A

Tutorial on the Center Manifold Theorem

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A.1 Review of Linear O.D.E's

Let A be an $n \times n$ matrix and consider the Cauchy problem for a linear system of O.D.E's with constant coefficients

$$\dot{x} = Ax\,, \qquad\qquad x(0) = \bar{x}\,. \tag{1.1}$$

The explicit solution (see [P]) can be written as

$$x(t) = e^{tA}\bar{x}\,, \qquad\qquad e^{tA} \doteq \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\,. \tag{1.2}$$

If $B = R^{-1}AR$ for some invertible matrix R, then

$$e^{tA} = e^{tRBR^{-1}} = Re^{tB}R^{-1}.$$

The actual computation of the exponential matrix e^{tA} can thus be carried out by reducing A to a more convenient canonical form B, and then computing e^{tB}. We give here an illustration of this procedure.

Example 1. Assume that A is a 6×6 matrix, with

$$\det(\zeta I - A) \;=\; (\zeta - \lambda)(\zeta - \mu)^3\big(\zeta - (\alpha + i\beta)\big)\big(\zeta - (\alpha - i\beta)\big),$$

so that λ is a simple real eigenvalue, μ is a multiple eigenvalue and $\alpha \pm i\beta$ are a pair of complex conjugate eigenvalues. Assume that the geometric multiplicity

of μ is 1. Then there exists an invertible matrix R that reduces A to the canonical form

$$B = R^{-1}AR = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 1 & 0 & 0 & 0 \\ 0 & 0 & \mu & 1 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 & \beta & \alpha \end{pmatrix}$$

In this case one has

$$e^{tB} = \begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\mu t} & te^{\mu t} & (t^2/2)e^{\mu t} & 0 & 0 \\ 0 & 0 & e^{\mu t} & te^{\mu t} & 0 & 0 \\ 0 & 0 & 0 & e^{\mu t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\alpha t}\cos\beta t & -e^{\alpha t}\sin\beta t \\ 0 & 0 & 0 & 0 & e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t \end{pmatrix}.$$

We say that a subspace $V \subset \mathbb{R}^n$ is **invariant** for the flow of (1.1) if $x \in V$ implies $e^{At}x \in V$ for all $t \in \mathbb{R}$. A natural way to decompose the space $\mathbb{R}^n$ as the sum of three invariant subspaces is now described.

Consider the eigenvalues of A, i.e. the zeroes of the polynomial $p(\zeta) \doteq \det(\zeta I - A)$. These are finitely many points in the complex plane (fig. 1).

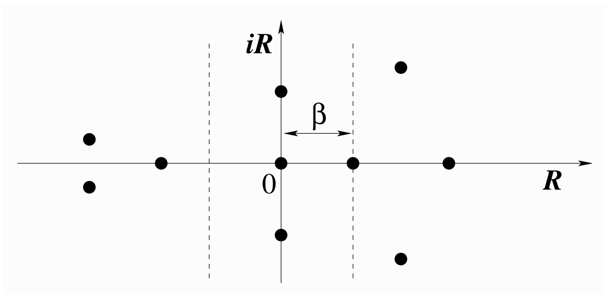

Fig. A.1.

The space $\mathbb{R}^n$ can then be decomposed as the sum of a stable, an unstable and a center subspace, respectively spanned by the (generalized) eigenvectors corresponding to eigenvalues with negative, positive and zero real part. We thus have

$$\mathbb{R}^n = V^s \oplus V^u \oplus V^c$$

with continuous projections

$$\pi_s : \mathbb{R}^n \mapsto V^s\,, \qquad \pi_u : \mathbb{R}^n \mapsto V^u\,, \qquad \pi_c : \mathbb{R}^n \mapsto V^c\,,$$

$$x = \pi_s x + \pi_c x + \pi_u x \,.$$

These projections commute with A and hence with the exponential e^{At} as well:

$$\pi_s e^{At} = e^{At}\pi_s \,, \qquad \pi_u e^{At} = e^{At}\pi_u \,, \qquad \pi_c e^{At} = e^{At}\pi_c \,. \tag{1.3}$$

In particular, these subspaces are invariant for the flow of (1.1). Defining the **spectral gap** of A as

$$\beta \doteq \min\big\{|\mathrm{Re}\,\lambda|\,; \quad \lambda \text{ is an eigenvalue with non-zero real part}\big\} \tag{1.4}$$

(see fig. 1), the following key estimates hold. For every $\varepsilon \in\,]0,\beta[$ there exists a constant C_ε such that

$$\begin{array}{lcll} \|e^{At}\pi_s\| & \leq & C_\varepsilon e^{-(\beta-\varepsilon)t} & t \geq 0\,, \\ \|e^{At}\pi_u\| & \leq & C_\varepsilon e^{(\beta-\varepsilon)t} & t \leq 0\,, \\ \|e^{At}\pi_c\| & \leq & C_\varepsilon e^{\varepsilon|t|} & t \in \mathbb{R}\,. \end{array} \tag{1.5}$$

A.2 Statement of the Center Manifold Theorem

Consider a nonlinear O.D.E. having an equilibrium point at the origin, say

$$\dot{x} = f(x)\,, \tag{2.1}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth function with $f(0) = 0$. The trajectory of (2.1) taking the initial value $x(0) = y$ will be denoted as

$$t \mapsto x(t) \doteq \tilde{x}(t,y)\,. \tag{2.2}$$

Calling $A = Df(0)$ the Jacobian matrix of f at the origin, we can write (2.1) in the form

$$\dot{x} = Ax + g(x)\,, \tag{2.3}$$

where $g(0) = 0$, $Dg(0) = 0$. It is reasonable to expect that, in a small neighborhood of the origin, the flow of (2.1) should look like the flow of the corresponding linearized system (1.1). The main result in this direction (fig. 2) is the famous

Theorem (Hartman-Grobman). *Let f be smooth. If all the eigenvalues of the matrix $A \doteq Df(0)$ have non-zero real part, then the flows of (1.1) and (2.1) are equivalent. More precisely, there exists a homeomorphism φ of a neighborhood $\mathbb{N}$ of the origin onto another neighborhood of the origin such that*

$$e^{At}\varphi(y) = \varphi\big(\tilde{x}(t,y)\big)$$

for all y, t such that y, $\tilde{x}(t,y) \in \mathbb{N}$.

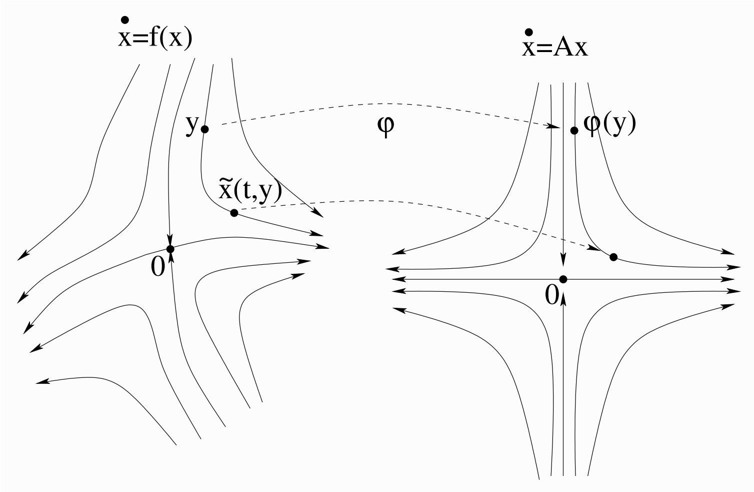

Fig. A.2.

For a proof, see [P]. This theorem settles the case where the center subspace vanishes: $V^c = \{0\}$. The Center Manifold Theorem, on the other hand, applies to the case where V^c is nontrivial. In essence, it says that near the origin all the interesting dynamics takes place on an invariant manifold $\mathcal{M}$, tangent to the center subspace V^c. Its main usefulness lies in this dimensional reduction: instead of studying a flow on the entire space $\mathbb{R}^n$, one can then restrict the analysis to a "center manifold" having the same dimension as V^c.

Center Manifold Theorem. *Let* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *be a vector field in* $\mathcal{C}^{k+1}$ *(here* $k \geq 1$*), with* $f(0) = 0$*. Consider the matrix* $A = Df(0)$*, and let* V^s, V^u, V^c *be the corresponding stable, unstable, center subspaces. Then there exists* $\delta > 0$ *and a local center manifold* $\mathcal{M}$ *with the following properties.*

(i) There exists a $\mathcal{C}^k$ *function* $\phi : V^c \mapsto \mathbb{R}^n$ *with* $\pi_c\, \phi(x_c) = x_c$ *such that*

$$\mathcal{M} = \big\{ \phi(x_c)\ ; \quad x_c \in V^c, \quad |x_c| < \delta \big\}. \tag{2.4}$$

(ii) The manifold $\mathcal{M}$ *is locally invariant for the flow of (2.1), i.e.* $x \in \mathcal{M}$ *implies* $\tilde{x}(t, x) \in \mathcal{M}$ *for* $|t|$ *small.*
(iii) $\mathcal{M}$ *is tangent to* V^c *at the origin.*
(iv) Every globally bounded orbit remaining in a suitably small neighborhood of the origin is entirely contained inside $\mathcal{M}$.
(v) Given any trajectory such that $x(t) \to 0$ *as* $t \to +\infty$*, there exists* $\eta > 0$ *and a trajectory* $t \mapsto y(t) \in \mathcal{M}$ *on the center manifold such that*

$$e^{\eta t}\, \big|x(t) - y(t)\big| \to 0 \qquad \textit{as} \quad t \to +\infty\,. \tag{2.5}$$

Remarks. By (i), the manifold $\mathcal{M}$ is parametrized by points on the center subspace V^c. In particular, it has the same dimension as V^c. The invariance

property (ii) means that the vector field f is tangent to $\mathcal{M}$ at every point $x \in \mathcal{M}$. By (v), every solution which approaches the origin as $t \to +\infty$ can be described as an exponentially small perturbation of some trajectory on the center manifold. An entirely similar statement holds for solutions which approach the origin as $t \to -\infty$. The proof will show that in (2.5) one can choose any constant $\eta \in\,]0, \beta[$ smaller than the spectral gap of A.

A.3 Proof of the Center Manifold Theorem

The proof, mainly following [V], will be given in several steps. Throughout the following, the Landau notation $\mathcal{O}(1)$ will be used to indicate a quantity depending only on the vector field f, whose absolute value remains uniformly bounded.

A.3.1 Reduction to the Case of a Compact Perturbation.

Set $g(x) \doteq f(x) - Ax$. As a first step we show that, by using a cutoff function, one can assume that g has compact support and that its $\mathbb{C}^1$ norm is arbitrarily small. Indeed, let $\rho : \mathbb{R} \mapsto [0,1]$ be a smooth, even function with compact support, such that

$$\rho(\zeta) = \begin{cases} 1 & if \quad |\zeta| \leq 1, \\ 0 & if \quad |\zeta| \geq 2. \end{cases}$$

For $\varepsilon > 0$ small, define the truncated function

$$g_\varepsilon(x) \doteq \rho\big(|x|/\varepsilon\big)\, g(x)\,.$$

Observing that

$$\big|g(x)\big| = \mathcal{O}(1) \cdot |x|^2\,, \qquad\qquad \big|Dg(x)\big| = \mathcal{O}(1) \cdot |x|\,,$$

we obtain

$$\begin{aligned} \big\|g_\varepsilon\big\|_{\mathbb{C}^1} &\leq \sup_{|x|<2\varepsilon} \Big\{ \big|g_\varepsilon(x)\big| + \big|Dg_\varepsilon(x)\big| \Big\} \\ &\leq \sup_{|x|<2\varepsilon} \Big\{ \big|g(x)\big| + \varepsilon^{-1} \Big|\rho'\big(|x|/\varepsilon\big)\Big|\, \big|g(x)\big| + \big|Dg(x)\big| \Big\} \\ &= \mathcal{O}(1) \cdot \varepsilon\,. \end{aligned}$$

By possibly replacing g with g_ε, we can thus assume that $g \in \mathbb{C}_c^{k+1}$ and that $\|g\|_{\mathbb{C}^1}$ is as small as we like. With these assumptions we shall prove the existence of a global center manifold:

$$\mathcal{M} = \big\{\phi(x_c)\ ; \quad x_c \in V^c\big\}\,, \tag{3.1}$$

parametrized by the whole subspace V^c, without any restriction on the size of x_c.

In the general case, the corresponding local properties (i)–(v) can then be easily obtained, observing that g_ε coincides with g for $|x| \le \varepsilon$.

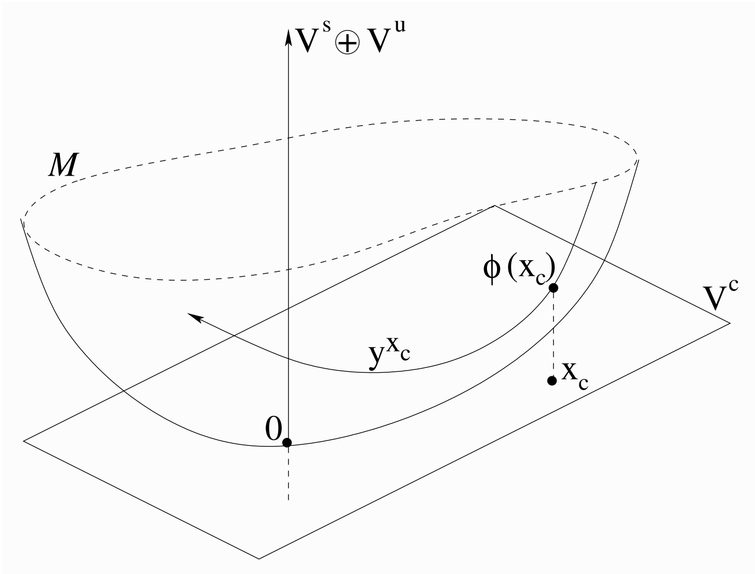

Fig. A.3.

A.3.2 Characterization of the Global Center Manifold.

We now come to the actual construction of points on the center manifold (fig. 3). Let $\beta > 0$ be the spectral gap of the matrix A, as in (1.4). Fix any number $\eta \in]0, \beta[$ and define the space of functions with "slow growth"

$$Y_\eta \doteq \Big\{ y : \mathbb{R} \mapsto \mathbb{R}^n \ ; \quad \big\|y(\cdot)\big\|_\eta \doteq \sup_t e^{-\eta|t|}\big|y(t)\big| < \infty \Big\} .$$

Of course, this implies

$$\big|y(t)\big| \le e^{\eta|t|}\|y\|_\eta . \tag{3.2}$$

For every $x_c \in V^c$ we seek a trajectory $y(\cdot) \in Y_\eta$ such that $\pi_c y(0) = x_c$. The following arguments, based on the Contraction Mapping Principle, will show the existence and uniqueness of such a trajectory.

Any solution $t \mapsto y(t)$ of (2.3) can be represented by the "variation of constant formula"

$$y(t) = e^{A(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} g\big(y(\tau)\big)\, d\tau\,. \tag{3.3}$$

We can decompose (3.3) as a sum of its center, stable and unstable components. Notice that here we can choose different starting times in connection with different components:

$$\begin{aligned} y(t) = \pi_c &\left(e^{A(t-t_c)}y(t_c) + \int_{t_c}^{t} e^{A(t-\tau)} g\big(y(\tau)\big)\, d\tau \right) \\ +\pi_s &\left(e^{A(t-t_s)}y(t_s) + \int_{t_s}^{t} e^{A(t-\tau)} g\big(y(\tau)\big)\, d\tau \right) \\ +\pi_u &\left(e^{A(t-t_u)}y(t_u) + \int_{t_u}^{t} e^{A(t-\tau)} g\big(y(\tau)\big)\, d\tau \right)\,. \end{aligned} \tag{3.4}$$

We now choose $t_c = 0$ and let $t_s \to -\infty$ while $t_u \to +\infty$. Recalling that e^{At} commutes with all three projections π_c, π_s, π_u, from (3.4) and the assumptions $y(0) = x_c$, $y(\cdot) \in Y_\eta$ we obtain

$$\begin{aligned} y(t) = e^{At}x_c &+ \int_0^t e^{A(t-\tau)}\pi_c g\big(y(\tau)\big)\, d\tau \\ &+ \int_{-\infty}^{t} e^{A(t-\tau)}\pi_s\, g\big(y(\tau)\big)\, d\tau - \int_t^{\infty} e^{A(t-\tau)}\pi_u\, g\big(y(\tau)\big)\, d\tau\,. \end{aligned} \tag{3.5}$$

Indeed, for fixed t, as $t_u \to \infty$ by (1.5) and (3.2) we have

$$\lim_{t_u\to\infty} \left|e^{A(t-t_u)}\pi_u\, y(t_u)\right| \le \lim_{t_u\to\infty} C_\varepsilon e^{(\beta-\varepsilon)(t-t_u)} e^{\eta|t_u|}\|y\|_\eta = 0\,.$$

Similarly, as $t_s \to -\infty$ we have

$$\lim_{t_s\to-\infty} \left|e^{A(t-t_s)}\pi_u\, y(t_s)\right| \le \lim_{t_s\to-\infty} C_\varepsilon e^{-(\beta-\varepsilon)(t-t_s)} e^{\eta|t_s|}\|y\|_\eta = 0\,.$$

Remark. The representation (3.3) is useful in connection with a Cauchy problem, with data assigned at time $t = t_0$. On the other hand, one can regard (3.5) as representing the solution to a three-point boundary value problem. We are here assigning the center component $\pi_c y(0) = x_c$ at time $t = 0$, while (the asymptotic behavior of) the stable component $\pi_s\, y$ is prescribed at time $t = -\infty$ and (the asymptotic behavior of) the unstable component $\pi_u\, y$ is prescribed at time $t = +\infty$.

A.3.3 Construction of the Center Manifold.

For each $x_c \in V^c$, a unique solution $y(\cdot) \in Y_\eta$ of (3.4) will be obtained by the contraction mapping principle. Define the map $\Gamma : V^c \times Y_\eta \mapsto Y_\eta$ by setting

$$
\begin{aligned}
\Gamma\big(x_c, y\big)(t) &= e^{At}x_c + \int_0^t e^{A(t-\tau)}\pi_c g\big(y(\tau)\big)\,d\tau \\
&+ \textstyle\int_{-\infty}^t e^{A(t-\tau)}\pi_s\, g\big(y(\tau)\big)\,d\tau - \int_t^\infty e^{A(t-\tau)}\pi_u\, g\big(y(\tau)\big)\,d\tau\,.
\end{aligned}
\tag{3.6}
$$

To show that $\Gamma\big(x_c, y\big) \in Y_\eta$ we use the inequalities (1.5), choosing $\varepsilon \doteq \eta$ to estimate the center component and $\varepsilon \doteq \beta - \eta$ to estimate the stable and unstable components.

$$
\begin{aligned}
\big|\Gamma\big(x_c, y\big)(t)\big| &\le C_\eta e^{\eta|t|}|x_c| + \int_0^t C_\eta e^{\eta|t-\tau|}\|g\|_{\mathbb{C}^0}\,d\tau \\
&\quad + \int_{-\infty}^t C_{\beta-\eta}e^{-\eta(t-\tau)}\|g\|_{\mathbb{C}^0}\,d\tau + \int_t^\infty C_{\beta-\eta}e^{\eta(t-\tau)}\|g\|_{\mathbb{C}^0}\,d\tau \\
&= C\,\|g\|_{\mathbb{C}^0}\,e^{\eta|t|}
\end{aligned}
$$

for a suitable constant C. We claim that, for every fixed x_c, the map $y \mapsto \Gamma(x_c, y)$ is a strict contraction. Indeed, call $\delta_0 \doteq \|y_1 - y_2\|_\eta$. By (3.2) this implies

$$
\big|y_1(t) - y_2(t)\big| \le \delta_0 e^{\eta|t|}\,, \qquad\qquad \big|g(y_1(t)) - g(y_2(t))\big| \le \delta_0 e^{\eta|t|}\|g\|_{\mathbb{C}^1}
$$

for all $t \in \mathbb{R}$. Therefore

$$
\begin{aligned}
&\big|\Gamma(x_c, y_1)(t) - \Gamma(x_c, y_2)(t)\big| \\
&\le \int_0^t C_\varepsilon e^{\varepsilon|t-\tau|}\delta_0 e^{\eta|\tau|}\|g\|_{\mathbb{C}^1}\,d\tau + \int_{-\infty}^t C_\varepsilon e^{-(\beta-\varepsilon)(t-\tau)}\delta_0 e^{\eta|\tau|}\|g\|_{\mathbb{C}^1}\,d\tau \\
&\quad + \int_t^\infty C_\varepsilon e^{(\beta-\varepsilon)(t-\tau)}\delta_0 e^{\eta|\tau|}\|g\|_{\mathbb{C}^1}\,d\tau \le C'\cdot\delta_0\|g\|_{\mathbb{C}^1}e^{\eta|t|}
\end{aligned}
$$

for some constant C' independent of y_1, y_2. Assuming that $\|g\|_{\mathbb{C}^1} \le 1/2C'$ we thus have

$$
\big\|\Gamma(x_c, y_1) - \Gamma(x_c, y_2)\big\|_\eta \le \frac{1}{2}\|y_1 - y_2\|_\eta\,, \tag{3.7}
$$

proving our claim. By the Contraction Mapping Theorem (see the Appendix), for each $x_c \in V^c$ the map $y \mapsto \Gamma(x_c, y)$ has a unique fixed point $y^{x_c} \in Y_\eta$, which provides a solution to (3.5). Since Γ is Lipschitz continuous (namely: linear) w.r.t. the variable x_c, it follows that the map $x_c \mapsto y^{x_c}$ is Lipschitz continuous.

For every $x_c \in V^c$ we now set

$$
\phi(x_c) \doteq y^{x_c}(0) \tag{3.8}
$$

and define the manifold $\mathcal{M}$ in terms of (3.1). By the previous analysis, the map $\phi : V^c \mapsto \mathbb{R}^n$ is Lipschitz continuous. By (3.5) it is clear that $\pi_c\phi(x_c) = \pi_c y^{x_c}(0) = x_c$.

A.3.4 Proof of the Invariance Property (ii).

To show that $\mathcal{M}$ is invariant for the flow of (2.3), fix any point $x_0 \in \mathcal{M}$. By construction, the trajectory starting at x, which we denote as $t \mapsto \tilde{x}(t, x_0)$, lies in Y_η. Fix any time t_1. To prove that the point $x_1 \doteq \tilde{x}(t_1, x_0)$ also lies on $\mathcal{M}$ we need to show that the trajectory $t \mapsto \tilde{x}(t, x_1)$ lies in Y_η. But this is clear because

$$\big|\tilde{x}(t, x_1)\big| = \big|\tilde{x}(t + t_1, x_0)\big| \le C\, e^{\eta|t+t_1|} \le \big(C\, e^{\eta|t_1|}\big)e^{\eta|t|}.$$

A.3.5 Proof of (iv).

By construction, every trajectory having slow growth at $\pm\infty$, i.e. with $y \in Y_\eta$, is entirely contained in the center manifold $\mathcal{M}$. This is certainly the case for all globally bounded trajectories.

A.3.6 Proof of the Tangency Property (iii).

Since $g(0) = 0$, the function $y(t) \equiv 0$ is (trivially) a globally bounded solution. Hence, by (iv), the manifold $\mathcal{M}$ contains the origin.

To prove that $\mathcal{M}$ is tangent to V^c at the origin, for any $x_c \in V^c$ consider the function $y(t) \doteq e^{At}x_c$. Since $g \neq 0$, we don't expect this to be a solution of (2.3). However (see the Appendix), by the contraction property (3.7), the distance between y and the unique fixed point y^{x_c} can be estimated as twice the distance between y and its first iterate:

$$\big\|y - y^{x_c}\big\|_\eta \le 2\big\|y - \Gamma(x_c, y)\big\|_\eta\,. \tag{3.9}$$

We now observe that

$$\big|g(y(\tau))\big| \le \big|y(\tau)\big|^2\, \|g\|_{\mathbb{C}^2} \le \big(C_\varepsilon e^{\varepsilon|\tau|}|x_c|\big)^2\|g\|_{\mathbb{C}^2}\,.$$

By the definition of Γ at (3.6) it now follows

$$\begin{aligned}\big|y(t) - \Gamma(x_c, y)(t)\big| \le & \int_0^t C_\varepsilon e^{\varepsilon|t-\tau|}\big(C_\varepsilon e^{\varepsilon|\tau|}|x_c|\big)^2\|g\|_{\mathbb{C}^2}\, d\tau \\ & + \int_{-\infty}^t C_\varepsilon e^{-(\beta-\varepsilon)(t-\tau)}\big(C_\varepsilon e^{\varepsilon|\tau|}|x_c|\big)^2\|g\|_{\mathbb{C}^2}\, d\tau \\ & + \int_t^\infty C_\varepsilon e^{(\beta-\varepsilon)(t-\tau)}\big(C_\varepsilon e^{\varepsilon|\tau|}|x_c|\big)^2\|g\|_{\mathbb{C}^2}\, d\tau \\ \le & \, C\,|x_c|^2 e^{\eta|t|}\end{aligned}$$

for some constant C independent of $|x_c|$. Therefore

$$\big|y(0) - y^{x_c}(0)\big| \le \|y - y^{x_c}\|_\eta \le 2\big\|y - \Gamma(x_c, y)\big\|_\eta \le 2C|x_c|^2\,. \tag{3.10}$$

Recalling that $y(0) = x_c$, $y^{x_c}(0) = \phi(x_c)$, an easy consequence of (3.10) is

$$\lim_{x_c \to 0} \frac{\big|\phi(x_c) - x_c\big|}{|x_c|} = 0.$$

Hence the manifold $\mathcal{M}$ is tangent to V^c at the origin.

A.3.7 Proof of the Asymptotic Approximation Property (v).

Let $x : [0, +\infty[\mapsto \mathbb{R}^n$ be a solution of (2.1) which approaches the origin as $t \to +\infty$. We extend $x(\cdot)$ to a bounded function $x^*(\cdot)$ defined on the whole real line by setting

$$x^*(t) = \begin{cases} x(t) & if \quad t \geq 0, \\ x(0) & if \quad t < 0. \end{cases}$$

Notice that x^* provides a globally bounded solution to

$$\dot{x}^*(t) = Ax^* + g(x^*) + \varphi(t) \qquad \varphi(t) = \begin{cases} 0 & if \quad t > 0, \\ -Ax(0) - g\big(x(0)\big) & if \quad t < 0. \end{cases}$$

Therefore, x^* can be represented by the "variation of constant formula"

$$\begin{aligned} x^*(t) = e^{A(t-t_0)}\pi_s x^*(t_0) + \int_{t_0}^t e^{A(t-\tau)}\pi_s g\big(x^*(\tau)\big)\,d\tau + \int_{t_0}^t e^{A(t-\tau)}\pi_s \varphi(\tau)\,d\tau \\ + e^{A(t-t_1)}\pi_{cu} x^*(t_1) + \int_{t_1}^t e^{A(t-\tau)}\pi_{cu} g\big(x^*(\tau)\big)\,d\tau + \int_{t_1}^t e^{A(t-\tau)}\pi_{cu}\varphi(\tau)\,d\tau\,. \end{aligned} \tag{3.11}$$

Here and in the sequel, $\pi_{cu} = \pi_c + \pi_u$ denotes the projection on the center-unstable space $V^c \oplus V^u$.

Consider now the space of functions

$$Z_\eta \doteq \Big\{ z : \mathbb{R} \mapsto \mathbb{R}^n \; ; \quad \big\|z(\cdot)\big\|_\eta \doteq \sup_t e^{\eta t}\big|z(t)\big| < \infty \Big\}.$$

We claim that there exists a function $z \in Z_\eta$ such that $y = x^* + z \in Y_\eta$ is a global solution of (2.3), contained in the center manifold $\mathcal{M}$. Recalling (3.11), for any choice of t_0, t_1 such a function $z(\cdot)$ should provide a solution to the integral equation

$$
\begin{aligned}
z(t) = & -\pi_s x^*(t) + e^{A(t-t_0)}\pi_s\big(x^*(t_0)+z(t_0)\big) \\
& + \int_{t_0}^{t} e^{A(t-\tau)}\pi_s g\big(x^*(\tau)+z(\tau)\big)\,d\tau \\
& -\pi_{cu}x^*(t) + e^{A(t-t_1)}\pi_{cu}\big(x^*(t_1)+z(t_1)\big) \\
& + \int_{t_1}^{t} e^{A(t-\tau)}\pi_{cu} g\big(x^*(\tau)+z(\tau)\big)\,d\tau \\
= & \; e^{A(t-t_0)}\pi_s z(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\pi_s\Big[g\big(x^*(\tau)+z(\tau)\big) - g\big(x^*(\tau)\big)\Big]\,d\tau \\
& - \int_{t_0}^{t} e^{A(t-\tau)}\pi_s\varphi(\tau)\,d\tau + e^{A(t-t_1)}\pi_{cu}z(t_1) \\
& + \int_{t_1}^{t} e^{A(t-\tau)}\pi_{cu}\Big[g\big(x^*(\tau)+z(\tau)\big) - g\big(x^*(\tau)\big)\Big]\,d\tau \\
& - \int_{t_1}^{t} e^{A(t-\tau)}\pi_{cu}\varphi(\tau)\,d\tau\,.
\end{aligned}
$$

Letting $t_0 \to -\infty$ and $t_1 \to +\infty$ we obtain

$$
\begin{aligned}
z(t) = & \int_{-\infty}^{t} e^{A(t-\tau)}\pi_s\Big[g\big(x^*(\tau)+z(\tau)\big) - g\big(x^*(\tau)\big)\Big]\,d\tau \\
& - \int_{-\infty}^{t} e^{A(t-\tau)}\pi_s\varphi(\tau)\,d\tau \\
& - \int_{t}^{\infty} e^{A(t-\tau)}\pi_{cu}\Big[g\big(x^*(\tau)+z(\tau)\big) - g\big(x^*(\tau)\big)\Big]\,d\tau \\
& + \int_{t}^{\infty} e^{A(t-\tau)}\pi_{cu}\varphi(\tau)\,d\tau \\
\doteq & \; \Lambda(z)(t)\,.
\end{aligned}
\tag{3.12}
$$

Recalling that $\varphi(\tau) = 0$ for $\tau > 0$ and using the basic inequalities (1.5), we see that the map $\Lambda : Z_\eta \mapsto Z_\eta$ is a strict contraction, provided that the norm $\|g\|_{\mathcal{C}^1}$ is suitably small. Therefore Λ admits a unique fixed point $z \in Z_\eta$, which satisfies (3.12). Since x^* is globally bounded and $z \in Z_\eta \subset Y_\eta$, it is clear that $y \doteq x^* + z \in Y_\eta$, hence it represents a trajectory contained in the center manifold. For all $t > 0$ we now have

$$
\big|x(t) - y(t)\big| = \big|z(t)\big| \le e^{-\eta t}\|z\|_\eta\,.
$$

This implies (2.5) for any smaller choice of the exponent η.

A.3.8 Smoothness of the Center Manifold.

To complete the proof, it remains to show that the map $x_c \mapsto \phi(x_c)$ is k times continuously differentiable. This fact would easily follow from the implicit function theorem, if we could prove that $\Gamma : V^c \times Y_\eta \mapsto Y_\eta$ is a $\mathcal{C}^k$ map.

Unfortunately this is not true. Indeed, for any non-trivial function $g \in \mathbb{C}_c^\infty$, the substitution operator $y \mapsto G(y)$ defined by

$$G(y)(t) \doteq g\big(y(t)\big) \tag{3.13}$$

is not differentiable as a map from Y_η into itself.

Example 2. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function with compact support, such that

$$g(x) = x^2 \qquad \text{for} \quad |x| \le \varepsilon\,.$$

If the map $G : Y_\eta \mapsto Y_\eta$ in (3.13) were differentiable at the origin $0 \in Y_\eta$, its differential could only be the identically zero map. However, consider the sequence of functions

$$y_n(t) \doteq \begin{cases} \varepsilon & if \quad t \in [n,\ n+1], \\ 0 & otherwise. \end{cases}$$

This is mapped into the sequence

$$G(y_n)(t) \doteq \begin{cases} \varepsilon^2 & if \quad t \in [n,\ n+1], \\ 0 & otherwise. \end{cases}$$

By the definition of the norm on the space Y_η, as $n \to \infty$ one has

$$\|y_n\|_\eta = \sup_t \ e^{-\eta|t|} y_n(t) = \varepsilon\, e^{-\eta n} \to 0\,.$$

We now have

$$\lim_{n\to\infty} \frac{\big\|G(y_n)\big\|_\eta}{\|y_n\|_\eta} = \lim_{n\to\infty} \frac{\varepsilon^2\, e^{-\eta n}}{\varepsilon\, e^{-\eta n}} = \varepsilon \neq 0\,,$$

showing that the zero linear map cannot be the differential of G at the origin.

To overcome the difficulty pointed out by the previous example, one can observe that G becomes k times differentiable if viewed as a map from a smaller space $Y_{\eta'}$ (with a stronger norm) into a Y_η. The proof of the regularity of the center manifold $\mathcal{M}$ strongly relies on this fact.

Lemma. *Let $g \in \mathbb{C}^{k+1}$ and assume $0 < \eta' < (k+1)\eta' \le \eta$. Then the substitution operator G at (3.13) is k times differentiable as a map from $Y_{\eta'}$ into Y_η.*

Proof. We begin by recalling Taylor's formula (see [D], p.190)

$$g(y+z) = T_k g(y,z) + R_k(y,z),$$

where

$$T_k g(y,z) \doteq \sum_{j=0}^{k} \frac{D^j g(y)}{j!}\, z^{[j]}\,,$$

and

$$R_k(y,z) = \left(\int_0^1 \frac{(1-\xi)^k}{k!} D^{k+1} g(y+\xi z)\, d\xi\right) z^{[k+1]}\,.$$

The j-th derivative of a function is here written as a multilinear symmetric operator, while $z^{[j]} = z \otimes \cdots \otimes z$ denotes the tensor product of j factors all equal to z. In order to prove that the map $y \mapsto G(y)$ is in $\mathbb{C}^k(Y_{\eta'}, Y_\eta)$, we need to check that

$$\big\|G(y+z) - T_k G(y,z)\big\|_\eta = \sup_t e^{-\eta|t|}\Big|g\big(y(t)+z(t)\big) - T_k g\big(y(t),\, z(t)\big)\Big| = \mathcal{O}(1)\cdot\|z\|_{\eta'}^{k+1}.$$

This is clear because

$$\begin{aligned} e^{-\eta|t|}\Big|R_k\big(y(t),\, z(t)\big)\Big| \ &\leq\ e^{-\eta|t|}\cdot\frac{1}{k!}\|g\|_{\mathbb{C}^{k+1}}\big\|z(t)\big\|^{k+1} \\ \leq e^{-\eta|t|}\cdot\frac{1}{k!}\|g\|_{\mathbb{C}^{k+1}}\, e^{(k+1)\eta'|t|}\|z\|_{\eta'}^{k+1} \ &\leq\ \frac{1}{k!}\|g\|_{\mathbb{C}^{k+1}}\,\|z\|_{\eta'}^{k+1}, \end{aligned}$$

provided that $(k+1)\eta' \leq \eta$. •

Corollary. *For every $j = 1, \ldots, k$, the operator Γ defined at (3.6) is a $\mathbb{C}^\ell$ map from $V^c \times Y_{\eta'}$ into Y_η, provided that $2\ell\eta' \leq \eta$.*

Indeed, Γ can be written in the form

$$\Gamma(x,y) \doteq Sx + K \circ G(y), \tag{3.14}$$

where

$$(Sx)(t) \doteq e^{At}x\,, \tag{3.15}$$

$$\big(Kv\big)(t) \doteq \int_0^t e^{A(t-\tau)}\pi_c v(\tau)\, d\tau + \int_{-\infty}^t e^{A(t-\tau)}\pi_s\, v(\tau)\, d\tau - \int_t^\infty e^{A(t-\tau)}\pi_u\, v(\tau)\, d\tau\,. \tag{3.16}$$

Since both $S : V^c \mapsto Y_\eta$ and $K : Y_\eta \mapsto Y_\eta$ are continuous linear mappings, by the previous lemma it follows that Γ is a $\mathbb{C}^\ell$ map from $V^c \times Y_{\eta'}$ into Y_η, provided that $(\ell+1)\eta' \leq \eta$. •

We now resume the proof of the main theorem. By induction, define the sequence of mappings $y_\nu : V^c \mapsto Y_\eta$, such that

$$y_0(x) \equiv 0 \qquad\qquad y_\nu(x) \doteq \Gamma(x, y_{\nu-1}) \qquad \nu \geq 1\,.$$

In particular, one has

$$y_0(x)(t) = 0 \qquad\qquad y_1(x)(t) \doteq e^{At}x\,.$$

By the argument at (3.7), we already know that the sequence y_ν converges pointwise to the function $x_c \mapsto y^{x_c}$ uniformly for $x_c \in V^c$ on bounded sets. We now show that the same is true also for all derivatives, up to order k. Recalling that $\phi(x_c) \doteq y^{x_c}(0)$, this will show that $\phi \in \mathbb{C}^k$, completing the proof.

Fix $\eta \in]0, \beta[$ and consider the numbers $0 < \eta_0 < \eta_1 < \ldots < \eta_k = \eta < \beta$, defined by

$$\eta_j \doteq e^{2j-2k}\eta \qquad\qquad j = 0, 1, \ldots, k.$$

By the previous Corollary, Γ is then a mapping of class $\mathbb{C}^\ell$ from $V^c \times Y_{\eta_i}$ into Y_{η_j}, provided that $i + \ell \le j$. For convenience, the higher derivatives of a map $y : V^c \mapsto Y_\eta$ will be denoted as $D^j y \doteq d^j y/dx^{[j]}$. We recall that these are elements of the space of j-linear mappings $L^j(V^c,\, Y_\eta)$.

As $\nu \to \infty$, the convergence of the sequence of derivatives $D^j y_\nu$ will be proved by induction on j. Assume that, for all $i = 0, 1, \ldots, j-1$, the sequence of derivatives

$$x \mapsto D^i y_\nu(x)$$

converges uniformly on bounded sets, in the space of maps

$$V^c \mapsto L^i(V^c;\, Y_{\eta_i})\,.$$

We claim that the sequence of j-derivatives also converges. To see this, we first compute the derivatives of the composite map $x \mapsto \Gamma\big(x, y(x)\big)$. Differentiating (3.14) several times, one finds

$$\begin{aligned}
\frac{d}{dx}\Gamma\big(x, y(x)\big) &= S + K \circ DG\, Dy\,,\\
\frac{d^2}{dx^{[2]}}\Gamma\big(x, y(x)\big) &= K \circ \Big[DG\, D^2 y + D^2 G\,(Dy \otimes Dy)\Big]\\
\frac{d^3}{dx^{[3]}}\Gamma\big(x, y(x)\big) &= K \circ \Big[DG\, D^3 y + 3D^2 G\,(Dy \otimes D^2 y) + D^3 G\,(Dy \otimes Dy \otimes Dy)\Big]
\end{aligned}$$

By induction (see [B]), it is clear that the j-th derivative of (3.14) has an expression of the form

$$\frac{d^j}{dx^{[j]}}\Gamma\big(x, y(x)\big) = K \circ \Big[DG\, D^j y + \Phi_j(y, Dy, \ldots, D^{j-1}y)\Big]$$

for a suitable function Φ_j involving only derivatives of lower order. The inductive assumption now guarantees that the sequence of mappings

$$x \mapsto \Phi_j\big(y_\nu(x),\, Dy_\nu(x), \ldots,\, D^{j-1}y_\nu(x)\big)$$

converges for all $x \in V^c$, uniformly on bounded sets. On the other hand, if $\|g\|_{\mathbb{C}^1}$ is small enough, for every $y \in Y_\eta$ the operator

$$\psi \mapsto \big[K \circ DG(y)\big]\,\psi$$

is a strict contraction in the space $L^j(V^c, Y_{\eta_j})$. An application of the Contraction Mapping Theorem (see the Appendix) now yields the convergence of the sequence $D^j y_\nu$, uniformly on bounded sets. This completes the proof of the theorem. •

Assuming that the norm $\|g\|_{\mathbb{C}^1}$ is sufficiently small, we proved the existence and uniqueness of a GLOBAL center manifold consisting of all trajectories $t \mapsto y(t)$ having "slow growth" at $\pm\infty$, so that $y \in Y_\eta$. In the general case, we could still prove the existence of a LOCAL center manifold, defined in a neighborhood of the origin. However, one should be aware that this local center manifold may not be unique, because its construction depends on the choice of the cut-off function.

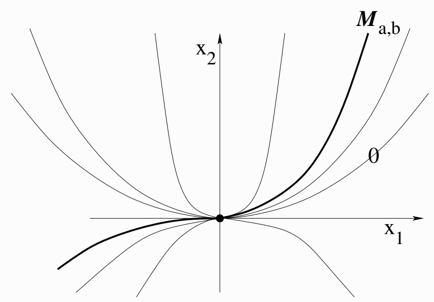

Fig. A.4.

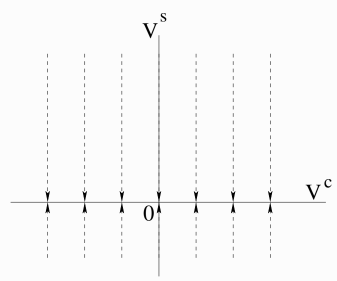

Fig. 5

Example 3. Consider the system (fig. 4)

$$\dot{x}_1 = -x_1^3, \qquad \dot{x}_2 = -x_2. \tag{3.17}$$

Linearizing at the origin, one obtains the system (fig. 5)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The corresponding stable, center and unstable subspaces are

$$V^s = \{(0, x_2); \ \ x_2 \in \mathbb{R}\}, \qquad V^c = \{(x_1, 0); \ \ x_1 \in \mathbb{R}\}, \qquad V^u = \{(0,0)\}.$$

By explicitly solving the equation

$$\frac{dx_1}{dx_2} = \frac{x_1^3}{x_2}$$

we find that, for any choice of $a, b \in \mathbb{R}$, the following manifold $\mathcal{M}_{a,b}$ is invariant w.r.t. the flow of (3.17):

$$\mathcal{M}_{a,b} \doteq \Big\{ \big(x, \psi(x)\big) \Big\} \qquad \psi(x) = \begin{cases} a\, e^{-1/2x^2} & if \quad x < 0, \\ 0 & if \quad x = 0, \\ b\, e^{-1/2x^2} & if \quad x > 0. \end{cases}$$

Notice that every $\mathcal{M}_{a,b}$ is smooth, and tangent to V^c at the origin, In this case, we thus have a continuum family of center manifolds.

A.4 The Contraction Mapping Theorem

For reader's convenience, we prove here the fixed point theorem which was used throughout these notes.

Contraction Mapping Theorem. *Let X, Y be Banach spaces and let $\Gamma : X \times Y \mapsto Y$ be a continuous mapping which is a strict contraction in the y variable, i.e.*

$$\big\|\Gamma(x,y) - \Gamma(x,y')\big\| \leq \kappa \|y - y'\| \qquad \textit{for all} \;\; x \in X, \quad y, y' \in Y, \quad (A.1)$$

for some constant $\kappa < 1$. Then the following holds.

(i) For every $x \in X$, there exists a unique $y(x) \in Y$ such that

$$y(x) = \Gamma\big(x,\, y(x)\big). \qquad (A.2)$$

(ii) For every $x \in X$, $y \in Y$ one has

$$\big\|y - y(x)\big\| \leq \frac{1}{1-\kappa} \big\|y - \Gamma(x,y)\big\|. \qquad (A.3)$$

(iii) If Γ is Lipschitz continuous w.r.t. x, say

$$\big\|\Gamma(x,y) - \Gamma(x',y)\big\| \leq L\, \|x - x'\| \qquad \textit{for all} \;\; x, x' \in X, \quad y \in Y, \quad (A.4)$$

then the same is true of the map $x \mapsto y(x)$. Namely

$$\big\|y(x) - y(x')\big\| \leq \frac{L}{1-\kappa}\, \|x - x'\|\,. \qquad (A.5)$$

(iv) Consider any convergent sequence $x_\nu \to \bar{x}$ in X. Then for every $y_0 \in Y$ the sequence of iterates

$$y_{\nu+1} \doteq \Gamma(x_\nu,\, y_\nu)$$

converges to the point $\bar{y} \doteq y(\bar{x})$.

Proof. Fix any point $y \in Y$. For each x, consider the sequence

$$y_0 \doteq y, \qquad y_1 \doteq \Gamma(x, y_0), \qquad \ldots \qquad y_{\nu+1} \doteq \Gamma(x, y_\nu), \qquad \ldots$$

By induction, for every $\nu \geq 0$ one checks that

$$\|y_{\nu+1} - y_\nu\| \leq \kappa \|y_\nu - y_{\nu-1}\| \leq \kappa^\nu \|y_1 - y_0\| = \kappa^\nu \big\|\Gamma(x,y) - y\big\|. \qquad (A.6)$$

Since $\kappa < 1$, the sequence y_ν is Cauchy and converges to some limit point, which we call $y(x)$. The continuity of Γ now implies

$$y(x) = \lim_{\nu\to\infty} y_{\nu+1} = \lim_{\nu\to\infty} \Gamma(x, y_\nu) = \Gamma\left(x, \lim_{\nu\to\infty} y_\nu\right) = \Gamma\big(x,\, y(x)\big).$$

Hence (A.2) holds. The uniqueness of the fixed point $y(x)$ is proved observing that, if $y_1 = \Gamma(x, y_1)$ and $y_2 = \Gamma(x, y_2)$, then (A.1) implies

$$\|y_1 - y_2\| = \big\|\Gamma(x, y_1) - \Gamma(x, y_2)\big\| \leq \kappa \|y_1 - y_2\|.$$

Hence $y_1 = y_2$.

To prove (A.3), it suffices to observe that (A.6) implies

$$\|y - y(x)\| \leq \sum_{\nu=0}^{\infty} \|y_{\nu+1} - y_\nu\| \leq \sum_{\nu=0}^{\infty} \kappa^\nu \big\|\Gamma(x,y) - y\big\| = \frac{\big\|\Gamma(x,y) - y\big\|}{1-\kappa}\,.$$

Toward a proof of (A.5), we use (A.3) with $y \doteq y(x')$ and obtain

$$\begin{aligned}
&\big\|y(x') - y(x)\big\| \leq \frac{1}{1-\kappa} \Big\|y(x') - \Gamma\big(x, y(x')\big)\Big\| \\
&= \frac{1}{1-\kappa} \Big\|\Gamma\big(x', y(x')\big) - \Gamma\big(x, y(x')\big)\Big\| \leq \frac{L}{1-\kappa} \|x' - x\|\,.
\end{aligned}$$

To prove the remaining statement (iv), observe that the quantities $\varepsilon_\nu \doteq \big\|\Gamma(x_\nu, \bar{y}) - \Gamma(\bar{x}, \bar{y})\big\|$ converge to zero as $\nu \to \infty$. Moreover, the contraction property implies

$$\begin{aligned}
\|y_{\nu+1} - \bar{y}\| &= \Big\|\Gamma(x_\nu, y_\nu) - \Gamma(\bar{x}, \bar{y})\big\| \\
&\leq \Big\|\Gamma(x_\nu, y_\nu) - \Gamma(x_\nu, \bar{y})\big\| + \big\|\Gamma(x_\nu, \bar{y}) - \Gamma(\bar{x}, \bar{y})\big\| \\
&\leq \kappa \|y_\nu - \bar{y}\| + \varepsilon_\nu\,.
\end{aligned} \qquad (A.7)$$

¿From (A.7) we deduce

$$\|y_\nu - \bar{y}\| \leq \kappa^\nu \, \|y_0 - \bar{y}\| + \sum_{j=1}^{\nu} \kappa^{\nu-j} \varepsilon_j$$

and therefore

$$\limsup_{\nu\to\infty} \|y_\nu - \bar{y}\| \leq \limsup_{\nu\to\infty} \frac{\varepsilon_\nu}{1-\kappa} = 0\,.$$

•

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