{14} + X^{10}Y^{14} - X^{11}Y^{14} + X^{13}Y^{14} + 5X^{11}Y^{15} - 2X^{12}Y^{15} \\
&- X^{14}Y^{15} + X^{11}Y^{16} + 5X^{12}Y^{16} - 2X^{14}Y^{16} + X^{16}Y^{16} + X^9Y^{17} - X^{12}Y^{17} \\
&+ X^{13}Y^{17} + 7X^{14}Y^{17} - X^{16}Y^{17} + X^{12}Y^{18} + 2X^{14}Y^{18} + 2X^{15}Y^{18} \\
&+ X^{16}Y^{18} + 2X^{12}Y^{19} - 4X^{14}Y^{19} + 6X^{15}Y^{19} + X^{16}Y^{19} + 3X^{17}Y^{19} \\
&- X^{12}Y^{20} + 4X^{13}Y^{20} - 3X^{15}Y^{20} + 2X^{16}Y^{20} + X^{17}Y^{20} + 3X^{18}Y^{20} \\
&- X^{12}Y^{21} - 2X^{13}Y^{21} + 4X^{14}Y^{21} + X^{15}Y^{21} - 2X^{17}Y^{21} + X^{18}Y^{21} \\
&+ 2X^{19}Y^{21} + X^{20}Y^{21} - 2X^{14}Y^{22} + X^{16}Y^{22} + X^{20}Y^{22} + X^{21}Y^{22} \\
&- 3X^{15}Y^{23} + X^{16}Y^{23} + 3X^{17}Y^{23} - 2X^{18}Y^{23} - X^{19}Y^{23} - X^{20}Y^{23} \\
&+ 2X^{21}Y^{23} - 5X^{16}Y^{24} - 3X^{18}Y^{24} + 6X^{19}Y^{24} - X^{20}Y^{24} - 6X^{21}Y^{24} \\
&+ X^{22}Y^{24} + X^{16}Y^{25} - 5X^{17}Y^{25} - X^{18}Y^{25} - 8X^{19}Y^{25} + 6X^{20}Y^{25} \\
&+ X^{21}Y^{25} - 2X^{22}Y^{25} + X^{17}Y^{26} - 2X^{18}Y^{26} - 4X^{19}Y^{26} - 6X^{20}Y^{26} \\
&+ X^{21}Y^{26} - X^{23}Y^{26} - 2X^{24}Y^{26} + X^{18}Y^{27} - X^{19}Y^{27} - 9X^{20}Y^{27} \\
&- 3X^{21}Y^{27} - 2X^{22}Y^{27} + X^{23}Y^{27} - X^{25}Y^{27} - X^{17}Y^{28} + 2X^{19}Y^{28} \\
&+ X^{20}Y^{28} - 7X^{21}Y^{28} - 8X^{22}Y^{28} - 3X^{23}Y^{28} + 2X^{24}Y^{28} - X^{26}Y^{28} \\
&- X^{27}Y^{28} - X^{19}Y^{29} + 4X^{21}Y^{29} + X^{22}Y^{29} - 11X^{23}Y^{29} - 4X^{24}Y^{29} \\
&- 2X^{25}Y^{29} + X^{26}Y^{29} - 2X^{21}Y^{30} + 4X^{22}Y^{30} - X^{23}Y^{30} - 5X^{24}Y^{30} \\
&- 5X^{25}Y^{30} - 4X^{26}Y^{30} + X^{27}Y^{30} - X^{21}Y^{31} - X^{22}Y^{31} + 8X^{23}Y^{31} \\
&- X^{24}Y^{31} - 9X^{26}Y^{31} - X^{27}Y^{31} + X^{20}Y^{32} + X^{21}Y^{32} - 2X^{22}Y^{32} \\
&- 2X^{23}Y^{32} + 3X^{24}Y^{32} + 5X^{25}Y^{32} + 7X^{26}Y^{32} - 10X^{27}Y^{32} - X^{28}Y^{32} \\
&- 2X^{29}Y^{32} + X^{21}Y^{33} + 2X^{22}Y^{33} - 4X^{25}Y^{33} + 6X^{26}Y^{33} + 8X^{27}Y^{33} \\
&- 5X^{28}Y^{33} - X^{29}Y^{33} - X^{30}Y^{33} + 3X^{23}Y^{34} + X^{24}Y^{34} + 3X^{25}Y^{34} \\
&- X^{26}Y^{34} + X^{27}Y^{34} + 6X^{28}Y^{34} + 2X^{29}Y^{34} - 2X^{30}Y^{34} - X^{31}Y^{34} \\
&+ 5X^{24}Y^{35} + X^{26}Y^{35} + X^{27}Y^{35} + 3X^{28}Y^{35} + 9X^{29}Y^{35} - X^{30}Y^{35} \\
&- X^{32}Y^{35} - 2X^{24}Y^{36} + 5X^{25}Y^{36} + 4X^{26}Y^{36} + 10X^{27}Y^{36} - 8X^{28}Y^{36} \\
&- 5X^{29}Y^{36} + 12X^{30}Y^{36} + X^{31}Y^{36} + 4X^{32}Y^{36} - X^{33}Y^{36} - X^{25}Y^{37} \\
&+ 2X^{26}Y^{37} + 15X^{28}Y^{37} - X^{29}Y^{37} - 2X^{30}Y^{37} + 4X^{31}Y^{37} + X^{32}Y^{37} \\
&+ 3X^{33}Y^{37} - X^{26}Y^{38} - X^{27}Y^{38} + 2X^{28}Y^{38} + 13X^{29}Y^{38} + 2X^{30}Y^{38} \\
&+ X^{31}Y^{38} - X^{32}Y^{38} + 4X^{33}Y^{38} + X^{34}Y^{38} + X^{35}Y^{38} - 2X^{27}Y^{39} - X^{28}Y^{39}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2X^{29}Y^{39} + 9X^{30}Y^{39} + 5X^{31}Y^{39} - X^{33}Y^{39} + 3X^{34}Y^{39} + X^{36}Y^{39} \\
&+ X^{27}Y^{40} - X^{28}Y^{40} - 6X^{29}Y^{40} - 5X^{30}Y^{40} + 13X^{31}Y^{40} + 12X^{32}Y^{40} \\
&- 3X^{33}Y^{40} - X^{34}Y^{40} + X^{35}Y^{40} + X^{36}Y^{40} + X^{37}Y^{40} + X^{29}Y^{41} - 2X^{30}Y^{41} \\
&- 10X^{31}Y^{41} + 8X^{33}Y^{41} + 9X^{34}Y^{41} - 2X^{35}Y^{41} - X^{36}Y^{41} + X^{30}Y^{42} \\
&- 5X^{31}Y^{42} - 4X^{32}Y^{42} - X^{33}Y^{42} + 3X^{34}Y^{42} + 7X^{35}Y^{42} - 2X^{36}Y^{42} \\
&+ X^{37}Y^{42} - X^{29}Y^{43} + 2X^{31}Y^{43} - 5X^{32}Y^{43} - 6X^{33}Y^{43} - 12X^{34}Y^{43} \\
&+ 6X^{35}Y^{43} + 6X^{36}Y^{43} - X^{39}Y^{43} - X^{30}Y^{44} - X^{31}Y^{44} + X^{32}Y^{44} \\
&+ 2X^{33}Y^{44} - 6X^{34}Y^{44} - 18X^{35}Y^{44} + 4X^{36}Y^{44} + 2X^{37}Y^{44} + 3X^{38}Y^{44} \\
&- X^{31}Y^{45} - 3X^{32}Y^{45} - 3X^{33}Y^{45} + 6X^{34}Y^{45} + X^{35}Y^{45} - 15X^{36}Y^{45} \\
&- 8X^{37}Y^{45} - X^{38}Y^{45} + 5X^{39}Y^{45} - X^{40}Y^{45} - 2X^{33}Y^{46} - 4X^{34}Y^{46} \\
&- X^{36}Y^{46} - 6X^{37}Y^{46} - 6X^{38}Y^{46} - 2X^{39}Y^{46} + X^{40}Y^{46} - 3X^{34}Y^{47} \\
&- 7X^{35}Y^{47} + 4X^{36}Y^{47} + 3X^{37}Y^{47} - 12X^{38}Y^{47} - 5X^{39}Y^{47} - 5X^{40}Y^{47} \\
&+ 2X^{41}Y^{47} + 2X^{35}Y^{48} - 12X^{36}Y^{48} + 9X^{38}Y^{48} - 6X^{39}Y^{48} - 10X^{41}Y^{48} \\
&+ X^{42}Y^{48} + X^{35}Y^{49} + 4X^{36}Y^{49} - 10X^{37}Y^{49} - 5X^{38}Y^{49} + X^{39}Y^{49} \\
&+ X^{41}Y^{49} - 7X^{42}Y^{49} - X^{43}Y^{49} + X^{36}Y^{50} + 4X^{37}Y^{50} - 4X^{38}Y^{50} \\
&- 5X^{39}Y^{50} - 5X^{40}Y^{50} + 6X^{41}Y^{50} - X^{42}Y^{50} - 2X^{43}Y^{50} - 2X^{44}Y^{50} \\
&+ X^{37}Y^{51} + 6X^{38}Y^{51} - 2X^{39}Y^{51} - 6X^{40}Y^{51} - 3X^{41}Y^{51} + 5X^{42}Y^{51} \\
&- X^{44}Y^{51} - 2X^{45}Y^{51} + X^{37}Y^{52} + 9X^{39}Y^{52} + 4X^{40}Y^{52} - 7X^{41}Y^{52} \\
&- 7X^{42}Y^{52} + X^{43}Y^{52} + 5X^{44}Y^{52} - X^{45}Y^{52} - X^{46}Y^{52} - 2X^{39}Y^{53} \\
&+ 5X^{40}Y^{53} + 8X^{41}Y^{53} + 4X^{42}Y^{53} - 8X^{43}Y^{53} - 2X^{44}Y^{53} + 3X^{45}Y^{53} \\
&+ X^{40}Y^{54} + 3X^{41}Y^{54} + 6X^{42}Y^{54} + 7X^{43}Y^{54} - 5X^{44}Y^{54} - X^{45}Y^{54} \\
&+ X^{46}Y^{54} + 2X^{47}Y^{54} - X^{48}Y^{54} + X^{39}Y^{55} + X^{40}Y^{55} - X^{41}Y^{55} + X^{42}Y^{55} \\
&+ 10X^{43}Y^{55} + 5X^{44}Y^{55} - X^{45}Y^{55} - 2X^{46}Y^{55} - 2X^{47}Y^{55} + 3X^{48}Y^{55} \\
&+ X^{41}Y^{56} - 2X^{42}Y^{56} - X^{43}Y^{56} + 8X^{44}Y^{56} + 7X^{45}Y^{56} + 5X^{46}Y^{56} \\
&- 4X^{47}Y^{56} - X^{48}Y^{56} + X^{49}Y^{56} + 3X^{42}Y^{57} - 4X^{44}Y^{57} + 2X^{45}Y^{57} \\
&+ 9X^{46}Y^{57} + 7X^{47}Y^{57} - 2X^{48}Y^{57} + 3X^{44}Y^{58} - 2X^{46}Y^{58} + 4X^{47}Y^{58} \\
&+ 5X^{48}Y^{58} + X^{44}Y^{59} + X^{45}Y^{59} - 3X^{46}Y^{59} + 2X^{48}Y^{59} + 7X^{49}Y^{59} \\
&- X^{44}Y^{60} + X^{45}Y^{60} + 2X^{46}Y^{60} - X^{48}Y^{60} - 6X^{49}Y^{60} + 7X^{50}Y^{60} \\
&+ X^{51}Y^{60} - X^{45}Y^{61} + 3X^{48}Y^{61} - X^{49}Y^{61} - 5X^{50}Y^{61} + 4X^{51}Y^{61} \\
&+ X^{52}Y^{61} - 2X^{46}Y^{62} - X^{47}Y^{62} - 3X^{48}Y^{62} + 4X^{49}Y^{62} - 4X^{51}Y^{62} \\
&- X^{52}Y^{62} + 2X^{53}Y^{62} - 2X^{47}Y^{63} + X^{48}Y^{63} - 2X^{50}Y^{63} + 2X^{51}Y^{63} \\
&- 3X^{52}Y^{63} - 2X^{53}Y^{63} + 2X^{54}Y^{63} + X^{47}Y^{64} - 3X^{48}Y^{64} - 5X^{49}Y^{64} \\
&- X^{50}Y^{64} + X^{51}Y^{64} + 3X^{52}Y^{64} - 2X^{53}Y^{64} - 2X^{54}Y^{64} - 6X^{51}Y^{65}
\end{aligned}
$$

$$+ X^{53}Y^{65} - X^{54}Y^{65} - X^{51}Y^{66} - 6X^{52}Y^{66} - X^{56}Y^{66} - X^{50}Y^{67} - 2X^{53}Y^{67}$$
$$- X^{54}Y^{67} - X^{55}Y^{67} + X^{56}Y^{67} - X^{57}Y^{67} + 2X^{52}Y^{68} - 4X^{54}Y^{68}$$
$$- 3X^{55}Y^{68} + X^{56}Y^{68} - X^{53}Y^{69} + 3X^{54}Y^{69} - 3X^{56}Y^{69} - X^{57}Y^{69}$$
$$- X^{58}Y^{70} + 2X^{55}Y^{71} + X^{57}Y^{71} - X^{59}Y^{71} + 2X^{58}Y^{72} - X^{59}Y^{72} + X^{56}Y^{73}$$
$$- X^{57}Y^{73} + 2X^{59}Y^{73} + 2X^{57}Y^{74} + X^{60}Y^{74} - X^{60}Y^{75} + 2X^{61}Y^{75}$$
$$+ X^{59}Y^{76} + X^{60}Y^{76} - X^{61}Y^{76} + X^{62}Y^{76} + X^{60}Y^{77} + X^{63}Y^{78} - X^{63}Y^{81}$$
$$- X^{66}Y^{83}.$$

A.8 $\mathfrak{g}_{1457A}$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{\mathfrak{g}_{1457A}}(X,Y)$, mentioned on p. 68:

$$1 - X^{4}Y^{5} - X^{4}Y^{8} + X^{5}Y^{8} - X^{5}Y^{9} - X^{8}Y^{10} + X^{8}Y^{11} - 2X^{9}Y^{11} + X^{8}Y^{12}$$
$$- X^{9}Y^{12} - X^{10}Y^{12} + 2X^{9}Y^{13} - 2X^{10}Y^{13} + X^{10}Y^{14} - X^{9}Y^{15} + 2X^{13}Y^{15}$$
$$- X^{14}Y^{15} + X^{9}Y^{16} - 2X^{10}Y^{16} - X^{13}Y^{16} + 2X^{14}Y^{16} + X^{10}Y^{17} - X^{11}Y^{17}$$
$$+ X^{14}Y^{17} + 2X^{13}Y^{18} - 2X^{14}Y^{18} + 3X^{14}Y^{19} - 2X^{15}Y^{19} - X^{13}Y^{20}$$
$$+ 3X^{14}Y^{20} - X^{14}Y^{21} + 4X^{15}Y^{21} - X^{16}Y^{21} + X^{18}Y^{21} - X^{15}Y^{22} + X^{16}Y^{22}$$
$$- X^{17}Y^{22} + X^{18}Y^{22} + X^{19}Y^{22} + X^{14}Y^{23} - X^{15}Y^{23} - 3X^{18}Y^{23} + 4X^{19}Y^{23}$$
$$+ 2X^{15}Y^{24} - X^{16}Y^{24} - X^{18}Y^{24} - 2X^{19}Y^{24} + 2X^{20}Y^{24} + X^{16}Y^{25}$$
$$+ X^{18}Y^{25} - X^{19}Y^{25} - X^{18}Y^{26} + 4X^{19}Y^{26} - 2X^{20}Y^{26} - X^{23}Y^{26} - X^{18}Y^{27}$$
$$- X^{19}Y^{27} + 4X^{20}Y^{27} - X^{23}Y^{27} - 3X^{19}Y^{28} + 3X^{20}Y^{28} + X^{21}Y^{28}$$
$$- X^{24}Y^{28} - 3X^{20}Y^{29} + 2X^{21}Y^{29} - X^{23}Y^{29} + X^{24}Y^{29} - X^{21}Y^{30}$$
$$- 3X^{23}Y^{30} + X^{24}Y^{30} + X^{20}Y^{31} - 5X^{24}Y^{31} + 2X^{25}Y^{31} - X^{20}Y^{32}$$
$$+ X^{21}Y^{32} + X^{23}Y^{32} - X^{24}Y^{32} - 3X^{25}Y^{32} - X^{23}Y^{33} + X^{24}Y^{33} - X^{25}Y^{33}$$
$$- X^{28}Y^{33} - 3X^{24}Y^{34} + 2X^{25}Y^{34} + X^{27}Y^{34} - X^{29}Y^{34} - X^{24}Y^{35}$$
$$- 2X^{25}Y^{35} + X^{26}Y^{35} + X^{27}Y^{35} + X^{28}Y^{35} - X^{29}Y^{35} - 3X^{25}Y^{36} - X^{26}Y^{37}$$
$$- X^{28}Y^{37} + X^{27}Y^{38} + X^{28}Y^{38} - 3X^{29}Y^{38} - X^{30}Y^{38} + X^{33}Y^{38} + 3X^{28}Y^{39}$$
$$- 3X^{30}Y^{39} - X^{25}Y^{40} + X^{28}Y^{40} + 3X^{29}Y^{40} - X^{30}Y^{40} - X^{31}Y^{40} + X^{30}Y^{41}$$
$$+ X^{32}Y^{41} + 3X^{33}Y^{42} + X^{29}Y^{43} - X^{30}Y^{43} - X^{31}Y^{43} - X^{32}Y^{43} + 2X^{33}Y^{43}$$
$$+ X^{34}Y^{43} + X^{29}Y^{44} - X^{31}Y^{44} - 2X^{33}Y^{44} + 3X^{34}Y^{44} + X^{30}Y^{45} + X^{33}Y^{45}$$
$$- X^{34}Y^{45} + X^{35}Y^{45} + 3X^{33}Y^{46} + X^{34}Y^{46} - X^{35}Y^{46} - X^{37}Y^{46} + X^{38}Y^{46}$$
$$- 2X^{33}Y^{47} + 5X^{34}Y^{47} - X^{38}Y^{47} - X^{34}Y^{48} + 3X^{35}Y^{48} + X^{37}Y^{48}$$
$$- X^{34}Y^{49} + X^{35}Y^{49} - 2X^{37}Y^{49} + 3X^{38}Y^{49} + X^{34}Y^{50} - X^{37}Y^{50}$$
$$- 3X^{38}Y^{50} + 3X^{39}Y^{50} + X^{35}Y^{51} - 4X^{38}Y^{51} + X^{39}Y^{51} + X^{40}Y^{51}$$

$$+ X^{35}Y^{52} + 2X^{38}Y^{52} - 4X^{39}Y^{52} + X^{40}Y^{52} + X^{39}Y^{53} - X^{40}Y^{53} - X^{42}Y^{53}$$
$$- 2X^{38}Y^{54} + 2X^{39}Y^{54} + X^{40}Y^{54} + X^{42}Y^{54} - 2X^{43}Y^{54} - 4X^{39}Y^{55}$$
$$+ 3X^{40}Y^{55} + X^{43}Y^{55} - X^{44}Y^{55} - X^{39}Y^{56} - X^{40}Y^{56} + X^{41}Y^{56} - X^{42}Y^{56}$$
$$+ X^{43}Y^{56} - X^{40}Y^{57} + X^{42}Y^{57} - 4X^{43}Y^{57} + X^{44}Y^{57} - 3X^{44}Y^{58} + X^{45}Y^{58}$$
$$+ 2X^{43}Y^{59} - 3X^{44}Y^{59} + 2X^{44}Y^{60} - 2X^{45}Y^{60} - X^{44}Y^{61} + X^{47}Y^{61}$$
$$- X^{48}Y^{61} - 2X^{44}Y^{62} + X^{45}Y^{62} + 2X^{48}Y^{62} - X^{49}Y^{62} + X^{44}Y^{63}$$
$$- 2X^{45}Y^{63} + X^{49}Y^{63} - X^{48}Y^{64} + 2X^{48}Y^{65} - 2X^{49}Y^{65} + X^{48}Y^{66}$$
$$+ X^{49}Y^{66} - X^{50}Y^{66} + 2X^{49}Y^{67} - X^{50}Y^{67} + X^{50}Y^{68} + X^{53}Y^{69} - X^{53}Y^{70}$$
$$+ X^{54}Y^{70} + X^{54}Y^{73} - X^{58}Y^{78}.$$

A.9 $\mathfrak{g}_{1457B}$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{\mathfrak{g}_{1457B}}(X,Y)$, mentioned on p. 68:

$$1 - X^{4}Y^{5} - X^{4}Y^{8} + X^{5}Y^{8} - X^{5}Y^{9} - X^{8}Y^{10} + X^{8}Y^{11} - 2X^{9}Y^{11} + X^{8}Y^{12}$$
$$- X^{9}Y^{12} - X^{10}Y^{12} + 2X^{9}Y^{13} - 2X^{10}Y^{13} + X^{10}Y^{14} + 2X^{13}Y^{15} - X^{14}Y^{15}$$
$$- X^{10}Y^{16} - X^{13}Y^{16} + 2X^{14}Y^{16} + X^{10}Y^{17} - X^{11}Y^{17} + X^{14}Y^{17} + X^{13}Y^{18}$$
$$- X^{14}Y^{18} + 3X^{14}Y^{19} - 2X^{15}Y^{19} + X^{15}Y^{20} + 3X^{15}Y^{21} - X^{16}Y^{21}$$
$$+ X^{18}Y^{21} - X^{15}Y^{22} + X^{16}Y^{22} + X^{19}Y^{22} - X^{17}Y^{23} - X^{18}Y^{23} + 3X^{19}Y^{23}$$
$$- X^{18}Y^{24} - 2X^{19}Y^{24} + 2X^{20}Y^{24} + X^{15}Y^{25} - X^{18}Y^{25} + X^{19}Y^{26} - X^{20}Y^{26}$$
$$- X^{23}Y^{26} + X^{20}Y^{27} - X^{22}Y^{27} - X^{19}Y^{28} + X^{20}Y^{28} + X^{21}Y^{28} + X^{22}Y^{28}$$
$$- X^{23}Y^{28} - X^{24}Y^{28} - X^{19}Y^{29} + X^{21}Y^{29} - X^{20}Y^{30} - X^{23}Y^{30} - 3X^{23}Y^{31}$$
$$+ X^{25}Y^{31} + 2X^{23}Y^{32} - 3X^{24}Y^{32} - X^{25}Y^{32} - X^{25}Y^{33} - X^{27}Y^{33} + X^{24}Y^{34}$$
$$- X^{25}Y^{34} + 2X^{27}Y^{34} - 2X^{28}Y^{34} - X^{24}Y^{35} + X^{27}Y^{35} + X^{28}Y^{35} - X^{29}Y^{35}$$
$$- X^{25}Y^{36} + 3X^{28}Y^{36} - 2X^{29}Y^{36} - X^{25}Y^{37} - 2X^{28}Y^{37} + 2X^{29}Y^{37}$$
$$- X^{29}Y^{38} + X^{32}Y^{38} + 2X^{28}Y^{39} - 2X^{29}Y^{39} - X^{30}Y^{39} - X^{32}Y^{39} + X^{33}Y^{39}$$
$$+ 4X^{29}Y^{40} - 3X^{30}Y^{40} + X^{29}Y^{41} + X^{30}Y^{41} - X^{31}Y^{41} + X^{32}Y^{41} - X^{33}Y^{41}$$
$$+ X^{30}Y^{42} - X^{32}Y^{42} + 4X^{33}Y^{42} - X^{34}Y^{42} + 2X^{34}Y^{43} - X^{35}Y^{43}$$
$$- 2X^{33}Y^{44} + 3X^{34}Y^{44} - 2X^{34}Y^{45} + 2X^{35}Y^{45} + X^{34}Y^{46} - X^{37}Y^{46}$$
$$+ X^{38}Y^{46} + 2X^{34}Y^{47} - X^{35}Y^{47} - 2X^{38}Y^{47} + X^{39}Y^{47} - X^{34}Y^{48}$$
$$+ 2X^{35}Y^{48} + X^{38}Y^{48} - X^{39}Y^{48} + X^{38}Y^{49} - 2X^{38}Y^{50} + 2X^{39}Y^{50}$$
$$- X^{38}Y^{51} - X^{39}Y^{51} + X^{40}Y^{51} - 2X^{39}Y^{52} + X^{40}Y^{52} - X^{40}Y^{53} - X^{43}Y^{54}$$
$$+ X^{43}Y^{55} - X^{44}Y^{55} - X^{44}Y^{58} + X^{48}Y^{63}.$$

A.10 $\mathfrak{tr}_6(\mathbb{Z})$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{\mathfrak{tr}_6(\mathbb{Z})}(Y)$ mentioned on p. 78:

$$
\begin{aligned}
&1 + 2Y^2 + 3Y^4 + 2Y^5 + 4Y^6 + 4Y^7 + 7Y^8 + 8Y^9 + 10Y^{10} + 13Y^{11} + 16Y^{12} \\
&+ 19Y^{13} + 24Y^{14} + 27Y^{15} + 34Y^{16} + 37Y^{17} + 44Y^{18} + 48Y^{19} + 56Y^{20} \\
&+ 59Y^{21} + 70Y^{22} + 72Y^{23} + 81Y^{24} + 83Y^{25} + 90Y^{26} + 91Y^{27} + 95Y^{28} \\
&+ 93Y^{29} + 99Y^{30} + 91Y^{31} + 92Y^{32} + 82Y^{33} + 80Y^{34} + 63Y^{35} + 62Y^{36} \\
&+ 38Y^{37} + 34Y^{38} + 9Y^{39} - 27Y^{41} - 38Y^{42} - 68Y^{43} - 75Y^{44} - 105Y^{45} \\
&- 115Y^{46} - 139Y^{47} - 146Y^{48} - 173Y^{49} - 171Y^{50} - 195Y^{51} - 188Y^{52} \\
&- 206Y^{53} - 194Y^{54} - 206Y^{55} - 188Y^{56} - 195Y^{57} - 171Y^{58} - 173Y^{59} \\
&- 146Y^{60} - 139Y^{61} - 115Y^{62} - 105Y^{63} - 75Y^{64} - 68Y^{65} - 38Y^{66} - 27Y^{67} \\
&+ 9Y^{69} + 34Y^{70} + 38Y^{71} + 62Y^{72} + 63Y^{73} + 80Y^{74} + 82Y^{75} + 92Y^{76} \\
&+ 91Y^{77} + 99Y^{78} + 93Y^{79} + 95Y^{80} + 91Y^{81} + 90Y^{82} + 83Y^{83} + 81Y^{84} \\
&+ 72Y^{85} + 70Y^{86} + 59Y^{87} + 56Y^{88} + 48Y^{89} + 44Y^{90} + 37Y^{91} + 34Y^{92} \\
&+ 27Y^{93} + 24Y^{94} + 19Y^{95} + 16Y^{96} + 13Y^{97} + 10Y^{98} + 8Y^{99} + 7Y^{100} \\
&+ 4Y^{101} + 4Y^{102} + 2Y^{103} + 3Y^{104} + 2Y^{106} + Y^{108}.
\end{aligned}
$$

A.11 $\mathfrak{tr}_7(\mathbb{Z})$, Counting Ideals

The following polynomial is $W^{\triangleleft}_{\mathfrak{tr}_7(\mathbb{Z})}(Y)$ mentioned on p. 78:

$$
\begin{aligned}
&1 + 3Y^2 + 5Y^4 + 3Y^5 + 7Y^6 + 9Y^7 + 13Y^8 + 18Y^9 + 25Y^{10} + 32Y^{11} \\
&+ 44Y^{12} + 56Y^{13} + 75Y^{14} + 94Y^{15} + 125Y^{16} + 153Y^{17} + 199Y^{18} + 242Y^{19} \\
&+ 305Y^{20} + 367Y^{21} + 459Y^{22} + 545Y^{23} + 673Y^{24} + 793Y^{25} + 958Y^{26} \\
&+ 1124Y^{27} + 1337Y^{28} + 1553Y^{29} + 1834Y^{30} + 2106Y^{31} + 2458Y^{32} \\
&+ 2806Y^{33} + 3228Y^{34} + 3656Y^{35} + 4172Y^{36} + 4668Y^{37} + 5290Y^{38} \\
&+ 5867Y^{39} + 6573Y^{40} + 7245Y^{41} + 8028Y^{42} + 8767Y^{43} + 9642Y^{44} \\
&+ 10421Y^{45} + 11360Y^{46} + 12183Y^{47} + 13136Y^{48} + 13963Y^{49} + 14921Y^{50} \\
&+ 15683Y^{51} + 16609Y^{52} + 17279Y^{53} + 18089Y^{54} + 18627Y^{55} + 19271Y^{56} \\
&+ 19582Y^{57} + 20023Y^{58} + 20038Y^{59} + 20192Y^{60} + 19882Y^{61} + 19663Y^{62} \\
&+ 18961Y^{63} + 18352Y^{64} + 17163Y^{65} + 16125Y^{66} + 14444Y^{67} + 12905Y^{68} \\
&+ 10732Y^{69} + 8700Y^{70} + 5995Y^{71} + 3517Y^{72} + 305Y^{73} - 2612Y^{74} \\
&- 6241Y^{75} - 9546Y^{76} - 13535Y^{77} - 17095Y^{78} - 21361Y^{79} - 25071Y^{80}
\end{aligned}
$$

$$
\begin{aligned}
&- 29441Y^{81} - 33196Y^{82} - 37522Y^{83} - 41121Y^{84} - 45290Y^{85} - 48557Y^{86}\\
&- 52361Y^{87} - 55180Y^{88} - 58427Y^{89} - 60607Y^{90} - 63191Y^{91} - 64544Y^{92}\\
&- 66322Y^{93} - 66778Y^{94} - 67583Y^{95} - 67068Y^{96} - 66871Y^{97} - 65267Y^{98}\\
&- 64071Y^{99} - 61396Y^{100} - 59142Y^{101} - 55484Y^{102} - 52239Y^{103}\\
&- 47622Y^{104} - 43560Y^{105} - 38095Y^{106} - 33306Y^{107} - 27241Y^{108}\\
&- 21857Y^{109} - 15362Y^{110} - 9666Y^{111} - 2883Y^{112} + 2883Y^{113} + 9666Y^{114}\\
&+ 15362Y^{115} + 21857Y^{116} + 27241Y^{117} + 33306Y^{118} + 38095Y^{119}\\
&+ 43560Y^{120} + 47622Y^{121} + 52239Y^{122} + 55484Y^{123} + 59142Y^{124}\\
&+ 61396Y^{125} + 64071Y^{126} + 65267Y^{127} + 66871Y^{128} + 67068Y^{129}\\
&+ 67583Y^{130} + 66778Y^{131} + 66322Y^{132} + 64544Y^{133} + 63191Y^{134}\\
&+ 60607Y^{135} + 58427Y^{136} + 55180Y^{137} + 52361Y^{138} + 48557Y^{139}\\
&+ 45290Y^{140} + 41121Y^{141} + 37522Y^{142} + 33196Y^{143} + 29441Y^{144}\\
&+ 25071Y^{145} + 21361Y^{146} + 17095Y^{147} + 13535Y^{148} + 9546Y^{149}\\
&+ 6241Y^{150} + 2612Y^{151} - 305Y^{152} - 3517Y^{153} - 5995Y^{154} - 8700Y^{155}\\
&- 10732Y^{156} - 12905Y^{157} - 14444Y^{158} - 16125Y^{159} - 17163Y^{160}\\
&- 18352Y^{161} - 18961Y^{162} - 19663Y^{163} - 19882Y^{164} - 20192Y^{165}\\
&- 20038Y^{166} - 20023Y^{167} - 19582Y^{168} - 19271Y^{169} - 18627Y^{170}\\
&- 18089Y^{171} - 17279Y^{172} - 16609Y^{173} - 15683Y^{174} - 14921Y^{175}\\
&- 13963Y^{176} - 13136Y^{177} - 12183Y^{178} - 11360Y^{179} - 10421Y^{180}\\
&- 9642Y^{181} - 8767Y^{182} - 8028Y^{183} - 7245Y^{184} - 6573Y^{185} - 5867Y^{186}\\
&- 5290Y^{187} - 4668Y^{188} - 4172Y^{189} - 3656Y^{190} - 3228Y^{191} - 2806Y^{192}\\
&- 2458Y^{193} - 2106Y^{194} - 1834Y^{195} - 1553Y^{196} - 1337Y^{197} - 1124Y^{198}\\
&- 958Y^{199} - 793Y^{200} - 673Y^{201} - 545Y^{202} - 459Y^{203} - 367Y^{204}\\
&- 305Y^{205} - 242Y^{206} - 199Y^{207} - 153Y^{208} - 125Y^{209} - 94Y^{210} - 75Y^{211}\\
&- 56Y^{212} - 44Y^{213} - 32Y^{214} - 25Y^{215} - 18Y^{216} - 13Y^{217} - 9Y^{218}\\
&- 7Y^{219} - 3Y^{220} - 5Y^{221} - 3Y^{223} - Y^{225}.
\end{aligned}
$$

B

Factorisation of Polynomials Associated to Classical Groups

In this appendix we are concerned with the proof of Theorem 6.9. The proof depends on extending the following classical identity on root systems: let w_i be the reflection in the root defined by α_i, then

$$\lambda(w_i w) = \begin{cases} \lambda(w) + 1 & \text{if } w^{-1}(\alpha_i) \in \Phi^+ \ , \\ \lambda(w) - 1 & \text{if } w^{-1}(\alpha_i) \in \Phi^- \ . \end{cases}$$

To explain our generalisation to the root systems $X_l = C_l$ or D_l, we set up some notation. Let Φ_{k+1} be the sub-root system generated by $\{\alpha_{l-k}, \ldots, \alpha_l\}$ of type X_{k+1}. Let $w_{\Phi_{k+1}}$ be the element sending Φ^+_{k+1} to Φ^-_{k+1}.

Let us recall the structure of the root systems C_l and D_l and their corresponding Weyl groups. Let $\mathbf{e}_i$ be the standard basis for the l-dimensional vector space $\mathbb{R}^l$.

$C_l^+ = \{ 2\mathbf{e}_i, \mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le l \}$ with simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \ldots, \alpha_{l-1} = \mathbf{e}_{l-1} - \mathbf{e}_l, \alpha_l = 2\mathbf{e}_l$. $W(C_l)$ is the semi-direct product of the symmetric group on $\mathbf{e}_i$ and the group $(\mathbb{Z}/2\mathbb{Z})^l$ operating by $\mathbf{e}_i \mapsto (\pm 1)_i \mathbf{e}_i$.

$D_l^+ = \{ \mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le l \}$ with simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \ldots, \alpha_{l-1} = \mathbf{e}_{l-1} - \mathbf{e}_l, \alpha_l = \mathbf{e}_{l-1} + \mathbf{e}_l$. $W(D_l)$ is the semi-direct product of the symmetric group on $\mathbf{e}_i$ and the group $(\mathbb{Z}/2\mathbb{Z})^{l-1}$ operating by $\mathbf{e}_i \mapsto (\pm 1)_i \mathbf{e}_i$ with $\prod_i (\pm 1)_i = 1$.

We shall write $w = \pi_w \sigma_w$ where π_w is the permutation and σ_w is the sign change (where we employ the convention that we implement the sign change followed by the permutation). For each $w \in W$, let $w(k)$ be the permutation of $\mathbf{e}_{\pi_{w^{-1}}(i)}$ for $i = k, \ldots, l$ which alters the order. For $k = 1, \ldots, r+1$ let

$$\begin{aligned} W(k) &= \left\{ \begin{array}{c} w = \pi_w \sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}} w w(k))^{-1}(\alpha_{k-1}) \\ \text{have the same sign and } (\sigma_{w^{-1}})_k = 1 \end{array} \right\} \\ &\qquad \cup \left\{ \begin{array}{c} w = \pi_w \sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}} w w(k))^{-1}(\alpha_{k-1}) \\ \text{have opposite signs and } (\sigma_{w^{-1}})_k = -1 \end{array} \right\} \\ &= W(k)^+ \cup W(k)^- \end{aligned}$$

and

$$\begin{aligned} J_k^-(w) &= \{\, j : k \le j \le r, (\sigma_{w^{-1}})_j = 1, (\sigma_{w^{-1}})_{j+1} = -1 \,\} \ , \\ J_k^+(w) &= \{\, j : k \le j \le r, (\sigma_{w^{-1}})_j = -1, (\sigma_{w^{-1}})_{j+1} = 1 \,\} \ . \end{aligned}$$

Note that we put $W(1) = \{\, w = \pi_w \sigma_w : (\sigma_{w^{-1}})_1 = 1 \,\} = W(1)^+$ since there is no α_0.

Theorem B.1. *For $k = 1, \dots, r+1$,*

1. *The map $w \mapsto w_{\Phi_{l-k+1}} w w(k)$ is a bijection from $W(k)$ to $W \setminus W(k)$;*
2. *If $w \in W(k)^+$ then*

$$\begin{aligned} &\lambda(w_{\Phi_{l-k+1}} w w(k)) \\ &= \lambda(w) - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + \sum_{j \in J_k^-(w)} b_j + (\sigma_{w^{-1}})_{r+1} b_{r+1} \ ; \end{aligned}$$

3. *If $w \in W(k)^-$ then*

$$\begin{aligned} &\lambda(w_{\Phi_{l-k+1}} w w(k)) \\ &= \lambda(w) - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + b_{k-1} + \sum_{j \in J_k^-(w)} b_j + (\sigma_{w^{-1}})_{r+1} b_{r+1} \ . \end{aligned}$$

Note that part 1 implies that parts 2 and 3 can be used to provide an identity valid on the whole of W. Although complicated, taking $X_l = C_l$ and $k = l$ reduces to the classical identity for $i = l$. To see this note that $J_l^+(w) = J_l^-(w) = \varnothing$, $b_l - b_{l-1}/2 = 1$, and $w_{\Phi_{l-k+1}} w w(k) = w_l w$.

Having set up this notation, we can extend Theorem 6.9 to describe more precisely the factorisation:

Theorem B.2. *If $G = \mathrm{GSp}_{2l}$ of type C_l or $G = \mathrm{GO}_{2l}^+$ of type D_l then for $k = 1, \dots, r+1$*

$$\begin{aligned} P_{G,\rho}(X,Y) &= (1 + X^{b_{k-1}/2} Y) \left(\sum_{w \in W(k)} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \right) \\ &= (1+Y) \left(\prod_{i=1}^{r} (1 + X^{b_i/2} Y) \right) R_G(X,Y) \ , \end{aligned}$$

where

$$R_G(X,Y) = \left(\sum_{w \in \tilde{W}} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \right)$$

and

$$\tilde{W} = \bigcap_{k=1}^{r+1} W(k) \ .$$

It is important therefore in establishing natural boundaries to remove the cyclotomic factors and provide a description of the resulting polynomial. This is precisely the goal of Theorem B.2 in the case of $P_{\mathrm{GSp}_{2l}}(X,Y)$ and $P_{\mathrm{GO}_{2l}^+}(X,Y)$. In this appendix we establish the following:

Theorem B.3. *If* $G = \mathrm{GSp}_{2l}$ *of type* C_l *or* $G = \mathrm{GO}_{2l}^+$ *of type* D_l *then* $P_G(X,Y)$ *has a factor of the form*

$$(1+Y)\prod_{i=1}^{r}(1+X^{b_i/2}Y)\ ,$$

where $r = l-1$ *for* $G = \mathrm{GSp}_{2l}$ *and* $r = l-2$ *for* $G = \mathrm{GO}_{2l}^+$.

Proof. For convenience, let us use the notation that $b_0 = 0$. Let X_l denote either the Dynkin diagram C_l or D_l. We shall use the following identities: for C_l we have

$$b_l - \mathrm{card}(C_{k+1}^+) + \mathrm{card}(A_k^+) = b_{l-(k+1)}/2 \text{ for } k = 0,\dots,l-1\ ,$$

for D_l we have

$$\begin{aligned} b_l - \mathrm{card}(D_{k+1}^+) + \mathrm{card}(A_k^+) &= b_{l-(k+1)}/2 \text{ for } k = 2,\dots,l-1\ ,\\ b_l - \mathrm{card}(D_1^+) &= b_{l-2}/2\ . \end{aligned}$$

The element $w_{\Phi_{k+1}}$ is the sign change $\mathbf{e}_i \mapsto \mathbf{e}_i$ for $i = 1,\dots,l-k-1$ and $\mathbf{e}_i \mapsto -\mathbf{e}_i$ for $i = l-k,\dots,r+1$ (note that in the case of D_l this then determines the sign change $\mathbf{e}_l$, namely $\mathbf{e}_l \mapsto (-1)^k\mathbf{e}_l$.) For each $w \in W$, let $w(k)$ be the permutation of $\mathbf{e}_{\pi_{w^{-1}}(i)}$ for $i = k,\dots,l$ which alters the order.

For $k = 1,\dots,r+1$ let

$$\begin{aligned} W(k) &= \left\{ \begin{matrix} w = \pi_w\sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}}ww(k))^{-1}(\alpha_{k-1}) \\ \text{have the same sign and } (\sigma_{w^{-1}})_k = 1 \end{matrix} \right\} \\ &\quad \cup \left\{ \begin{matrix} w = \pi_w\sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}}ww(k))^{-1}(\alpha_{k-1}) \\ \text{have opposite signs and } (\sigma_{w^{-1}})_k = -1 \end{matrix} \right\} \\ &= W(k)^+ \cup W(k)^-\ . \end{aligned}$$

Note that we shall put $W(1) = \{\, w = \pi_w\sigma_w : (\sigma_{w^{-1}})_1 = 1 \,\} = W(1)^+$ since there is no α_0. The point is that things are going to work out because this means that in the second case actually it forces $\alpha_{k-1} \in w(\Phi_l^+)$. We're trying to divide W up into two pieces so that $w \mapsto w_{\Phi_{l-k+1}}ww(k)$ is a bijection and the difference in the polynomial is effected by multiplication by $X^{b_{k-1}/2}Y$.

Then the claim is that $w \mapsto w_{\Phi_{l-k+1}}ww(k)$ is a bijection between $W(k)$ and $W \setminus W(k)$.

Note first of all that $w_{\Phi_{l-k+1}}(w_{\Phi_{l-k+1}}ww(k))(w_{\Phi_{l-k+1}}ww(k))(k) = w$, since $(w_{\Phi_{l-k+1}}ww(k))(k) = w(k)$. Secondly, since $w(k)$ is just a permutation and

$w_{\Phi_{l-k+1}}$ changes the sign of $\mathbf{e}_k$, $(\sigma_{w^{-1}})_k = -(\sigma_{(w_{\Phi_{l-k+1}} ww(k))^{-1}})_k$. Hence $w \mapsto w_{\Phi_{l-k+1}} ww(k)$ maps $W(k)$ into $W \setminus W(k)$ and also maps $W \setminus W(k)$ into $W(k)$. It is straightforward to see, using this second map, that $w \mapsto w_{\Phi_{l-k+1}} ww(k)$ is then a bijection between $W(k)$ and $W \setminus W(k)$.

We claim now that the correspondence $w \mapsto w_{\Phi_{l-k+1}} ww(k)$ behaves in the following manner:

$$
\begin{aligned}
&X^{b_{k-1}/2} Y \left(X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \right) \\
&= X^{-\lambda(w_{\Phi_{l-k+1}} ww(k))} \prod_{\alpha_j \in w_{\Phi_{l-k+1}} ww(k)(\Phi^-)} X^{b_j} Y^{c_j} \ .
\end{aligned}
$$

Let

$$
\begin{aligned}
J_k^-(w) &= \{\, j : k \le j \le r, (\sigma_{w^{-1}})_j = 1, (\sigma_{w^{-1}})_{j+1} = -1 \,\} \ , \\
J_k^+(w) &= \{\, j : k \le j \le r, (\sigma_{w^{-1}})_j = -1, (\sigma_{w^{-1}})_{j+1} = 1 \,\} \ .
\end{aligned}
$$

Then divide $J(w) = \left\{\, j \le r : w^{-1}\alpha_j \in \Phi^- \,\right\}$ into $J_k^+(w)$ and its complement $J(w) \setminus J_k^+(w)$. The first claim is then that for $w \in W(k)^+$

$$
J(w_{\Phi_{l-k+1}} ww(k)) = (J(w) \setminus J_k^+(w)) \cup J_k^-(w)
$$

and for $w \in W(k)^-$,

$$
J(w_{\Phi_{l-k+1}} ww(k)) = (J(w) \setminus J_k^+(w)) \cup J_k^-(w) \cup \{k-1\} \ .
$$

For $1 \le j \le r$,

$$
w^{-1}\alpha_j = w^{-1}(\mathbf{e}_j - \mathbf{e}_{j+1}) = (\sigma_{w^{-1}})_j \, \mathbf{e}_{\pi_{w^{-1}}(j)} - (\sigma_{w^{-1}})_{j+1} \mathbf{e}_{\pi_{w^{-1}}(j+1)} \ .
$$

So firstly $J(w) \supset J_k^+(w)$ and $J(w) \cap J_k^-(w) = \varnothing$.

For $k \le j \le r$,

$$
\begin{aligned}
&\left(w_{\Phi_{l-k+1}} ww(k)\right)^{-1} \alpha_j \\
&= -\left(\sigma_{w^{-1}}\right)_j w(k) \left(\mathbf{e}_{\pi_{w^{-1}}(j)}\right) + \left(\sigma_{w^{-1}}\right)_{j+1} w(k) \left(\mathbf{e}_{\pi_{w^{-1}}(j+1)}\right) \ .
\end{aligned}
$$

If $\left(\sigma_{w^{-1}}\right)_j = -\left(\sigma_{w^{-1}}\right)_{j+1}$, (i.e. $j \in J_k^+(w) \cup J_k^-(w)$) then $w^{-1}\alpha_j \in \Phi^-$ if and only if $\left(w_{\Phi_{l-k+1}} ww(k)\right)^{-1} \alpha_j \notin \Phi^-$. If $\left(\sigma_{w^{-1}}\right)_j = \left(\sigma_{w^{-1}}\right)_{j+1}$, then

$$
\left(\sigma_{w^{-1}}\right)_j \mathbf{e}_{\pi_{w^{-1}}(j)} - \left(\sigma_{w^{-1}}\right)_{j+1} \mathbf{e}_{\pi_{w^{-1}}(j+1)} = \left(\sigma_{w^{-1}}\right)_j \left(\mathbf{e}_{\pi_{w^{-1}}(j)} - \mathbf{e}_{\pi_{w^{-1}}(j+1)}\right) .
$$

The point of using $w(k)$ now comes into effect because

$$
\left(w_{\Phi_{l-k+1}} ww(k)\right)^{-1} \alpha_j \quad = \left(\sigma_{w^{-1}}\right)_j \left(-w(k)\mathbf{e}_{\pi_{w^{-1}}(j)} + w(k)\mathbf{e}_{\pi_{w^{-1}}(j+1)}\right)
$$

will have the same sign as $w^{-1}\alpha_j$. This is because $\mathbf{e}_{i_1} - \mathbf{e}_{i_2} \in \Phi^-$ if and only if $i_2 < i_1$ and $w(k)$ has the effect of altering the order of $\pi_{w^{-1}}(i)$ for $i = k, \dots, l$.

$w \mapsto w_{\Phi_{l-k+1}} w w(k)$ has no effect on those $j < k-1$ since $w^{-1}\alpha_j = \left(w_{\Phi_{l-k+1}} w w(k)\right)^{-1} \alpha_j$.

The only root we haven't taken account of is $\alpha_{k-1} = \mathbf{e}_{k-1} - \mathbf{e}_k$. If $w \in W(k)^+$ then we are assuming that $k-1 \in J(w)$ if and only if $k-1 \in J(w_{\Phi_{l-k+1}} w w(k))$. So the only issue here is that if $w \in W(k)^-$, then $w^{-1}\alpha_{k-1} \notin \Phi^-$, i.e. $k-1 \notin J(w)$. Then by definition of $W(k)^-$, $k-1 \in J(w_{\Phi_{l-k+1}} w w(k))$. Now

$$\begin{aligned} w^{-1}\alpha_{k-1} &= (\sigma_{w^{-1}})_{k-1}\, \mathbf{e}_{\pi_{w^{-1}}(k-1)} - (\sigma_{w^{-1}})_k\, \mathbf{e}_{\pi_{w^{-1}} k} \\ \left(w_{\Phi_{l-k+1}} w w(k)\right)^{-1} \alpha_{k-1} &= (\sigma_{w^{-1}})_{k-1}\, \mathbf{e}_{\pi_{w^{-1}}(k-1)} + (\sigma_{w^{-1}})_k\, w(k) \left(\mathbf{e}_{\pi_{w^{-1}} k}\right) \end{aligned}$$

Then $w \in W(k)^-$ (i.e. that these two elements have different signs) implies that the sign of $w^{-1}\alpha_{k-1}$ is $-(\sigma_{w^{-1}})_k = 1$, by definition of $W(k)^-$.

We start with the case C_l. For ease of notation, set $\Phi^1 = \Phi^+$, $\Phi^{-1} = \Phi^-$. Let us suppose first that $w \in W(k)^+$. We have to prove that:

$$\lambda(w_{\Phi_{l-k+1}} w w(k)) = \lambda(w) - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + \sum_{j \in J_k^-(w)} b_j + \varepsilon_w b_l\ , \quad \text{(B.1)}$$

where $w^{-1}\alpha_l \in \Phi^{\varepsilon_w}$ and $\varepsilon_w \in \{\pm 1\}$. Notice that the powers of Y are correct since if $\varepsilon_w = 1$, then $\operatorname{card} J_k^+(w) = \operatorname{card} J_k^-(w)$ (look at the string of signs in $\sigma_{w^{-1}}$ from k to l which by hypothesis begins and ends with $+$, then $\operatorname{card} J_k^+(w)$ is the number of sign changes $-$ to $+$, and $\operatorname{card} J_k^-(w)$ is the number of sign changes $+$ to $-$). Then the degree of Y in the monomial corresponding to w is $2 \operatorname{card} J(w)$ and to $w_{\Phi_{l-k+1}} w w(k)$ is

$$\begin{aligned} 2 \operatorname{card} J(w_{\Phi_{l-k+1}} w w(k)) + 1 &= 2 \operatorname{card} \left((J(w) \setminus J_k^+(w)) \cup J_k^-(w)\right) + 1 \\ &= 2 \operatorname{card} J(w) + 1\ . \end{aligned}$$

If $\varepsilon_w = -1$, then $\operatorname{card} J_k^-(w) = \operatorname{card} J_k^+(w) - 1$, and the degree of Y in the monomial corresponding to w is $2 \operatorname{card} J(w) + 1$ and to $w_{\Phi_{l-k+1}} w w(k)$ is

$$\begin{aligned} 2 \operatorname{card} J(w_{\Phi_{l-k+1}} w w(k)) &= 2 \operatorname{card} \left((J(w) \setminus J_k^+(w)) \cup J_k^-(w)\right) \\ &= 2 \left(\operatorname{card} J(w) + 1\right) \\ &= 2 \operatorname{card} J(w) + 2\ . \end{aligned}$$

Recall that the length of a word is the number of positive roots sent to negative roots by that word. It is the same as the length of its inverse. We look first at the effect of w^{-1} and $(w_{\Phi_{l-k+1}} w w(k))^{-1}$ on $\mathbf{e}_i \pm \mathbf{e}_j$ for $k \le i \le j \le l$. Define

$$K_i(w) = \left\{\, \alpha = \mathbf{e}_i \pm \mathbf{e}_j : i \le j \le l, w^{-1}(\alpha) \in \Phi^- \,\right\}\ .$$

Lemma B.4. *If $i \geq k$ and $(\sigma_{w^{-1}})_i = \varepsilon_i$ then*

$$\operatorname{card} K_i(w) = \operatorname{card} K_i(w_{\Phi_{l-k+1}} w w(k)) + \varepsilon_i(l-i+1)\ .$$

Proof. The point here is that in $w^{-1}(\mathbf{e}_i \pm \mathbf{e}_j) = \varepsilon_i \mathbf{e}_{i'} \pm \mathbf{e}_{j'}$ there is always one root with both signs of the basis elements equal to ε_i, and one with alternate signs. As we have explained the first root then changes sign under $(w_{\Phi_{l-k+1}} w w(k))^{-1}$ whilst the second retains its sign. So if $(\sigma_{w^{-1}})_i = \varepsilon_i$ then there are $l-i+1$ roots (including $2\mathbf{e}_i$) which get mapped by w^{-1} into Φ^{ε_i} but get mapped by $(w_{\Phi_{l-k+1}} w w(k))^{-1}$ into $-\Phi^{\varepsilon_i}$; the other $l-i$ roots will keep the same sign. □

So in the roots $\mathbf{e}_i \pm \mathbf{e}_j$ for $k \leq i \leq j \leq l$ we get a change of length

$$\sum_{i=k}^{l} \varepsilon_i(l-i+1) = \sum_{i=k}^{l}(l-i+1) - 2\sum_{k \leq i, \varepsilon_i = -1}(l-i+1)\ .$$

Now, $b_l - b_{k-1}/2 = \sum_{i=1}^{l}(l-i+1) - \sum_{i=1}^{k-1}(l-i+1) = \sum_{i=k}^{l}(l-i+1)$. Also we have

$$\begin{aligned} &2\sum_{k \leq i, \varepsilon_i = -1}(l-i+1) \\ &= \sum_{j \in J_k^+(w)} 2\sum_{i=1}^{j}(l-i+1) - \sum_{j \in J_k^-(w)} 2\sum_{i=1}^{j}(l-i+1) + 2\delta\sum_{i=1}^{l}(l-i+1)\ , \end{aligned}$$

where $\delta = 0$ if $\varepsilon_w = 1$ and $\delta = 1$ if $\varepsilon_w = -1$. One can see this by looking at the string of +s and −s. A string of −s starts at a $j_1 + 1$ where $j_1 \in J_k^-(w)$ and ends at a j_2 where $j_2 \in J_k^+(w)$. If the last term in the string is a − then since $l \notin J(w)$ we need to add the last term as appropriate. But

$$\begin{aligned} &\sum_{j \in J_k^+(w)} 2\sum_{i=1}^{j}(l-i+1) - \sum_{j \in J_k^-(w)} 2\sum_{i=1}^{j}(l-i+1) + 2\delta\sum_{i=1}^{l}(l-i+1) \\ &= \sum_{j \in J_k^+(w)} b_j - \sum_{j \in J_k^-(w)} b_j + 2\delta b_l\ . \end{aligned}$$

Hence we have got a contribution to the change in length between w and $w_{\Phi_{l-k+1}} w w(k)$ by looking at the roots $\mathbf{e}_i \pm \mathbf{e}_j$ for $k \leq i \leq j \leq l$ of

$$b_l - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + \sum_{j \in J_k^-(w)} b_j - 2\delta b_l\ .$$

So our claim is that the other roots don't contribute any change in length. That is certainly true of $\mathbf{e}_i \pm \mathbf{e}_j$ for $i \leq j \leq k-1$ since the elements w^{-1} and $\left(w_{\Phi_{l-k+1}} w w(k)\right)^{-1}$ act in the same way on these roots.

The last case where $i \le k-1 < j$, if $\mathbf{e}_{\pi_{w^{-1}}(j)}$ and $w(k)\mathbf{e}_{\pi_{w^{-1}}(j)}$ are both on the same side of i then there is no change in the number of roots being sent to negative roots. If however they are on different sides then to see that there is no change in the number of positive roots changing sign we have to consider the four positive roots $\mathbf{e}_i \pm \mathbf{e}_j$ and $\mathbf{e}_i \pm \mathbf{e}_{\pi_{w^{-1}}(j)}$ if $\pi_{w^{-1}}(j) > i$ (and otherwise $\mathbf{e}_i \pm \mathbf{e}_j$ and $\mathbf{e}_i \pm w(k)\mathbf{e}_{\pi_{w^{-1}}(j)}$).

Let us suppose now that $w \in W(k)^-$. We have to prove that:

$$\lambda(w_{\Phi_{l-k+1}} w w(k)) = \lambda(w) - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + b_{k-1} + \sum_{j \in J_k^-(w)} b_j + \varepsilon_w b_l \,. \tag{B.2}$$

Check first that the powers of Y match up again. If $\varepsilon_w = 1$, then $\operatorname{card} J_k^+(w) - 1 = \operatorname{card} J_k^-(w)$ (look at the string of signs in $\sigma_{w^{-1}}$ from k to l which by hypothesis begins with $-$ and ends with $+$, then $\operatorname{card} J_k^+(w)$ is the number of sign changes $-$ to $+$, and $\operatorname{card} J_k^-(w)$ is the number of sign changes $+$ to $-$). Then the degree of Y in the monomial corresponding to w is $2\operatorname{card} J(w)$ and to $w_{\Phi_{l-k+1}} w w(k)$ is

$$\begin{aligned}
&2\operatorname{card} J(w_{\Phi_{l-k+1}} w w(k)) + 1 \\
&= 2\operatorname{card}\big((J(w) \setminus J_k^+(w)) \cup J_k^-(w) \cup \{k-1\}\big) + 1 \\
&= 2\operatorname{card} J(w) + 1 \,.
\end{aligned}$$

If $\varepsilon_w = -1$, then $\operatorname{card} J_k^-(w) = \operatorname{card} J_k^+(w)$, and the degree of Y in the monomial corresponding to w is $2\operatorname{card} J(w) + 1$ and to $w_{\Phi_{l-k+1}} w w(k)$ is

$$\begin{aligned}
2\operatorname{card} J(w_{\Phi_{l-k+1}} w w(k)) &= 2\operatorname{card}\big((J(w) \setminus J_k^+(w)) \cup J_k^-(w) \cup \{k-1\}\big) \\
&= 2\,(\operatorname{card} J(w) + 1) \\
&= 2\operatorname{card} J(w) + 2 \,.
\end{aligned}$$

Again we look at the effect of w^{-1} and $(w_{\Phi_{l-k+1}} w w(k))^{-1}$ on $\mathbf{e}_i \pm \mathbf{e}_j$ for $k \le i \le j \le l$ and with the same argument we get a change of length

$$\sum_{i=k}^{l} \varepsilon_i (l-i+1) = \sum_{i=k}^{l} (l-i+1) - 2 \sum_{k \le i, \varepsilon_i = -1} (l-i+1) \,.$$

Now, this time since the string of $+$'s and $-$'s starts with a $-$ we need to add an extra term to get

$$\begin{aligned}
2 \sum_{k \le i, \varepsilon_i = -1} (l-i+1) &= \sum_{j \in J_k^+(w)} 2 \sum_{i=1}^{j} (l-i+1) - \sum_{j \in J_k^-(w)} 2 \sum_{i=1}^{j} (l-i+1) \\
&\quad - 2 \sum_{i=1}^{k-1} (l-i+1) + 2\delta \sum_{i=1}^{l} (l-i+1) \\
&= \sum_{j \in J_k^+(w)} b_j - \sum_{j \in J_k^-(w)} b_j - b_{k-1} + 2\delta b_l \,,
\end{aligned}$$

where $\delta = 0$ if $\varepsilon_w = 1$ and $\delta = 1$ if $\varepsilon_w = -1$.

Hence we have got a contribution to the change in length between w and $w_{\Phi_{l-k+1}} ww(k)$ by looking at the roots $\mathbf{e}_i \pm \mathbf{e}_j$ for $k \leq i \leq j \leq l$ of

$$b_l - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + \sum_{j \in J_k^-(w)} b_j - 2\delta b_l \ .$$

The same argument as above shows that the other roots don't contribute to a change in length.

Note that these identities B.1 and B.2 are generalisations of the classical identities:

$$\begin{aligned} \lambda(w_l w) &= \lambda(w) + 1 \text{ if } w^{-1}(\alpha_l) \in \Phi^+ \ , \\ \lambda(w_l w) &= \lambda(w) - 1 \text{ if } w^{-1}(\alpha_l) \in \Phi^- \ . \end{aligned}$$

This establishes the proof of Theorem B.1 detailed in the Introduction.

These identities therefore suffice in the case of C_l to show that our claim that the correspondence $w \mapsto w_{\Phi_{l-k+1}} ww(k)$ behaves in the following manner:

$$\begin{aligned} & X^{b_{k-1}/2} Y \left(X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \right) \\ &= X^{-\lambda(w_{\Phi_{l-k+1}} ww(k))} \prod_{\alpha_j \in w_{\Phi_{l-k+1}} ww(k)(\Phi^-)} X^{b_j} Y^{c_j} \ . \end{aligned}$$

Hence

$$P_G(X,Y) = (1 + X^{b_{k-1}/2} Y) \left(\sum_{w \in W(k)} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \right) \ ,$$

where

$$\begin{aligned} W(k) = & \left\{ \begin{array}{c} w = \pi_w \sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } \left(w_{\Phi_{l-k+1}} ww(k) \right)^{-1} (\alpha_{k-1}) \\ \text{have the same sign and } (\sigma_{w^{-1}})_k = 1 \end{array} \right\} \\ & \cup \left\{ \begin{array}{c} w = \pi_w \sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } \left(w_{\Phi_{l-k+1}} ww(k) \right)^{-1} (\alpha_{k-1}) \\ \text{have opposite signs and } (\sigma_{w^{-1}})_k = -1 \end{array} \right\} \end{aligned}$$

and

$$P_G(X,Y) = (1+Y) \prod_{i=1}^{r} (1 + X^{b_i/2} Y) R_G(X,Y) \ ,$$

where

$$R_G(X,Y) = \left(\sum_{w \in \tilde{W}} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \right)$$

and

$$\tilde{W} = \bigcap_{k=1}^{r+1} W(k) .$$

This concludes the proof of Theorem B.3 for the case of C_l and establishes the description of the resulting factor detailed in B.2.

Next consider the case D_l. We start with looking at the effect of w and $w_{\Phi_{l-k+1}} ww(k)$ on the roots $\mathbf{e}_i \pm \mathbf{e}_j$ for $k \le i < j \le l$. Define again

$$K_i(w) = \{\, \alpha = \mathbf{e}_i \pm \mathbf{e}_j : i < j \le l, w^{-1}(\alpha) \in \Phi^- \,\} .$$

Lemma B.5. *If $i \ge k$ and $(\sigma_{w^{-1}})_i = \varepsilon_i$ then*

$$\operatorname{card} K_i(w) = \operatorname{card} K_i(w_{\Phi_{l-k+1}} ww(k)) + \varepsilon_i (l-i) .$$

If $i < k$ then $\operatorname{card} K_i(w) = \operatorname{card} K_i(w_{\Phi_{l-k+1}} ww(k))$.

The same proof works here with the observation that in D_l we don't have roots $2\mathbf{e}_i$ so our counting arguments for C_l here and elsewhere will generally be effected by a drop of one everywhere.

Note taking $i = l-1$, that this lemma implies in particular for the roots simple α_{l-1} and α_l that we get one more or one less of these roots in the monomial corresponding to $w_{\Phi_{l-k+1}} ww(k)$ according to whether ε_{l-1} is respectively 1 or -1. Note that in the combinatorial data for D_l, $c_{l-1} = c_l = 1$. Therefore the proof that the degree of Y in the monomial corresponding to $w_{\Phi_{l-k+1}} ww(k)$ is one more than that for $w \in W(k)$ is the same as for C_l except that we look just at the string of $+$'s and $-$'s in $\sigma_{w^{-1}}$ from k to $l-1$.

Let us suppose first that $w \in W(k)^+$. We have to prove that:

$$\lambda(w_{\Phi_{l-k+1}} ww(k)) = \lambda(w) - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + \sum_{j \in J_k^-(w)} b_j + \varepsilon_{l-1} b_{l-1} . \tag{B.3}$$

Note that since $b_{l-1} = b_l$, the last term takes account of the change of degree in X corresponding to the action of w and $w_{\Phi_{l-k+1}} ww(k)$ on the roots α_{l-1} and α_l.

By Lemma B.5,

$$\begin{aligned}
\lambda(w_{\Phi_{l-k+1}} ww(k)) - \lambda(w) &= \sum_{i=k}^{l-1} \varepsilon_i \,(l-i) \\
&= \sum_{i=k}^{l-1} (l-i) - 2 \sum_{k \le i < l, \varepsilon_i = -1} (l-i) \\
&= b_{l-1} - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + \sum_{j \in J_k^-(w)} b_j - 2\delta b_{l-1} ,
\end{aligned}$$

where $\delta = 0$ if $\varepsilon_{l-1} = 1$ and $\delta = 1$ if $\varepsilon_{l-1} = -1$. The last equality just follows the same argument as for C_l with the observation that for D_l

$$b_{l-1} = \sum_{i=1}^{l-1}(l-i) \, ,$$
$$b_j = 2\sum_{i=1}^{j}(l-i) \, .$$

If $w \in W(k)^-$ then by a similar adaptation of the argument for C_l one can prove that

$$\begin{aligned}&\lambda(w_{\Phi_{l-k+1}} w w(k)) \\ &= \lambda(w) - b_{k-1}/2 - \sum_{j \in J_k^+(w)} b_j + b_{k-1} + \sum_{j \in J_k^-(w)} b_j + \varepsilon_{l-1} b_{l-1} \, . \end{aligned} \tag{B.4}$$

Again the identities B.3 and B.4 prove that in the case of D_l,

$$P_G(X,Y) = (1+Y)\prod_{i=1}^{r}(1 + X^{b_i/2}Y)R_G(X,Y) \, ,$$

where

$$R_G(X,Y) = \left(\sum_{w \in \tilde{W}} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \right)$$

and

$$\tilde{W} = \bigcap_{k=1}^{r+1} W(k) \, .$$

This concludes the proof of Theorem B.3 for the case of D_l and establishes the description of the resulting factor detailed in B.2. □

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Index

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