A. The Bochner Integral

This chapter is a slight modification of Chap. A in [FK01].

Let $\big(X, \| \ \|\big)$ be a Banach space, $\mathcal{B}(X)$ the Borel σ-field of X and $(\Omega, \mathcal{F}, \mu)$ a measure space with finite measure μ.

A.1. Definition of the Bochner integral

Step 1: As first step we want to define the integral for simple functions which are defined as follows. Set

$$\mathcal{E} := \Big\{ f : \Omega \to X \ \Big| \ f = \sum_{k=1}^{n} x_k 1_{A_k},\ x_k \in X,\ A_k \in \mathcal{F},\ 1 \leqslant k \leqslant n,\ n \in \mathbb{N} \Big\}$$

and define a semi-norm $\| \ \|_{\mathcal{E}}$ on the vector space $\mathcal{E}$ by

$$\|f\|_{\mathcal{E}} := \int \|f\| \, \mathrm{d}\mu, \quad f \in \mathcal{E}.$$

To get that $\big(\mathcal{E}, \| \ \|_{\mathcal{E}}\big)$ is a normed vector space we consider equivalence classes with respect to $\| \ \|_{\mathcal{E}}$. For simplicity we will not change the notations.

For $f \in \mathcal{E}$, $f = \sum_{k=1}^{n} x_k 1_{A_k}$, A_k's *pairwise disjoint* (such a representation is called *normal* and always exists, because $f = \sum_{k=1}^{n} x_k 1_{A_k}$, where $f(\Omega) = \{x_1, \ldots, x_k\}$, $x_i \neq x_j$, and $A_k := \{f = x_k\}$) and we now define the Bochner integral to be

$$\int f \, \mathrm{d}\mu := \sum_{k=1}^{n} x_k \mu(A_k).$$

(Exercise: This definition is independent of representations, and *hence* linear.) In this way we get a mapping

$$\begin{aligned} \mathrm{int}: \quad \big(\mathcal{E}, \| \ \|_{\mathcal{E}}\big) \quad &\to \quad \big(X, \| \ \|\big) \\ f \quad &\mapsto \quad \int f \, \mathrm{d}\mu \end{aligned}$$

which is linear and uniformly continuous since $\big\| \int f \, \mathrm{d}\mu \big\| \leqslant \int \|f\| \, \mathrm{d}\mu$ for all $f \in \mathcal{E}$.

Therefore we can extend the mapping int to the abstract completion of $\mathcal{E}$ with respect to $\| \ \|_{\mathcal{E}}$ which we denote by $\overline{\mathcal{E}}$.

Step 2: We give an explicit representation of $\overline{\mathcal{E}}$.

Definition A.1.1. A function $f : \Omega \to X$ is called strongly measurable if it is $\mathcal{F}/\mathcal{B}(X)$-measurable and $f(\Omega) \subset X$ is separable.

Definition A.1.2. Let $1 \leqslant p < \infty$. Then we define

$$\begin{aligned}\mathcal{L}^p(\Omega, \mathcal{F}, \mu; X) &:= \mathcal{L}^p(\mu; X)\\ &:= \Big\{ f : \Omega \to X \,\Big|\, f \text{ is strongly measurable with}\\ &\qquad\qquad \text{respect to } \mathcal{F}, \text{ and } \int \|f\|^p \,\mathrm{d}\mu < \infty \Big\}\end{aligned}$$

and the semi-norm

$$\|f\|_{L^p} := \left(\int \|f\|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{L}^p(\Omega, \mathcal{F}, \mu; X).$$

The space of all equivalence classes in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu; X)$ with respect to $\| \ \|_{L^p}$ is denoted by $L^p(\Omega, \mathcal{F}, \mu; X) := L^p(\mu; X)$.

Claim: $L^1(\Omega, \mathcal{F}, \mu; X) = \overline{\mathcal{E}}$.

Step 2.a: $\big(L^1(\Omega, \mathcal{F}, \mu; X), \| \ \|_{L^1}\big)$ is complete.

The proof is just a modification of the proof of the Fischer–Riesz theorem by the help of the following proposition.

Proposition A.1.3. *Let $(\Omega, \mathcal{F})$ be a measurable space and let X be a Banach space. Then:*

(i) the set of $\mathcal{F}/\mathcal{B}(X)$-measurable functions from Ω to X is closed under the formation of pointwise limits, and

(ii) the set of strongly measurable functions from Ω to X is closed under the formation of pointwise limits.

Proof. Simple exercise or see [Coh80, Proposition E.1, p. 350]. □

Step 2.b: $\mathcal{E}$ is a dense subset of $L^1(\Omega, \mathcal{F}, \mu; X)$ with respect to $\| \ \|_{L^1}$.

This can be shown by the help of the following lemma.

Lemma A.1.4. *Let E be a metric space with metric d and let $f : \Omega \to E$ be strongly measurable. Then there exists a sequence f_n, $n \in \mathbb{N}$, of simple E-valued functions (i.e. f_n is $\mathcal{F}/\mathcal{B}(E)$-measurable and takes only a finite number of values) such that for arbitrary $\omega \in \Omega$ the sequence $d\big(f_n(\omega), f(\omega)\big)$, $n \in \mathbb{N}$, is monotonely decreasing to zero.*

Proof. [DPZ92, Lemma 1.1, p. 16] Let $\{e_k \mid k \in \mathbb{N}\}$ be a countable dense subset of $f(\Omega)$. For $m \in \mathbb{N}$ define

$$\begin{aligned} d_m(\omega) &:= \min\{d(f(\omega), e_k) \mid k \leqslant m\} \quad \big(= \operatorname{dist}(f(\omega), \{e_k, k \leqslant m\})\big), \\ k_m(\omega) &:= \min\{k \leqslant m \mid d_m(\omega) = d(f(\omega), e_k)\}, \\ f_m(\omega) &:= e_{k_m(\omega)}. \end{aligned}$$

Obviously f_m, $m \in \mathbb{N}$, are simple functions since they are $\mathcal{F}/\mathcal{B}(E)$-measurable (exercise) and

$$f_m(\Omega) \subset \{e_1, e_2, \ldots, e_m\}.$$

Moreover, by the density of $\{e_k \mid k \in \mathbb{N}\}$, the sequence $d_m(\omega)$, $m \in \mathbb{N}$, is monotonically decreasing to zero for arbitrary $\omega \in \Omega$. Since $d\big(f_m(\omega), f(\omega)\big) = d_m(\omega)$ the assertion follows. □

Let now $f \in L^1(\mu; X)$. By the Lemma A.1.4 above we get the existence of a sequence of simple functions f_n, $n \in \mathbb{N}$, such that

$$\big\|f_n(\omega) - f(\omega)\big\| \downarrow 0 \quad \text{for all } \omega \in \Omega \text{ as } n \to \infty.$$

Hence $f_n \xrightarrow{n\to\infty} f$ in $\| \ \|_{L^1}$ by Lebesgue's dominated convergence theorem.

A.2. Properties of the Bochner integral

Proposition A.2.1 (Bochner inequality). *Let $f \in L^1(\Omega, \mathcal{F}, \mu; X)$. Then*

$$\left\|\int f \,\mathrm{d}\mu\right\| \leqslant \int \|f\| \,\mathrm{d}\mu.$$

Proof. We know the assertion is true for $f \in \mathcal{E}$, i.e. $\operatorname{int} : \mathcal{E} \to X$ is linear, continuous with $\|\operatorname{int} f\| \leqslant \|f\|_{\mathcal{E}}$ for all $f \in \mathcal{E}$, so the same is true for its unique continuous extension $\overline{\operatorname{int}} : \overline{\mathcal{E}} = L^1(\mu; X) \to X$, i.e. for all $f \in L^1(X, \mu)$

$$\left\|\int f \,\mathrm{d}\mu\right\| = \big\|\overline{\operatorname{int}} f\big\| \leqslant \|f\|_{\overline{\mathcal{E}}} = \int \|f\| \,\mathrm{d}\mu.$$ □

Proposition A.2.2. *Let $f \in L^1(\Omega, \mathcal{F}, \mu; X)$. Then*

$$\int L \circ f \,\mathrm{d}\mu = L\Big(\int f \,\mathrm{d}\mu\Big)$$

holds for all $L \in L(X, Y)$, where Y is another Banach space.

Proof. Simple exercise or see [Coh80, Proposition E.11, p. 356]. □

Proposition A.2.3 (**Fundamental theorem of calculus**). *Let* $-\infty < a < b < \infty$ *and* $f \in C^1\big([a,b];X\big)$. *Then*

$$f(t) - f(s) = \int_s^t f'(u)\,\mathrm{d}u := \begin{cases} \int 1_{[s,t]}(u) f'(u)\,\mathrm{d}u & \text{if } s \leqslant t \\ -\int 1_{[t,s]}(u) f'(u)\,\mathrm{d}u & \text{otherwise} \end{cases}$$

for all $s, t \in [a,b]$ *where* $\mathrm{d}u$ *denotes the Lebesgue measure on* $\mathcal{B}(\mathbb{R})$.

Proof. Claim 1: If we set $F(t) = \int_s^t f'(u)\,\mathrm{d}u$, $t \in [a,b]$, we get that $F'(t) = f'(t)$ for all $t \in [a,b]$.

For that we have to prove that

$$\left\| \frac{1}{h}\big(F(t+h) - F(t)\big) - f'(t) \right\|_X \xrightarrow{h \to 0} 0.$$

To this end we fix $t \in [a,b]$ and take an arbitrary $\varepsilon > 0$. Since f' is continuous on $[a,b]$ there exists $\delta > 0$ such that $\big\|f'(u) - f'(t)\big\|_X < \varepsilon$ for all $u \in [a,b]$ with $|u - t| < \delta$. Then we obtain that

$$\begin{aligned} \left\| \frac{1}{h}\big(F(t+h) - F(t)\big) - f'(t) \right\|_X &= \left\| \frac{1}{h} \int_t^{t+h} \big(f'(u) - f'(t)\big)\,\mathrm{d}u \right\|_X \\ \leqslant \frac{1}{h} \int_t^{t+h} \big\|f'(u) - f'(t)\big\|_X \,\mathrm{d}u < \varepsilon \end{aligned}$$

if $t + h \in [a,b]$ and $|h| < \delta$.

Claim 2: If $\tilde{F} \in C^1\big([a,b];X\big)$ is a further function with $\tilde{F}' = F' = f'$ then there exists a constant $c \in X$ such that $F - \tilde{F} = c$.

For all $L \in X^* = L(X,\mathbb{R})$ we define $g_L := L(F - \tilde{F})$. Then $g_L' = 0$ and therefore g_L is constant. Since X^* separates the points of X by the Hahn–Banach theorem (see [Alt92, Satz 4.2, p. 114]) this implies that $F - \tilde{F}$ itself is constant. □

B. Nuclear and Hilbert–Schmidt Operators

This chapter is identical to Chap. B in [FK01].

Let $\big(U, \langle\ ,\ \rangle_U\big)$ and $\big(H, \langle\ ,\ \rangle\big)$ be two separable Hilbert spaces. The space of all bounded linear operators from U to H is denoted by $L(U,H)$; for simplicity we write $L(U)$ instead of $L(U,U)$. If we speak of the adjoint operator of $L \in L(U,H)$ we write $L^* \in L(H,U)$. An element $L \in L(U)$ is called symmetric if $\langle Lu, v\rangle_U = \langle u, Lv\rangle_U$ for all $u, v \in U$. In addition, $L \in L(U)$ is called nonnegative if $\langle Lu, u\rangle \geqslant 0$ for all $u \in U$.

Definition B.0.1 (Nuclear operator). An element $T \in L(U,H)$ is said to be a nuclear operator if there exists a sequence $(a_j)_{j\in\mathbb{N}}$ in H and a sequence $(b_j)_{j\in\mathbb{N}}$ in U such that

$$Tx = \sum_{j=1}^{\infty} a_j \langle b_j, x\rangle_U \quad \text{for all } x \in U$$

and

$$\sum_{j\in\mathbb{N}} \|a_j\| \cdot \|b_j\|_U < \infty.$$

The space of all nuclear operators from U to H is denoted by $L_1(U,H)$.
If $U = H$, $T \in L_1(U,H)$ is nonnegative and symmetric, then T is called *trace class.*

Proposition B.0.2. *The space $L_1(U,H)$ endowed with the norm*

$$\|T\|_{L_1(U,H)} := \inf\Big\{\sum_{j\in\mathbb{N}} \|a_j\| \cdot \|b_j\|_U \ \Big|\ Tx = \sum_{j=1}^{\infty} a_j \langle b_j, x\rangle_U,\ x \in U\Big\}$$

is a Banach space.

Proof. [MV92, Corollar 16.25, p. 154]. □

Definition B.0.3. Let $T \in L(U)$ and let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U. Then we define

$$\operatorname{tr} T := \sum_{k\in\mathbb{N}} \langle Te_k, e_k\rangle_U$$

if the series is convergent.

One has to notice that this definition could depend on the choice of the orthonormal basis. But there is the following result concerning nuclear operators.

Remark B.0.4. *If $T \in L_1(U)$ then $\operatorname{tr} T$ is well-defined independently of the choice of the orthonormal basis e_k, $k \in \mathbb{N}$. Moreover we have that*

$$|\operatorname{tr} T| \leqslant \|T\|_{L_1(U)}.$$

Proof. Let $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ be sequences in U such that

$$Tx = \sum_{j\in\mathbb{N}} a_j \langle b_j, x\rangle_U$$

for all $x \in U$ and $\sum\limits_{j\in\mathbb{N}} \|a_j\|_U \cdot \|b_j\|_U < \infty$.

Then we get for any orthonormal basis e_k, $k \in \mathbb{N}$, of U that

$$\langle Te_k, e_k\rangle_U = \sum_{j\in\mathbb{N}} \langle e_k, a_j\rangle_U \cdot \langle e_k, b_j\rangle_U$$

and therefore

$$\begin{aligned}
\sum_{k\in\mathbb{N}} \big|\langle Te_k, e_k\rangle_U\big| &\leqslant \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \big|\langle e_k, a_j\rangle_U \cdot \langle e_k, b_j\rangle_U\big| \\
&\leqslant \sum_{j\in\mathbb{N}} \Big(\sum_{k\in\mathbb{N}} \big|\langle e_k, a_j\rangle_U\big|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{k\in\mathbb{N}} \big|\langle e_k, b_j\rangle_U\big|^2\Big)^{\frac{1}{2}} \\
&= \sum_{j\in\mathbb{N}} \|a_j\|_U \cdot \|b_j\|_U < \infty.
\end{aligned}$$

This implies that we can exchange the summation to get that

$$\sum_{k\in\mathbb{N}} \langle Te_k, e_k\rangle_U = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \langle e_k, a_j\rangle_U \cdot \langle e_k, b_j\rangle_U = \sum_{j\in\mathbb{N}} \langle a_j, b_j\rangle_U,$$

and the assertion follows. □

Definition B.0.5 (Hilbert–Schmidt operator). A bounded linear operator $T : U \to H$ is called Hilbert–Schmidt if

$$\sum_{k\in\mathbb{N}} \|Te_k\|^2 < \infty$$

where e_k, $k \in \mathbb{N}$, is an orthonormal basis of U.

The space of all Hilbert–Schmidt operators from U to H is denoted by $L_2(U, H)$.

Remark B.0.6. *(i) The definition of Hilbert–Schmidt operator and the number*

$$\|T\|^2_{L_2(U,H)} := \sum_{k\in\mathbb{N}} \|Te_k\|^2$$

does not depend on the choice of the orthonormal basis e_k, $k \in \mathbb{N}$, and we have that $\|T\|_{L_2(U,H)} = \|T^*\|_{L_2(H,U)}$. For simplicity we also write $\|T\|_{L_2}$ instead of $\|T\|_{L_2(U,H)}$.

(ii) $\|T\|_{L(U,H)} \leqslant \|T\|_{L_2(U,H)}$.

(iii) Let G be another Hilbert space and $S_1 \in L(H,G)$, $S_2 \in L(G,U)$, $T \in L_2(U,H)$. Then $S_1T \in L_2(U,G)$ and $TS_2 \in L_2(G,H)$ and

$$\|S_1T\|_{L_2(U,G)} \leqslant \|S_1\|_{L(H,G)}\|T\|_{L_2(U,H)},$$

$$\|TS_2\|_{L_2(G,H)} \leqslant \|T\|_{L(U,H)}\|S_2\|_{L_2(G,U)}.$$

Proof. (i) If e_k, $k \in \mathbb{N}$, is an orthonormal basis of U and f_k, $k \in \mathbb{N}$, is an orthonormal basis of H we obtain by the Parseval identity that

$$\sum_{k\in\mathbb{N}} \|Te_k\|^2 = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} \big|\langle Te_k, f_j\rangle\big|^2 = \sum_{j\in\mathbb{N}} \|T^* f_j\|^2_U$$

and therefore the assertion follows.

(ii) Let $x \in U$ and f_k, $k \in \mathbb{N}$, be an orthonormal basis of H. Then we get that

$$\|Tx\|^2 = \sum_{k\in\mathbb{N}} \langle Tx, f_k\rangle^2 \leqslant \|x\|^2_U \sum_{k\in\mathbb{N}} \|T^* f_k\|^2_U = \|T\|^2_{L_2(U,H)} \cdot \|x\|^2_U.$$

(iii) Let e_k, $k \in \mathbb{N}$ be an orthonormal basis of U. Then

$$\sum_{k\in\mathbb{N}} \|S_1Te_k\|^2_G \leqslant \|S_1\|^2_{L(H,G)}\|T\|^2_{L_2(U,H)}.$$

Furthermore, since $(TS_2)^* = S_2^*T^*$, it follows that by the above and (i) that $TS_2 \in L_2(G,H)$ and

$$\begin{aligned}\|TS_2\|_{L_2(G,H)} &= \|(TS_2)^*\|_{L_2(H,G)}\\ &= \|S_2^*T^*\|_{L_2(H,G)}\\ &\leqslant \|S_2\|_{L(G,U)} \cdot \|T\|_{L_2(U,H)}.\end{aligned}$$

□

Proposition B.0.7. *Let $S, T \in L_2(U,H)$ and let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U. If we define*

$$\langle T, S\rangle_{L_2} := \sum_{k\in\mathbb{N}} \langle Se_k, Te_k\rangle$$

we obtain that $\big(L_2(U,H), \langle\ ,\ \rangle_{L_2}\big)$ is a separable Hilbert space.

If f_k, $k \in \mathbb{N}$, is an orthonormal basis of H we get that $f_j \otimes e_k := f_j\langle e_k, \cdot\rangle_U$, $j,k \in \mathbb{N}$, is an orthonormal basis of $L_2(U,H)$.

Proof. We have to prove the completeness and the separability.

1. $L_2(U,H)$ is complete:

   Let T_n, $n \in \mathbb{N}$, be a Cauchy sequence in $L_2(U,H)$. Then it is clear that it is also a Cauchy sequence in $L(U,H)$. Because of the completeness of $L(U,H)$ there exists an element $T \in L(U,H)$ such that $\|T_n - T\|_{L(U,H)} \longrightarrow 0$ as $n \to \infty$. But by the lemma of Fatou we also have for any orthonormal basis e_k, $k \in \mathbb{N}$, of U that

$$\begin{aligned}\|T_n - T\|_{L_2}^2 &= \sum_{k\in\mathbb{N}} \big\langle (T_n - T)e_k, (T_n - T)e_k\big\rangle \\ &= \sum_{k\in\mathbb{N}} \liminf_{m\to\infty} \big\|(T_n - T_m)e_k\big\|^2 \\ &\leqslant \liminf_{m\to\infty} \sum_{k\in\mathbb{N}} \big\|(T_n - T_m)e_k\big\|^2 = \liminf_{m\to\infty} \|T_n - T_m\|_{L_2}^2 < \varepsilon\end{aligned}$$

   for all $n \in \mathbb{N}$ big enough. Therefore the assertion follows.

2. $L_2(U,H)$ is separable:

   If we define $f_j \otimes e_k := f_j\langle e_k, \cdot\rangle_U$, $j,k \in \mathbb{N}$, then it is clear that $f_j \otimes e_k \in L_2(U,H)$ for all $j,k \in \mathbb{N}$ and for arbitrary $T \in L_2(U,H)$ we get that

$$\langle f_j \otimes e_k, T\rangle_{L_2} = \sum_{n\in\mathbb{N}} \langle e_k, e_n\rangle_U \cdot \langle f_j, Te_n\rangle = \langle f_j, Te_k\rangle.$$

   Therefore it is obvious that $f_j \otimes e_k$, $j,k \in \mathbb{N}$, is an orthonormal system. In addition, $T = 0$ if $\langle f_j \otimes e_k, T\rangle_{L_2} = 0$ for all $j,k \in \mathbb{N}$, and therefore $\operatorname{span}(f_j \otimes e_k \mid j,k \in \mathbb{N})$ is a dense subspace of $L_2(U,H)$. □

Proposition B.0.8. *Let $\big(G, \langle\ ,\ \rangle_G\big)$ be a further separable Hilbert space. If $T \in L_2(U,H)$ and $S \in L_2(H,G)$ then $ST \in L_1(U,G)$ and*

$$\|ST\|_{L_1(U,G)} \leqslant \|S\|_{L_2} \cdot \|T\|_{L_2}.$$

Proof. Let f_k, $k \in \mathbb{N}$, be an orthonormal basis of H. Then we have that

$$STx = \sum_{k\in\mathbb{N}} \langle Tx, f_k\rangle Sf_k, \quad x \in U$$

and therefore

$$\|ST\|_{L_1(U,G)} \leqslant \sum_{k\in\mathbb{N}} \|T^* f_k\|_U \cdot \|Sf_k\|_G$$
$$\leqslant \Big(\sum_{k\in\mathbb{N}} \|T^* f_k\|_U^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{k\in\mathbb{N}} \|Sf_k\|_G^2\Big)^{\frac{1}{2}} = \|S\|_{L_2} \cdot \|T\|_{L_2}. \qquad \square$$

Remark B.0.9. *Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U. If $T \in L(U)$ is symmetric, nonnegative with $\sum_{k\in\mathbb{N}} \langle Te_k, e_k\rangle_U < \infty$ then $T \in L_1(U)$.*

Proof. The result is obvious by the previous proposition and the fact that there exists $T^{\frac{1}{2}} \in L(U)$ nonnegative and symmetric such that $T = T^{\frac{1}{2}} T^{\frac{1}{2}}$ (see Proposition 2.3.4). Then $T^{\frac{1}{2}} \in L_2(U)$. $\square$

Proposition B.0.10. *Let $L \in L(H)$ and $B \in L_2(U, H)$. Then $LBB^* \in L_1(H)$, $B^*LB \in L_1(U)$ and we have that*

$$\operatorname{tr} LBB^* = \operatorname{tr} B^*LB.$$

Proof. We know by Remark B.0.6 (iii) and Proposition B.0.8 that $LBB^* \in L_1(H)$ and $B^*LB \in L_1(U)$. Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U and let f_k, $k \in \mathbb{N}$, be an orthonormal basis of H. Then the Parseval identity implies that

$$\sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} \big|\langle f_k, Be_n\rangle \cdot \langle f_k, LBe_n\rangle\big|$$
$$\leqslant \sum_{n\in\mathbb{N}} \Big(\sum_{k\in\mathbb{N}} \big|\langle f_k, Be_n\rangle\big|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{k\in\mathbb{N}} \big|\langle f_k, LBe_n\rangle\big|^2\Big)^{\frac{1}{2}}$$
$$= \sum_{n\in\mathbb{N}} \|Be_n\| \cdot \|LBe_n\| \leqslant \|L\|_{L(H)} \cdot \|B\|_{L_2}^2.$$

Therefore, it is allowed to interchange the sums to obtain that

$$\operatorname{tr} LBB^* = \sum_{k\in\mathbb{N}} \langle LBB^* f_k, f_k\rangle = \sum_{k\in\mathbb{N}} \langle B^* f_k, B^* L^* f_k\rangle_U$$
$$= \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} \langle B^* f_k, e_n\rangle_U \cdot \langle B^* L^* f_k, e_n\rangle_U = \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \langle f_k, Be_n\rangle \cdot \langle f_k, LBe_n\rangle$$
$$= \sum_{n\in\mathbb{N}} \langle Be_n, LBe_n\rangle = \sum_{n\in\mathbb{N}} \langle e_n, B^*LBe_n\rangle_U = \operatorname{tr} B^*LB. \qquad \square$$

C. Pseudo Inverse of Linear Operators

This chapter is a slight modification of Chapter C in [FK01].

Let $\big(U, \langle\ ,\ \rangle_U\big)$ and $\big(H, \langle\ ,\ \rangle\big)$ be two Hilbert spaces.

Definition C.0.1 (Pseudo inverse). Let $T \in L(U,H)$ and $\mathrm{Ker}(T) := \{x \in U \mid Tx = 0\}$. The pseudo inverse of T is defined as

$$T^{-1} := \big(T\,|_{\mathrm{Ker}(T)^\perp}\big)^{-1} : T\big(\mathrm{Ker}(T)^\perp\big) = T(U) \to \mathrm{Ker}(T)^\perp.$$

(Note that T is one-to-one on $\mathrm{Ker}(T)^\perp$.)

Remark C.0.2. *(i) There is an equivalent way of defining the pseudo inverse of a linear operator $T \in L(U,H)$. For $x \in T(U)$ one sets $T^{-1}x \in U$ to be the solution of minimal norm of the equation $Ty = x$, $y \in U$.*

(ii) If $T \in L(U,H)$ then $T^{-1} : T(U) \to \mathrm{Ker}(T)^\perp$ is linear and bijective.

Proposition C.0.3. *Let $T \in L(U)$ and T^{-1} the pseudo inverse of T.*

(i) If we define an inner product on $T(U)$ by

$$\langle x, y\rangle_{T(U)} := \langle T^{-1}x, T^{-1}y\rangle_U \quad \text{for all } x, y \in T(U),$$

then $\big(T(U), \langle\ ,\ \rangle_{T(U)}\big)$ is a Hilbert space.

(ii) Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of $(\mathrm{Ker}\,T)^\perp$. Then Te_k, $k \in \mathbb{N}$, is an orthonormal basis of $\big(T(U), \langle\ ,\ \rangle_{T(U)}\big)$.

Proof. $T : (\mathrm{Ker}\,T)^\perp \to T(U)$ is bijective and an isometry if $(\mathrm{Ker}\,T)^\perp$ is equipped with $\langle\ ,\ \rangle_U$ and $T(U)$ with $\langle\ ,\ \rangle_{T(U)}$. □

Now we want to present a result about the images of linear operators. To this end we need the following lemma.

Lemma C.0.4. *Let $T \in L(U,H)$. Then the set $T\overline{B_c(0)}$ $\big(= \big\{Tu \;\big|\; u \in U,\ \|u\|_U \leqslant c\big\}\big)$, $c \geqslant 0$, is convex and closed.*

Proof. Since T is linear it is obvious that the set is convex.

Since a convex subset of a Hilbert space is closed (with respect to the norm) if and only if it is weakly closed, it suffices to show that $T\overline{B_c(0)}$ is weakly closed. Since $T : U \to H$ is linear and continuous (with respect to the norms

on U, H respectively) it is also obviously continuous with respect to the weak topologies on U, H respectively. But by the Banach–Alaoglou theorem (see e.g. [RS72, Theorem IV.21, p. 115]) closed balls in a Hilbert space are weakly compact. Hence $\overline{B_c(0)}$ is weakly compact, and so is its continuous image, i.e. $T\overline{B_c(0)}$ is weakly compact, therefore weakly closed. □

Proposition C.0.5. *Let $\big(U_1, \langle\ ,\ \rangle_1\big)$ and $\big(U_2, \langle\ ,\ \rangle_2\big)$ be two Hilbert spaces. In addition, we take $T_1 \in L(U_1, H)$ and $T_2 \in L(U_2, H)$. Then the following statements hold.*

(i) If there exists a constant $c \geqslant 0$ such that $\|T_1^* x\|_1 \leqslant c\|T_2^* x\|_2$ for all $x \in H$ then $\big\{T_1 u \mid u \in U_1,\ \|u\|_1 \leqslant 1\big\} \subset \big\{T_2 v \mid v \in U_2,\ \|v\|_2 \leqslant c\big\}$. In particular, this implies that $\operatorname{Im} T_1 \subset \operatorname{Im} T_2$.

(ii) If $\|T_1^* x\|_1 = \|T_2^* x\|_2$ for all $x \in H$ then $\operatorname{Im} T_1 = \operatorname{Im} T_2$ and $\|T_1^{-1} x\|_1 = \|T_2^{-1} x\|_2$ for all $x \in \operatorname{Im} T_1$.

Proof. [DPZ92, Proposition B.1, p. 407]

(i) Assume that there exists $u_0 \in U_1$ such that

$$\|u_0\|_1 \leqslant 1 \quad \text{and} \quad T_1 u_0 \notin \big\{T_2 v \mid v \in U_2,\ \|v\|_2 \leqslant c\big\}.$$

By Lemma C.0.4 we know that the set $\big\{T_2 v \mid v \in U_2,\ \|v\|_2 \leqslant c\big\}$ is closed and convex. Therefore, we get by the separation theorem (see [Alt92, 5.11 Trennungssatz, p. 166]) there exists $x \in H$, $x \neq 0$, such that

$$1 < \langle x, T_1 u_0\rangle \quad \text{and} \quad \langle x, T_2 v\rangle \leqslant 1 \ \text{ for all } v \in U_2 \text{ with } \|v\|_2 \leqslant c.$$

Thus $\|T_1^* x\|_1 > 1$ and $c\|T_2^* x\|_2 = \sup\limits_{\|v\|_2 \leqslant c} \big|\langle T_2^* x, v\rangle_2\big| \leqslant 1$, a contradiction.

(ii) By (i) we know that $\operatorname{Im} T_1 = \operatorname{Im} T_2$. It remains to verify that

$$\|T_1^{-1} x\|_1 = \|T_2^{-1} x\|_2 \quad \text{for all } x \in \operatorname{Im} T_1.$$

If $x = 0$ then $\|T_1^{-1} 0\|_1 = 0 = \|T_2^{-1} 0\|_2$.

If $x \in \operatorname{Im} T_1 \setminus \{0\}$ then there exist $u_1 \in (\operatorname{Ker} T_1)^\perp$ and $u_2 \in (\operatorname{Ker} T_2)^\perp$ such that $x = T_1 u_1 = T_2 u_2$. We have to show that $\|u_1\|_1 = \|u_2\|_2$.

Assume that $\|u_1\|_1 > \|u_2\|_2 > 0$. Then (i) implies that

$$\begin{aligned}\frac{x}{\|u_2\|_2} &= T_2\Big(\frac{u_2}{\|u_2\|_2}\Big)\\ &\in \big\{T_2 v \mid v \in U_2,\ \|v\|_2 \leqslant 1\big\} = \big\{T_1 u \mid u \in U_1,\ \|u\|_1 \leqslant 1\big\}.\end{aligned}$$

But

$$\frac{x}{\|u_2\|_2} = T_1\Big(\frac{u_1}{\|u_2\|_2}\Big) \quad \text{and} \quad \Big\|\frac{u_1}{\|u_2\|_2}\Big\|_1 > 1,$$

therefore, there exists $\tilde{u}_1 \in U_1$, $\|\tilde{u}_1\|_1 \leqslant 1$, so that for $\tilde{u}_2 := \frac{u_1}{\|u_2\|_2} \in (\operatorname{Ker} T_1)^\perp$ we have

$$T_1 \tilde{u}_1 = \frac{x}{\|u_2\|_2} = T_1 \tilde{u}_2 \,, \quad \text{i.e. } \tilde{u}_1 - \tilde{u}_2 \in \operatorname{Ker} T_1.$$

Therefore,

$$\begin{aligned} 0 &= \langle \tilde{u}_1 - \tilde{u}_2, \tilde{u}_2 \rangle_1 = \langle \tilde{u}_1, \tilde{u}_2 \rangle_1 - \|\tilde{u}_2\|_1^2 \\ &\leqslant \|\tilde{u}_1\|_1 \|\tilde{u}_2\|_1 - \|\tilde{u}_2\|_1^2 = \big(1 - \|\tilde{u}_2\|_1\big)\|\tilde{u}_2\|_1. \end{aligned}$$

This is a contradiction. □

Corollary C.0.6. *Let $T \in L(U,H)$ and set $Q := TT^* \in L(H)$. Then we have*

$$\operatorname{Im} Q^{\frac{1}{2}} = \operatorname{Im} T \quad \textit{and} \quad \big\|Q^{-\frac{1}{2}} x\big\| = \|T^{-1}x\|_U \ \textit{for all } x \in \operatorname{Im} T,$$

where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$.

Proof. Since by Lemma 2.3.4 $Q^{\frac{1}{2}}$ is symmetric we have for all $x \in H$ that

$$\Big\|\big(Q^{\frac{1}{2}}\big)^* x\Big\|^2 = \big\|Q^{\frac{1}{2}} x\big\|^2 = \langle Qx, x\rangle = \langle TT^*x, x\rangle = \|T^*x\|_U^2.$$

Therefore the assertion follows by Proposition C.0.5. □

D. Some Tools from Real Martingale Theory

We need the following Burkholder–Davis inequality for real-valued continuous local martingales.

Proposition D.0.1. *Let $(N_t)_{t\in[0,T]}$ be a real-valued continuous local martingale on a probability space (Ω, E, P) with respect to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Then for all stopping times $\tau(\leqslant T)$*

$$E(\sup_{t\in[0,\tau]}|N_t|) \leqslant 3E(\langle N\rangle_\tau^{1/2}).$$

Proof. See e.g. [KS88, Theorem 3.28]. □

Corollary D.0.2. *Let $\varepsilon, \delta \in]0,\infty[$. Then for N as in Proposition D.0.1*

$$P(\sup_{t\in[0,T]}|N_t| \geqslant \varepsilon) \leqslant \frac{3}{\varepsilon}E(\langle N\rangle_T^{1/2} \wedge \delta) + P(\langle N\rangle_T^{1/2} > \delta).$$

Proof. Let

$$\tau := \inf\{t \geqslant 0 |\, \langle N\rangle_t^{1/2} > \delta\} \wedge T.$$

Then $\tau(\leqslant T)$ is an $\mathcal{F}_t$-stopping time. Hence by Proposition D.0.1

$$\begin{aligned}
&P\left(\sup_{t\in[0,T]}|N_t| \geqslant \varepsilon\right)\\
=&P\left(\sup_{t\in[0,T]}|N_t| \geqslant \varepsilon, \tau = T\right) + P\left(\sup_{t\in[0,T]}|N_t| \geqslant \varepsilon, \tau < T\right)\\
\leqslant&\frac{3}{\varepsilon}E(\langle N\rangle_\tau^{1/2}) + P\left(\sup_{t\in[0,T]}|N_t| \geqslant \varepsilon, \langle N\rangle_T^{1/2} > \delta\right)\\
\leqslant&\frac{3}{\varepsilon}E(\langle N\rangle_T^{1/2} \wedge \delta) + P(\langle N\rangle_T^{1/2} > \delta).
\end{aligned}$$

□

E. Weak and Strong Solutions: the Yamada-Watanabe Theorem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with normal filtration $\mathcal{F}_t$, $t \in [0, \infty[$. Below we shall call $((\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ a *stochastic basis.* Let $d, d_1 \in \mathbb{N}$ and let $M(d \times d_1, \mathbb{R})$ denote the set of all real $d \times d_1$-matrices equipped with the norm (3.1.2). Let

$$W^d := C([0, \infty[\to \mathbb{R}^d) \tag{E.0.1}$$

and

$$W_0^d := \{w \in W^d | w(0) = 0\}. \tag{E.0.2}$$

W^d is equipped with metric

$$\varrho(w_1, w_2) := \sum_{k=1}^{\infty} 2^{-k} (\max_{0 \leqslant t \leqslant k} |w_1(t) - w_2(t)| \wedge 1), \quad w_1, w_2 \in W^d, \tag{E.0.3}$$

which makes it a Polish space. Its Borel σ-algebra is denoted by $\mathcal{B}(W^d)$. Let $\mathcal{B}_t(W^d)$ denote the σ-Algebra generated by all maps π_s, $0 \leqslant s \leqslant t$, where $\pi_s(w) := w(s)$, $w \in W^d$. Let $\mathcal{A}^{d,d_1}$ denote the set of all $\mathcal{B}([0, \infty[) \otimes \mathcal{B}(W^d)/\mathcal{B}(M(d \times d_1, \mathbb{R}))$-measurable maps $\alpha : [0, \infty[\times W^d \to M(d \times d_1, \mathbb{R})$ such that for each t$\in [0, \infty[$ the map

$$W^d \ni w \mapsto \alpha(t, w) \in M(d \times d_1, \mathbb{R})$$

is $\mathcal{B}_t(W^d)/\mathcal{B}(M(d \times d_1, \mathbb{R}))$-measurable.

E.1. The main result

Fix $\sigma \in \mathcal{A}^{d,d_1}$ and $b \in \mathcal{A}^{d,1}$ and consider the following stochastic differential equation:

$$\mathrm{d}X(t) = b(t, X)\, \mathrm{d}t + \sigma(t, X)\, \mathrm{d}W(t), \quad t \in [0, \infty[. \tag{E.1.1}$$

Definition E.1.1. An $\mathbb{R}^d$-valued continuous, $(\mathcal{F}_t)$-adapted process $X(t)$, $t \in [0, \infty[$, on some stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ is called a *(weak) solution* to (E.1.1), if

(i)
$$\int_0^t |b(s,X)| \, \mathrm{d}s < \infty \quad P\text{-a.e. for all } t \in [0,\infty[.$$

(ii)
$$\int_0^t \|\sigma(s,X)\|^2 \, \mathrm{d}s < \infty \quad P\text{-a.e. for all } t \in [0,\infty[.$$

(iii) There exists an $\mathbb{R}^{d_1}$-valued standard $(\mathcal{F}_t)$-Wiener process $W(t)$, $t \in [0,\infty[$, on $(\Omega,\mathcal{F},P)$ such that P-a.e.

$$X(t) = X(0) + \int_0^t b(s,X) \, \mathrm{d}s + \int_0^t \sigma(s,X) \, \mathrm{d}W(s), \quad t \in [0,\infty[. \tag{E.1.2}$$

Remark E.1.2. *(i) Clearly, by the measurability assumption on elements in $\mathcal{A}^{d,d_1}$ it follows that if X is a solution, then $[0,t] \times \Omega \ni (s,\omega) \mapsto \sigma(s,X(\omega))$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}/\mathcal{B}(M(d \times d_1,\mathbb{R}))$-measurable and $\sigma(t,X)$ is $\mathcal{F}_t$-measurable for $t \in [0,\infty[$. Likewise for $b(\cdot,X)$. The $(\mathcal{F}_t)$-adaptedness for $\sigma(\cdot,X)$ and $b(\cdot,X)$ follows since the $(\mathcal{F}_t)$-adaptiveness of X is equivalent to the $\mathcal{F}_t/\mathcal{B}_t(W^d)$ measurability of X.*

(ii) Below we shall briefly say (X,W) in Definition E.1.1 is a (weak) solution to (E.1.1) *not always mentioning explicitly the stochastic basis, that comes with it.*

Definition E.1.3. We say that *(weak) uniqueness* holds for (E.1.1) if whenever X and X' are two (weak) solutions (with stochastic bases $(\Omega,\mathcal{F},P,(\mathcal{F}_t))$, $(\Omega',\mathcal{F}',P',(\mathcal{F}'_t))$ and associated Wiener processes $W(t)$, $W'(t)$, $t \in [0,\infty[$) such that

$$P \circ X(0)^{-1} = P' \circ X'(0)^{-1},$$

(as measures on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$), then

$$P \circ X^{-1} = P' \circ (X')^{-1}$$

(as measures on $(W^d,\mathcal{B}(W^d))$).

Definition E.1.4. We say that *pathwise uniqueness* holds for (E.1.1), if whenever X and X' are two (weak) solutions on the same stochastic basis $(\Omega,\mathcal{F},P,(\mathcal{F}_t))$ and with the same $(\mathcal{F}_t)$-Wiener process $W(t)$, $t \in [0,\infty[$ on $(\Omega,\mathcal{F},P)$ such that $X(0) = X'(0)$ P-a.e., then P-a.e.

$$X(t) = X'(t),\, t \in [0,\infty[.$$

To define strong solutions we need to introduce the following class $\hat{\mathcal{E}}$ of maps:

Let $\hat{\mathcal{E}}$ denote the set of all maps $F : \mathbb{R}^d \times W_0^{d_1} \to W^d$ such that for every probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ there exists a $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})}^{\mu \otimes P^W} / \mathcal{B}(W^d)$-measurable map $F_\mu : \mathbb{R}^d \times W_0^{d_1} \to W^d$ such that for μ-a.e. $x \in \mathbb{R}^d$

$$F(x, w) = F_\mu(x, w) \text{ for } P^W\text{-a.e. } w \in W_0^{d_1}.$$

Here $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})}^{\mu \otimes P^W}$ denotes the completion of $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})$ with respect to $\mu \otimes P^W$, and P^W denotes classical Wiener measure on $(W_0^{d_1}, \mathcal{B}(W_0^{d_1}))$.

Let $F \in \hat{\mathcal{E}}$. For an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable map $\xi : \Omega \to \mathbb{R}^d$ on some probability space $(\Omega, \mathcal{F}, P)$ and an $\mathbb{R}^{d_1}$-valued, standard Wiener process $W(t)$, $t \in [0, \infty[$, on $(\Omega, \mathcal{F}, P)$ independent of ξ, we set

$$F(\xi, W) := F_{P \circ \xi^{-1}}(\xi, W).$$

Definition E.1.5. A (weak) solution X to (E.1.1) on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ and associated Wiener process $W(t)$, $t \in [0, \infty[$, is called a *strong solution* if there exists $F \in \hat{\mathcal{E}}$ such that for $x \in \mathbb{R}^d$, $w \mapsto F(x, w)$ is $\overline{\mathcal{B}_t(W_0^{d_1})}^{P^W} / \mathcal{B}_t(W^d)$-measurable for every $t \in [0, \infty[$ and

$$X = F(X(0), W) \quad P\text{-a.e.},$$

where $\overline{\mathcal{B}_t(W_0^{d_1})}^{P^W}$ denotes the completion with respect to P^W in $\mathcal{B}(W_0^{d_1})$.

Definition E.1.6. Equation (E.1.1) is said to have a *unique strong solution*, if there exists $F \in \hat{\mathcal{E}}$ satisfying the adaptiveness condition in Definition E.1.5 and such that:

1. For every $\mathbb{R}^{d_1}$-valued standard $(\mathcal{F}_t)$-Wiener process $W(t)$, $t \in [0, \infty[$, on a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ and any $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable $\xi : \Omega \to \mathbb{R}^d$ the continuous process

   $$X := F(\xi, W)$$

   satisfies (i), (ii) and (E.1.2) in Definition E.1.1, i.e. $(F(\xi, W), W)$ is a (weak) solution to (E.1.1), and $X(0) = \xi$ P-a.e..

2. For any (weak) solution (X, W) to (E.1.1) we have

   $$X = F(X(0), W) \ P\text{-a.e.}.$$

Remark E.1.7. *Since $X(0)$ in the above definition is P-independent of W, thus*

$$P \circ (X(0), W)^{-1} = \mu \otimes P^W,$$

we have that the existence of a unique strong solution for (E.1.1) *implies that also (weak) uniqueness holds.*

Now we can formulate the main result of this section.

Theorem E.1.8. *Let $\sigma \in \mathcal{A}^{d,d_1}$ and $b \in \mathcal{A}^{d,1}$. Then equation* (E.1.1) *has a unique strong solution if and only if both of the following properties hold:*

(i) For every probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ there exists a (weak) solution (X, W) of (E.1.1) *such that μ is the distribution of $X(0)$.*

(ii) Pathwise uniqueness holds for (E.1.1).

Proof. Suppose (E.1.1) has a unique strong solution. Then (ii) obviously holds. To show (i) one only has to take the classical Wiener space $(W_0^{d_1}, \mathcal{B}(W_0^{d_1}), P^W)$ and consider $(\mathbb{R}^d \times W_0^{d_1}, \overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})}^{\mu \otimes P^W}, \mu \otimes P^W)$ with filtration

$$\bigcap_{\varepsilon > 0} \sigma(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_{t+\varepsilon}(W_0^{d_1}), \mathcal{N}), \quad t \geqslant 0,$$

where $\mathcal{N}$ denotes all $\mu \otimes P^W$-zero sets in $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})}^{\mu \otimes P^W}$. Let $\xi : \mathbb{R}^d \times W_0^{d_1} \to \mathbb{R}^d$ and $W : \mathbb{R}^d \times W_0^{d_1} \to W_0^{d_1}$ be the canonical projections. Then $X := F(\xi, W)$ is the desired weak solution in (i).

Now let us suppose that (i) and (ii) hold. The proof that then there exists a unique strong solution for (E.1.1) is quite technical. We structure it through a series of lemmas.

Lemma E.1.9. *Let $(\Omega, \mathcal{F})$ be a measurable space such that $\{\omega\} \in \mathcal{F}$ for all $\omega \in \Omega$ and such that*

$$D := \{(\omega, \omega) | \omega \in \Omega\} \in \mathcal{F} \otimes \mathcal{F}$$

(which is e.g. the case if Ω is a Polish space and $\mathcal{F}$ its Borel σ-algebra). Let P_1, P_2 be probability measures on $(\Omega, \mathcal{F})$ such that $P_1 \otimes P_2(D) = 1$. Then $P_1 = P_2 = \delta_{\omega_0}$ for some $\omega_0 \in \Omega$.

Proof. Let $f : \Omega \to [0, \infty[$ be $\mathcal{F}$-measurable. Then

$$\begin{aligned}
\int f(\omega_1) P_1(d\omega_1) &= \iint f(\omega_1) P_1(d\omega_1) P_2(d\omega_2) \\
&= \iint 1_D(\omega_1, \omega_2) f(\omega_1) P_1(d\omega_1) P_2(d\omega_2) \\
&= \iint 1_D(\omega_1, \omega_2) f(\omega_2) P_1(d\omega_1) P_2(d\omega_2) = \int f(\omega_2) P_2(d\omega_2),
\end{aligned}$$

so $P_1 = P_2$. Furthermore,

$$1 = \iint 1_D(\omega_1, \omega_2) P_1(d\omega_1) P_2(d\omega_2) = \int P_1(\{\omega_2\}) P_2(d\omega_2),$$

hence $1 = P_1(\{\omega_2\})$ for P_2-a.e. $\omega_2 \in \Omega$. Therefore, $P_1 = \delta_{\omega_0}$ for some $\omega_0 \in \Omega$. □

Fix a probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and let (X, W) with stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a (weak) solution to (E.1.1) with initial distribution μ. Define a probability measure P_μ on $(\mathbb{R}^d \times W^d \times W_0^{d_1}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1}))$, by

$$P_\mu := P \circ (X(0), X, W)^{-1}.$$

Lemma E.1.10. *There exists a family $K_\mu((x,w), dw_1), x \in \mathbb{R}^d, w \in W_0^{d_1}$, of probability measures on $(W^d, \mathcal{B}(W^d))$ having the following properties:*

(i) For every $A \in \mathcal{B}(W^d)$ the map

$$\mathbb{R}^d \times W_0^{d_1} \ni (x, w) \mapsto K_\mu((x,w), A)$$

is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})$-measurable.

(ii) For every $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1})$-measurable map $f : \mathbb{R}^d \times W^d \times W_0^{d_1} \to [0, \infty[$ we have

$$\begin{aligned} &\int f(x, w_1, w) P_\mu(dx\, dw_1\, dw) \\ &= \int_{\mathbb{R}^d} \int_{W_0^{d_1}} \int_{W^d} f(x, w_1, w) K_\mu((x,w), dw_1) P^W(dw) \mu(dx). \end{aligned}$$

(iii) If $t \in [0, \infty[$ and $f : W^d \to [0, \infty[$ is $\mathcal{B}_t(W^d)$-measurable, then

$$\mathbb{R}^d \times W_0^{d_1} \ni (x, w) \mapsto \int f(w_1) K_\mu((x,w), dw_1)$$

is $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_t(W_0^{d_1})}^{\mu \otimes P^W}$-measurable, where $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_t(W_0^{d_1})}^{\mu \otimes P^W}$ denotes the completion with respect to $\mu \otimes P^W$ in $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})$.

Proof. Let $\Pi : \mathbb{R}^d \times W^d \times W_0^{d_1} \to \mathbb{R}^d \times W_0^{d_1}$ be the canonical projection. Since $X(0)$ is $\mathcal{F}_0$-measurable, hence P-independent of W, it follows that

$$P_\mu \circ \Pi^{-1} = P \circ (X(0), W)^{-1} = \mu \otimes P^W.$$

Hence by the existence result on regular conditional distributions (cf. e.g. [IW81, Corollary to Theorem 3.3 on p. 15]), the existence of the family $K_\mu((x, w),\ \mathrm{d}w_1)$, $x \in \mathbb{R}^d$, $w \in W_0^{d_1}$, satisfying (i) and (ii) follows.

To prove (iii) it suffices to show that for $t \in [0, \infty[$ and for all $A_0 \in \mathcal{B}(\mathbb{R}^d)$, $A_1 \in \mathcal{B}_t(W^d)$, $A \in \mathcal{B}_t(W_0^{d_1})$ and

$$A' := \{\pi_{r_1} - \pi_t \in B_1, \ldots, \pi_{r_k} - \pi_t \in B_k\},$$

$$t \leqslant r_1 < \ldots < r_k,\ B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}^{d_1}),$$

$$\begin{aligned}
&\int_{A_0}\int_{W_0^{d_1}} 1_{A\cap A'}(w)K_\mu((x,w),A_1)P^W(dw)\mu(dx)\\
&=\int_{A_0}\int_{W_0^{d_1}} 1_{A\cap A'}(w)E_{\mu\otimes P^W}(K_\mu(\cdot,A_1)|\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}_t(W_0^{d_1}))P^W(dw)\mu(dx),
\end{aligned} \tag{E.1.3}$$

since the system of all $A\cap A'$, $A\in\mathcal{B}_t(W_0^{d_1})$, A' as above generates $\mathcal{B}(W_0^{d_1})$. But by part (ii) above, the left-hand side of (E.1.3) is equal to

$$\begin{aligned}
&\int 1_{A_0}(x)1_{A\cap A'}(w)1_{A_1}(w_1)P_\mu(dx\,dw_1\,dw)\\
&=\int 1_{A_0}(X(0))1_{A_1}(X)1_A(W)1_{A'}(W)\,\mathrm{d}P \qquad\qquad \text{(E.1.4)}\\
&=\int 1_{A_0}(X(0))1_{A_1}(X)1_A(W)E_P(1_{A'}(W)|\mathcal{F}_t)\,\mathrm{d}P.
\end{aligned}$$

But $1_{A'}(W)$ is P-independent of $\mathcal{F}_t$, since W is an $(\mathcal{F}_t)$-Wiener process on $(\Omega,\mathcal{F},P)$, so

$$E_P(1_{A'}(W)|\mathcal{F}_t)=E_P(1_{A'}(W)).$$

Hence the right-hand side of (E.1.4) is equal to

$$\begin{aligned}
&P^W(A')\int 1_{A_0}(x)1_A(w)1_{A_1}(w_1)P_\mu(dx\,dw_1\,dw)\\
&=P^W(A')\int_{A_0}\int_A K_\mu((x,w),A_1)P^W(dw)\mu(dx)\\
&=P^W(A')\int_{A_0}\int_A E_{\mu\otimes P^W}(K_\mu(\cdot,A_1)|\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}_t(W_0^{d_1}))((x,w))\\
&\qquad P^W(dw)\mu(dx)\\
&=\int_{A_0}\int_{W_0^{d_1}} 1_{A\cap A'}(w)E_{\mu\otimes P^W}(K_\mu(\cdot,A_1)|\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}_t(W_0^{d_1}))((x,w))\\
&\qquad P^W(dw)\mu(dx),
\end{aligned}$$

since A' is P^W-independent of $\mathcal{B}_t(W_0^{d_1})$. □

For $x\in\mathbb{R}^d$ define a measure Q_x on

$$(\mathbb{R}^d\times W^d\times W^d\times W_0^{d_1},\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(W^d)\otimes\mathcal{B}(W^d)\otimes\mathcal{B}(W_0^{d_1}))$$

by

$$\begin{aligned}
Q_x(A):=&\int_{R^d}\int_{W_0^{d_1}}\int_{W^d}\int_{W^d} 1_A(z,w_1,w_2,w)\\
&\qquad K_\mu((z,w),dw_1)K_\mu((z,w),dw_2)P^W(dw)\delta_x(dz).
\end{aligned}$$

Define the stochastic basis

$$
\begin{aligned}
\tilde{\Omega} &:= \mathbb{R}^d \times W^d \times W^d \times W_0^{d_1} \\
\tilde{\mathcal{F}}^x &:= \overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1})}^{Q_x} \\
\tilde{\mathcal{F}}_t^x &:= \bigcap_{\varepsilon>0} \sigma(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_{t+\varepsilon}(W^d) \otimes \mathcal{B}_{t+\varepsilon}(W^d) \otimes \mathcal{B}_{t+\varepsilon}(W_0^{d_1}), \mathcal{N}_x),
\end{aligned}
$$

where

$$\mathcal{N}_x := \{N \in \tilde{\mathcal{F}}^x | Q_x(N) = 0\},$$

and define maps

$$
\begin{aligned}
&\Pi_0 : \tilde{\Omega} \to \mathbb{R}^d,\ (x, w_1, w_2, w) \mapsto x, \\
&\Pi_i : \tilde{\Omega} \to W^d,\ (x, w_1, w_2, w) \mapsto w_i \in W^d, \quad i = 1, 2, \\
&\Pi_3 : \tilde{\Omega} \to W_0^{d_1},\ (x, w_1, w_2, w) \mapsto w \in W_0^{d_1}.
\end{aligned}
$$

Then, obviously,

$$Q_x \circ \Pi_0^{-1} = \delta_x \tag{E.1.5}$$

and

$$Q_x \circ \Pi_3^{-1} = P^W (= P \circ W^{-1}). \tag{E.1.6}$$

Lemma E.1.11. *There exists $N_0 \in \mathcal{B}(\mathbb{R}^d)$ with $\mu(N_0) = 0$ such that for all $x \in N_0^c$ we have that Π_3 is an $(\tilde{\mathcal{F}}_t^x)$-Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}^x, Q_x)$ taking values in $\mathbb{R}^{d_1}$.*

Proof. By definition Π_3 is $(\tilde{\mathcal{F}}_t^x)$-adapted for every $x \in \mathbb{R}^d$. Furthermore, for $0 \leqslant s < t$, $y \in \mathbb{R}^d$, and $A_0 \in \mathcal{B}(\mathbb{R}^d)$, $A_i \in \mathcal{B}_s(W^d)$, $i = 1, 2$, $A_3 \in \mathcal{B}_s(W_0^{d_1})$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} E_{Q_x}(\exp(i\langle y, \Pi_3(t) - \Pi_3(s)\rangle) 1_{A_0 \times A_1 \times A_2 \times A_3}) \mu(dx) \\
&= \int_{\mathbb{R}^d} \int_{W_0^{d_1}} \exp(i\langle y, w(t) - w(s)\rangle) 1_{A_0}(x) 1_{A_3}(w) \\
&\qquad K_\mu((x, w), A_1) K_\mu((x, w), A_2) P^W(dw) \mu(dx) \\
&= \int_{W_0^{d_1}} \exp(i\langle y, w(t) - w(s)\rangle) P^W(dw) \int_{\mathbb{R}^d} Q_x(A_0 \times A_1 \times A_2 \times A_3) \mu(dx),
\end{aligned}
$$

where we used Lemma E.1.10(iii) in the last step. Now the assertion follows by (E.1.6), a monotone class argument and the same reasoning as in the proof of Proposition 2.1.13. $\square$

Lemma E.1.12. *There exists $N_1 \in \mathcal{B}(\mathbb{R}^d)$, $N_0 \subset N_1$, with $\mu(N_1) = 0$ such that for all $x \in N_1^c$, (Π_1, Π_3) and (Π_2, Π_3) with stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}^x, Q_x, (\tilde{\mathcal{F}}_t^x))$ are (weak) solutions of* (E.1.1) *such that*

$$\Pi_1(0) = \Pi_2(0) = x \quad Q_x\text{-a.e.},$$

therefore, $\Pi_1 = \Pi_2$ Q_x-a.e.

Proof. For $i = 1, 2$ consider the set $A_i \in \tilde{\mathcal{F}}^x$ defined by

$$A_i := \Big\{ \Pi_i(t) - \Pi_i(0) = \int_0^t b(s, \Pi_i)\, \mathrm{d}s + \int_0^t \sigma(s, \Pi_i)\, \mathrm{d}\Pi_3(s) \\ \text{for all } t \in [0, \infty[\Big\} \cap \{\Pi_i(0) = \Pi_0\}.$$

Define $A \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1})$ analogously with Π_i replaced by the canonical projection from $\mathbb{R}^d \times W^d \times W_0^{d_1}$ onto the second and Π_0, Π_3 by the canonical projection onto the first and third coordinate respectively. Then by Lemma E.1.10 (ii) for $i = 1, 2$

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{W_0^{d_1}} \int_{W^d} \int_{W^d} 1_{A_i}(x, w_1, w_2, w) \\ &\qquad K_\mu((x, w), dw_1) K_\mu((x, w), dw_2) P^W(dw)\mu(dx) \\ &= P_\mu(A) = P(\{(X(0), X, W) \in A\}) = 1. \end{aligned} \tag{E.1.7}$$

Since all measures in the left-hand side of (E.1.7) are probability measures, it follows that for μ-a.e. $x \in \mathbb{R}^d$

$$1 = Q_x(A_i) = Q_x(A_{i,x}),$$

where for $i = 1, 2$

$$A_{i,x} := \Big\{ \Pi_i(t) - x = \int_0^t b(s, \Pi_i)\, \mathrm{d}s + \int_0^t \sigma(s, \Pi_i)\, \mathrm{d}\Pi_3(s),\ \forall t \in [0, \infty[\Big\}!.$$

Hence the first assertion follows. The second then follows by the pathwise uniqueness assumption in condition (ii) of the theorem. □

Lemma E.1.13. *There exists a $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})}^{\mu \otimes P^W} / \mathcal{B}(W^d)$- measurable map*

$$F_\mu : \mathbb{R}^d \times W_0^{d_1} \to W^d$$

such that

$$K_\mu((x, w), \cdot) = \delta_{F_\mu(x,w)}$$

$$\text{(= Dirac measure on } \mathcal{B}(W^d) \text{ with mass in } F_\mu(x, w))$$

for $\mu \otimes P^W$*-a.e.* $(x,w) \in \mathbb{R}^d \times W_0^{d_1}$. *Furthermore,* F_μ *is* $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_t(W_0^{d_1})}^{\mu\otimes P^W}/\mathcal{B}_t(W^d)$*-measurable for all* $t \in [0,\infty[$, *where* $\overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_t(W_0^{d_1})}^{\mu\otimes P^W}$ *denotes the completion with respect to* $\mu \otimes P^W$ *in* $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_t(W_0^{d_1})$.

Proof. By Lemma E.1.12 for all $x \in N_1^c$, we have

$$\begin{aligned}1 &= Q_x(\{\Pi_1 = \Pi_2\})\\ &= \int_{W_0^{d_1}} \int_{W^d} \int_{W^d} 1_D(w_1,w_2) K_\mu((x,w),dw_1) K_\mu((x,w),dw_2) P^W(dw),\end{aligned}$$

where $D := \{(w_1,w_1) \in W^d \times W^d | w_1 \in W^d\}$. Hence by Lemma E.1.9 there exists $N \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W_0^{d_1})$ such that $\mu \otimes P^W(N) = 0$ and for all $(x,w) \in N^c$ there exists $F_\mu(x,w) \in W^d$ such that

$$K_\mu((x,w),dw_1) = \delta_{F_\mu(x,w)}(dw_1).$$

Set $F_\mu(x,w) := 0$, if $(x,w) \in N$. Let $A \in \mathcal{B}(W^d)$. Then

$$\{F_\mu \in A\} = (\{F_\mu \in A\} \cap N) \cup (\{K_\mu(\cdot, A) = 1\} \cap N^c)$$

and the measurability properties of F_μ follow from Lemma E.1.10. □

Having defined the mapping F_μ let us check the conditions of Definition E.1.5 and Definition E.1.6. We start with condition 2.

Lemma E.1.14. *We have*

$$X = F_\mu(X(0),W) \quad P\text{-a.e.}.$$

Proof. By Lemmas E.1.10 and E.1.13 we have

$$\begin{aligned}&P(\{X = F_\mu(X(0),W)\})\\ =&\int_{\mathbb{R}^d} \int_{W_0^{d_1}} \int_{W^d} 1_{\{w_1 = F_\mu(x,w)\}}(x,w_1,w) \delta_{F_\mu(x,w)}(dw_1) P^W(dw)\mu(dx)\\ =&1.\end{aligned}$$

□

Now let us check condition 1. Let W' be another $\mathbb{R}^{d_1}$-valued standard $(\mathcal{F}'_t)$-Wiener process on a stochastic basis $(\Omega', \mathcal{F}', P', (\mathcal{F}'_t))$ and $\xi : \Omega' \to \mathbb{R}^d$ an $\mathcal{F}'_0/\mathcal{B}(\mathbb{R}^d)$-measurable map and $\mu := P' \circ \xi^{-1}$. Let F_μ be as above and set

$$X' := F_\mu(\xi, W').$$

Lemma E.1.15. *(X', W') is a (weak) solution to* (E.1.1) *with $X'(0) = \xi$ P'-a.s..*

Proof. We have

$$\begin{aligned} P'(\{\xi = X'(0)\}) &= P'(\{\xi = F_\mu(\xi, W')(0)\}) \\ &= \mu \otimes P^W(\{(x,w) \in \mathbb{R}^d \times W_0^{d_1} | x = F_\mu(x,w)\}) \\ &= P(\{X(0) = F_\mu(X(0), W)(0)\}) = 1, \end{aligned}$$

where we used Lemma E.1.14 in the last step.

To see that (X', W') is a (weak) solution we consider the set $A \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1})$ defined in the proof of Lemma E.1.12. We have to show that

$$P'(\{(X'(0), X', W') \in A\}) = 1.$$

But since $X'(0) = \xi$ is P'-independent of W', we have

$$\begin{aligned} &\int 1_A(X'(0), F_\mu(X'(0), W'), W')\, \mathrm{d}P' \\ =&\int_{\mathbb{R}^d} \int_{W_0^{d_1}} 1_A(x, F_\mu(x,w), w) P^W(dw)\mu(dx) \\ =&\int_{\mathbb{R}^d} \int_{W_0^{d_1}} \int_{W^d} 1_A(x, w_1, w) \delta_{F_\mu(x,w)}(dw_1) P^W(dw)\mu(dx) \\ =&\int 1_A(x, w_1, w) P_\mu(dx\, dw_1\, dw) \\ =&P(\{(X(0), X, W) \in A\}) = 1, \end{aligned}$$

where we used E.1.10 and E.1.11 in the second to last step. □

To complete the proof we still have to construct $F \in \hat{\mathcal{E}}$ and to check the adaptiveness conditions on the corresponding mappings F_μ. Below we shall apply what we have obtained above now also to δ_x replacing μ. So, for each $x \in \mathbb{R}^d$ we have a function F_{δ_x}. Now define

$$F(x,w) := F_{\delta_x}(x,w),\ x \in \mathbb{R}^d,\ w \in W_0^{d_1}. \tag{E.1.8}$$

The proof of Theorem E.1.8 is then completed by the following lemma.

Lemma E.1.16. *Let μ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $F_\mu : \mathbb{R}^d \times W_0^{d_1} \to W^d$ as constructed in Lemma E.1.13. Then for μ-a.e. $x \in \mathbb{R}^d$*

$$F(x,\cdot) = F_\mu(x,\cdot) \quad P^W - a.e..$$

Furthermore, $F(x,\cdot)$ is $\overline{\mathcal{B}_t(W_0^{d_1})}^{P^W} / \mathcal{B}_t(W^d)$-measurable for all $x \in \mathbb{R}^d$, $t \in [0,\infty[$, where $\overline{\mathcal{B}_t(W_0^{d_1})}^{P^W}$ denotes the completion of $\mathcal{B}_t(W_0^{d_1})$ with respect to P^W in $B(W_0^{d_1})$.

Proof. Let

$$\bar{\Omega} := \mathbb{R}^d \times W^d \times W_0^{d_1}$$

$$\bar{\mathcal{F}} := \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1})$$

and fix $x \in \mathbb{R}^d$. Define a measure $\bar{Q}_x$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ by

$$\bar{Q}_x(A) := \int_{\mathbb{R}^d} \int_{W_0^{d_1}} \int_{W^d} 1_A(z, w_1, w) K_\mu((z, w), dw_1) P^W(dw) \delta_x(dz)$$

with K_μ as in Lemma E.1.10. Consider the stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}^x, \bar{Q}_x, (\bar{\mathcal{F}}_t^x))$ where

$$\bar{\mathcal{F}}^x := \overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1})}^{\bar{Q}_x},$$

$$\bar{\mathcal{F}}_t^x := \bigcap_{\varepsilon > 0} \sigma(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_{t+\varepsilon}(W^d) \otimes \mathcal{B}_{t+\varepsilon}(W_0^{d_1}), \bar{\mathcal{N}}_x),$$

where $\bar{\mathcal{N}}_x := \{N \in \bar{\mathcal{F}}^x | \bar{Q}_x(N) = 0\}$. As in the proof of Lemma E.1.12 one shows that (Π, Π_3) on $(\bar{\Omega}, \bar{\mathcal{F}}^x, \bar{Q}_x, (\bar{\mathcal{F}}_t^x))$ is a (weak) solution to (E.1.1) with $\Pi(0) = x$ $\bar{Q}_x$-a.e. Here

$$\Pi_0 : \mathbb{R}^d \times W^d \times W_0^{d_1} \to \mathbb{R}^d,\ (x, w_1, w) \mapsto x,$$

$$\Pi : \mathbb{R}^d \times W^d \times W_0^{d_1} \to W^d,\ (x, w_1, w) \mapsto w_1,$$

$$\Pi_3 : \mathbb{R}^d \times W^d \times W_0^{d_1} \to W_0^{d_1},\ (x, w_1, w) \mapsto w.$$

By Lemma E.1.15 $(F_{\delta_x}(x, \Pi_3), \Pi_3)$ on the stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}^x, \bar{Q}_x, (\bar{\mathcal{F}}_t^x))$ is a (weak) solution to (E.1.1) with

$$F_{\delta_x}(x, \Pi_3)(0) = x.$$

Hence by our pathwise uniqueness assumption (ii), it follows that

$$F_{\delta_x}(x, \Pi_3) = \Pi \quad \bar{Q}_x\text{-a.s.}. \tag{E.1.9}$$

Hence for all $A \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(W^d) \otimes \mathcal{B}(W_0^{d_1})$ by Lemma E.1.13 and (E.1.9)

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{W^d} \int_{W_0^{d_1}} 1_A(x, w_1, w) \delta_{F_\mu(x,w)}(dw_1) P^W(dw) \mu(dx) \\
&= \int_{\mathbb{R}^d} \bar{Q}_x(A) \mu(dx) \\
&= \int_{\mathbb{R}^d} \int_{\bar{\Omega}} 1_A(\Pi_0, F_{\delta_x}(x, \Pi_3), \Pi_3)\, \mathrm{d}\bar{Q}_x \mu(dx) \\
&= \int_{\mathbb{R}^d} \int_{W_0^{d_!}} 1_A(x, F_{\delta_x}(x, w), w) P^W(dw) \mu(dx) \\
&= \int_{\mathbb{R}^d} \int_{W_0^{d_1}} \int_{W^d} 1_A(x, w_1, w) \delta_{F_{\delta_x}(x,w)}(dw_1) P^W(dw) \mu(dx),
\end{aligned}$$

which implies the assertion.

Let $x \in \mathbb{R}^d$, $t \in [0, \infty[$, $A \in \mathcal{B}_t(W^d)$, and define

$$\bar{F}_{\delta_x} := 1_{\{x\} \times W_0^{d_1}} F_{\delta_x}.$$

Then

$$\bar{F}_{\delta_x} = F_{\delta_x} \quad \delta_x \otimes P^W - \text{a.e.},$$

hence

$$\{\bar{F}_{\delta_x} \in A\} \in \overline{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}_t(W_0^{d_1})}^{\delta_x \otimes P^W}. \tag{E.1.10}$$

But

$$\{\bar{F}_{\delta_x} \in A\} = \{x\} \times \{F_{\delta_x}(x, \cdot) \in A\} \cup (\mathbb{R}^d \backslash \{x\}) \times \{0 \in A\},$$

so by (E.1.10) it follows that

$$\{F_{\delta_x}(x, \cdot) \in A\} \in \overline{\mathcal{B}_t(W_0^{d_1})}^{P^W}.$$

$\square$

F. Strong, Mild and Weak Solutions

This chapter is a short version of Chapter 2 in [FK01]. We only state the results and refer to [FK01], [DPZ92] for the proofs.

As in previous chapters let $(U, \| \|_U)$ and $(H, \| \|)$ be separable Hilbert spaces. We take $Q = I$ and fix a cylindrical Q-Wiener process $W(t)$, $t \geqslant 0$, in U on a probability space $(\Omega, \mathcal{F}, P)$ with a normal filtration $\mathcal{F}_t$, $t \geqslant 0$. Moreover, we fix $T > 0$ and consider the following type of stochastic differential equations in H:

$$\begin{aligned} dX(t) &= [CX(t) + F(X(t))]\,dt + B(X(t))\,dW(t), \quad t \in [0, T], \\ X(0) &= \xi, \end{aligned} \tag{F.0.1}$$

where:

- $C : D(C) \to H$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geqslant 0$, of linear operators on H,
- $F : H \to H$ is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable,
- $B : H \to L(U, H)$,
- ξ is a H-valued, $\mathcal{F}_0$-measurable random variable.

Definition F.0.1 (mild solution). An H-valued predictable process $X(t)$, $t \in [0, T]$, is called a *mild solution* of problem (F.0.1) if

$$\begin{aligned} X(t) = S(t)\xi &+ \int_0^t S(t-s)F(X(s))\,ds \\ &+ \int_0^t S(t-s)B(X(s))\,dW(s) \quad P\text{-a.s.} \end{aligned} \tag{F.0.2}$$

for each $t \in [0, T]$. In particular, the appearing integrals have to be well-defined.

Definition F.0.2 (analytically strong solutions). A $D(C)$-valued predictable process $X(t)$, $t \in [0, T]$, (i.e. $(s, \omega) \mapsto X(s, \omega)$ is $\mathcal{P}_T/\mathcal{B}(H)$-measurable) is called an *analytically strong solution* of problem (F.0.1) if

$$X(t) = \xi + \int_0^t CX(s) + F(X(s))\,ds + \int_0^t B(X(s))\,dW(s) \quad P\text{-a.s.} \tag{F.0.3}$$

for each $t \in [0, T]$. In particular, the integrals on the right-hand side have to be well-defined, that is, $CX(t)$, $F(X(t))$, $t \in [0, T]$, are P-a.s. Bochner integrable and $\mathcal{B}(X) \in \mathcal{N}_W$.

Definition F.0.3 (analytically weak solution). An H-valued predictable process $X(t)$, $t \in [0, T]$, is called an *analytically weak solution* of problem (F.0.1) if

$$\begin{aligned} \langle X(t), \zeta \rangle = \langle \xi, \zeta \rangle &+ \int_0^t \langle X(s), C^* \zeta \rangle + \langle F(X(s)), \zeta \rangle \, ds \\ &+ \int_0^t \langle \zeta, B(X(s)) dW(s) \rangle \quad P\text{-a.s.} \end{aligned} \tag{F.0.4}$$

for each $t \in [0, T]$ and $\zeta \in D(C^*)$. Here $(C^*, D(C^*))$ is the adjoint of $(C, D(C))$ on H.
In particular, as in Definitions F.0.2 and F.0.1, the appearing integrals have to be well-defined.

Proposition F.0.4 (analytically weak versus analytically strong solutions).

(i) Every analytically strong solution of problem (F.0.1) *is also an analytically weak solution.*

(ii) Let $X(t)$, $t \in [0, T]$, be an analytically weak solution of problem (F.0.1) *with values in $D(C)$ such that $B(X(t))$ takes values in $L_2(U, H)$ for all $t \in [0, T]$. Besides we assume that*

$$P\left(\int_0^T \|CX(t)\| \, dt < \infty \right) = 1$$

$$P\left(\int_0^T \|F(X(t))\| \, dt < \infty \right) = 1$$

$$P\left(\int_0^T \|B(X(t))\|_{L_2}^2 \, dt < \infty \right) = 1.$$

Then the process is also an analytically strong solution.

Proposition F.0.5 (analytically weak versus mild solutions).

(i) Let $X(t)$, $t \in [0, T]$, be an analytically weak solution of problem (F.0.1) *such that $B(X(t))$ takes values in $L_2(U, H)$ for all $t \in [0, T]$. Besides*

we assume that

$$P\left(\int_0^T \|X(t)\|\,dt < \infty\right) = 1$$
$$P\left(\int_0^T \|F(X(t))\|\,dt < \infty\right) = 1$$
$$P\left(\int_0^T \|B(X(t))\|_{L_2}^2\,dt < \infty\right) = 1.$$

Then the process is also a mild solution.

(ii) Let $X(t)$, $t \in [0,T]$, be a mild solution of problem (F.0.1) *such that the mappings*

$$(t,\omega) \mapsto \int_0^t S(t-s)F(X(s,\omega))\,ds$$
$$(t,\omega) \mapsto \int_0^t S(t-s)B(X(s))\,dW(s)(\omega)$$

have predictable versions. In addition, we require that

$$P(\int_0^T \|F(X(t))\|\,dt < \infty) = 1$$
$$\int_0^T E(\int_0^t \|\langle S(t-s)B(X(s)), C^*\zeta\rangle\|_{L_2(U,\mathbb{R})}^2\,ds)\,dt < \infty$$

for all $\zeta \in D(C^*)$.
Then the process is also an analytically weak solution.

Remark F.0.6. *The precise relation of mild and analytically weak solutions with the variational solutions from Definition 4.2.1 is obviously more difficult to describe in general. We shall concentrate just on the following quite typical special case:*
Consider the situation of Subsection 4.2, but with A and B independent of t and ω. Assume that there exist a self-adjoint operator $(C, D(C))$ on H such that $-C \geqslant const. > 0$ and $F : H \to H$ $\mathcal{B}(H)/\mathcal{B}(H)$-measurable such that

$$A(x) = C(x) + F(x), \quad x \in V,$$

and

$$V := D((-C)^{\frac{1}{2}}),$$

equipped with the graph norm of $(-C)^{\frac{1}{2}}$. Then it is easy to see that C extends to a continuous linear operator form V to V^*, again denoted by C such that for $x \in V$, $y \in D(C)$

$$_{V^*}\langle Cx, y\rangle_V = \langle x, Cy\rangle. \tag{F.0.5}$$

Now let X be a (variational) solution in the sense of Definition 4.2.1, then it follows immediately from (F.0.5) *that X is an analytically weak solution in the sense of Definition F.0.3.*

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Index

Symbols

$N(m, Q)$	Gaussian measure with mean m and covariance Q, 6
$W(t)$, $t \in [0, T]$	standard Wiener process, 13
	cylindrical Wiener process, 39
$E(X\|\mathcal{G})$	conditional expectation of X given $\mathcal{G}$, 18
$\mathcal{M}_T^2(E)$	space of all continuous E-valued, square integrable martingales, 20
$\mathcal{E}$	class of all $L(U, H)$-valued elementary processes, 22
Ω_T	$[0, T] \times \Omega$, 21
$\mathrm{d}x$	Lebesgue measure, 21
P_T	$\mathrm{d}x_{\|[0,T]} \otimes P$, 21
$\mathcal{P}_T$	predictable σ-field on Ω_T, 27
$\int \Phi(s)\ \mathrm{d}W(s)$	stochastic integral w.r.t. W, 22
$L^p(\Omega, \mathcal{F}, \mu; X)$	set of all with respect to μ p-integrable mappings from Ω to X, 106
$L^p(\Omega, \mathcal{F}, \mu)$	$L^p(\Omega, \mathcal{F}, \mu; \mathbb{R})$
L_0^p	$L^p(\Omega, \mathcal{F}_0, P; H)$
$L^p([0, T]; H)$	$L^p([0, T], \mathcal{B}([0, T]),\ \mathrm{d}x; H)$
$L^p([0, T],\ \mathrm{d}x)$	$L^p([0, T], \mathcal{B}([0, T]),\ \mathrm{d}x; \mathbb{R})$
$\\| \ \\|_T$	L^2-norm on $L^2(\Omega_T, \mathcal{P}_T, P_T; L_2^0)$, 25
$\mathcal{N}_W^2(0, T; H)$	$L^2(\Omega_T, \mathcal{P}_T, P_T; L_2^0)$, 28
$\mathcal{N}_W^2(0, T)$	$\mathcal{N}_W^2(0, T; H)$
$\mathcal{N}_W^2$	$\mathcal{N}_W^2(0, T; H)$
$\mathcal{N}_W(0, T; H)$	space of all stochastically integrable processes, 30
$\mathcal{N}_W(0, T)$	$\mathcal{N}_W(0, T; H)$
$L(U, H)$	space of all bounded and linear operators from U to H, 109
$L(U)$	$L(U, U)$
$L_1(U, H)$	space of all nuclear operators from U to H, 109
$\mathrm{tr}\ Q$	trace of Q, 109
$L_2(U, H)$	space of all Hilbert–Schmidt operators from U to H, 110
$\\| \ \\|_{L_2}$	Hilbert–Schmidt norm, 111

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