Appendix on

Schensted correspondence and Littelmann paths

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A

Introduction

A.1 Preamble

These lectures describe some combinatorial properties of the set $I(n, r)$ of all "words" $i_1 i_2 \ldots i_r$ of length r, whose "letters" $i_1, i_2, \ldots, i_r$ are drawn from the "alphabet" $\underline{n} = \{1, \ldots, n\}$. Clearly $I(n, r)$ is a finite set, with n^r elements.

Let $\Lambda(n, r)$ be the set of all vectors $\beta = (\beta_1, \ldots, \beta_n)$ whose coefficients are non-negative integers satisfying $\sum_{\nu \in \underline{n}} \beta_\nu = r$. The elements $\beta \in \Lambda(n, r)$ are sometimes called *weights* (see section 3.1). Let $\Lambda^+(n, r)$ be the subset of $\Lambda(n, r)$ consisting of all β which are *dominant*, i.e. which satisfy $\beta_1 \geq \cdots \geq \beta_n (\geq 0)$. A dominant weight in this sense is often referred to as a *partition of r with no more than n parts.*

Example. $\Lambda^+(2, 4) = \{(4, 0), (3, 1), (2, 2)\}$.

The set $I(n, r)$ plays a humble rôle in the representation theory of the general linear group $\mathsf{GL}(n, K)$ (see section 2.6), because it indexes the basis $\{\, v_i = v_{i_1} \otimes \cdots \otimes v_{i_r} \,:\, i \in I(n, r)\}$ of the r-fold tensor power $V^{\otimes r}$ of a vector space V of dimension n, with respect to a given basis $\{v_1, \ldots, v_n\}$ of V.

But the present work is not based on linear algebra. We shall see that $I(n, r)$ has a rich combinatorial structure in its own right, based on two operations which may be performed on any word $i \in I(n, r)$; namely

(A.1a) the Robinson–Schensted algorithm, and

(A.1b) the application of maps $\tilde{e}_c$, $\tilde{f}_c$ which are essentially Littelmann's "*root operators*" e_α, f_α (see [35] and (A.3g)(2)).

Peter Littelmann uses the root operators as foundation of a remarkable theory [35], sometimes called the "path model" of the classical representation theory of GL_n; this is more combinatorial, and simpler in some ways, than the classical theory. Our work is an attempt to understand this "proto-representation theory" of GL_n.

A striking feature of Littelmann's theory is that it applies to arbitrary complex, symmetrizable Kac-Moody algebras. Our work, which applies only to sl_n, is therefore restricted to the special case of algebras of type A_{n-1}. But there is some advantage in this restriction; Littelmann's "paths" become "words", and we may work in the familiar combinatorial context of this set of lecture notes.

(A.1a) and (A.1b) will be described briefly in §A.2, §A.3, and discussed in more detail later.

A.2 The Robinson-Schensted algorithm

This algorithm (henceforth referred to as the Schensted process) turns a word $i \in I(n,r)$ into a triple $(\lambda(i), P(i), Q(i))$, where

(A.2a) $\lambda(i) = (\lambda_1(i), \ldots, \lambda_n(i))$ is a dominant weight; i.e. $\lambda(i)$ is a partition of r into at most n parts,

(A.2b) $P(i)$ is a standard tableau of "shape" $\lambda(i)$ (see section 4.2 and (4.5a)). The entries in the tableau $P(i)$ are the letters $i_1, i_2, \ldots, i_r$ in the word i, permuted in such a way that $P(i)$ is *standard*, i.e. so that the entries in each row of $P(i)$ are weakly increasing ($\leq$) from left to right, and the entries in each column are strictly increasing ($<$) from top to bottom.

(A.2c) $Q(i)$ is a standard tableau of "shape" $\lambda(i)$, whose entries are the integers $1, \ldots, r$ permuted in such a way that $Q(i)$ is standard[1].

Schensted calls $P(i)$ and $Q(i)$ the *P-symbol* and the *Q-symbol* (respectively) of the word i (see [46, p. 181]).

Schensted's rules which define $\lambda(i)$, $P(i)$ and $Q(i)$ will be given in §B.2. But it may be useful to look at the special case $r = 2$, where $\lambda(i)$, $P(i)$ and $Q(i)$ are easy to describe.

Take $r = 2$, and n any integer ≥ 2. A typical word in $I(n,2)$ is $i_1 i_2$. The only dominant weights are $(2,0,0,\ldots,0)$ and $(1,1,0,\ldots,0)$. The only values for $Q(i)$ are the tableaux $\boxed{1\,|\,2}$ and $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$.

Table A.1 below (which is made using the rules in §B.2; see (B.4b)) produces a partition $I(n,2) = I_{\boxed{1|2}} \,\dot\cup\, I_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}}$, where

(A.2d) $I_{\boxed{1|2}} = \{\, i \in I(n,2) \,:\, i_1 \leq i_2 \,\}$ and $I_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}} = \{\, i \in I(n,2) \,:\, i_1 > i_2 \,\}$.

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[1]Standard tableaux were first defined by A. Young [59] in his representation theory of the symmetric group $\mathsf{Sym}\{1, \ldots, r\}$. For this reason, standard tableaux are often called *Young tableaux*, or *generalized Young tableaux* [34].

From this table we see that the set $I_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}$ is the set of all $i \in I(n,2)$ such that $Q(i) = \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$, and $I_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}$ is the set of all $i \in I(n,2)$ such that $Q(i) = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$. These sets are therefore the equivalence classes for the equivalence $\approx$ which we shall define in §A.4. The general case will be discussed in §C.1.

i	$\lambda(i)$	$P(i)$	$Q(i)$
$i_1 \leq i_2$	$(2,0,0,\ldots,0)$	$\begin{array}{\|c\|c\|}\hline i_1&i_2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}$
$i_1 > i_2$	$(1,1,0,\ldots,0)$	$\begin{array}{\|c\|}\hline i_2\\\hline i_1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$

Table A.1. The Schensted process in case $n \geq r = 2$.

A.3 The operators $\tilde{\mathbf{e}}_{\mathbf{c}}$, $\tilde{\mathbf{f}}_{\mathbf{c}}$

Let $a, b \in \underline{n}$, $a \neq b$, and let $\alpha_{a,b} = (0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0)$ denote the element of $\mathbb{Z}^n$ which has 1, -1 at the places a, b respectively, and zero at all other places. These $n(n-1)$ vectors are called the *roots* of a system of type A_{n-1}. Define $\Sigma = \{\alpha_{1,2}, \alpha_{2,3}, \ldots, \alpha_{n-1,n}\}$. This is a subset of the set of all roots; its elements are called the *simple roots*.[2]

Choose an element $c \in \{1,2,\ldots,n-1\}$. To define Littelmann's operators $\tilde{e}_c$ and $\tilde{f}_c$ we need some preliminary definitions.

- Define the map $\omega = \omega_{c,c+1} : \underline{n} \to \mathbb{Z}$ by the rule $\omega(\nu) = 1$, -1 or zero, according as $\nu = c$, $\nu = c+1$, or $\nu \notin \{c, c+1\}$.
- Define the map $h_c^i : \{0,1,\ldots,r\} \to \mathbb{Z}$ by the rule:

  (A.3a) $h_c^i(0) = 0$, and $h_c^i(t) = \omega(i_1) + \cdots + \omega(i_t)$ for all $t \in \{1,\ldots,r\}$.

  This means for any $t \in \{1,\ldots,r\}$,

  (A.3b) $h_c^i(t)$ is the number of c's in the initial segment $i_1 i_2 \ldots i_t$ of the word i, minus the number of $c+1$'s in this segment.[3]

- Next let $M = M_c^i$ denote the largest of the integers $h_c^i(0), h_c^i(1), \ldots, h_c^i(r)$. Notice that M_c^i is always ≥ 0 since $h_c^i(0) = 0$.

[2] To read this Appendix, it is *not* necessary to know the theory of roots and root systems!

[3] h_c^i is sometimes called the *height function*.

- There may be several values of $t \in \{0, 1, \ldots, r\}$ such that $h_c^i(t) = M_c^i$; let $q = q_c^i$ be the least of these values, and let $\overline{q} = \overline{q}_c^i$ be the greatest.

(A.3c) Lemma. (i) *If* $q \neq 0$*, then* $i_q = c$.
(ii) *If* $\overline{q} \neq r$*, then* $i_{\overline{q}+1} = c + 1$.

Proof. (i) Suppose $q \neq 0$. We know that $h_c^i(q) = M$. Let $\mu = h_c^i(q-1)$. By (A.3a) $M = h_c^i(q) = \mu + \omega(i_q)$. The possible values for $\omega(i_q)$ are 1, -1 and 0. But if $\omega(i_q) = -1$ then $M = \mu - 1$, hence $\mu > M$ against the definition of M. If $\omega(i_q) = 0$, then $\mu = M$, against the definition of q, which says that q is the *least* value of t for which $h_c^i(t) = M$. Hence $\omega(i_q) = 1$, which implies that $i_q = c$. The proof of (ii) is similar, and is left to the reader.

(A.3d) Definition (see [35, §1]). With the notation given above, define maps $\tilde{e}_c, \tilde{f}_c : I(n,r) \to I(n,r) \cup \{\infty\}$ as follows.

(A.3e) If $M^i = 0$, define $\tilde{f}_c(i) = \infty$ (or say "$\tilde{f}_c(i)$ is undefined"). If $M^i \neq 0$, define $\tilde{f}_c(i)$ to be the word $s \in I(n,r)$ given by $s_t = i_t$ if $t \neq q$, and $s_q = c+1$.

(A.3f) If $M^i = h_c^i(r)$, define $\tilde{e}_c(i) = \infty$ (or say "$\tilde{e}_c(i)$ is undefined"). If $M^i \neq h_c^i(r)$, define $\tilde{e}_c(i)$ to be the word $s \in I(n,r)$ given by $s_t = i_t$ if $t \neq \overline{q} + 1$, and $s_{\overline{q}+1} = c$.

(A.3g) Remarks.

(1) We have labelled these operators with the index c, rather than with the corresponding simple root $\alpha = \alpha_{c,c+1}$.

(2) Let $B : I(n,r) \to I(n,r)$ be the operator which turns each word $i_1 i_2 \ldots i_r$ into its "reverse" $i_r i_{r-1} \ldots i_2 i_1$. Then the maps just defined are related to Littelmann's "root operators" f_α, e_α (see [35, §1]) as follows: $\tilde{f}_c = B f_\alpha B$, $\tilde{e}_c = B e_\alpha B$.

(3) Let $i \in I(n,r)$. Then each of $\tilde{f}_c$, $\tilde{e}_c$ takes i either to ∞, or to a word which is identical to i except at one place. At this "critical place", $\tilde{f}_c(i)$ changes the entry from c to $c+1$, and $\tilde{e}_c(i)$ changes the entry from $c+1$ to c (see (A.3c)).

(4) The *weight* $\mathsf{wt}(i)$ of a word $i \in I(n,r)$ is the vector $\beta \in \mathbb{Z}^n$ defined as follows: for each $\nu \in \underline{n}$, β_ν is the number of places $\varrho \in \underline{r}$ for which $i_\varrho = \nu$ (see section 3.1). Then (3) shows that $\mathsf{wt}(\tilde{f}_c(i)) = \mathsf{wt}(i) - \alpha_{c,c+1}$, if $\tilde{f}_c(i) \neq \infty$. Similarly $\mathsf{wt}(\tilde{e}_c(i)) = \mathsf{wt}(i) + \alpha_{c,c+1}$, if $\tilde{e}_c(i) \neq \infty$.

(5) The maps $\tilde{f}_c$, $\tilde{e}_c$ are "inverse" to each other in the sense: if $\tilde{f}_c(i) \neq \infty$, then $\tilde{e}_c \tilde{f}_c(i) = i$, while if $\tilde{e}_c(i) \neq \infty$, then $\tilde{f}_c \tilde{e}_c(i) = i$.

(6) *Concatenation.* If $i \in I(n,r)$ and $j \in I(n,s)$, define the *concatenation* of i and j to be the word $i \mid j = (i_1, \ldots, i_r, j_1, \ldots, j_s) \in I(n, r+s)$. Then for any $c \in \{1, \ldots, n-1\}$ we have

$$\tilde{f}_c(i \mid j) = \begin{cases} \tilde{f}_c(i) \mid j & \text{if } M_c^i \geq h_c^i(r) + M_c^j, \text{ and} \\ i \mid \tilde{f}_c(j) & \text{if } M_c^i < h_c^i(r) + M_c^j, \end{cases}$$

and

$$\tilde{e}_c(i \mid j) = \begin{cases} \tilde{e}_c(i) \mid j & \text{if } M_c^i > h_c^i(r) + M_c^j, \text{ and} \\ i \mid \tilde{e}_c(j) & \text{if } M_c^i \leq h_c^i(r) + M_c^j. \end{cases}$$

All of the statements in (A.3g) are due (and in much greater generality) to Littelmann; see [35, §2]. However these statements are also easily verified directly from the definitions above.

As an example, we prove the second of the two statements in (A.3g)(5), namely

(5*) If $i \in I(n, r)$ and $c \in \{1, \ldots, n-1\}$ such that $\tilde{e}_c(i) \neq \infty$, then $\tilde{f}_c \tilde{e}_c(i) = i$.

Proof. To calculate $\tilde{e}_c(i)$, we first calculate the height function h_c^i. This function was defined in (A.3a): $h_c^i(0) = 0$, and $h_c^i(t) = \omega(i_1) + \cdots + \omega(i_t)$ for all $t \in \{1, \ldots, r\}$; it is given as the third line of table A.2 below. Let $M = M_c^i$; recall the definition of $\overline{q} = \overline{q}_c^i$ (see (A.3b) and (A.3c)), and notice that in our case $\overline{q} < r$, because $\tilde{e}_c(i) \neq \infty$. By (A.3c)(ii), $i_{\overline{q}+1} = c + 1$. In the fourth row of table A.2 are inequalities (e.g. $h_c^i(t) \leq M$) which, taken together, express that $\overline{q}$ is the largest value of t such that $h_c^i(t) = M$.

<table>
<tr><td>t</td><td>0</td><td>1</td><td>2</td><td>$\cdots$</td><td>$\overline{q} = \overline{q}_c^i$</td><td>$\overline{q}+1$</td><td>$\cdots$</td><td>$r$</td></tr>
<tr><td>i_t</td><td></td><td>i_1</td><td>i_2</td><td>$\cdots$</td><td>$i_{\overline{q}}$</td><td>$i_{\overline{q}+1} = c+1$</td><td>$\cdots$</td><td>i_r</td></tr>
<tr><td>$h_c^i(t)$</td><td>0</td><td>$\omega(i_1)$</td><td>$\omega(i_1) + \omega(i_2)$</td><td>$\cdots$</td><td>$M = M_c^i$</td><td>$M - 1$</td><td>$\cdots$</td><td>$h_c^i(r)$</td></tr>
<tr><td></td><td colspan="4">$\leq M$</td><td>M</td><td colspan="3">$< M$</td></tr>
<tr><td>s_t</td><td></td><td>i_1</td><td>i_2</td><td>$\cdots$</td><td>$i_{\overline{q}}$</td><td>c</td><td>$\cdots$</td><td>i_r</td></tr>
<tr><td>$h_c^s(t)$</td><td>0</td><td colspan="4">$< M + 1$</td><td>$M + 1$</td><td colspan="2">$\leq M + 1$</td></tr>
<tr><td>$\tilde{f}_c(s)_t$</td><td></td><td>i_1</td><td>i_2</td><td>$\cdots$</td><td>$i_{\overline{q}}$</td><td>$c+1$</td><td>$\cdots$</td><td>i_r</td></tr>
</table>

Table A.2. The height functions of i and $s = \tilde{e}_c(i)$.

According to definition (A.3f), the word $s = \tilde{e}_c(i)$ coincides with i except at the place $\overline{q}+1$. At this place $i_{\overline{q}+1} = c+1$, and $s_{\overline{q}+1} = c$. To calculate $\tilde{f}_c(s)$, we must know the function h_c^s. Clearly $h_c^s(t) = h_c^i(t)$ for all $t \in \{0, \ldots, \overline{q}\}$, and it is easy to see that $h_c^s(t) = h_c^i(t) + 2$ for all $t \in \{\overline{q} + 1, \ldots, r\}$.

Now $h_c^i(\overline{q}+1) = h_c^i(\overline{q}) + \omega(i_{\overline{q}+1}) = M - 1$, hence $h_c^s(\overline{q}+1) = M+1$. We may now check the inequalities in the penultimate line of table A.2. These show that $M_c^s = M + 1 > 0$, and that $q_c^s = \overline{q} + 1$. But then $\tilde{f}_c(s) = \tilde{f}_c(\tilde{e}_c(i)) = i$, as required.

A.4 What is to be done

The Schensted process associates to each $i \in I(n,r)$ a triple $(\lambda(i), P(i), Q(i))$. In §C.1 we shall define two equivalences on the set $I(n,r)$.

Definitions.

(A.4a) $i \sim j$ means that $P(i) = P(j)$, and

(A.4b) $i \approx j$ means that $Q(i) = Q(j)$.

Donald Knuth [34] introduced the relation $\sim$, and proved that $\sim$ is the equivalence on $I(n,r)$ generated by a collection of *basic moves* $i \to j$; see §C.3. The main result of these lectures is the following analogue to Knuth's theorem:

(A.4c) Theorem A. *Let $i, j \in I(n,r)$. Then $i \approx j$ if and only if there is a finite sequence of words (elements of $I(n,r)$):*

$$i(1),\ i(2),\ \dots,\ i(s)$$

such that $i(1) = i$, $i(s) = j$ and for each adjacent pair $i(\nu)$, $i(\nu+1)$ ***either*** *there exists an element $c \in \{1,\dots,n-1\}$ such that $\tilde{f}_c(i(\nu)) = i(\nu+1)$,* ***or*** *there exists an element $c \in \{1,\dots,n-1\}$ such that $\tilde{e}_c(i(\nu)) = i(\nu+1)$.*

Expressed less formally, Theorem A says that $\approx$ is the equivalence relation on $I(n,r)$ generated by a collection of basic moves $i \Rightarrow j$, where $i \Rightarrow j$ means that $\tilde{e}_c(i) = j$ for some $c \in \{1, 2, \dots, n-1\}$, or that $\tilde{f}_c(i) = j$.

Theorem A will be proved in §D.2. Chapter D also contains some notes on the representation theory of the "Littelmann algebra" $L(n,r)$, which is an analogue of the Schur algebra $S(n,r)$.

The proof of Theorem A depends on the following

(A.4d) Proposition B. *Let $c \in \{1, 2, \dots, n-1\}$. Then the operation $\tilde{f}_c$ commutes with the Schensted process, in the following sense:*

(A.4e) $KP(\tilde{f}_c(i)) = \tilde{f}_c(KP(i))$ for all $i \in I(n,r)$.

To explain the symbols KP which appear in (A.4e), we need the

Definition. Suppose given a standard tableau

$$P = \begin{array}{lllll} y_{1,1} & y_{1,2} & \cdots & \cdots & y_{1,\lambda_1} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,\lambda_2} & \\ \vdots & & & & \\ y_{m,1} & \cdots & y_{m,\lambda_m} & & \end{array}$$

of shape $\lambda \in \Lambda^+(n,r)$, with all its entries $y_{a,b} \in \underline{n}$. Define the "Knuth unwinding" of P to be the following word:

$$KP = y_{m,1} \cdots y_{m,\lambda_m} y_{m-1,1} \cdots y_{m,\lambda_{m-1}} \cdots y_{1,1} \cdots y_{1,\lambda_1} \,;$$

see §C.2, or [34, page 173]. This is an element of $I(n,r)$, therefore we can apply the operators $\tilde{f}_c$, $\tilde{e}_c$ to it, and (A.4e) makes sense[4].

Proposition B will be proved in §C.4.

[4]Some authors identify the tableau P with the word KP, but to be cautious, we shall not make this identification in this Appendix.

B

The Schensted Process

B.1 Notations for tableaux

Choose a dominant weight $\lambda \in \Lambda^+(n, r)$. The *shape* of λ is the following subset of $\mathbb{Z} \times \mathbb{Z}$ (see, for example, section 4.2):

(B.1a) $[\lambda] := \{ (a, b) \,:\, 1 \leq a \leq n,\, 1 \leq b \leq \lambda_a \}$.

A *λ-tableau*, or a *tableau of shape* λ, is a map $U : [\lambda] \to \mathbb{Z}$. We may think of U as a "partial matrix", with $U((a, b))$ as the *entry* in U at the place (a, b). These entries are elements of $\mathbb{Z}$, and U has entries only at the *places* $(a, b) \in [\lambda]$. The entry $U((a, b))$ is often denoted $u_{a,b}$.

We usually assume that a tableau $U = (u_{a,b})_{(a,b) \in [\lambda]}$ is *standard*; i.e. that each row (a) is weakly increasing from left to right: $u_{a,1} \leq u_{a,2} \leq \cdots \leq u_{a,\lambda_a}$, and that each column (b) is strictly increasing from top to bottom: $u_{1,b} < u_{2,b} < \cdots < u_{\beta_b,b}$ (β_b is the length of column (b)). Notice that the latter condition implies that if the entries $u_{a,b}$ of a tableau U all lie in $\underline{n}$, then the number of rows of U cannot exceed n.

B.2 The map $\mathsf{Sch} : \mathbf{I(n, r)} \longrightarrow \mathbf{T(n, r)}$

Let $T(n, r)$ be the set of all triples (λ, P, Q) such that $\lambda \in \Lambda^+(n, r)$, P is a λ-tableau whose entries are drawn from the set $\underline{n} = \{1, 2, \ldots, n\}$, and Q is a λ-tableau whose entries are $1, 2, \ldots, r$ in some order. (The r entries of Q are distinct; the r entries of P may include repetitions.)

The subject of this chapter is Schensted's map [46, pp. 180–181]

(B.2a) $\mathsf{Sch} : I(n, r) \to T(n, r)$.

We shall define Sch in §B.3, and prove in §B.6 that it is bijective [46, p. 182].

The map Sch is defined by induction on r. For $r = 1$, let

(B.2b) $\mathsf{Sch}(i) = \Big((1,0,\dots,0), \boxed{i_1}, \boxed{1}\Big)$

for any one-letter word $i = i_1$ in $I(n,1)$. Note that $\boxed{i_1}$, $\boxed{1}$ are tableaux of shape $(1,0,\dots,0)$, which means that each can be regarded as a 1×1 matrix.

From now on we assume $r > 1$.

Insertion. Fundamental for Schensted's work [46] is a process (or algorithm) which "inserts" a given element x_1 of $\underline{n}$ into a given tableau U. The result of this process is a tableau $U \leftarrow x_1$ whose entries are the entries of U (although perhaps in a different order), together with one extra entry x_1.

Example. Using the methods to be explained in §B.4, we shall show that if

$$U = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \\ \cline{1-1}\end{array} \text{ and } x_1 = 1, \text{ then } U \leftarrow x_1 \text{ is the tableau } \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} 4 & \\ \cline{1-1}\end{array} \quad \text{(see (B.4b))}.$$

The insertion process will be described in the next section; see (B.3b) and (B.3d).

Now suppose U is the middle term of an element (μ, U, V) of $T(n, r-1)$. As soon as we have calculated $P = U \leftarrow x_1$, we shall be able (see (B.3e)) to construct a dominant weight $\lambda \in \Lambda^+(n,r)$, and also a λ-tableau Q, such that (λ, P, Q) is an element of $T(n,r)$. We shall denote this element $(\mu, U, V) \leftarrow x_1$. See also [46, p. 181] and [34, pp. 712, 713].

The process provides the inductive step needed to define $\mathsf{Sch}(i)$ for any $i = i_1 i_2 \dots i_{r-1} i_r \in I(n,r)$ $(r > 1)$, namely

(B.2c) Definition. $\mathsf{Sch}(i) := \mathsf{Sch}(i') \leftarrow i_r$, *where* $i' = i_1 i_2 \dots i_{r-1}$.

Therefore we have a formula (which can also be used as a definition of $\mathsf{Sch}(i)$, see [46, p. 181]).

(B.2d) Formula. $\mathsf{Sch}(i) := (\cdots((\mathsf{Sch}(i_1) \leftarrow i_2) \leftarrow i_3) \quad \cdots \quad) \leftarrow i_r$.

B.3 Inserting a letter into a tableau

Suppose we have (1) an element (μ, U, V) of $T(n, r-1)$, where $r > 1$, and (2) an element x_1 of $\underline{n}$. In this section we define the element $(\mu, U, V) \leftarrow x_1$ of $T(n,r)$. To do this, we first define the tableau $U \leftarrow x_1$.

Schensted does this by modifying U row by row. For this purpose, it is convenient[1] to supplement each row (a) of U with two "virtual entries" $u_{a,0} = 0$ and $u_{a,\mu_a+1} = \infty$. Note that $(a, 0)$ and $(a, \mu_a + 1)$ are *not* elements of $[\mu]$, and therefore $u_{a,0}$ and u_{a,μ_a+1} are not true entries in row (a).

Take any $y \in \underline{n}$. Even though y may not be equal to any of the entries $u_{a,k}$ of row (a), we may "position" y into row (a), using the following elementary lemma.

[1] See [34, p. 711].

(B.3a) Lemma. *For any $y \in \underline{n}$ and any $a \in \underline{n}$, there is a unique element $k(a) = k(a,y)$ in $\{1, 2, \ldots, \mu_a, \mu_a + 1\}$ such that $u_{a,k(a,y)-1} \le y < u_{a,k(a,y)}$.*

Proof. Define $k(a,y)$ to be the smallest $k \in \{1, 2, \ldots, \mu_a, \mu_a + 1\}$ such that $y < u_{a,k}$.

Example. Suppose $\mu_a = 5$, and that row (a) (including the virtual entries) is

$$(0)\ 2\ 2\ 2\ 3\ 7\ (\infty)\,.$$

If $y = 2$, then $k(a) = 4$, because $u_{a,3} \le 2 < u_{a,4}$.
If $y = 1$, then $k(a) = 1$, because $u_{a,0} \le 1 < u_{a,1}$.
If $y = 4$, then $k(a) = 5$, because $u_{a,4} \le 4 < u_{a,5}$.
The situation $k(a,y) = \mu_a + 1 = 6$ occurs if and only if $u_{a,5} \le y < \infty$, that is, if and only if $y \ge 7$.

The insertion sequence. Let $\mu \in \Lambda^+(n,r)$, let U be a μ-tableau whose entries all lie in $\underline{n}$, and let $x_1 \in \underline{n}$. In order to define $P := U \leftarrow x_1$ first make the "insertion sequence"

(B.3b) x_1, $k(1)$, x_2, $k(2)$, ..., x_z, $k(z)$,

which contains all the data needed to construct $U \leftarrow x_1$.

Definition of the insertion sequence.

Step 1.

- x_1 is the given element of $\underline{n}$.
- $k(1)$ is the smallest $k \in \{1, \ldots, \mu_1, \mu_1 + 1\}$ such that $x_1 < u_{1,k}$. Equivalently, $k(1)$ is the unique element of $\{1, \ldots, \mu_1, \mu_1 + 1\}$ such that $u_{1,k(1)-1} \le x_1 < u_{1,k(1)}$. The case $k(1) = \mu_1 + 1$ occurs if and only if $u_{1,\mu(1)} \le x_1$ $(< x_{1,\mu(1)+1} = \infty)$, i.e. if and only if $x_1 \ge u_{1,\mu_1}$ (hence x_1 is $\ge$ all entries in row (1) of U).
- **If $k(1) = \mu_1 + 1$, the sequence is ended.**

Step 2. Now assume that $k(1) \ne \mu_1 + 1$. Then continue the definition of the insertion sequence.

- $x_2 := u_{1,k(1)}$.
- $k(2)$ is the smallest $k \in \{1, \ldots, \mu_2, \mu_2 + 1\}$ such that $x_2 < u_{2,k}$. Equivalently, $k(2)$ is the unique element of $\{1, \ldots, \mu_2, \mu_2 + 1\}$ such that $u_{2,k(2)-1} \le x_2 < u_{2,k(2)}$. The case $k(2) = \mu_2 + 1$ occurs if and only if $x_2 \ge u_{2,\mu(2)}$.
- **If $k(2) = \mu_2 + 1$, the sequence is ended.**

Step 3. Now assume that $k(2) \ne \mu_2 + 1$. Then continue

- $x_3 := u_{2,k(2)}$, etc.

Inductive Step. The general step is as follows: after x_{a-1} $(:= u_{a-2,k(a-2)})$ and $k(a-1)$ have been defined, then

- **If $k(a-1) = \mu_{a-1} + 1$, the sequence is ended.**

Now assume that $k(a-1) \ne \mu_{a-1} + 1$, and proceed to define

- $x_a := u_{a-1,k(a-1)}$,
- $k(a)$ is the least $k \in \{1, \ldots, \mu_a, \mu_a + 1\}$ such that $x_a < u_{a,k}$. Equivalently, $k(a)$ is the unique element of $\{1, \ldots, \mu_a, \mu_a + 1\}$ such that

(B.3c) $u_{a,k(a)-1} \leq x_a < u_{a,k(a)}$.

Definition of z. For each a such that $k(a-1) \neq \mu_{a-1} + 1$, (B.3c) shows that $x_a < u_{a,k(a)} = x_{a+1}$. Therefore the sequence $x_1 < x_2 < \cdots$ is finite ($x_1, x_2, \ldots$ are all elements of $\underline{n}$). Define z to be the largest element of $\underline{n}$ such that $k(z-1) \neq \mu_{z-1} + 1$. Then we must have $k(z) = \mu_z + 1$ (otherwise we could go on to define $k(z+1)$), and $u_{z,\mu_z} \leq x_z$ $(< \infty)$, i.e. x_z is $\geq$ every entry in row (z) of U.

(B.3d) Definition of $\mathbf{U} \leftarrow \mathbf{x_1}$. Let $\lambda = \mu + \varepsilon_z$, where ε_z is the n-vector with 1 in place z, and zero at all other places. We shall show in (B.5b) that $\lambda \in \Lambda^+(n, r)$. Define $U \leftarrow x_1$ to be the λ-tableau $P = (p_{a,b})_{(a,b)\in[\lambda]}$ whose entry $p_{a,b}$ is identical with the corresponding entry $u_{a,b}$ of U, **except**

1° at the places $(a, k(a))$ for $a = 1, 2, \ldots, z-1$. At these places we define $p_{a,k(a)} = x_a$ (whereas $u_{a,k(a)} = x_{a+1}$), **and**
2° at place $(z, \mu_z + 1)$, where U has no entry, we define P to have entry $p_{z,\mu_z+1} = x_z$.

The shape of P is $\lambda = \mu + \varepsilon_z = (\mu_1, \ldots, \mu_{z-1}, \mu_z + 1, \mu_{z+1}, \ldots, \mu_n)$, because row (a) of P has the same length as row (a) of U, for all $a \neq z$, while the length of row (z) of P is one more than the length of row (z) of U.

Note that, for all $a > z$, the row (a) of P is identical to row (a) of U.

(B.3e) Definition of $(\mu, \mathbf{U}, \mathbf{V}) \leftarrow \mathbf{x_1}$. Let $(\mu, U, V) \in T(n, r-1)$, and let x_1 be an element of $\underline{n}$. Let P be the tableau $U \leftarrow x_1$ defined in (B.3d). Let λ be the weight $\mu + \varepsilon_z$. Define Q by enlarging the μ-tableau V, giving it a new entry r in place $(z, \mu_z + 1)$. Then $(\mu, U, V) \leftarrow x_1$ is by definition the triple (λ, P, Q).

(B.3f) Exercise. Prove that $k(1) \geq k(2) \geq \cdots \geq k(z)$ in any case.

[Hint. Let $a \in \{2, \ldots, z\}$. We must prove that $k(a) \leq k(a-1)$. By definition $k(a)$ lies in $\{1, \ldots, \mu_a + 1\}$, therefore $k(a) \leq \mu_a + 1$, which is $\leq k(a-1)$ if $k(a-1) \geq \mu_a + 1$. But if $k(a-1) < \mu_a + 1$, i.e. $k(a-1) \leq \mu_a$, then there exists an entry $u_{a,k(a-1)}$ in row (a) of U. Column standardness of U shows that $u_{a,k(a-1)} > u_{a-1,k(a-1)}$. Therefore $x_a = u_{a-1,k(a-1)} < u_{a,k(a-1)}$. But $k(a)$ is the least $k \in \{1, \ldots, \mu_a + 1\}$ such that $x_a < u_{a,k}$. It follows that $k(a) \leq k(a-1)$.]

Note. We have not yet proved that the triple (λ, P, Q) belongs to $T(n, r)$. For this we must show that $\lambda \in \Lambda^+(n, r)$, and that P, Q are standard. These things will be proved in §B.5, but we first look at some examples.

B.4 Examples of the Schensted process

The basic operation for the Schensted process is the insertion of a letter into a tableau. So suppose $r > 1$, let μ be an element of $\Lambda^+(n, r-1)$, let U be a μ-tableau, and let x_1 be any element of $\underline{n}$. We want to find the tableau $P = U \leftarrow x_1$.

The tableau $P = U \leftarrow x_1$ (see (B.3d)) can be made by modifying the rows $(1), (2), \ldots$ of U, in turn. First "position" x_1 (which may or may not be equal to one of the entries of U) into row (1) of U (see Lemma (B.3a)). This means, find the (unique) element $k(1)$ such that $u_{1,k(1)-1} \le x_1 < u_{1,k(1)}$. Assume that $k(1) \neq \mu_1 + 1$. Let $x_2 := u_{1,k(1)}$. Now let x_1 "bump"[2] x_2 into row (2), which means:

(i) change the entry $x_2 = u_{1,k(1)}$ in place $(1, k(1))$ to $x_1 = p_{1,k(1)}$, and then
(ii) "position" x_2 into row (2), that is: find the unique index $k(2)$ such that $u_{2,k(2)-1} \le x_2 < u_{2,k(2)}$.

Then row (1) of U, changed by (i), is row (1) of P.

Now we are ready to change row (2) of U into row (2) of P. In general, when row $(a-1)$ of U has been changed into row $(a-1)$ of P, we define $x_a := k_{a-1,k(a-1)}$ and "bump" x_a into row (a). This process goes on until we reach row (z), where $k(z) = \mu_z + 1$. Then row (z) of P is made by adjoining an entry x_z to row (z) of U, in the new place $(z, \mu_z + 1)$ (which was not a place for U). All subsequent rows of P are the same as the corresponding rows of U.

It is sometimes better to use a slightly different "technology", to construct $U \leftarrow x_1$ from U. Here one makes the parameters $x_1, k(1), x_2, k(2), \ldots$ as before, and records these on the tableau U; for each a we put bracketed (x_a) between the entries $u_{a,k(a-1)}$ and $u_{a,k(a)}$ of row (a). We do not change any of the entries of U. The resulting diagram (it is not a tableau in our sense) is called "U prepared for insertion of x_1". We pass from this diagram to $P = U \leftarrow x_1$ by replacing $\ldots (x_a)\ x_{a+1} \ldots$ by $\ldots x_a \ldots$, for each $a \in \{1, \ldots, z-1\}$; for row (z), replace $\ldots u_{z,\mu_z}\ (x_z)$ by $\ldots u_{z,\mu_z}\ x_z$.

(B.4a) Example. Suppose we want to insert $x_1 = 2$ into the tableau U shown in the left-hand column of table B.1 below. The second column shows U "prepared" for this insertion. This means that we have put bracketed (x_a) between the entries $u_{a,k(a)-1}$, $u_{a,k(a)}$ for each $a = 1, 2, \ldots, z$; we have not yet changed any of the entries of U. Once U has been prepared, make $P = P(i)$ from U by changing the entry $u_{a,k(a)} = x_{a+1}$ to $p_{a,k(a)} = x_a$, for all $a = 1, 2, \ldots, z-1$, i.e. the term x_a in the bracketed (x_a) replaces its right-hand neighbour $x_{a+1} = u_{a,k(a)}$. This procedure determines row (z), namely it is the first row where (x_z) does not have a right-hand neighbour. The row (z) of P, is

[2] The verb "bump" was introduced, in this context, by Knuth [34, p. 713]. Alternatives would be "dump", or even "jump".

found by replacing (x_z) by x_z. In the example shown, we have $x_1 = 2$, $x_2 = 3$, $z = 2$, $k(1) = 5$, $k(2) = 3$. Note that 3 in row (1) of U, is bumped into row (2).

U	U prepared for insertion of 2	$P = U \leftarrow 2$
$\begin{array}{\|c\|c\|c\|c\|c\|} 1 & 1 & 2 & 2 & 3 \\ 3 & 3 \\ 4 \end{array}$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|} 1 & 1 & 2 & 2 & (2) & 3 \\ 3 & 3 & (3) \\ 4 \end{array}$	$\begin{array}{\|c\|c\|c\|c\|c\|} 1 & 1 & 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 \end{array}$

Table B.1. Example for the insertion process.

(B.4b) Example. Consider two "extreme" possibilities.

(i) It can happen that, for some a, the element $x_a < u_{a,1}$. In that case $k(a) = 1$, and (B.3a) shows that $0 \le x_a < u_{a,1}$. When U is "prepared" as above, row (a) looks like this:

$$(x_a)\ u_{a,1}\ u_{a,2}\ \cdots\ u_{a,\mu_a}\,.$$

(ii) It can happen that, for some a, we have $k(a) \neq \mu_a + 1$, and $\mu_{a+1} = 0$. This means that we must "bump" x_{a+1} $(= u_{a,k(a)})$ into an empty row $(a+1)$. We have $0 < x_{a+1} < \infty$, which allows us to say that $k(a+1) = 1$. Then $k(a+1) = \mu_{a+1} + 1$. So $a + 1 = z$, and P has an entry x_{a+1} in place $(a+1, 1)$.

The following example illustrates both possibilities (i) and (ii). Suppose we insert $x_1 = 1$ into the tableau $U = \begin{array}{|c|c|} 1 & 2 \\ 4 \end{array}$. Prepared for this insertion, U becomes $\begin{array}{|c|c|c|} 1 & (1) & 2 \\ (2) & 4 \\ (4) \end{array}$. Hence $P = U \leftarrow 1 = \begin{array}{|c|c|} 1 & 1 \\ 2 \\ 4 \end{array}$. In this example x_1, x_2, x_3 are 1, 2, 4, respectively, $z = 3$; and $k(1) = 2$, $k(2) = 1$, $k(3) = 1$.

We are now in a position to calculate the P-symbol $P(i)$ and the Q-symbol $Q(i)$ of a given word $i \in I(n, r)$. To find $P(i) = P(i_1 i_2, \cdots i_r)$, we must calculate, successively, the P-symbols of the words i_1, $i_1 i_2$, $\ldots$, $i_1 i_2 \cdots i_r$, starting with $P(i_1) = \boxed{i_1}$, and using the insertion process

$$P(i_1 i_2 \cdots i_t) = P(i_1 i_2 \cdots i_{t-1}) \leftarrow i_t\,.$$

It follows that the entries of $P(i)$ are the entries $i_1, \ldots, i_r$ of i, in some order. The construction of $Q(i)$ is different, $Q(i_1 \cdots i_t)$ is *not* made by inserting i_t into $Q(i_1 \cdots i_{t-1})$; the construction follows definition (B.3e), see Example (B.4c) below.

(B.4c) Example. Calculate $P(i)$, where $i = 1\,4\,2\,1\,2$. Calculate also $\lambda(i)$ and $Q(i)$.

First we must work out the successive tableaux $P_t(i) = P(i_1 \ldots i_t)$, for $t = 1, 2, \ldots, 5$. We find (the operator $\xrightarrow{y}$ means "insert y into the tableau on the left")

$$P_1(i) = \begin{array}{|c|}\hline 1 \\ \hline\end{array} \xrightarrow{4} P_2(i) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array} \xrightarrow{2} P_3(i) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}$$

$$\xrightarrow{1} P_4(i) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \xrightarrow{2} P_5(i) = P(i) = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}.$$

At the same time we get the dominant weight at each stage, namely $\lambda(i_1 \cdots i_t)$ is just the shape of the tableau $P_t(i)$. In particular, $\lambda(i) = (3, 1, 1, 0, \ldots, 0)$. Now we make the tableaux $Q_t(i) = Q(i_1 \ldots i_t)$ as follows: if we know $Q_{t-1}(i)$, then $Q_t(i)$ is got by putting "t" in the place which was new, when $P_t(i)$ was constructed from $P_{t-1}(i)$. Thus

$$Q_1(i) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \quad Q_2(i) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \quad Q_3(i) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array},$$

$$Q_4(i) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \quad Q_5(i) = Q(i) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}.$$

(B.4d) Example. Calculate $\lambda(i)$, $P(i)$, $Q(i)$ for any word $i = i_1 i_2 \in I(n, 2)$, and so verify the table given in §A.2.

To find $P(i)$, we must insert i_2 into the tableau $U = \begin{array}{|c|}\hline i_1 \\ \hline\end{array}$. When U is prepared for this insertion, it becomes $\begin{array}{|c|c|}\hline i_1 & (i_2) \\ \hline\end{array}$ in case $i_1 \leq i_2$, and it becomes $\begin{array}{|c|c|}\hline (i_2) & i_1 \\ \hline (i_1) \\ \cline{1-1}\end{array}$ in case $i_1 > i_2$. Therefore $P(i) = \begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline\end{array}$ in case $i_1 \leq i_2$, and $P(i) = \begin{array}{|c|}\hline i_2 \\ \hline i_1 \\ \hline\end{array}$ in case $i_1 > i_2$. It follows that $\lambda(i)$ is $(2, 0, 0, \ldots, 0)$ or $(1, 1, 0, \ldots, 0)$, in these respective cases.

To find $Q(i)$, we must add 2 to $V = Q(i_1) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$ in the place $(z, \mu_z + 1)$. In the case $i_1 \leq i_2$, we have $z = 1$, so $Q(i) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$. In case $i_1 > i_2$, we have $z = 2$, so $Q(i) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$.

B.5 Proof that $(\mu, U, V) \leftarrow x_1$ belongs to $T(n, r)$

We keep the notations of §§B.2, B.3, B.4. Suppose that $r > 1$ and that $(\mu, U, V) \in T(n, r-1)$. Let $x_1 \in \underline{n}$. The triple $(\lambda, P, Q) = (\mu, U, V) \leftarrow x_1$ is defined in (B.3e). In this section we shall prove that $(\lambda, P, Q) \in T(n, r)$. For this we must show that $\lambda \in \Lambda^+(n, r)$, and that P, Q are both standard.

Write $u_{a,b}$ for the (a, b)-entry of U, and $p_{a,b}$ for the (a, b)-entry of P.

(B.5a) Proposition. *The weight*

$$\lambda = \mu + \varepsilon_z = (\mu_1, \ldots, \mu_{z-1}, \mu_z + 1, \mu_{z+1}, \ldots, \mu_n)$$

is dominant. It follows that Q is standard.

Proof. We already know that $\mu_1 \geq \cdots \geq \mu_{z-1} \geq \mu_z \geq \mu_{z+1} \geq \cdots \geq \mu_n$ because μ is dominant. If λ is not dominant, it must be that $\mu_{z-1} = \mu_z$. But this leads to a contradiction. We know that $x_z \geq u_{z,\mu_z}$, and that $u_{z,\mu_z} > u_{z-1,\mu_z}$ because U is standard. But $u_{z-1,\mu_z} \geq u_{z-1,k(z-1)} = x_z$ (the last equality is the definition of x_z), and putting these inequalities together gives the contradiction $x_z > x_z$.

By the definition (B.3e), Q is made by adding an entry r at the end of row (z) of V. It is clear that Q is a standard λ-tableau, whose entries are $1, 2, \ldots, r$ in some order.

(B.5b) Proposition. *The λ-tableau P is standard.*

Proof. First we shall show that P is "row standard", i.e. that

(i) $p_{a,h-1} \leq p_{a,h}$

for all adjacent pairs $(a, h-1)$, (a, h) of places in any row (a) of $[\lambda]$.

If $(a, k(a))$ is not one of $(a, h-1)$, (a, h) then by (B.5a) $p_{a,h} = u_{a,h}$ and $p_{a,h-1} = u_{a,h-1}$, therefore (i) follows from the corresponding fact for row (a) of U. If $(a, h) = (a, k(a))$ then $p_{a,h-1} = u_{a,h-1} \leq x_a$, and $x_a = p_{a,k(a)} = p_{a,h}$; thus (i) holds. There remains the case $(a, h-1) = (a, k(a))$. Then (i) says $p_{a,k(a)} \leq p_{a,k(a)+1}$. But $p_{a,k(a)} = x_a$ and $p_{a,k(a)+1} = u_{a,k(a)+1}$. Thus (i) follows from $x_a < x_{a+1} = u_{a,k(a)} \leq u_{a,k(a)+1}$.

To complete the proof of Proposition (B.5b), we must show that P is "column standard", i.e. that if (a, h) and $(a+1, h)$ are adjacent places in the same column of $[\lambda]$, then

(ii) $p_{a+1,h} > p_{a,h}$.

If $h \neq k(a)$ and $h \neq k(a+1)$ then $p_{a,h} = u_{a,h}$ and $p_{a+1,h} = u_{a+1,h}$, hence (ii) follows from $u_{a+1,h} > u_{a,h}$, which holds because U is column standard.

If $h = k(a)$, $h \neq k(a+1)$ then $u_{a,h} = u_{a,k(a)} = x_{a+1} > x_a$. But $u_{a+1,k(a)} > u_{a,k(a)}$ because U is column standard. Therefore $p_{a+1,k(a)} = u_{a+1,k(a)} > x_a = p_{a,k(a)}$, which proves (ii) in this case.

Now suppose that $h = k(a+1)$, $h \neq k(a)$. Then $p_{a+1,k(a+1)} = x_{a+1}$, and $p_{a,k(a+1)} = u_{a,k(a+1)}$. In place $(a, k(a))$ of P we have x_a (see (B.3d)). Since P is "row standard" (just proved, above) and $k(a+1) \leq k(a)$ (see (B.3f)) we have $u_{a,k(a+1)} \leq x_a$; also $x_a < x_{a+1}$ by (B.3b). So $p_{a,k(a+1)} = u_{a,k(a+1)} \leq x_a < x_{a+1} = p_{a+1,k(a+1)}$. This proves (ii) in case $h = k(a+1)$, $h \neq k(a)$.

There remains only the case $h = k(a) = k(a+1)$. In this case $p_{a+1,h} = p_{a+1,k(a+1)} = x_{a+1}$ and $p_{a,h} = p_{a,k(a)} = x_a$. But $x_{a+1} > x_a$, therefore (ii) holds. The proof of Proposition (B.5b) is now complete.

B.6 The inverse Schensted process

This section and the next are devoted to Schensted's fundamental

(B.6a) Theorem (see [46, p. 182]; [34, pp. 715–716]). *The map*

$$\mathsf{Sch} : I(n,r) \to T(n,r)$$

is bijective.

This will be proved by constructing a map $\mathsf{M} : T(n,r) \to I(n,r)$ which is a two-sided inverse to Sch (see (B.7b)).

If $r = 1$, it is easy to make a map M inverse to Sch. The only element in $\Lambda^+(n,1)$ is $\lambda = (1,0,\ldots,0)$, hence any element in $T(n,1)$ has the form $\big(\lambda, \boxed{x}, \boxed{1}\big)$ for some $x \in \underline{n}$. We define $\mathsf{M}\big(\big(\lambda, \boxed{x}, \boxed{1}\big)\big)$ to be x (regarded as a 1-letter word). By (B.2b), $\mathsf{Sch}(x) = \big(\lambda, \boxed{x}, \boxed{1}\big)$. It is easy to check now that $\mathsf{M} : T(n,1) \to I(n,1)$ is a two-sided inverse to $\mathsf{Sch} : I(n,1) \to T(n,1)$. It follows that Sch is bijective in case $r = 1$.

From now on in this section, assume that $r > 1$.

The process given in §B.3 delivers a map, which we call *insertion*,

(B.6b) $\mathsf{J} : T(n,r-1) \times \underline{n} \to T(n,r)$,

which takes a pair $((\mu, U, V), x_1)$ to the element $(\mu, U, V) \leftarrow x_1$ of $T(n,r)$.

Next define another map, called *extrusion*,

(B.6c) $\mathsf{E} : T(n,r) \to T(n,r-1) \times \underline{n}$.

To make E, we need an "inverse Schensted process", which will turn any $(\lambda, P, Q) \in T(n,r)$ into a pair consisting of a triple $(\mu, U, V) \in T(n,r-1)$ and an element $w_1 \in \underline{n}$.

How to define E. Let $(\lambda, P, Q) \in T(n,r)$. Let $(a,b) \in [\lambda]$ be the (unique) place where $q_{a,b} = r$. Since Q is standard, r must be at the end of its row. Therefore if $a = z$, then b must be λ_z, so that $q_{z,\lambda_z} = r$. But r is also at the end of its column, which implies that $\lambda_z > \lambda_{z+1}$. This proves

(B.6d) The weight $\mu = \lambda - \varepsilon_z$ is dominant. Hence $\mu \in \Lambda^+(n, r-1)$.

Definition of the extrusion sequence. We shall next define the *extrusion sequence*

(B.6e) $l(z),\ w_z,\ l(z-1),\ w_{z-1},\ \ldots,\ l(1),\ w_1$.

To make this sequence, we must know Q (which determines z) as well[3] as the λ-tableau P.

Step 1.
- $l(z) = \lambda_z$;
- $w_z := p_{z,\lambda_z}$ (this is the entry in P, at the place (z, λ_z) where Q has entry r).

Step 2.
- $l(z-1)$ is the largest $l \in \{1, 2, \ldots, \lambda_{z-1}\}$ such that $p_{z-1,l} < w_z$. Equivalently, $l(z-1)$ is the unique element in $\{1, 2, \ldots, \lambda_{z-1}\}$ such that $p_{z-1,l(z-1)} < w_z \leq p_{z-1,l(z-1)+1}$.
- $w_{z-1} := p_{z-1,l(z-1)}$.

Inductive Step. When $l(a+1)$ and $w_{a+1} := p_{a+1,l(a+1)}$ have been defined, we go on to define
- $l(a)$ is the largest $l \in \{1, \ldots, \lambda_a\}$ such that $p_{a,l} < w_{a+1}$. Equivalently, $l(a)$ is the unique element in $\{1, \ldots, \lambda_a\}$ such that

(B.6f) $p_{a,l(a)} < w_{a+1} \leq p_{a,l(a)+1}$.

- $w_a := p_{a,l(a)}$.

Note that if $a < z$ there is always at least one $l \in \{1, 2, \ldots, \lambda_a\}$ such that $p_{a,l} < w_{a+1}$, namely $l = l(a+1)$; this is because P is column standard, hence $p_{a,l(a+1)} < p_{a+1,l(a+1)} = w_{a+1}$. So the extrusion sequence (B.6e) always ends with $\ldots,\ l(1),\ w_1$.

Final Step. The last two terms are as follows.
- $l(1)$ is the largest $l \in \{1, 2, \ldots, \lambda_1\}$ such that $p_{1,l} < w_2$, and
- $w_1 := p_{1,l(1)}$.

We say that the element $w_1 \in \underline{n}$ has been "extruded"[4] from P (or more precisely from the given element (λ, P, Q) in $T(n, r)$). But the extrusion process also defines an element (μ, U, V), see below.

(B.6g) Definition of E. Let $(\lambda, P, Q) \in T(n, r)$. Then $\mathsf{E}((\lambda, P, Q))$ is the pair $((\mu, U, V), w_1)$, where

- $\mu = \lambda - \varepsilon_z = (\lambda_1, \ldots, \lambda_{z-1}, \lambda_z - 1, \lambda_{z+1}, \ldots, \lambda_n)$,

[3] To define $P = U \leftarrow x_1$, we did not need to know Q, because z is defined by the insertion sequence; see (B.3c), (B.3d).

[4] In the way that a small amount (w_1) of toothpaste is *extruded* from its tube (λ, P, Q).

- V is Q, with the entry $q_{z,\lambda_z} = r$ removed,
- w_1 is the extruded element of $\underline{n}$ defined above, and
- $U = (u_{a,b})_{(a,b)\in[\mu]}$ is the following μ-tableau: $u_{a,b} = p_{a,b}$ for all $(a,b) \in [\mu]$, **except**
  1° at the places $(a, l(a))$ for $a = 1, 2, \dots, z-1$. At these places we define $u_{a,l(a)} = w_{a+1}$ (whereas $p_{a,l(a)} = w_a$), **and**
  2° there is no entry in U at the place (z, λ_z), because U is a μ-tableau and $(z, \lambda_z) \notin [\mu]$.

To complete the definition of E, the following lemma is required.

(B.6h) Lemma. *The triple (μ, U, V) belongs to $T(n, r-1)$.*

Proof. From (B.6d) we know that $\mu \in \Lambda^+(n, r-1)$. It is clear that V is a μ-tableau whose entries are $1, 2, \dots, r-1$, in some order. It remains only to show that U is standard. The proof of this is very similar to that of Proposition (B.5b), and we leave to the reader.

(B.6i) Proposition. *The maps J, E are inverse to each other.*

Proof. We shall first prove that

(i) $\mathsf{E} \circ \mathsf{J} = \mathsf{id}_{T(n,r-1)\times\underline{n}}$.

Take any element $((\mu, U, V), x_1)$ in $T(n, r-1) \times \underline{n}$. Let

(B.6j) $x_1,\ k(1),\ \dots,\ x_z,\ k(z),$

be the insertion sequence used to define $(\lambda, P, Q) = \mathsf{J}((\mu, U, V), x_1) = (\mu, U, V) \leftarrow x_1$. Here z is such that $k(z) = \mu_z + 1 = \lambda_z$, where $\lambda = \mu + \varepsilon_z$. Note that Q has r in place (z, λ_z), and $p_{z,\lambda_z} = x_z$ (see (B.3d) and (B.3e)).

To prove (i) it is enough to show that $\mathsf{E}((\lambda, P, Q)) = ((\mu, U, V), x_1)$. Now $\mathsf{E}((\lambda, P, Q))$ is determined by the extrusion sequence (see (B.6e))

(B.6k) $l(z),\ w_z,\ l(z-1),\ w_{z-1},\ \dots,\ l(1),\ w_1.$

The "z" which appears in (B.6k) indexes the row of Q which contains the entry r, see (B.6d). Therefore this "z" is the same as the z in (B.6j). From (B.6e) we have $l(z) = \lambda_z$ and $w_z = p_{z,\lambda_z}$. But from the definition of P, $p_{z,\lambda_z} = p_{z,\mu_z+1} = x_z$. Therefore

(B.6l) $l(z) = k(z)$ *and* $w_z = x_z$.

Our ambition is to prove

(B.6m) $l(a) = k(a)$ *and* $w_a = x_a$

for all $a \in \{z, z-1, \dots, 1\}$. Suppose $a < z$ and that (using "upward" induction)

(B.6n) $l(a+1) = k(a+1)$ *and* $w_{a+1} = x_{a+1}$.

By (B.3c), there holds $x_a < x_{a+1} = u_{a,k(a)} \leq u_{a,k(a)+1}$. However the definitions in §B.3 show that $x_a = p_{a,k(a)}$, and (B.6n) gives $w_{a+1} = x_{a+1}$. So $x_a = p_{a,k(a)} < w_{a+1} \leq u_{a,k(a)+1} = p_{a,k(a)+1}$. But comparing this with (B.6f), we see that $l(a) = k(a)$. Hence $w_a = p_{a,l(a)} = p_{a,k(a)} = x_a$; thus (B.6m) holds for all a. In particular, $w_1 = x_1$, and we find easily that

$$\mathsf{E}(\mathsf{J}(((\mu, U, V), x_1)))) = \mathsf{E}((\lambda, P, Q)) = ((\mu, U, V), x_1);$$

in other words we have proved (i).

To complete the proof of (B.6i) we must prove

(ii) $\mathsf{J} \circ \mathsf{E} = \mathsf{id}_{T(n,r)}$.

Take any element $(\lambda, P, Q) \in T(n,r)$. Let (B.6k) be the extrusion sequence which defines $\mathsf{E}((\lambda, P, Q)) = ((\mu, U, V), w_1)$ (see (B.6g)). Let

(B.6o) $(w_1 =)\ x_1,\ k(1),\ x_2,\ k(2),\ \ldots,\ x_z,\ k(z)$

be the insertion sequence which defines $\mathsf{J}((\mu, U, V), w_1)$. In order to prove (ii) we must show that $\mathsf{J}((\mu, U, V), w_1) = (\lambda, P, Q)$.

The first step is to prove

(B.6p) $w_a = x_a$

for all $a \in \{1, 2, \ldots, z\}$. This holds for $a = 1$, by definition. Suppose that (B.6p) holds for some a. By (B.6f), $l(a)$ is the unique element of $\{1, 2, , \ldots, z\}$ such that

(B.6q) $p_{a,l(a)} = w_a < w_{a+1} \leq p_{a,l(a)+1}$.

From this follows that $p_{a,l(a)-1} \leq w_a < w_{a+1}$. Hence, using the definition (B.6g) of U, we have $u_{a,l(a)-1} \leq w_a < u_{a,l(a)}$, and since $w_a = x_a$, there holds

$$u_{a,l(a)-1} \leq x_a < u_{a,l(a)}\,.$$

However this proves that $l(a) = k(a)$, from (B.3c). Consequently $w_{a+1} = p_{a,l(a)} = p_{a,k(a)} = x_{a+1}$ (see (B.3b); we are here using the insertion of x_1 into (μ, U, V)). Now we can prove, by induction on a, that

(B.6r) $w_a = x_a$ and $l(a) = k(a)$, for all $a \in \{1, 2, \ldots, z\}$.

Using (B.6r) and the definitions (B.3d) and (B.3e) (applied to the insertion of x_1 into (μ, U, V)), it is quite easy to show that $\mathsf{J}((\mu, U, V), w_1) = (\lambda, P, Q)$. This concludes the proof of Proposition (B.6i).

B.7 The ladder

We shall define a map $\mathsf{M} : T(n,r) \to I(n,r)$ inverse to $\mathsf{Sch} : I(n,r) \to T(n,r)$, and hence prove Schensted's Theorem (B.6a). M will be given as the product of maps $\mathsf{E}_0, \mathsf{E}_1, \ldots, \mathsf{E}_{r-1}$ displayed in table B.2 ("The ladder") below.

Set	Typical element of set
$I(n,r)$	$i_1 i_2 \dots i_s i_{s+1} \dots i_{r-1} i_r$
$\mathsf{J}_1 \downarrow \qquad \uparrow \mathsf{E}_{r-1}$	
$T(n,1) \times I(n,r-1)$	$\Big((\lambda_1, P_1, Q_1), i_2 \dots i_s i_{s+1} \dots i_{r-1} i_r\Big)$
$\mathsf{J}_2 \downarrow \qquad \uparrow \mathsf{E}_{r-2}$	
$\vdots$	$\vdots$
$\mathsf{J}_{s-1} \downarrow \qquad \uparrow \mathsf{E}_{r-s+1}$	
$T(n,s-1) \times I(n,r-s+1)$	$\Big((\lambda_{s-1}, P_{s-1}, Q_{s-1}), i_s i_{s+1} \dots i_{r-1} i_r\Big)$
$\mathsf{J}_s \downarrow \qquad \uparrow \mathsf{E}_{r-s}$	
$T(n,s) \times I(n,r-s)$	$\Big((\lambda_s, P_s, Q_s), i_{s+1} \dots i_{r-1} i_r\Big)$
$\mathsf{J}_{s+1} \downarrow \qquad \uparrow \mathsf{E}_{r-s-1}$	
$\vdots$	$\vdots$
$\mathsf{J}_{r-1} \downarrow \qquad \uparrow \mathsf{E}_1$	
$T(n,r-1) \times I(n,1)$	$\Big((\lambda_{r-1}, P_{r-1}, Q_{r-1}), i_r\Big)$
$\mathsf{J}_r \downarrow \qquad \uparrow \mathsf{E}_0$	
$T(n,r)$	(λ_r, P_r, Q_r)

Table B.2. The ladder.

Notations and Explanations. To define

$$\mathsf{E}_s : T(n, r-s) \times I(n, s) \longrightarrow T(n, r-s-1) \times I(n, s+1),$$

first apply E to a typical element $(\lambda_{r-s}, P_{r-s}, Q_{r-s})$ of the set $T(n, r-s)$: this gives a pair $((\lambda_{r-s-1}, P_{r-s-1}, Q_{r-s-1}), i_{r-s})$ where i_{r-s} is some element of $\underline{n}$. By definition, E_s takes the element $((\lambda_{r-s}, P_{r-s}, Q_{r-s}), i_{r-s+1} \dots i_{r-1} i_r)$ of $T(n, r-s) \times I(n, s)$ to $((\lambda_{r-s-1}, P_{r-s-1}, Q_{r-s-1}), i_{r-s} i_{r-s+1} \dots i_{r-1} i_r)$.

The map

$$\mathsf{J}_{r-s} : T(n, r-s-1) \times I(n, s+1) \longrightarrow T(n, r-s) \times I(n, s) :$$

takes (by definition)

$$((\lambda_{r-s-1}, P_{r-s-1}, Q_{r-s-1}), i_{r-s} i_{r-s+1} \dots i_{r-1} i_r) \\ \longmapsto ((\lambda_{r-s}, P_{r-s}, Q_{r-s}), i_{r-s+1} \dots i_{r-1} i_r),$$

where $(\lambda_{r-s}, P_{r-s}, Q_{r-s}) = (\lambda_{r-s-1}, P_{r-s-1}, Q_{r-s-1}) \leftarrow i_{r-s}$.

Note. To explain the top step of the ladder, take $T(n, 0)$ to be the 1-element set which contains only the triple (λ, P, Q), where $\lambda = (0, 0, \dots, 0)$ and P, Q are empty tableaux. Then identify $T(n, 0) \times I(n, r)$ with $I(n, r)$. In the same way, the bottom step is $T(n, r) \times I(n, 0) = T(n, r)$, where $I(n, 0)$ consists of the empty word only.

(B.7a) Exercise. Prove that $\mathsf{J}_{r-s} = \mathsf{E}_s^{-1}$. [Hint: use Proposition (B.6i).]

As we go up the ladder, the successive operators E_s erode $T(n, r)$, step by step, until it becomes $I(n, r)$. This progress is inverted as we go down from $I(n, r)$ to $T(n, r)$, using the operators J_s. But this "going down" is exactly described by the formula (B.2c), which means that $\mathsf{Sch} = \mathsf{J}_r \circ \mathsf{J}_{r-1} \circ \dots \circ \mathsf{J}_1$. Define

(B.7b) $\mathsf{M} := \mathsf{E}_{r-1} \circ \dots \circ \mathsf{E}_0$.

By (B.7a), M is a two-sided inverse to Sch. This proves Theorem (B.6a).

C

Schensted and Littelmann operators

C.1 Preamble

Schensted (see §§B.3, B.6) associates to every word $i \in I(n,r)$ a unique triple $(\lambda(i), P(i), Q(i)) \in T(n,r)$. This provides the following decomposition (disjoint union) of the set $I(n,r)$:

(C.1a) $I(n,r) = \bigcup_{\lambda \in \Lambda^+(n,r)} I_\lambda(n,r),$

where $I_\lambda(n,r)$ is the set of all $i \in I(n,r)$ such that $\lambda(i) = \lambda$, for each dominant weight $\lambda \in \Lambda^+(n,r)$. We define the *shape of a word* i to be the shape of $P(i)$ (which is also the shape of $Q(i)$). So $I_\lambda(n,r)$ is the set of all words of shape λ. In a case where n, r are supposed known, we may write $I_\lambda(n,r) = I_\lambda$.

Example. The set $I(3,3)$ is decomposed into three subsets $I_{(300)}$, $I_{(210)}$ and $I_{(111)}$; this decomposition of $I(3,3)$ is illustrated in §E.1.

Assume from now on that $\lambda \in \Lambda^+(n,r)$ is fixed.

Definition. Define two equivalence relations $\sim$ and $\approx$ on I_λ: if $i, j \in I_\lambda$ then

(C.1b) $i \sim j$ means that $P(i) = P(j)$, and

(C.1c) $i \approx j$ means that $Q(i) = Q(j)$.

We will use the following notation.

(C.1d) For any (standard) λ-tableau P whose entries are drawn from the set $\underline{n}$, let $I_\lambda(P, \sim)$ be the $\sim$ equivalence class $\{\, i \in I_\lambda \,:\, P(i) = P \,\}$, and

(C.1e) For any (standard) λ-tableau Q whose entries are $1, 2, \dots, r$ (in some order), let $I_\lambda(Q, \approx)$ be the $\approx$ equivalence class $\{\, i \in I_\lambda \,:\, Q(i) = Q \,\}$.

Remark. If either of the tableaux P, Q is given, its shape λ is known. For this reason we will usually omit the suffix λ, and write $I_\lambda(P,\sim) = I(P,\sim)$ and $I_\lambda(Q,\approx) = I(Q,\approx)$.

The equivalence relation $\sim$ was introduced by Knuth, who proved that $\sim$ is the equivalence relation on I_λ generated by a certain collection of basic (or "elementary") moves $i \to j$, each of which affects only two places in i and j. Knuth's theorem will be proved in §§C.3, C.4. This proof is based on Knuth's paper [34, Theorem 6, p. 723].

Littelmann defines a graph G, in a wider context than here [35, p. 504]. Theorem A (see (A.4c) and Chapter D) will show that (in our present context) the equivalence relation determined by G is equal to $\approx$. We regard Theorem A as an analogue to Knuth's theorem; it says that $\approx$ is the equivalence relation on I_λ generated by a certain collection of elementary moves $i \Rightarrow j$, where $i \Rightarrow j$ means that there exists $c \in \{1, 2, \ldots, n-1\}$ such that $\tilde{f}_c(i) = j$ or such that $\tilde{e}_c(i) = j$. Notice that if $i \Rightarrow j$, then the words i and j differ in exactly one place; see (A.3g)(2).

Example. The tables in §E.1 show the $\sim$ and $\approx$ classes for the case $n = r = 3$. The $\approx$ classes are given as vertical columns in these tables; for example

$$I\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \approx\right) = \{211,\ 212,\ 311,\ 213,\ 312,\ 313,\ 322,\ 323\},$$

and $I\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}, \approx\right)$ is the one-word set $\{321\}$. The $\sim$ classes are given as horizontal rows in the tables in §E.1; for example

$$I\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \sim\right) = \{231,\ 213\},$$

and $I\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline\end{array}, \sim\right)$ is the one-word set $\{112\}$. The one-word set $\{321\}$ is both a $\sim$ and a $\approx$ class.

C.2 Unwinding a tableau

To each tableau Y we shall associate a word KY, which may be called the *(Knuth) unwinding of* Y, as follows (see [34, p. 723] or [18, p. 17]).

Let $\lambda \in \Lambda^+(n,r)$ be a dominant weight, and let m be the number of rows of $[\lambda]$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$. Define the *Knuth ordering* $<$ on $[\lambda]$ as follows (see [34, p. 723]):

(C.2a)
$$\begin{aligned}(m,1) < (m,2) < \cdots < (m,\lambda_m)& \\ < (m-1,1) < (m-1,2) < \cdots < (m-1,\lambda_{m-1})& \\ < \cdots& \\ < (2,1) < (2,2) < \cdots < (2,\lambda_2)& \\ < (1,1) < (1,2) < \cdots < (1,\lambda_1).&\end{aligned}$$

Now let $Y = (y_{a,b})_{(a,b)\in[\lambda]}$ be any λ-tableau. Define KY to be the word (C.2b) of length r obtained by writing out the entries $y_{a,b}$ according to the order (C.2a):

(C.2b) $KY := y_{m,1}y_{m,2}\cdots y_{m,\lambda_m}y_{m-1,1}y_{m-1,2}\cdots y_{m-1,\lambda_{m-1}}$
$\cdots\ y_{2,1}y_{2,2}\cdots y_{2,\lambda_2}y_{1,1}y_{1,2}\cdots y_{1,\lambda_1}.$

The word KY is (by definition) the *unwinding* of the tableau Y. So KY is the word obtained by writing out the entries of each row of Y from left to right, starting with the bottom row, and working up to the first row.

Example. If $Y = \begin{array}{cccc} \boxed{1}&\boxed{1}&\boxed{2}&\boxed{2}\\ \boxed{2}&\boxed{3}&&\\ \boxed{3}&&& \end{array}$, then $KY = 3231122$.

Suppose that $i \in I(n,r)$. The Schensted process (see §B.3) constructs an element $(\lambda(i), P(i), Q(i))$ of $T(n,r)$, where $\lambda \in \Lambda^+(n,r)$ and $P(i)$ is a λ-tableau. Then the "unwinding" $KP(i)$ of $P(i)$ is a word, an element of $I(n,r)$. Thus we have an operation $KP : I(n,r) \to I(n,r)$, which takes each i in $I(n,r)$ to $KP(i)$. However, if we apply the Schensted process to $KP(i)$, we just get $P(i)$ again; this follows from Proposition (C.2c) below.

(C.2c) Proposition. *Let $\lambda \in \Lambda^+(n,r)$ with $\lambda_1 \geq \cdots \geq \lambda_m > 0$, and let Y be a λ-tableau.*

(i) *$P(KY) = Y$, and*

(ii) *$Q(KY)$ is completely determined by the shape λ of Y; it is the same for all λ-tableaux Y.*

The tableau $Q(KY)$ is described in (C.2h).

Proof of part (i) of Proposition (C.2c). We shall prove (i) by induction on the number m of rows of Y. If $m = 1$, then Y is a one-rowed tableau $\boxed{y_{1,1}}\boxed{y_{1,2}}\boxed{\quad\cdots\quad}\boxed{y_{1,\lambda_1}}$ of shape $(\lambda_1, 0, \ldots, 0)$, and $KY = y_{1,1}y_{1,2}\ldots y_{1,\lambda_1}$. We make $P(KY)$ by successively inserting $y_{1,2}, \ldots, y_{1,\lambda_1}$ into the tableau $\boxed{y_{1,1}}$ (see (B.2d) and (B.3d)). But since $y_{1,1} \leq y_{1,2} \leq \cdots \leq y_{1,\lambda_1}$, each insertion simply adds a new entry to the first row. Therefore $P(KY) = \boxed{y_{1,1}}\boxed{y_{1,2}}\boxed{\quad\cdots\quad}\boxed{y_{1,\lambda_1}} = Y$. Thus (i) holds if $m = 1$. By the definition (B.3e), we have $Q(KY) = \boxed{1}\boxed{2}\boxed{\ \cdots\ }\boxed{\lambda_1}$; this proves that (ii) also holds.

Now suppose that $m > 1$ and that Proposition (C.2c) holds for any tableau with $m-1$ rows. In particular it holds for the tableau X made by removing the first row of Y; therefore $P(KX) = X$. It is clear that $KY = KX \mid y_{1,1}\cdots y_{1,\lambda_1}$, hence $P(KY) = P(KX) \leftarrow y_{1,1} \leftarrow \cdots \leftarrow y_{1,\lambda_1} = X \leftarrow y_{1,1} \leftarrow \cdots \leftarrow y_{1,\lambda_1}$.

Diagram C.1 shows X, and above it, in parentheses, are the entries of row (1) of Y; these are *not* entries of X.

$$X = \begin{array}{ccccccccc}
(y_{1,1}) & (y_{1,2}) & \cdots & (y_{1,t-1}) & (y_{1,t}) & \cdots & (y_{1,\lambda_2}) & \cdots & (y_{1,\lambda_1}) \\
y_{2,1} & y_{2,2} & \cdots & y_{2,t-1} & y_{2,t} & \cdots & y_{2,\lambda_2} & \cdots & 0 \\
y_{3,1} & y_{3,2} & \cdots & y_{3,t-1} & y_{3,t} & \cdots & y_{3,\lambda_2} & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \\
\vdots & \vdots & & \vdots & \vdots & & y_{\beta_{\lambda_2},\lambda_2} & & \\
\vdots & \vdots & & \vdots & \vdots & & & & \\
\vdots & \vdots & & \vdots & y_{\beta_t,t} & & & & \\
\vdots & \vdots & & y_{\beta_{t-1},t-1} & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & & & & & & & \\
\vdots & y_{\beta_2,2} & & & & & & & \\
y_{\beta_1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0
\end{array}$$

Diagram C.1. The tableau X, made by removing the first row from the tableau Y.

In diagram C.1, the number β_s denotes the length of column s, including the term $(y_{1,s})$. Therefore $\beta_1 = m$, and $\beta = (\beta_1, \beta_2, \dots, \beta_{\lambda_1}, 0, \dots, 0)$ can be regarded as the partition of r conjugate to $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\beta_1}, 0, \dots, 0)$. There holds $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{t-1} \geq \beta_t \geq \cdots \geq \beta_{\lambda_1}$, but for ease of drawing, diagram C.1 illustrates a case where the β_s are distinct.

(C.2d) Definition. Let $t \in \{0, 1, 2, \dots, \lambda_1\}$. Let $X[0] := X$, and if $t \geq 1$, define $X[t]$ to be the diagram obtained from diagram C.1 by removing the parentheses from $(y_{1,1}), (y_{1,2}), \dots, (y_{1,t})$, and then pushing columns $(1), (2), \dots, (t)$ down by one place.

(C.2e) Remark. For each $s \in \{1, \dots, t\}$, column (s) of $X[t]$ is the same as column (s) of Y. In particular, $X[\lambda_1] = Y$.

(C.2f) Lemma. *Let $t \in \{0, 1, 2, \dots, \lambda_1\}$. Then*

$$P(KX \,|\, y_{1,1} y_{1,2} \dots y_{1,t}) = X[t].$$

In particular, $P(KY) = P(KX \,|\, y_{1,1} y_{1,2} \dots y_{1,\lambda_1}) = X[\lambda_1] = Y$.

Thus (C.2f) will complete the proof of part (i) of (C.2c).

Proof of Lemma (C.2f). We use induction on t. If $t = 0$, the lemma claims that $X = X[0]$, which is true. So let $t \in \{1, 2, \dots, \lambda_1\}$, and suppose that the lemma is true when t is replaced by $t-1$; that is $P(KX \,|\, y_{1,1} y_{1,2} \dots y_{1,t-1}) = X[t-1]$. Now $P(KX \,|\, y_{1,1} y_{1,2} \dots y_{1,t}) = P(KX \,|\, y_{1,1} y_{1,2} \dots y_{1,t-1}) \leftarrow y_{1,t}$. So to prove Lemma (C.2f), it will be enough to prove that $X[t-1] \leftarrow y_{1,t}$ is the tableau $X[t]$ defined in (C.2d). The tableau $X[t-1]$ is displayed in

$$X[t-1] = \begin{array}{ccccccccc}
 & & & & (y_{1,t}) & \cdots & (y_{1,\lambda_2}) & \cdots & (y_{1,\lambda_1}) \\
y_{1,1} & y_{1,2} & \cdots & y_{1,t-1} & y_{2,t} & \cdots & y_{2,\lambda_2} & \cdots & 0 \\
y_{2,1} & y_{2,2} & \cdots & y_{2,t-1} & y_{3,t} & \cdots & y_{3,\lambda_2} & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \\
\vdots & \vdots & & \vdots & \vdots & & y_{\beta_{\lambda_2},\lambda_2} & & \\
\vdots & \vdots & & \vdots & \vdots & & & & \\
\vdots & \vdots & & \vdots & y_{\beta_t,t} & & & & \\
\vdots & y_{\beta_2-1,2} & \cdots & y_{\beta_{t-1},t-1} & 0 & \cdots & 0 & \cdots & 0 \\
y_{\beta_1-1,1} & y_{\beta_2,2} & & & & & & & \\
y_{\beta_1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0
\end{array}$$

Diagram C.2. The tableau $X[t-1] = P(KX \mid y_{1,1}y_{1,2}\dots y_{1,t-1})$.

diagram C.2. We calculate $X[t-1] \leftarrow y_{1,t}$ by the general insertion procedure given in §B.3. To make the present notation conform with that in §B.3, take $U := X[t-1]$, $x_1 := y_{1,t}$ and μ to be the shape of U. Calculate the insertion parameters $x_1, k(1), x_2, k(2), \dots$ by the inductive rule: given the element $x_a = y_{a,t}$ for some $a \geq 1$, define $k(a)$ to be the unique element k in $\{1, \dots, \mu_a, \mu_a + 1\}$ such that

(1) $u_{a,k-1} \leq x_a < u_{a,k}$.

Then define $x_{a+1} := u_{a,k(a)}$.

It is very easy to find $k(a)$ in our case. There holds

(2) $y_{a,t-1} \leq y_{a,t} < y_{a+1,t}$,

for any $a \in \{1, \dots, \beta_t - 1\}$; the inequalities in (2) follow from the fact that Y is standard. But (2) is the same as (1), if we take $k = t$ and $x_{a+1} = y_{a+1,t}$. This shows that $k(a) = t$ and $x_{a+1} = y_{a+1,t}$. So starting with $x_1 = y_{1,t}$, which is given, we may find all insertion parameters for the insertion $X[t-1] \leftarrow y_{1,t}$. The result is given in table C.1 below. The parameter z (see the line under (B.3c)) is equal to β_t. Notice that all the $k(a)$ are equal to t, which means

a	1	2	$\cdots$	a	$\cdots$	β_t
x_a	$y_{1,t}$	$y_{2,t}$	$\cdots$	$y_{a,t}$	$\cdots$	$y_{\beta_t,t}$
$k(a)$	t	t	$\cdots$	t	$\cdots$	t

Table C.1. Insertion parameters for the insertion $X[t-1] \leftarrow y_{1,t}$.

that all the x_a lie in column t. Use (B.3d) to find $X[t-1] \leftarrow y_{1,t}$; the result is $X[t]$ as claimed in Lemma (C.2f).

Proof of part (ii) of Proposition (C.2c). We are dealing with the word $j = KY$, whose letters are indexed by the set $[\lambda]$. Fix an element $(a,s) \in [\lambda]$. Then the segment $j_{(m,1)} j_{(m,2)} \cdots j_{(a,s)}$ of j has length

(C.2g) $\psi^{(\lambda)}(a,s) := \lambda_m + \cdots + \lambda_{a+1} + s.$

This gives a bijective, order-preserving map $\psi = \psi^{(\lambda)} : [\lambda] \to \underline{r}$. We may regard $\psi = \psi^{(\lambda)}$ as the λ-tableau whose "unwinding" $K\psi$ is the word[1] $1\,2\,3 \cdots (r-1)\,r$. Notice that the tableau $\psi = \psi^{(\lambda)}$ is, in general, not standard.

Example. If $\lambda = (3,3,2,0,\ldots,0)$, then $\psi = \psi^{(\lambda)} = \begin{array}{|c|c|c|}\hline 6 & 7 & 8 \\ \hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}$.

We shall prove part (ii) of Proposition (C.2c) by proving the following much stronger result:

(C.2h) Proposition (see [3, Appendix C]). *Let Y be any λ-tableau. Then*

$$Q(KY) = \mathsf{Q}^{(\lambda)},$$

where $\mathsf{Q}^{(\lambda)}$ is the λ-tableau given by $\mathsf{Q}^{(\lambda)}(a,s) := \psi^{(\lambda)}(\beta_s + 1 - a, s)$, for all $(a,s) \in [\lambda]$.

Expressed in words: $\mathsf{Q}^{(\lambda)}$ is obtained by reversing each column of the tableau $\psi^{(\lambda)}$.

Example. If $\lambda = (3,3,2,0,\ldots,0)$, then $\mathsf{Q}^{(\lambda)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 \\ \cline{1-2}\end{array}$.

If Proposition (C.2h) is true, the tableau $\mathsf{Q}^{(\lambda)}$ *must* be standard, since it is the Q-symbol of a word KY (see §B.5).

Proof of Proposition (C.2h). We use induction on m. The case $m = 1$ is easy; if $Y = \begin{array}{|c|c|c|c|}\hline y_{1,1} & y_{1,2} & \quad\cdots\quad & y_{1,\lambda_1} \\ \hline\end{array}$, then $Q(KY) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & \lambda_1 \\ \hline\end{array}$, which is the same as $\mathsf{Q}^{(\lambda)}$ (in this case we have $\mathsf{Q}^{(\lambda)} = \psi^{(\lambda)}$).

So now suppose that $m > 1$ and that Proposition (C.2h) is true when Y is replaced by any tableau with $m-1$ rows. In particular it is true for the tableau X obtained by removing the first row from Y, so that $Q(KX) = \mathsf{Q}^{(\lambda^*)}$, where $\lambda^* = (\lambda_2, \ldots, \lambda_m, 0, \ldots, 0) \in \Lambda^+(n, r - \lambda_1)$ is the shape of X.

---

[1] This word belongs to $I(r,r)$. To define $\psi^{(\lambda)}$, we should regard λ as an element of $\Lambda^+(r,r)$, but notice that $[\lambda] = [\lambda']$ if λ' is obtained from λ by adding zeros: $\lambda' = (\lambda_1, \ldots, \lambda_m, 0, 0, \ldots, 0)$.

We proved part (i) of Proposition (C.2c), namely that $P(KY) = Y$, by calculating in turn the P-symbols of the words

$$KX, \quad KX \,|\, y_{1,1}, \quad KX \,|\, y_{1,1}y_{1,2}, \quad \ldots, \quad KX \,|\, y_{1,1}y_{1,2}\cdots y_{1,\lambda_1} = KY.$$

So we shall do the same for the Q-symbols.

The first step is to find $Q(KX \,|\, y_{1,1})$. Use the procedure described in (B.3e), taking $U = X$, $x_1 = y_{1,1}$ and $r = \psi(1,1)$. (Notice that r is the length of the word $KX \,|\, y_{1,1}$). To go from $X = P(KX)$ to $P(KX \,|\, y_{1,1}) = X \leftarrow y_{1,1}$ is very easy; push down the first column of X by one place, and then put $y_{1,1}$ into the top place of that column (see proof of part (i) of Proposition (C.2c)). The tableaux X and $X \leftarrow y_{1,1}$ are shown in table C.2. To find $Q(KX \,|\, y_{1,1})$, use the recipe in (B.3e). Our induction hypothesis gives $Q(KX) = \mathsf{Q}^{(\lambda^*)}$.

The λ^*-tableau X	The $(\lambda^* + \varepsilon_1)$-tableau $X \leftarrow y_{1,1}$
$\begin{array}{ccccc} y_{2,1} & y_{2,2} & \cdots & y_{2,\lambda_2} & 0 \\ y_{3,1} & y_{3,2} & \cdots & y_{3,\lambda_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & y_{\beta_{\lambda_2},\lambda_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & y_{\beta_2,2} & \cdots & 0 & 0 \\ y_{\beta_1,1} & 0 & \cdots & 0 & 0 \end{array}$	$\begin{array}{ccccc} y_{1,1} & y_{2,2} & \cdots & y_{2,\lambda_2} & 0 \\ y_{2,1} & y_{3,2} & \cdots & y_{3,\lambda_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & y_{\beta_{\lambda_2},\lambda_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & y_{\beta_2,2} & \cdots & 0 & 0 \\ y_{\beta_1-1,1} & 0 & \cdots & 0 & 0 \\ y_{\beta_1,1} & 0 & \cdots & 0 & 0 \end{array}$

Table C.2. Inserting $y_{1,1}$ into the tableau X (P-symbol).

Diagram C.3 shows $\psi^{(\lambda^*)}$. To make $\mathsf{Q}^{(\lambda^*)}$ we reverse each column of $\psi^{(\lambda^*)}$; this

$$\psi^{(\lambda^*)} = \begin{array}{ccccc} \psi(2,1) & \psi(2,2) & \cdots & \psi(2,\lambda_2) & 0 \\ \psi(3,1) & \psi(3,2) & \cdots & \psi(3,\lambda_2) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \cdots & \psi(\beta_{\lambda_2},\lambda_2) & 0 \\ \vdots & \vdots & & \vdots & 0 \\ \vdots & \psi(\beta_2,2) & & 0 & 0 \\ \psi(\beta_1,1) & 0 & \cdots & 0 & 0 \end{array} .$$

Diagram C.3. The tableau $\psi^{(\lambda^*)}$, where $\lambda^* = (\lambda_2, \ldots, \lambda_m, 0, \ldots, 0)$.

gives the left-hand pane in table C.3. Now follow the instructions in (B.3e): to

The Q-symbol of the word KX	The Q-symbol of the word $KX \mid y_{1,1}$
$\begin{matrix} \psi(\beta_1,1) & \psi(\beta_2,2) & \cdots & \psi(\beta_{\lambda_2},\lambda_2) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \psi(4,1) & \psi(3,2) & \cdots & \psi(3,\lambda_2) & 0 \\ \psi(3,1) & \psi(2,2) & \cdots & \psi(2,\lambda_2) & 0 \\ \psi(2,1) & 0 & \cdots & 0 & 0 \end{matrix}$	$\begin{matrix} \psi(\beta_1,1) & \psi(\beta_2,2) & \cdots & \psi(\beta_{\lambda_2},\lambda_2) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \psi(4,1) & \psi(3,2) & \cdots & \psi(3,\lambda_2) & 0 \\ \psi(3,1) & \psi(2,2) & \cdots & \psi(2,\lambda_2) & 0 \\ \psi(2,1) & 0 & \cdots & 0 & 0 \\ \psi(1,1) & 0 & \cdots & 0 & 0 \end{matrix}$

Table C.3. Inserting $y_{1,1}$ into the tableau X (Q-symbol).

go from the left-hand pane to the right-hand pane, we adjoin a new place—this must be the place which is new when we go from $P(KX)$ to $P(KX) \leftarrow y_{1,1}$, namely the place $(\beta_1, 1)$. And in this place we must put "r", which in our case is $\psi(1,1)$. This gives $Q(KX \mid y_{1.1})$ shown in the right-hand pane of table C.3.

We go on to insert $y_{1,2}, \ldots, y_{1,\lambda_1}$ in turn. At each insertion, say $y_{1,t}$, we adjoin $\psi(1,t)$ to the bottom of column (t). But the new column (t) so made is the same as column (t) of $\mathsf{Q}^{(\lambda)}$. When we have inserted y_{1,λ_1} we have the complete tableau $\mathsf{Q}^{(\lambda)}$. This finishes the proof of Proposition (C.2h).

Hence we have proved Proposition (C.2c).

(C.2i) Exercise. Let $\lambda \in \Lambda^+(n,r)$, and let i be any element of $I_\lambda(n,r)$. Then $i = KP(i)$ if and only if $Q(i) = \mathsf{Q}^{(\lambda)}$. In other words, the $\approx$ class of $\mathsf{Q}^{(\lambda)}$ consists of all $i \in I_\lambda(n,r)$ which satisfy $i = KP(i)$.

[Hint. Let $i \in I(n,r)$. Schensted's Theorem (B.6a) tells us that

(i) $i = KP(i)$ if and only if $\mathsf{Sch}(i) = \mathsf{Sch}(KP(i))$.

Now assume that $i \in I_\lambda(n,r)$, which means that $\lambda(i) = \lambda$ (see (C.1a)). Hence

(ii) $\mathsf{Sch}(i) = (\lambda, P(i), Q(i))$.

To calculate $\mathsf{Sch}(KP(i))$ we take $Y = P(i)$ in (C.2c). This shows that $P(KP(i)) = P(i)$. Since $P(i)$ has shape λ, it follows that $\lambda(KP(i)) = \lambda$. But (C.2c)(ii) and (C.2h) tell us that $Q(KP(i)) = \mathsf{Q}^{(\lambda)}$. Therefore

(iii) $\mathsf{Sch}(KP(i)) = (\lambda, P(i), \mathsf{Q}^{(\lambda)})$.

Now (i), together with (ii) and (iii), give the desired result: $i = KP(i)$ if and only if $Q(i) = \mathsf{Q}^{(\lambda)}$.]

C.3 Knuth's theorem

(C.3a) Theorem (see [34, p. 723]). *Let i, j be words in $I(n,r)$. Then $i \sim j$ (i.e. $P(i) = P(j)$) if and only if there is a finite sequence of words*

(C.3b) $i(1), i(2), \ldots, i(s)$

such that $i(1) = i$, $i(s) = j$ and each consecutive pair of words $i(\sigma - 1)$, $i(\sigma)$ is connected by a basic *(or* elementary*)* move *of type K' or K''.*

These basic moves are as follows [34, p. 723].

Definition. A move of type K' changes a word

(C.3c) $\ldots b\,c\,a \ldots$ to $\ldots b\,a\,c \ldots,$

where a, b, c are letters (i.e. elements of $\underline{n}$) such that $a < b \leq c$.

A move of type K'' changes a word

(C.3d) $\ldots a\,c\,b \ldots$ to $\ldots c\,a\,b \ldots,$

where a, b, c are letters (i.e. elements of $\underline{n}$) such that $a \leq b < c$.

Remarks.

(i) Each basic move is assumed to be *symmetric*, i.e. if a move takes a word w to another word w', then it also takes w' to w.

(ii) In (C.3c) and (C.3d), the symbol $\ldots$ stands for a word (possibly empty) which is not changed in the move. For example the type K' move (C.3c) changes $BbcaC$ to $BbacC$, where B, C are fixed words. (By (i), this move also takes $BbacC$ to $BbcaC$.)

The "only if" part of Knuth's theorem will be proved in this section, and the "if" part will be proved in §C.4. So in this section (§C.3) we must prove

(C.3e) *If $i, j \in I(n,r)$ are such that $P(i) = P(j)$, then i can be connected to j by a finite sequence of basic moves.*

The essence of this is that *every insertion operation U to $U \leftarrow x$ can be broken down into a sequence of basic moves.* The next proposition puts this fact in a form suitable for our purposes; in (C.3p) it will be shown that (C.3f) implies (C.3e).

(C.3f) Proposition. *Let $r > 1$, $\mu \in \Lambda^+(n, r-1)$. Let U be any μ-tableau and x any element of $\underline{n}$. Regard KU and x as words of lengths $r-1$ and 1 respectively, so that the "concatenation" $w = KU \mid x$ of these may be regarded as an element in $I(n,r)$. Then there is a finite sequence of basic moves in $I(n,r)$ which takes w to the word $K(U \leftarrow x)$.*

Proof. We shall give in (C.3i)–(C.3k) an explicit sequence of basic moves which takes w to $K(U \leftarrow x)$[2].

It is desirable to fix first some notation for the words which will be used in the proof of (C.3f).

(C.3g) Notation for words and places. All the words in this section have length r, and their entries are labelled by the set of places $[\mu] \cup \{(r)\}$. The $r-1$ elements of $[\mu]$ are arranged according to the Knuth order (see (C.2a)), and (r) is the last place. Therefore if $\mu = (\mu_1, \ldots, \mu_m, 0, \ldots, 0) \in \Lambda^+(n, r-1)$ (with $\mu_1 \geq \cdots \geq \mu_m > 0$ and $\sum \mu_j = r-1$), then a typical word looks like this: $y = y_{m,1} \ldots y_{m,\mu_m} \; \ldots \; y_{1,1} \ldots y_{1,\mu_1} y_r$.

To resume the proof of (C.3f), write $x = x_1$ and $w = KU \mid x$, so that $w = u_{m,1} \ldots u_{m,\mu_m} \; \ldots \; u_{1,1} \ldots u_{1,\mu_1} x_1$. Recall from (B.3b) the "parameters" of the insertion of x_1 into the tableau U: for each $a \in \{1, \ldots, z\}$, define $x_a := u_{a-1,k(a-1)}$ if $a > 1$, or if $a = 1$ define $x_1 = x$. Define $k(a)$ to be the smallest $k \in \{1, 2, \ldots, \mu_a, \mu_a + 1\}$ such that $x_a < u_{a,k}$. If $k(a) = \mu_a + 1$ (which means that $x_a \geq u_{a,\mu_a}$), then the insertion sequence stops at this stage. Define z to be the first a such that $x_a \geq u_{a,\mu_a}$. The tableau $U \leftarrow x_1$ is denoted $P = (p_{a,b})_{(a,b) \in [\lambda]}$, where $\lambda = \mu + \varepsilon_z$ (see (B.3d)).

According to (B.3d), each row $a > z$ of the tableau U, is identical to the corresponding row of $P = U \leftarrow x_1$; and (B.3d)(1°) shows that also $u_{z,t} = p_{z,t}$ for all $t \in \{1, \ldots, \mu_z\}$. Therefore

(C.3h) $u_{a,t} = p_{a,t}$ for all places $(a,t) \leq (z, \mu_z)$.

Next define a sequence of words $\xi(a,t)$, one for each place $(a,t) \in [\mu]$, which will "interpolate" between the words $w = KU \mid x_1$ and $KP = K(U \leftarrow x_1)$. Use the following notation: if $\tau \in [\mu]$, then $\tau+$ (respectively $\tau-$) denotes the place immediately after (respectively immediately before) τ in the order (C.3g) of $[\mu] \cup \{(r)\}$. For example, if $a \in \{2, \ldots, z\}$, then $(a,t)+$ is $(a, t+1)$ for all $1 \leq t < \mu_a$, and $(a, \mu_a)+ = (a-1, 1)$.

Definition of the words $\boldsymbol{\xi(a,t)}$.

(C.3i) If $(a,t) \leq (z, \mu_z)$, then define $\xi(a,t) := KP$.

(C.3j) If $(a,t) > (z, \mu_z)$ and $k(a)+1 \leq t \leq \mu_a$, then define $\xi(a,t)_{(a,t)+} := x_a$, $\xi(a,t)_\tau := u_\tau$ if $\tau \leq (a,t)$, and $\xi(a,t)_\tau := p_{\tau-}$ if $\tau > (a,t)+$.

(C.3k) If $(a,t) > (z, \mu_z)$ and $1 \leq t \leq k(a)$, then define $\xi(a,t)_{(a,t)} := x_{a+1}$, $\xi(a,t)_\tau := u_\tau$ if $\tau < (a,t)$, and $\xi(a,t)_\tau := p_{\tau-}$ if $\tau > (a,t)$.

[2] In fact the construction of these basic moves is an essential part of Knuth's proof of his theorem; see [34, end of p. 723, and first 7 lines of p. 724].

(C.3l) Pivot of $\xi(\mathbf{a},\mathbf{t})$. Assume that $(a,t) > (z,\mu_z)$. Then the word $\xi(a,t)$ can be described as follows: define the *pivot of* $\xi(a,t)$ to be the pair $(x_a,(a,t)+)$ in case (C.3j), and to be $(x_{a+1},(a,t))$ in case (C.3k). Then in both cases the rule is: at every place τ left of the pivot, let $\xi(a,t)_\tau = u_\tau$, and at every place τ right of the pivot, let $\xi(a,t)_\tau = p_{\tau-}$. At the pivot itself, $\xi(a,t)$ has entry x_a in case (C.3j), or x_{a+1} in case (C.3k). If a word is among the $\xi(a,t)$, it is completely determined by its pivot.

(C.3m) Proposition.

(i) $\xi(1,\mu_1) = w = KU \mid x_1$.

(ii) $\xi(z-1,1) = KP = K(U \leftarrow x_1)$.

(iii) $\xi(a,1) = \xi(a+1,\mu_{a+1})$, *for all* $a \in \{1,\dots,m-1\}$.

(iv) *If* $k(a)+1 \le t \le \mu_a$, *there is a basic move of type* K' *which takes* $\xi(a,t)$ *to* $\xi(a,t-1)$.

(v) *If* $2 \le t \le k(a)$, *there is a basic move of type* K'' *which takes* $\xi(a,t)$ *to* $\xi(a,t-1)$.

Proof. (i) The pivot of $\xi(1,\mu_1)$ is $(x_1,(r))$, therefore $\xi(1,\mu_1)$ has u_τ in each place $\tau \in [\mu]$, and x_1 in place (r). Hence $\xi(1,\mu_1) = KU \mid x_1$.

(ii) The pivot of $\xi(z-1,1)$ is $(x_z,(z-1,1))$ since $1 \le t \le k(z-1)$ for $t=1$. Thus $\xi(z-1,1)$ has x_z at place $(z-1,1)$. At a place $\tau < (z-1,1)$, the entry in $\xi(z-1,1)$ is u_τ, and this equals p_τ, by (C.3h). At $t > (z-1,1)$, the entry is $p_{\tau-}$. Therefore $\xi(z-1,1) = KP$.

(iii) All that is needed, is to show that both $\xi(a,1)$ and $\xi(a+1,\mu_{a+1})$ have the same pivot $(x_{a+1},(a,1))$. We leave this as an exercise for the reader.

(iv) Suppose that $k(a)+1 \le t \le \mu_a$. By definition (C.3j), the entries of $\xi(a,t)$ at the places $(a,t)-$, (a,t), $(a,t)+$ are $u_{(a,t)-}$, $u_{a,t}$, x_a, respectively. If these entries are denoted b, c, a, then we shall prove that $a < b \le c$. The inequality $b \le c$, i.e. $u_{(a,t)-} \le u_{(a,t)}$, follows from $(a,t)- = (a,t-1)$ and the standardness of U (note that $k(a)+1 \le t$ implies that $2 \le t$). To see that $a < b$, use the inequality $u_{a,k(a)-1} \le x_a < u_{a,k(a)}$ (see (B.3c)). Standardness of U gives $u_{a,k(a)} \le u_{a,t-1}$, since $k(a)+1 \le t$. Therefore $x_a < u_{a,k(a)} \le u_{a,t}$, hence $a < b$.

We may now make a move of type K' which interchanges a and c, and leaves $\xi(a,t)$ otherwise unchanged. At the places (a,t), $(a,t)+$, the word $K'\xi(a,t)$ has entries x_a, $u_{a,t}$. However $u_{a,t} = p_{a,t}$ since $t \ne k(a)$. It is now easy to see that $K'\xi(a,t) = \xi(a,t-1)$.

(v) The proof is on the same lines as that of (iv). Suppose $2 \le t \le k(a)$. By (C.3k) the entries in $\xi(a,t)$ at the places $(a,t)- = (a,t-1)$, (a,t), $(a,t)+$ are $u_{a,t-1}$, x_{a+1}, $p_{a,t}$, respectively. If these entries are denoted a, c, b, we leave it to the reader to prove that $a \le b < c$. This shows that there is a move of type K'' which interchanges a, c, and leaves $\xi(a,t)$ otherwise unchanged. Therefore the entries in $K''\xi(a,t)$ at places $(a,t-1)$, (a,t), are x_{a+1}, $u_{a,t-1}$. But $u_{a,t-1} = p_{a,t-1}$ because $t-1 \ne k(a)$. It follows that $K''\xi(a,t) = \xi(a,t-1)$.

This completes the proof of Proposition (C.3m).

And this proves Proposition (C.3f).

(C.3n) Example. Take $\mu = (4,2,1,1) \in \Lambda^+(4,8)$, and U as given in (C.3o). Then U is a μ-tableau. Now take $x_1 = 1$. We calculate $P = U \leftarrow x_1$ by the methods of §B.4. This tableau also is given in (C.3o). It is a λ-tableau, where $\lambda = (4,2,2,1) \in \Lambda^+(4,9)$.

(C.3o) $$U = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&4\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\;, \qquad P = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2\\\cline{1-2} 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array}\;.$$

The parameters for the insertion of $x_1 = 1$ are as follows (see (B.3b)): $z = 3$ and

$$\begin{array}{lll} x_1 = 1\ (= p_{1,k(1)}), & x_2 = 2\ (= u_{1,k(1)} = p_{2,k(2)}), & x_3 = 4\ (= u_{2,k(2)} = p_{3,k(3)}), \\ k(1) = 3, & k(2) = 2, & k(3) = 2. \end{array}$$

It is rather easy to display the words $\xi(a,t)$, for all $(a,t) \in [\mu]$ (see table C.4). By (C.3m)(i),(ii) we know that $\xi(1,4) = KU \mid x_1$, and $\xi(2,1) = KP$; so write in these words. If $(a,t) \le (z,\mu_z)$ then $\xi(a,t) = KP$ by (C.3i), and this gives us $\xi(3,1)$ and $\xi(4,1)$. If $(a,t) > (z,\mu_z)$, determine the pivot of $\xi(a,t)$, using (C.3j) or (C.3k) as appropriate. For example the pivot of $\xi(1,4)$ is $(x_1,(1,4)+) = (x_1,(9))$, and the pivot of $\xi(1,2)$ is $(x_2,(1,2))$. For each $\xi(a,t)$, we have underlined the first term (x_a or x_{a+1}) in the pivot of $\xi(a,t)$ in table C.4.

Proof of the "only if" part of Knuth's theorem. Suppose $i \in I(n,r)$, and for each $s \in \{0,1,\dots,r\}$ define $P_s(i) = P(i_1 \dots i_s)$ (take $P_0(i)$ to be the empty tableau). Let $s \in \{1,2,\dots,r\}$ and let $U = P_{s-1}(i)$ and $x = i_s$. Then Proposition (C.3f) provides a sequence of words $\xi(a,t)$, and hence a sequence of basic moves taking the word $KP_{s-1}(i) \,|\, i_s$ to the word $K(P_{s-1}(i) \leftarrow i_s) = KP_s(i)$. Using the notation of §B.7 (the "ladder"), we may now construct a sequence of basic moves taking $KP_{s-1} \,|\, i_s i_{s+1} \dots i_r$ to $KP_s \,|\, i_{s+1} \dots i_r$; we simply use the sequence of words $\xi(a,t) \,|\, i_{s+1} \dots i_r$ in place of the $\xi(a,t)$. Since we can do this for each $s = 1,2,\dots,r$, we deduce the following fundamental proposition:

(C.3p) Proposition. *Given $i \in I(n,r)$, there is a sequence of basic moves in $I(n,r)$ which takes i to $KP(i)$.*

It is now easy to prove (C.3e), which is the "only if" part of Knuth's theorem (C.3a). For given $i,j \in I(n,r)$ such that $P(i) = P(j)$, we use (C.3p) to make two sequences of basic moves, one taking i to $KP(i)$ and one taking j to $KP(j) = KP(i)$. Then the first of these sequences, followed by the "reverse" of the second, takes i to j.

Place	$(4,1)$	$(3,1)$	$(2,1)$	$(2,2)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	(9)	Move
$\xi(1,4)$	$u_{4,1}$	$u_{3,1}$	$u_{2,1}$	$u_{2,2}$	$u_{1,1}$	$u_{1,2}$	$u_{1,3}$	$u_{1,4}$	$\underline{x_1}$	K'
$\xi(1,3)$	$u_{4,1}$	$u_{3,1}$	$u_{2,1}$	$u_{2,2}$	$u_{1,1}$	$u_{1,2}$	$\underline{x_2}$	x_1 $=p_{1,3}$	$p_{1,4}$ $=u_{1,4}$	K''
$\xi(1,2)$	$u_{4,1}$	$u_{3,1}$	$u_{2,1}$	$u_{2,2}$	$u_{1,1}$	$\underline{x_2}$	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	K''
$\xi(1,1)$	$u_{4,1}$	$u_{3,1}$	$u_{2,1}$	$u_{2,2}$	$\underline{x_2}$	$p_{1,1}$ $=u_{1,1}$	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	–
$\xi(2,2)$	$u_{4,1}$	$u_{3,1}$	$u_{2,1}$	$\underline{x_3}$	x_2 $=p_{2,2}$	$p_{1,1}$	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	K''
$\xi(2,1)$	$p_{4,1}$ $=u_{4,1}$	$p_{3,1}$ $=u_{3,1}$	$\underline{x_3}$ $=p_{3,2}$	$p_{2,1}$	$p_{2,2}$	$p_{1,1}$	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	–
$\xi(3,1)$	$p_{4,1}$	$p_{3,1}$	$p_{3,2}$	$p_{2,1}$	$p_{2,2}$	$p_{1,1}$	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	–
$\xi(4,1)$	$p_{4,1}$	$p_{3,1}$	$p_{3,2}$	$p_{2,1}$	$p_{2,2}$	$p_{1,1}$	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	–

Table C.4. The sequence of words associated to Schensted insertion.

C.4 The “if” part of Knuth’s theorem

Let n, r be positive integers. In this section we will prove the “if” part of Knuth’s theorem (C.3a), that is: if $i, j \in I(n,r)$ can be connected by a sequence of basic moves as in (C.3b), then $i \sim j$ (i.e. $P(i) = P(j)$).

Clearly it will be enough to prove:

(C.4a) If i and j in $I(n,r)$ are connected by a basic move, then $P(i) = P(j)$.

For a standard tableau U and letters $x_1, \ldots, x_k$, we write

$$U \leftarrow x_1 x_2 \cdots x_k = (\cdots((U \leftarrow x_1) \leftarrow x_2) \cdots) \leftarrow x_k$$

to ease the reading. We will prove the following

(C.4b) Proposition (see [34, pp. 721, 722]). *Let U be a standard tableau with entries drawn from $\underline{n}$, and let $a, b, c \in \underline{n}$.*

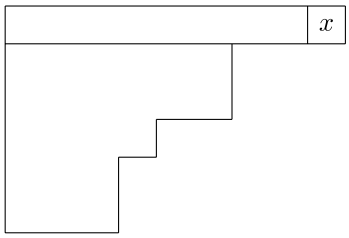
(1) *If $a < b \leq c$, then $U \leftarrow bac = U \leftarrow bca$.*

(2) *If $a \leq b < c$, then $U \leftarrow acb = U \leftarrow cab$.*

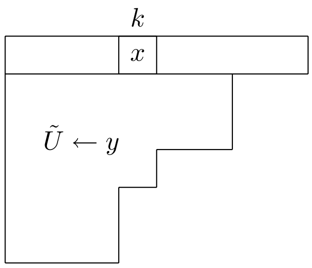
This proposition implies (C.4a), by means of a simple induction on the length r of i and j.

Our proof of the proposition builds on Schensted's original description of the insertion process $U \leftarrow x$, which reads as follows. Let U be a μ-tableau and x be a letter.

If U is the empty tableau or $u_{1,\mu_1} \leq x$ (so that the insertion sequence (B.3b) has length $z = 1$), then $U \leftarrow x$ is obtained from U by appending the letter x to the first row of U:

$U \leftarrow x =$ 

If U is not empty and $x < u_{1,\mu_1}$ (so that the insertion sequence (B.3b) has length $z > 1$), then choose $k \leq \mu_1$ minimal with $x < u_{1,k}$ and set $y = u_{1,k}$. The tableau $U \leftarrow x$ has first row $(u_{1,1}, \ldots, x, \ldots, u_{1,\mu_1})$ (with x in column k), while the remaining rows of $U \leftarrow x$ are given by $\tilde{U} \leftarrow y$. Here $\tilde{U}$ is the "sub-tableau" of U obtained from U by removing the first row. In illustrative terms:

$U \leftarrow x =$ 

Of course, this description of $U \leftarrow x$ follows directly from our description (B.3d).

The idea of proof for the proposition is this. In either case (1) or (2), check that the first rows of the two tableaux shown coincide. Then consider the tableaux obtained by removing the first rows.

If any of the three letters a, b, c does not bump a letter into the second row, then it is fairly easy to see that these "sub-tableaux" are equal. If all three letters a, b, c bump letters into the second row—x, y, z, say—then the letters x, y, z can be shown to satisfy either (1) or (2). We can then conclude by induction on the number of rows of U.

Before we give the proof of (C.4b), let's look at two examples.

Example 1. Let $n = 5$, $a = 1$, $b = 1$ and $c = 3$ (so that part (2) of the proposition applies), and consider the tableau

$$U = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 2 & 3 & 4 \\ \cline{1-4} 3 & 4 & 4 & 5 \\ \cline{1-4}\end{array}$$

We have

$$\tilde{U} = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & 4 & 5 \\ \hline\end{array}.$$

Let us concentrate on the first rows of $U \leftarrow acb$ and $U \leftarrow cab$. We get

$$U \leftarrow acb = U \leftarrow 131 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 3 & 3 \\ \hline \multicolumn{4}{|c|}{\tilde{U} \leftarrow 243} \\ \cline{1-4}\end{array}$$

from Schensted's inductive description, because $a = 1$ bumps $x = 2$ from the first row of U, $c = 3$ bumps $z = 4$ from the first row of $U \leftarrow a$, and $b = 1$ bumps $y = 3$ from the first row of $U \leftarrow ac$.

Similarly, we get

$$U \leftarrow cab = U \leftarrow 311 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 3 & 3 \\ \hline \multicolumn{4}{|c|}{\tilde{U} \leftarrow 423} \\ \cline{1-4}\end{array}$$

Applying (C.4b)(2), it follows that $\tilde{U} \leftarrow 243 = \tilde{U} \leftarrow 423$ (by induction on the number of rows), hence also $U \leftarrow 131 = U \leftarrow 311$.

Example 2. Let $n = 5$ again, $a = 1$, $b = 1$ and $c = 2$ (so that part (2) of proposition (C.4b) applies again), and consider the tableau

$$U = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 & 4 \\ \cline{1-5} 4 & 5 & 5 & 5 \\ \cline{1-4}\end{array}$$

We have

$$\tilde{U} = \begin{array}{|c|c|c|c|c|}\hline 2 & 3 & 4 & 4 & 4 \\ \hline 4 & 5 & 5 & 5 \\ \cline{1-4}\end{array}.$$

In this case, we get

$$U \leftarrow acb = U \leftarrow 121 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 3 & 3 \\ \hline \multicolumn{5}{|c|}{\tilde{U} \leftarrow 332} \\ \cline{1-5}\end{array}$$

from Schensted's inductive description, because $a = 1$ bumps $x = 3$ from the first row of U, $c = 2$ bumps $z = 3$ from the first row of $U \leftarrow a$, and $b = 1$ bumps $y = c = 2$ from the first row of $U \leftarrow ac$. Similarly, we get

$$U \leftarrow cab = U \leftarrow 211 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 3 & 3 \\ \hline \multicolumn{5}{|c|}{\tilde{U} \leftarrow 323} \\ \cline{1-5}\end{array}$$

This time applying (C.4b)(1), it follows that $\tilde{U} \leftarrow 332 = \tilde{U} \leftarrow 323$ (by induction on the number of rows), hence also $U \leftarrow 121 = U \leftarrow 211$.

Proof of Proposition (C.4b). The proof is done by induction on the number m of rows of U.

If $m = 0$, then U is the empty tableau and

$$(1) \quad U \leftarrow bac = \begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array} = U \leftarrow bca \text{ if } a < b \leq c,$$

$$(2) \quad U \leftarrow acb = \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array} = U \leftarrow cab \text{ if } a \leq b < c.$$

Thus (C.4b) holds in case $m = 0$.

Suppose $m > 0$. Let μ denote the shape of U and write $U = (u_{x,y})_{(x,y)\in[\mu]}$. Furthermore, let $\tilde{U}$ be the tableau obtained from U by removing the first row.

We shall prove the parts (1) and (2) separately.

Proof of part (1) of Proposition (C.4b). Assume we have $a, b, c \in \underline{n}$ such that $a < b \leq c$. We want to prove that $U \leftarrow bac = U \leftarrow bca$.

To find the tableau $W := U \leftarrow b$, let b bump $u_{1,l}$ into row (2), where l is the smallest element of $\{1, \ldots, \mu_1, \mu_1 + 1\}$ such that $b < u_{1,l}$. This means:

(i) row (1) of W is the same as row (1) of U except at place $(1, l)$, where $w_{1,l} = b$, and

(ii) the tableau obtained by removing the first row of W, is $\tilde{W} = \tilde{U} \leftarrow y$, where $y := u_{1,l}$. (If $l = \mu_1 + 1$, so that $y = \infty$, we make the convention that inserting ∞ into row (2) has no effect on $\tilde{U}$.)

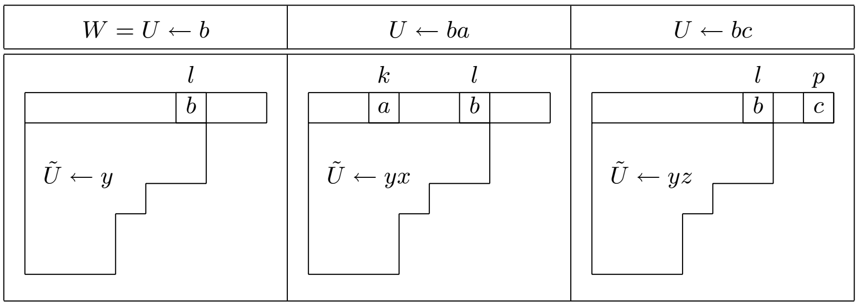


Table C.5. Inserting b, a and b, c into U when $a < b \leq c$.

To find the tableaux $U \leftarrow ba = W \leftarrow a$ and $U \leftarrow bc = W \leftarrow c$, define k to be the smallest element of $\{1, \ldots, \mu_1, \mu_1 + 1\}$ and p to be the smallest element of $\{1, \ldots, \mu_1 + 1, \mu_1 + 2\}$ such that $a < w_{1,k}$ and $c < w_{1,p}$, respectively. (The case $p = \mu_1 + 2$ only occurs when $k = \mu_1 + 1$.)

Then

(iii) $k \leq l$ (because $a < b = w_{1,l}$), and

(iv) $l < p$ (because $w_{1,l} = b \leq c < w_{1,p}$).

Make $U \leftarrow ba$ from W by letting a bump $x := w_{1,k}$ into row (2); make $U \leftarrow bc$ from W by letting c bump $z := w_{1,p}$ into row (2). The resulting tableaux are shown in table C.5.

To find $U \leftarrow bac$, insert c into the tableau $W' := U \leftarrow ba$. First find the smallest p' in $\{1, \ldots, \mu_1, \mu_1 + 1, \mu_1 + 2\}$ such that $c < w'_{1,p'}$. But any p' such that $c < w'_{1,p'}$ is $> l$ (because $w'_{1,l} = b \leq c$), and all the entries $w'_{1,s}$ in row (1) of W' such that $s \geq l$, coincide with the corresponding entries $w_{1,s}$ in row (1) of W, because the process which takes W to $W' = W \leftarrow a$ affects only the part of row (1) to the left of $(1, l)$. Therefore $p = p'$, and $U \leftarrow bac$ is shown in the left pane of table C.6. An entirely similar argument gives $U \leftarrow bca$, using the fact that the process which takes W to $W'' = W \leftarrow b$ affects only the part of row (1) to the right of $(1, l)$; this tableau is shown in the right pane of table C.6. We next prove that $x < y \leq z$. First, to prove $x < y$, observe

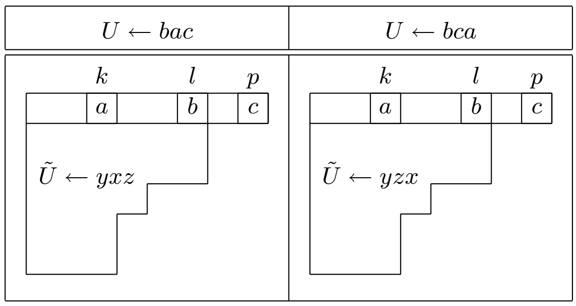
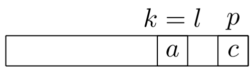


Table C.6. Inserting b, a, c and b, c, a into U when $a < b \leq c$.

that (iii) gives $x = w_{1,k} \leq w_{1,l} = b < u_{1,l} = y$. Second, to prove $y \leq z$, observe that (iv) gives $y = u_{1,l} \leq u_{1,p} = z$. This argument is also valid when $y = \infty$, because then $z = \infty$ as well.

It follows that $\tilde{U} \leftarrow yxz = \tilde{U} \leftarrow yzx$ by the induction hypothesis; and table C.6 gives the desired result $U \leftarrow bac = U \leftarrow bca$.

There is still one "loose end" to be tidied up! Namely it can happen that $k = l$; the argument above still works, but we must re-draw the first rows of the tableaux shown in table C.6. Each of these rows (which are equal) looks like



This concludes the proof of Proposition (C.4b)(1).

Proof of part (2) of Proposition (C.4b). We now assume that $a \leq b < c$ and show that $U \leftarrow acb = U \leftarrow cab$.

To find the tableaux $U \leftarrow a$ and $U \leftarrow c$, define k and p to be the smallest elements of $\{1, \ldots, \mu_1, \mu_1 + 1\}$ such that $a < u_{1,k}$ and $c < u_{1,p}$, respectively.

As usual, make $V := U \leftarrow a$ from U by letting a bump $x := u_{1,k}$ into row (2), and make $W := U \leftarrow c$ from U by letting c bump $z := u_{1,p}$ into row (2). Then $k \le p$ (because $a < c$). We consider two cases:

Case 1. $k < p$. Then $k \le \mu_1$, hence the first row of V has μ_1 entries. To find the tableau $U \leftarrow ac = V \leftarrow c$ define p' to be the smallest element of $\{1, \ldots, \mu_1, \mu_1 + 1\}$ such that $c < v_{1,p'}$. But any p' with $c < v_{1,p'}$ is $> k$ (because $v_{1,k} = a < c$), and all the entries $v_{1,s}$ in row (1) of V with $s > k$, coincide with the corresponding entries $u_{1,s}$ of U (because the first rows of U and V differ only at place $(1,k)$). It follows that $p = p'$, and c bumps the letter $z = u_{1,p}$ of V into row (2). An entirely similar argument shows that a bumps the letter $x = u_{1,k} = w_{1,k}$ of W into row (2).

The tableaux $V \leftarrow c$ and $W \leftarrow a$ are shown in table C.7.

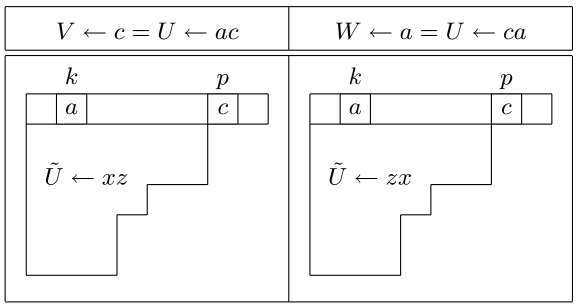


Table C.7. Inserting a, c and c, a into U when $a \le b < c$ and $k < p$.

The first rows of $V \leftarrow c$ and $W \leftarrow a$ coincide. Hence b bumps the same letter $y = v_{1,l} = w_{1,l}$ into row (2) of $V \leftarrow c$ and $W \leftarrow a$. Furthermore, it follows from $a \le b < c$ that $k < l \le p$. The tableaux $U \leftarrow acb = V \leftarrow cb$ and $U \leftarrow cab = W \leftarrow ab$ are displayed in table C.8.

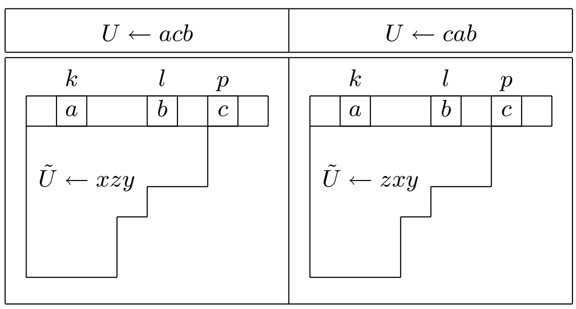


Table C.8. Inserting a, c, b and c, a, b into U when $a \le b < c$ and $k < p$.

We next prove that $x \leq y < z$. First, we have $x = u_{1,k} \leq u_{1,l} = y$ if $l < p$, and $x = v_{1,k} \leq v_{1,p-1} \leq c = y$ if $l = p$. Second, $y = v_{1,l} \leq v_{1,p} = c < u_{1,p} = z$.

It follows that $\tilde{U} \leftarrow xzy = \tilde{U} \leftarrow zxy$ by the induction hypothesis; and table C.8 gives $U \leftarrow acb = U \leftarrow cab$ in case $k < p$.

Case 2. $k = p$. Then $x = u_{1,k} = u_{1,p} = z$. To find the tableau $V \leftarrow c$ in this case, define p' to be the smallest element of $\{1, \ldots, \mu_1 + 1, \mu_1 + 2\}$ such that $c < v_{1,p'}$. But $v_{1,k} = a < c < u_{1,k} \leq u_{1,k+1} = v_{1,k+1}$, hence $p' = k + 1$. Therefore c bumps the letter $w = v_{1,k+1} = u_{1,k+1}$ of V into row (2).

To find the tableau $W \leftarrow a$, note that $w_{1,p-1} = u_{1,p-1} \leq a < c = w_{1,p}$. Therefore a bumps the letter $c = w_{1,p}$ of W into row (2).

The tableaux $V \leftarrow c$ and $W \leftarrow a$ are shown in table C.9.

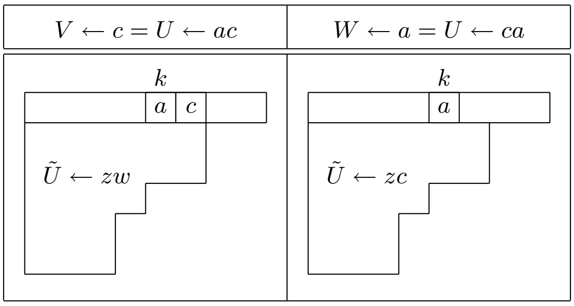


Table C.9. Inserting a, c and c, a into U when $a \leq b < c$ and $k = p$.

From $v_{1,k} = a \leq b < c = v_{1,k+1}$ it follows that b bumps the letter c (in column $k + 1$) of $V \leftarrow c$ into row (2).

From $w_{1,k} = a \leq b < c < u_{1,k} \leq u_{1,k+1} = w_{1,k+1}$ it follows that b bumps the letter $w = u_{1,k+1}$ into row (2) of $W \leftarrow a$.

The resulting tableaux $V \leftarrow cb = U \leftarrow acb$ and $W \leftarrow ab = U \leftarrow cab$ are shown in table C.10.

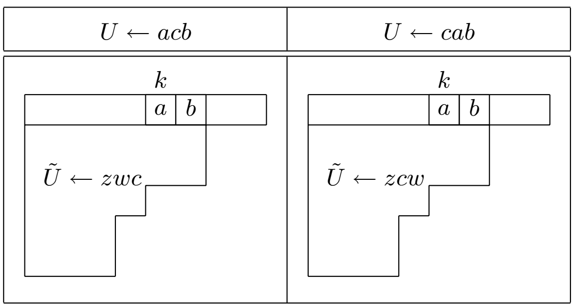


Table C.10. Inserting a, c, b and c, a, b into U when $a \leq b < c$ and $k = p$.

It is clear that $c < z \leq w$, because $c < u_{1,p} = z \leq u_{1,p+1} = u_{1,k+1} = w$. This argument is also valid when $z = \infty$, for then $w = \infty$ as well. Therefore part (1) of Proposition (C.4b) implies that $\tilde{U} \leftarrow zwc = \tilde{U} \leftarrow zcw$, and so $U \leftarrow acb = U \leftarrow cab$.

This concludes the proof of Proposition (C.4b), hence of Knuth's theorem (C.3a).

C.5 Littelmann operators on tableaux

Suppose we have $\lambda \in \Lambda^+(n,r)$, $c \in \{1, 2, \ldots, n-1\}$ and a λ-tableau P. The operator $\tilde{f}_c$ does not act on P, but it does act on the word KP. Theorem (C.5b) below will show that there is a unique tableau $\tilde{P}$ such that $K\tilde{P} = \tilde{f}_c(KP)$. It is reasonable to define $\tilde{f}_c(P)$ to be $\tilde{P}$.

In (C.3g), we regarded the entries in the words $KU \mid x_1$ and $K(U \leftarrow x_1)$ as indexed by the r-element set $[\mu] \cup \{(r)\}$. More generally, we can take any ordered r-element set $\mathsf{T} = \{\tau_1, \tau_2, \ldots, \tau_r\}$ such that $\tau_1 < \tau_2 < \cdots < \tau_r$, and use T to index the entries in a word i of length r. This means that if i is a word of length r, we write $i = i_{\tau_1} i_{\tau_2} \ldots i_{\tau_r}$. In this section, it will be convenient to take $\mathsf{T} = [\lambda]$, because $[\lambda]$ indexes the entries of the word KP (see §C.2).

For the moment, let $\mathsf{T} = \{\tau_1, \tau_2, \ldots, \tau_r\}$ be an arbitrary r-element set with $\tau_1 < \tau_2 < \cdots < \tau_r$. Then all the definitions for $\tilde{f}_c$ given in §A.3 translate into definitions for words indexed by T in a trivial manner (we leave it to the reader to make the analogous translations for $\tilde{e}_c$). In case $\mathsf{T} = [\lambda]$ these definitions appear as follows.

First define $\omega_{c,c+1} = \omega : \underline{n} \to \mathbb{Z}$ as in §A.3, so that $\omega(\nu) = 1$, -1 or 0 according as $\nu = c$, $c+1$ or $\nu \notin \{c, c+1\}$.

The map $h_c^{KP} = h^{KP} : [\lambda] \cup \{0\} \to \mathbb{Z}$ is given so that $h^{KP}(0) = 0$, while for any $t \in [\lambda]$ we define

$$\textbf{(C.5a)} \quad h^{KP}(t) := \sum_{(a,b) \leq t} \omega(p_{a,b}),$$

the order $\leq$ being that given by (C.2a).

Let $M = M_c^{KP}$ be the largest element of the set $\{0\} \cup \{\, h_c^{KP} \,:\, t \in [\lambda] \,\}$. If $M = 0$ define $\tilde{f}_c(KP) := \infty$, or say that "$\tilde{f}_c(KP)$ is undefined". If $M \neq 0$, let $q = q_c^{KP}$ be the least element t of $[\lambda]$ such that $h^{KP}(t) = M$. Then there must hold $p_{a,b} = c$, where $q = (a,b)$; see (A.3c). In this case we define $\tilde{f}_c(KP)$ to be the word obtained from KP by changing the entry $p_{a,b} = c$ to $c+1$; all other entries in KP are left unchanged.

The next theorem shows that if $\tilde{f}_c(KP) \neq \infty$, it is possible to define a *tableau* $\tilde{f}_c(P)$ in such a way that $K(\tilde{f}_c P) = \tilde{f}_c(KP)$ [3]

[3] Some authors identify the tableau P with the word KP, and view theorem (C.5b) as justification of this practice. But in this Appendix we will be cautious (perhaps over-cautious!) and we do not make this identification.

(C.5b) Theorem. *Let $\lambda \in \Lambda^+(n, r)$, $c \in \{1, 2, \ldots, n-1\}$ and P be a λ-tableau. Using the definitions above, assume $M \neq 0$, and define $q = (a, b)$ to be the least place (in the order (C.2a)) such that $h((a, b)) = M$. We know from (A.3c) that $p_{a,b} = c$. Then we have also*

(1) *If $(a, b+1) \in [\lambda]$, then $p_{a,b+1} \geq c+1$.*

(2) *If $(a+1, b) \in [\lambda]$, then $p_{a+1,b} > c+1$.*

(3) *If we change P to $\tilde{P}$ by changing the entry $p_{a,b} = c$ to $\tilde{p}_{a,b} = c+1$, and leaving unchanged all the other entries in P, we get a λ-tableau $\tilde{P}$ which is standard.*

(4) *$K\tilde{P} = \tilde{f}_c(KP)$.*

Proof. (1) Since P is standard, $p_{a,b+1} \geq p_{a,b} = c$. If $p_{a,b+1} < c+1$, we would have $p_{a,b+1} = c$. This gives $h_c^{KP}((a, b+1)) = h_c^{KP}((a, b)) + \omega(c) = M + 1$, contradicting the definition of M. So there must hold $p_{a,b+1} \geq c+1$.

(2) Since P is standard, we must have $p_{a+1,b} > c$. Unless $p_{a+1,b} > c+1$, we have $p_{a+1,b} = c+1$. We shall show that this leads to a contradiction. Table C.11 shows the rows (a) and $(a+1)$ of P, and their entries in certain columns. Let b'

	$\cdots$	$b'-1$	b'	$\cdots$	b	$b+1$	$\cdots$
a	$\cdots$	$< c$	c	$\cdots$	c	$\geq c$	$\cdots$
$a+1$	$\cdots$	$p_{a+1,b'+1}$	$c+1$	$\cdots$	$c+1$	$> c+1$	$\cdots$

Table C.11. Rows (a) and $(a+1)$ of P.

denote the leftmost of all columns such that $p_{a,b'} = c$. Since $a \geq c+1$, entries in row $(a+1)$ to the right of column (b) are all $> c+1$. The entries in the same row, in columns $(b'), \ldots, (b)$, are all equal to $c+1$. This is because such an entry $p_{a+1,b''}$ is left of $p_{a+1,b} = c+1$, and is also $> p_{a,b''} = c$.

From the definition (C.5a) we deduce

(C.5c) $h^{KP}(a, b) = h^{KP}(a+1, b'-1) + X + Y + Y^* + Z,$

where

$$X = \sum_{b' \leq x \leq b} \omega(p_{a+1,x}), \qquad Y = \sum_{b+1 \leq x \leq \lambda_{a+1}} \omega(p_{a+1,x}),$$
$$Y^* = \sum_{1 \leq x \leq b'} \omega(p_{a,x}) \qquad Z = \sum_{b' \leq x \leq b} \omega(p_{a,x}).$$

But for $b+1 \leq x \leq \lambda_{a+1}$ all the entries $p_{a+1,x}$ are $> c+1$, hence all the summands $\omega(p_{a+1,x}) = 0$, therefore $Y = 0$. Similarly $Y^* = 0$ because all the elements $p_{a,x}$ (for $1 \leq x \leq b'-1$) are $< c$. Finally $X + Z = 0$ because $X + Z$ is a sum of pairs $\omega(c) + \omega(c+1) = 0$. Therefore (C.5c) implies that $h^{KP}(a, b) = h^{KP}(a+1, b'-1)$. But this contradicts our definition

of (a,b) as the *least* place in $[\lambda]$ such that $h^{KP}(a,b) = M$. This proves part (2) of Theorem (C.5b).

Part (3) is now proved, since (1) and (2) show that $\tilde{P}$ is standard. Then (4) follows.

(C.5d) Example. Let $\lambda = (2,2,0,\ldots,0)$, regarded as an element of $\Lambda^+(n,4)$ for some $n \geq 4$. Consider the λ-tableau $P = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$. Then $KP = 3422$. Now

r	1	2	3	4
t	$(2,1)$	$(2,2)$	$(1,1)$	$(1,2)$
$(KP)_r$	3	4	2	2
$h^{KP}(r)$	-1	-1	0	1
$(\tilde{f}_c(KP))_r$	3	4	2	3

Table C.12. Illustration of Theorem (C.5b).

let $c = 2$. Calculate $\tilde{f}_c(KP)$ using table C.12. Notice that we have shown two set $\underline{4}$ and $[\lambda]$, either of which can be used to index the letters of the word KP. We see that $q^{KP} = 4$, or equivalently, $q^{KP} = (1,2)$. Therefore $\tilde{f}_c(P) = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline\end{array}$.

C.6 The proof of Proposition B

In this section we shall prove the fact, fundamental for our work, *that the operation KP commutes with all the Littelmann operators $\tilde{f}_c$*. In other words, we shall prove the

Proposition B. *Let $i \in I(n,r)$ and $c \in \{1,2,\ldots,n-1\}$ such that $\tilde{f}_c(i) \neq \infty$. Then $\tilde{f}_c(KP(i)) \neq \infty$ and*

(C.6a) $\tilde{f}_c(KP(i)) = KP(\tilde{f}_c(i))$.

For $i, j \in I(n,r)$ we write $iK'j$ (respectively, $iK''j$) if i and j are connected by a basic move of type K' (respectively, K''); see (C.3c), (C.3d). The proof of Proposition B is based on the following two lemmas.

(C.6b) Lemma. *Let $i, j \in I(n,r)$, and suppose that j is obtained from i by a basic move, say,*

$$i = (\ldots, i_k, i_{k+1}, \ldots), \quad j = (\ldots, i_{k+1}, i_k, \ldots), \quad i_k < i_{k+1}\,.$$

Then $M^i = M^j$, and there are the following alternatives for q^i and q^j:

(a) *If* $q^i \notin \{k, k+1\}$, *then* $q^j = q^i$.

(b) *If* $q^i = k+1$, *then* $q^j = k$.

(c) *If* $q^i = k$, *then either* $i_{k+1} = c+1$, $iK''j$ *and* $q^j = k+2$, *or* $i_{k+1} \neq c+1$ *and* $q^j = k+1$.

Proof. Set $x := i_k$ and $z := i_{k+1}$, so that $x < z$. We observe that

(i) $h_c^j(\nu) = h_c^i(\nu)$ for all $\nu \neq k$.

This follows directly from the definition (A.3a) of h_c^i and the fact that the words i and j are identical at all places except $\nu = k$ and $\nu = k+1$.

Next we show that

(ii) $M^i = M^j$.

Suppose first that $q^i \neq k$. Then $M^j \geq h_c^j(q^i) = h_c^i(q^i) = M^i$, by (i). Assume that $M^j > M^i$. Then $q^j = k$, by (i). This implies $z = c$ and $x < c$, by (A.3c)(i). It follows that $M^i \geq h_c^i(k+1) = h_c^j(k+1) = h_c^j(k) = M^j$, a contradiction. This shows $M^i = M^j$ in case $q^i \neq k$.

Suppose now $q^i = k$. Then $x = c$, by (A.3c)(i). Hence $M^i \geq M^j$, by (i). Assume that $M^i > M^j$. Then $M^i > h_c^j(k+1) = h_c^i(k+1) = M^i + \omega_c(z)$, which implies $z = c+1$. If $iK'j$, that is, $x < i_{k-1} \leq z$, then $i_{k-1} = c+1$ and thus $h_c^i(k) = h_c^i(k-2) + \omega_c(i_{k-1}) + \omega_c(x) = h_c^i(k-2)$. This contradicts the minimal choice of q^i. If $iK''j$, that is, if $x \leq i_{k+2} < z$, then $i_{k+2} = c$ and thus $M^j \geq h_c^j(k+2) = h_c^i(k+2) = h_c^i(k) = M^i$, again a contradiction. This shows $M^i = M^j$ also in case $q^i = k$, and (ii) is proved.

Now, for the proof of (a), suppose that $q^i \notin \{k, k+1\}$. Assume $q^j \neq q^i$, then $q^j = k$, by (i), and thus $z = c$. It follows that $x < c$ and therefore, by (ii), $h_c^i(k+1) = h_c^j(k) = M^j = M^i$. This implies $q^i < k$, since $q^i \neq k, k+1$, hence also $q^j < k$, by (i)—a contradiction. Part (a) is proved.

Now let $q^i = k+1$. Then $z = c$ and $x < c$, and we get from (ii) that $h_c^j(k) = h_c^j(k+1) = h_c^i(k+1) = M^i = M^j$. It follows that $q^j \leq k$. In fact, by (i), $q^j = k$. This implies (b).

Consider finally the case where $q^i = k$. Then $x = c$, and $q^j \geq k$, by (ii).

Suppose additionally that $z \neq c+1$. Then $z > c+1$, and it follows that $h_c^j(k) < h_c^j(k+1) = h_c^i(k+1) = h_c^i(k) = M^i = M^j$. But this implies $q^j = k+1$.

To conclude, let $z = c+1$. Assume $iK'j$, so that $x < i_{k-1} \leq z$. Then $i_{k-1} = c+1$ and $h_c^i(k) = h_c^i(k-2) + \omega_c(i_{k-1}) + \omega_c(x) = h_c^i(k-2)$. This contradicts the minimal choice of q^i. It follows that $iK''j$ as asserted, that is $x \leq i_{k+2} < z$. Hence $i_{k+2} = c$. Direct verification gives $h_c^j(k) = h_c^i(k) - 2 = M^i - 2 = M^j - 2$, $h_c^j(k+1) = M^j - 1$ and $h_c^j(k+2) = M^j$. Therefore $q^j = k+2$ as claimed in (c).

(C.6c) Lemma. *Let* $i, j \in I(n, r)$, *and suppose* j *is obtained from* i *by a basic move. If* $\tilde{f}_c(i) \neq \infty$, *then* $\tilde{f}_c(j) \neq \infty$, *and* $\tilde{f}_c(j)$ *is obtained from* $\tilde{f}_c(i)$ *by a basic move.*

There is a corresponding statement (and proof), with $\tilde{e}_c$ *replacing* $\tilde{f}_c$.

Proof. Thanks to symmetry in i and j, we may assume that either there exist components $y = i_{k-1}$, $x = i_k$, $z = i_{k+1}$ of i such that

$$(K') \quad i = (\ldots, y, x, z, \ldots), \quad j = (\ldots, y, z, x, \ldots), \quad x < y \le z,$$

or components $x = i_k$, $z = i_{k+1}$, $y = i_{k+2}$ of i such that

$$(K'') \quad i = (\ldots, x, z, y, \ldots), \quad j = (\ldots, z, x, y, \ldots), \quad x \le y < z.$$

Suppose $\tilde{f}_c(i) \neq \infty$. Then $M^j = M^i > 0$, by Lemma (C.6b). This implies that $\tilde{f}_c(j) \neq \infty$ as well.

We now consider the three cases listed in Lemma (C.6b).

Case (a). $q^i \notin \{k, k+1\}$. Then $q^i = q^j$, by Lemma (C.6b). Hence x and z remain unchanged when we apply $\tilde{f}_c$ to i and j. The claim follows directly if y is not changed, either.

Suppose that $y = c$, $iK'j$ and $q^i = k-1$. Then we get[4]

$$\tilde{f}_c(i) = (\ldots, \underline{c+1}, x, z, \ldots), \quad \tilde{f}_c(j) = (\ldots, \underline{c+1}, z, x, \ldots), \quad x < c \le z.$$

However, we have $z \ge c+1$, since in case $z = c$ we get $h_c^j(k) = h_c^j(k-1)+1$, and this contradicts the maximality of $h_c^j(q^j)$. Hence $\tilde{f}_c(i)K'\tilde{f}_c(j)$.

Suppose now that $y = c$, $iK''j$ and $q^i = k+2$. Then we get

$$\tilde{f}_c(i) = (\ldots, x, z, \underline{c+1}, \ldots), \quad \tilde{f}_c(j) = (\ldots, z, x, \underline{c+1}, \ldots), \quad x \le c < z.$$

However, we have $z > c+1$, since in case $z = c+1$ we get $h_c^i(k) = h_c^i(k+2)$, and this contradicts the minimal choice of q^i. Hence $\tilde{f}_c(i)K''\tilde{f}_c(j)$.

Case (b). $q^i = k+1$. Then $q^j = k$ and $z = c$, hence

$$\tilde{f}_c(i) = (\ldots, x, \underline{c+1}, \ldots), \quad \tilde{f}_c(j) = (\ldots, \underline{c+1}, x, \ldots).$$

If $iK''j$, then $x \le y < z < c+1$, therefore $\tilde{f}_c(i)K''\tilde{f}_c(j)$. In case $iK'j$, we get $x < y \le z < c+1$, hence $\tilde{f}_c(i)K'\tilde{f}_c(j)$.

Case (c). $q^i = k$. Here $x = c$, and we need to consider the alternative given in Lemma (C.6b)(c).

Suppose first that $z = c+1$, that $iK''j$ and $q^j = k+2$. Then $y = c$ since $c = x \le y < z = c+1$. Hence

$$\tilde{f}_c(i) = (\ldots, \underline{c+1}, c+1, c, \ldots), \quad \tilde{f}_c(j) = (\ldots, c+1, c, \underline{c+1}, \ldots).$$

We get that $\tilde{f}_c(j)K'\tilde{f}_c(i)$.

The case where $z \neq c+1$ and $q^j = k+1$ remains. Here $z > c+1$ and

$$\tilde{f}_c(i) = (\ldots, \underline{c+1}, z, \ldots), \quad \tilde{f}_c(j) = (\ldots, z, \underline{c+1}, \ldots).$$

Suppose $iK''j$, then $y \ge c+1$ since otherwise $h_c^i(k+2) = h_c^i(k)+1$. Therefore $c+1 \le y < z$ and $\tilde{f}_c(i)K''\tilde{f}_c(j)$. Similarly, if $iK'j$, then $y > c+1$, since otherwise $h_c^i(k-2) = h_c^i(k)$. Hence $c+1 < y \le z$ and $\tilde{f}_c(i)K'\tilde{f}_c(j)$.

[4]Those values which were changed by $\tilde{f}_c$ are underlined.

We are now in a position to give the

Proof of Proposition B. From Proposition (C.2c), we get $P(KP(i)) = P(i)$. Hence, by Theorem (C.3a), there exist words

$$i^{(0)},\ i^{(1)},\ \cdots,\ i^{(k-1)},\ i^{(k)} \in I(n,r)$$

such that $i^{(0)} = i$, $i^{(k)} = KP(i)$, and $i^{(\nu)}$ is obtained from $i^{(\nu-1)}$ by a basic move. From Lemma (C.6c), it follows that $\tilde{f}_c(i^{(\nu)}) \neq \infty$ and that $\tilde{f}_c(i^{(\nu)})$ is obtained from $\tilde{f}_c(i^{(\nu-1)})$ by a basic move, for all $\nu \in \{1, \ldots, k\}$.

Applying Theorem (C.3a) again, we get

$$P\Big(\tilde{f}_c(i)\Big) = P\Big(\tilde{f}_c(i^{(0)})\Big) = P\Big(\tilde{f}_c(i^{(k)})\Big) = P\Big(\tilde{f}_c(KP(i))\Big),$$

hence

$(*)$ $KP(\tilde{f}_c(i)) = KP(\tilde{f}_c(KP(i)))$.

There is a standard tableau $\tilde{P}$ such that $K\tilde{P} = \tilde{f}_c(KP(i))$, by Theorem (C.5b)(4). And by Proposition (C.2c)(i), $KP(K\tilde{P}) = K\tilde{P}$. Therefore $(*)$ becomes

$$KP(\tilde{f}_c(i)) = K\tilde{P} = \tilde{f}_c(KP(i))).$$

D

Theorem A and some of its consequences

In what follows, n, r are fixed positive integers.

D.1 Ingredients for the proof of Theorem A

We shall prove Theorem A in the next section, but we must first study some words in $I(n,r)$ which play a special role for the action of the Littelmann operators. To describe these words, we need the following lemma, which is an immediate consequence of the definitions in §A.3.

(D.1a) Lemma. *If* $i \in I(n,r)$ *and* $c \in \{1,\ldots,n-1\}$*, then*
(i) $\tilde{f}_c(i) = \infty$ *if and only if*

$$\#\{\,\nu \le t \,:\, i_\nu = c\,\} \le \#\{\,\nu \le t \,:\, i_\nu = c+1\,\}$$

for all $t \in \{1,\ldots,r\}$*, and*
(ii) $\tilde{e}_c(i) = \infty$ *if and only if*

$$\#\{\,\nu \ge s \,:\, i_\nu = c\,\} \ge \#\{\,\nu \ge s \,:\, i_\nu = c+1\,\}$$

for all $s \in \{1,\ldots,r\}$.

We are interested in the words which satisfy (i) for all c. So we set

$$\Upsilon := \big\{\, i \in I(n,r) \,:\, \tilde{f}_c(i) = \infty \text{ for all } c \in \{1,\ldots,n-1\} \,\big\}.$$

Define an operator $W : I(n,r) \to I(n,r)$ by

$$W(i_1 i_2 \ldots i_r) = (n+1-i_1, n+1-i_2, \ldots, n+1-i_r).$$

Then a word i belongs to Υ if and only if $W(i)$ is a "lattice permutation"[1].

[1] This term is rather confusing, because we shall use it for words which may not be permutations! A word $j \in I(n,r)$ is called a *permutation* if $n = r$ and the entries in j are $1, 2, \ldots, r$ in some order. Lattice permutations in this sense are used by D.E. Littlewood in the character theory of the symmetric group $\mathsf{Sym}(r)$ (see [36, page 67]). Lattice permutations in the present sense appear in [40] and [37].

Definition. A *lattice permutation*, in our language, is a word $j \in I(n,r)$ such that

(D.1b) $$\begin{aligned}\#\{\,\nu \le s \,:\, j_\nu = 1\,\} &\ge \#\{\,\nu \le s \,:\, j_\nu = 2\,\} \\ &\ge \cdots \\ &\ge \#\{\,\nu \le s \,:\, j_\nu = n-1\,\} \\ &\ge \#\{\,\nu \le s \,:\, j_\nu = n\,\},\end{aligned}$$

for all $s \in \{1,\dots,r\}$.

For example, the word $j = 11122132$, an element of $I(3,8)$, is a lattice permutation. The word $i = 33322312$ belongs to Υ, because $W(i) = j$.

Similarly, we are interested in the words which satisfy condition (ii) in Lemma (D.1a) for all c, and we set

$$\mathsf{T} := \big\{\, i \in I(n,r) \,:\, \tilde{e}_c(i) = \infty \text{ for all } c \in \{1,\dots,n-1\} \,\big\}.$$

In (A.3g)(2), the operator $B : I(n,r) \to I(n,r)$ was defined by

$$B(i_1 i_2 \dots i_{r-1} i_r) = i_r i_{r-1} \cdots i_2 i_1.$$

Thus a word i belongs to T if and only if $B(i)$ is a lattice permutation.

Define an operator $C : I(n,r) \to I(n,r)$ by $C = BW = WB$. Explicitly,

$$C(i_1 i_2 \dots i_{r-1} i_r) = (n+1-i_r, n+1-i_{r-1}, \dots, n+1-i_2, n+1-i_1).$$

Remarks.

(i) All these operators have square equal to the identity in $I(n,r)$.

(ii) If $i \in I(n,r)$ and $\mathsf{Sch}(i) = (\lambda(i), P(i), Q(i))$, then $\lambda(i)$ is the shape of i (see §C.1). The operator C preserves shape (i.e. $\lambda(Ci) = \lambda(i)$, see (D.3g)), but the operators B and W do not. For example, using the tables in §E.1, we see that $i = 221$ has shape $(2,1,0)$, but $B(i) = 122$ and $W(i) = 223$ both have shape $(3,0,0)$. However, $C(i) = 322$ has the same shape as i.

(D.1c) Lemma. *The operator C induces a bijection $\mathsf{T} \to \Upsilon$. Hence $|\mathsf{T}| = |\Upsilon|$.*

Proof. Let $i \in \mathsf{T}$. Then $B(i)$ is a lattice permutation, and $W(C(i)) = B(i)$. This shows that $C(i) \in \Upsilon$. Prove similarly that $i \in \mathsf{T}$ implies that $C(i) \in \Upsilon$.

From now on in this section, we fix $\lambda \in \Lambda^+(n,r)$.

The tableaux $\mathbf{T}_\lambda$ and $\mathbf{Z}_\lambda$. Define two λ-tableaux as follows:

(D.1d) $T_\lambda = (T_{s,t})_{(s,t)\in[\lambda]}$ where $T_{s,t} = s$ for all $(s,t) \in [\lambda]$; we denote the word KT_λ by i^λ.

(D.1e) $Z_\lambda = (Z_{s,t})_{(s,t)\in[\lambda]}$ where $Z_{s,t} = n - \beta_t + s$ for all $(s,t) \in [\lambda]$, and β_t denotes the length of column t of Z_λ; we denote the word KZ_λ by i_λ.

Example. If $\lambda = (5,3,2,0,0) \in \Lambda^+(5,10)$, then

$$\textbf{(D.1f)}\quad T_\lambda = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&2&2\\\cline{1-3} 3&3\\\cline{1-2}\end{array} \quad\text{and } Z_\lambda = \begin{array}{|c|c|c|c|c|}\hline 3&3&4&5&5\\\hline 4&4&5\\\cline{1-3} 5&5\\\cline{1-2}\end{array}\;.$$

Notice that T_λ is our old friend from (4.3b), where it is called T_l. It is useful to think of Z_λ as the tableau obtained from T_λ by subjecting it to two successive operations: first reverse each column of T_λ, and secondly replace each entry x in the tableau by $n+1-x$. In our example,

$$T_\lambda = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&2&2\\\cline{1-3} 3&3\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\hline 3&3&2&1&1\\\hline 2&2&1\\\cline{1-3} 1&1\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\hline 3&3&4&5&5\\\hline 4&4&5\\\cline{1-3} 5&5\\\cline{1-2}\end{array} = Z_\lambda.$$

Notation. Define $\mathcal{Q}(\lambda)$ to be the set of all (standard) λ-tableaux whose entries are $1,2,\ldots,r$ in some order.

Recall from (C.1e) that $I(Q,\approx)$ is the set of all words $i \in I(n,r)$ such that $Q(i) = Q$, for each $Q \in \mathcal{Q}(\lambda)$. These sets are the equivalence classes for $\approx$.

(D.1g) Theorem. *Let $Q \in \mathcal{Q}(\lambda)$. Then:*

(i) *There is a unique word $i \in I(n,r)$ such that $Q(i) = Q$ and i belongs to T. Moreover $P(i) = T_\lambda$.*

(ii) *There is a unique word $i \in I(n,r)$ such that $Q(i) = Q$ and i belongs to Υ. Moreover $P(i) = Z_\lambda$.*

(D.1h) Notation. For each $Q \in \mathcal{Q}(\lambda)$, denote the word i in (i) by i^Q, and the word i in (ii) by i_Q.

Proof of Theorem (D.1g). (i) By Schensted's Theorem (B.6a) there is a unique word i such that $P(i) = T_\lambda$ and $Q(i) = Q$. We claim that this i belongs to T.

Let $c \in \{1,2,\ldots,n\}$. From §A.3, we know that $\tilde{e}_c(i) = \infty$ if and only if the function h_c^i attains its maximum M_c^i at the last place in the word i, i.e. $h_c^i(r) = M_c^i$. Now $h_c^i(r)$ is the sum of the $\omega(i_\nu)$ for $\nu = 1,2,\ldots,r$ (see (A.3a)). But the r entries in the word $KP(i)$ form a permutation of the r entries of i. Hence $h_c^{KP(i)}(r) = h_c^i(r) = M_c^i$. By Lemma (C.6b) and Proposition (C.3p), the words i and $KP(i)$ give the same maximum, i.e. $M_c^i = M_c^{KP(i)}$. We can calculate $M_c^{KP(i)} = M_c^{KT_\lambda}$ easily; it is $\lambda_c - \lambda_{c+1}$, and it is attained at the last place $(1,\lambda_1)$ of $KP(i)$. Therefore the maximum M_c^i of h_c^i is also attained at the last place of i. This shows that $\tilde{e}_c(i) = \infty$ for all c, hence $i \in \mathsf{T}$.

Now we must prove *uniqueness*: if $j \in I(Q,\approx) \cap \mathsf{T}$, then $j = i$. It is enough to prove that $P(j) = T_\lambda$. We know $\tilde{e}_c(KP(j)) = KP(\tilde{e}_c(j))$, by Proposition B, hence $\tilde{e}_c(KP(j)) = \infty$ for all c. So the height function $h_c^{KP(j)}$ takes its maximum value at "place r", i.e. at place $(1,\lambda_1) \in [\lambda]$.

Consider the last entry t in the first row of $P(j)$; we must show that $t = 1$. If this is false, then $t > 1$. Consider the height functions for $c = t - 1$. The entry in $P(j)$ at place $(1, \lambda_1)$ (which corresponds to place r in the word $KP(j)$) is $c + 1$. So we have $h_c^{KP(j)}(r) < h_c^{KP(j)}(r - 1)$, a contradiction.

Hence all entries of the first row of $P(j)$ are equal to 1. Next consider the last entry, t say, in the s^{th} row of $P(j)$. We have $t \geq s$ since $P(j)$ is standard. Suppose $t > s$, and set $c = t - 1$. As before, the height function $h_c^{KP(j)}$ does not take its maximum value at r: it is constant on the letters of rows 1 up to $s - 1$, and its value at the last place of row s, say x, is less than its value at the place immediately preceding this last place. This is a contradiction. We have proved that $P(j) = T_\lambda$, and since we have assumed $Q(j) = Q$, the words j and i must be equal.

If we let λ vary over all partitions in $\Lambda^+(n, r)$, then this shows that $|\mathsf{T}|$ is equal to the number of standard tableaux having entries $1, 2, \dots, r$ in some order; this is also the total number of $\approx$-classes in $I(n, r)$.

(ii) By Schensted's theorem (B.6a) there is a unique word $i \in I(n, r)$ with $Q(i) = Q$ and $P(i) = Z_\lambda$. Using Lemma (C.6b) and Proposition B, as in the proof of (i), it is quite easy to see that $i \in \Upsilon$.

Therefore each $\approx$-class of words of shape λ contains at least one element of Υ. But as was noted above, if we let λ vary, the number of $\approx$-classes is $|\mathsf{T}|$, which is equal to $|\Upsilon|$ by Lemma (D.1c). This implies that each $\approx$-class contains a unique element of Υ; this must be the word i having $P(i) = Z_\lambda$ and $Q(i) = Q$.

This completes the proof of Theorem (D.1g).

(D.1i) Proposition. *$i^\lambda = i^{\mathsf{Q}(\lambda)}$ and $i_\lambda = i_{\mathsf{Q}(\lambda)}$.*

Proof. Taking $Y = T_\lambda$ in propositions (C.2c) and (C.2h), we get $P(i^\lambda) = T_\lambda$ and $Q(i^\lambda) = \mathsf{Q}^{(\lambda)}$. However (D.1g) and (D.1h) say that $P(i^{\mathsf{Q}(\lambda)}) = T_\lambda$ and $Q(i^{\mathsf{Q}(\lambda)}) = \mathsf{Q}^{(\lambda)}$. Therefore $i^\lambda = i^{\mathsf{Q}(\lambda)}$, by Schensted's theorem (B.6a). A similar proof, using Z_λ in place of T_λ, gives $i_\lambda = i_{\mathsf{Q}(\lambda)}$.

D.2 Proof of Theorem A

We shall now prove the Theorem A described in the introduction (see (A.4a)):

(D.2a) Theorem A. *Let $i, j \in I(n, r)$. Then $i \approx j$ if and only if there is a finite sequence of words (elements of $I(n, r)$):*

$$i(1),\ i(2),\ \dots,\ i(s)$$

such that $i(1) = i$, $i(s) = j$ and for each adjacent pair $i(\nu)$, $i(\nu + 1)$ ***either*** *there exists an element $c \in \{1, \dots, n - 1\}$ such that $\tilde{f}_c(i(\nu)) = i(\nu + 1)$,* ***or*** *there exists an element $c \in \{1, \dots, n - 1\}$ such that $\tilde{e}_c(i(\nu)) = i(\nu + 1)$.*

It is clear that the "if" part of this theorem is equivalent to the following

(D.2b) Proposition. *If $i, j \in I(n,r)$ and $c \in \{1, \ldots, n-1\}$ such that either $\tilde{f}_c(i) = j$, or $\tilde{e}_c(i) = j$, then $Q(i) = Q(j)$.*

Proof. Suppose there is an element $c \in \{1, \ldots, n-1\}$ such that $j = \tilde{f}_c(i)$. This implies that $\tilde{f}_c(i) \neq \infty$.

(a) We claim that $Q(i)$ and $Q(j)$ have the same shape. Equivalently, we claim that $P(i)$ and $P(j)$ have the same shape. Let $P(i)$ have shape λ. By Proposition B (see (C.6b)) we know that $KP(j) = KP(\tilde{f}_c(i)) = \tilde{f}_c(KP(i))$. Now take $P = P(i)$ in Theorem (C.5b). This says that there is a λ-tableau $\tilde{P}$ such that $K\tilde{P} = \tilde{f}_c(KP)$. Therefore $KP(j) = K\tilde{P}$. This shows that $P(j)$ has the same shape λ as $\tilde{P}$, which is the shape of $P(i)$. This proves claim (a).

(b) We shall use induction on r to prove that $Q(i) = Q(j)$. If $r = 1$ then i and j are one-letter words, and $Q(i) = Q(j)$ follows from (B.2b). Assume now that $r > 1$. Write $i = i'i_r$ and $j = j'j_r$, where $i' = i_1 \cdots i_{r-1}$ and $j' = j_1 \cdots j_{r-1}$ lie in $I(n, r-1)$.

There is a place $q \in \{1, 2, \ldots, r\}$ such that $i_q = c$, $j_q = c+1$ and $i_\nu = j_\nu$ for all $\nu \neq q$ (see (A.3e)). We either have $q < r$, or $q = r$. If $q < r$, then $j' = \tilde{f}_c(i')$, hence $Q(i') = Q(j')$ by the induction hypothesis. If $q = r$, then $j' = i'$ and clearly $Q(i') = Q(j')$.

It follows that $Q(i')$ and $Q(j')$ have the same shape, μ say, in either case. Let λ be the shape of $Q(i)$. By (a), λ is also the shape of $Q(j)$. To get $Q(i)$ from $Q(i')$, one puts r into the unique place which, when added to $[\mu]$, gives $[\lambda]$. To get $Q(j)$ from $Q(j') = Q(i')$, one puts r in the unique place which, when added to $[\mu]$, gives $[\lambda]$. Hence $Q(j) = Q(i)$. This completes the proof of Proposition (D.2b); and this proves the "if" part of Theorem A.

Proof of the "only if" part of Theorem A. We assume that $i, j \in I(n,r)$ are such that $Q(i) = Q(j)$; we must prove that there exists a sequence of words $i = i(1), (2), \ldots, i(s) = j$ with the properties listed in (D.2a). First make the

(D.2c) Definition. Let $i \in I(n,r)$. We define the *size* of i, denoted $\mathsf{sz}(i)$, by $\mathsf{sz}(i) := i_1 + i_2 + \cdots + i_r$. This is a positive integer, and from the definitions of $\tilde{f}_c$, $\tilde{e}_c$ it is clear that if $\tilde{f}_c(i) \neq \infty$ then $\mathsf{sz}(\tilde{f}_c(i)) = \mathsf{sz}(i) + 1$, and if $\tilde{e}_c(i) \neq \infty$ then $\mathsf{sz}(\tilde{e}_c(i)) = \mathsf{sz}(i) - 1$.

We make the convention $\mathsf{sz}(\infty) = 0$.

Let $\lambda \in \Lambda^+(n,r)$ and $Q \in \mathcal{Q}(\lambda)$. Let $w \in I(Q, \approx)$ (see (C.1e)), and let $\mathcal{S}(w)$ denote the set of all words of the form $\tilde{e}_{c_1}\tilde{e}_{c_2} \cdots \tilde{e}_{c_t}(w)$, where $c_1, c_2, \ldots, c_t$ are arbitrary elements of $\{1, 2, \ldots, n-1\}$. We allow that t may be zero, in which case $\tilde{e}_{c_1}\tilde{e}_{c_2} \cdots \tilde{e}_{c_t}(w) = w$. In general, $\tilde{e}_{c_1}\tilde{e}_{c_2} \cdots \tilde{e}_{c_t}(w)$ has size $\mathsf{sz}(w) - t$. Let S be minimal amongst the sizes of the elements of $\mathcal{S}(w)$, and choose an element $w' := \tilde{e}_{c_1}\tilde{e}_{c_2} \cdots \tilde{e}_{c_t}(w)$ of $\mathcal{S}(w)$ of size S (there may be many such w'). Then $\tilde{e}_c(w') = \infty$ for all $c \in \{1, 2, \ldots, n-1\}$, because if $\tilde{e}_c(w') \neq \infty$, then $\tilde{e}_c(w')$ would be an element of $\mathcal{S}(w)$ of size $S - 1$.

By Proposition (D.2b), all the elements of $\mathcal{S}(w)$ lie in $I(Q,\approx)$. But then Theorem (D.1g) tells us that $w' = \tilde{e}_{c_1}\tilde{e}_{c_2}\cdots\tilde{e}_{c_t}(w) = i^Q$. This implies that $w = \tilde{f}_{c_t}\cdots\tilde{f}_{c_2}\tilde{f}_{c_1}(i^Q)$. In other words, any element $w \in I(Q,\approx)$ can be joined to i^Q by a finite sequence of steps of the form $i(\nu) \xrightarrow{\tilde{f}} i(\nu+1)$; equivalently i^Q can be joined to w by a sequence of steps of the form $i(\nu+1) \xrightarrow{\tilde{e}} i(\nu)$. So given $i, j \in I(Q,\approx)$, we can join i to j by a sequence of the type described in the statement of Theorem A, by first joining i to i^Q and then joining i^Q to j. This completes the proof of Theorem A.

The arguments just given, together with Theorem (D.1g), provide valuable information on the $\approx$-classes. We summarize this in the

(D.2d) Proposition. *Let $\lambda \in \Lambda^+(n,r)$ and $Q \in \mathcal{Q}(\lambda)$. Then:*

(i) *There is a unique word i^Q in $I(Q,\approx)$ lying in T, i.e. such that $\tilde{e}_c(i) = \infty$ for all $c \in \{1,\ldots,n-1\}$. This word is specified by $\mathsf{Sch}(i^Q) = (\lambda, T_\lambda, Q)$.*

(ii) *There is a unique word i_Q in $I(Q,\approx)$ lying in Υ, i.e. such that $\tilde{f}_c(i) = \infty$ for all $c \in \{1,\ldots,n-1\}$. This word is specified by $\mathsf{Sch}(i_Q) = (\lambda, Z_\lambda, Q)$.*

(iii) *The following three conditions on a word $i \in I(n,r)$ are equivalent (i.e. each condition implies the other two):*

(1) $i \in I(Q,\approx)$.

(2) *There exist $c_1,\ldots,c_t \in \{1,\ldots,n-1\}$ such that $i = \tilde{f}_{c_1}\cdots\tilde{f}_{c_t}(i^Q)$.*

(3) *There exist $d_1,\ldots,d_s \in \{1,\ldots,n-1\}$ such that $i = \tilde{e}_{d_1}\cdots\tilde{e}_{d_s}(i_Q)$.*

In (2) and (3), we allow t and s to be $= 0$, respectively. In these cases we interpret $\tilde{f}_{c_1}\cdots\tilde{f}_{c_t}(i^Q)$ to be i^Q and $\tilde{e}_{d_1}\cdots\tilde{e}_{d_s}(i_Q)$ to be i_Q, respectively.

Proof. All the statements above can be deduced easily from Theorem (D.1g), Theorem (D.2a) (i.e. Theorem A) and the proof of Theorem (D.2a).

Weights. Remember (see (A.3g)(3), or §3.1) that the *weight* $\mathsf{wt}(i)$ of a word i is the n-vector $(w_1,\ldots,w_n)$, where for each $\nu \in \underline{n}$, w_ν is the number of $\rho \in \underline{r}$ such that $i_\rho = \nu$. Classical representation theory of $\mathsf{GL}_n(\mathbb{C})$, which can be regarded as a sequel to classical invariant theory, uses weights extensively—they describe the (polynomial) representations K_λ of the diagonal subgroup $T_n(\mathbb{C})$, see §3.2; then these are "induced" to give irreducible (polynomial) representations of $\mathsf{GL}_n(\mathbb{C})$, see the end of Chapter 4.

(D.2e) Remark. It is clear that $\mathsf{wt}(i) = \mathsf{wt}(j)$, if $i, j \in I(n,r)$ are such that $j = i\pi$ for some π in the symmetric group $\mathsf{Sym}(r)$ (the symmetric group is denoted $G(r)$ in §2.1). In particular, $\mathsf{wt}(i) = \mathsf{wt}(KP(i))$ for any $i \in I(n,r)$, because the entries in $KP(i)$ are the same as the entries in i, apart from a place permutation $\pi \in \mathsf{Sym}(r)$.

In the classical representation theory of $\mathsf{GL}_n(\mathbb{C})$, which is essentially the representation theory of the Schur algebra $S(n,r)$, the (isomorphism types of) simple modules are indexed by dominant weights, i.e. by the elements of $\Lambda^+(n,r)$. We shall see in §D.4 that this holds also for the (isomorphism

types of) simple modules for the Littelmann algebra $L = L(n,r)$, although the argument is different from that which applies to $S(n,r)$.

The weights of the elements of $I(Q,\approx)$ have properties given in the next proposition.

(D.2f) Proposition. *Let $\lambda \in \Lambda^+(n,r)$ and $Q \in \mathcal{Q}(\lambda)$, then*

(i) $\mathsf{wt}(i^Q) = \lambda = (\lambda_1, \dots, \lambda_n)$,

(ii) $\mathsf{wt}(i_Q) = (\lambda_n, \dots, \lambda_1)$,

(iii) *i^Q (respectively, i_Q) is the only word in $I(Q,\approx)$ having weight $(\lambda_1, \dots, \lambda_n)$ (respectively, $(\lambda_n, \dots, \lambda_1)$), and*

(iv) *the weight ω of any word in $I(Q,\approx)$ satisfies the inequalities*

$$(\lambda_1, \dots, \lambda_n) \trianglerighteq \omega \trianglerighteq (\lambda_n, \dots, \lambda_1).$$

(If $\xi, \eta \in \Lambda(n,r)$, we write $\xi \trianglerighteq \eta$ to mean that the difference $\xi - \eta$ lies in the set $U = \sum_{\alpha\in\Sigma} \mathbb{Z}_+\alpha$; see [33, page 3]).

Proof. (i) From (D.1g)(i) we know that $P(i^Q) = T_\lambda$. Therefore $\mathsf{wt}(i^Q)$ is the same as the weight of KT_λ (see Remark (D.2e)). It is very easy to see that $\mathsf{wt}(KT_\lambda) = (\lambda_1, \dots, \lambda_n)$.

(ii) In the same way, we deduce from (D.1g)(ii) that $\mathsf{wt}(i_Q)$ is the same as the weight $(u_1, \dots, u_n)$ of KZ_λ. So for each $\delta \in \underline{n}$, u_δ is the number of pairs $(s,t) \in [\lambda]$ such that $n - \beta_t + s = \delta$. For each $t \in \underline{n}$, there is exactly one entry δ in column t of Z_λ, if and only if $1 \le \beta_t - (n - \delta)$. Therefore u_δ equals the number of columns of Z_λ of lengths greater than or equal to $n + 1 - \delta$. But this number is $\lambda_{n+1-\delta}$.

(iii) and (iv) Let i be any word in $I(Q,\approx)$. By (D.2d) we know that there exist integers $c_1, \dots, c_t$ in $\{1, \dots, n-1\}$ such that $i = \tilde{f}_{c_1} \cdots \tilde{f}_{c_t}(i^Q)$. From (A.3g)(3), we know that $\mathsf{wt}(i) = \mathsf{wt}(i^Q) - \alpha_{c_1,c_1+1} - \cdots - \alpha_{c_t,c_t+1}$. Therefore $i \trianglelefteq i^Q$; moreover the case $\mathsf{wt}(i) = \mathsf{wt}(i^Q) = (\lambda_1, \dots, \lambda_n)$ occurs only if $t = 0$ i.e. only if $i = i^Q$. A similar argument shows that the weight of any word $i \in I(Q,\approx)$ is $\trianglerighteq \mathsf{wt}(i_Q)$, with equality only if $i = i_Q$.

D.3 Properties of the operator C

First we want to understand how the action of C is related to the action of the Littelmann operators.

Comparing the height functions of i and Ci. Fix $i \in I(n,r)$, and consider the height function h_c^i for some $c \in \{1, 2, \dots, n-1\}$. This depends on the i_ν which are equal to c or $c+1$. The operator C turns c and $c+1$ into $n-c+1$ and $n-c$, respectively. This suggests comparing h_c^i with h_{n-c}^{Ci}.

Take some $s \in \{1, \dots, r\}$, then by definition

(D.3a) $h_c^i(s) = \#\{\, \nu \le s \,:\, i_\nu = c \,\} - \#\{\, \nu \le s \,:\, i_\nu = c+1 \,\}.$

Now write $Ci = j_1 \dots j_r$ for a moment and consider

(D.3b) $h^{Ci}_{n-c}(r-s) = \#\{\,\rho \le r-s \,:\, j_\rho = n-c\,\}$
$$-\#\{\,\rho \le r-s \,:\, j_\rho = n-c+1\,\}.$$

We have $j_\rho = n - i_{r-\rho+1} + 1$ for all $\rho \in \{1, \dots, r\}$. Furthermore, $n - i_\nu + 1 = n - c$ if and only if $i_\nu = c+1$, and $n - i_\nu = n - c$ if and only if $i_\nu = c$, for all $\nu \in \{1, \dots, r\}$. So (D.3b) gives

(D.3c) $h^{Ci}_{n-c}(r-s) = \#\{\,\nu \ge s+1 \,:\, i_\nu = c+1\,\} - \#\{\,\nu \ge s+1 \,:\, i_\nu = c\,\}.$

By using the notation:

$$\Pi_b := \#\{\,\nu \in \Pi \,:\, i_\nu = b\,\},$$

for every subset Π of $\{1, \dots, s\}$, and every element b of $\underline{n}$, formula (D.3c) becomes

(i) $h^{Ci}_{n-c}(r-s) = -\{s+1, \dots, r\}_c + \{s+1, \dots, r\}_{c+1}\,.$

Also, by definition of the height function h^i_c, we have

(ii) $h^i_c(s) = \{1, \dots, s\}_c - \{1, \dots, s\}_{c+1}\,.$

If we subtract (i) from (ii) we get

(D.3d) *If $i \in I(n,r)$ and $Y = h^i_c(r)$, then $h^i_c(s) - h^{Ci}_{n-c}(r-s) = Y$ for all $s \in \{0, \dots, r\}$.*

Example. Let $n = 3$, $r = 5$, $c = 1$, and consider $i = 22111 \in I(n,r)$. Then $Ci = 33322$ and $n - c = 2$. The height functions h^i_1 and h^{Ci}_2 are

$$\Big(\ h^i_1(0),\ \ h^i_1(1),\ \ h^i_1(2),\ \ h^i_1(3),\ \ h^i_1(4),\ \ h^i_1(5)\Big) = (\ \ 0, -1, -2, -1,\ \ 0,\ \ 1),$$

$$\Big(h^{Ci}_2(5), h^{Ci}_2(4), h^{Ci}_2(3), h^{Ci}_2(2), h^{Ci}_2(1), h^{Ci}_2(0)\Big) = (-1, -2, -3, -2, -1,\ \ 0).$$

Note that h^i_1 has the maximum at place r and h^{Ci}_2 has maximum value zero.

(D.3e) Lemma. *Let $c \in \{1, 2, \dots, n-1\}$. Then, for each $i \in I(n,r)$, we have $C(\tilde{e}_c(i)) = \tilde{f}_{n-c}(Ci)$ and $C(\tilde{f}_c(i)) = \tilde{e}_{n-c}(Ci)$.*

Proof. Since C^2 is the identity, the second part follows from the first. We prove the first part.

By (D.3d), we have a geometric description of how the height functions are related: given $h = h^i_c$, then to find $\tilde{h} = h^{Ci}_{n-c}$, one reflects the graph of h in the vertical line $x = r$, and translates it in the "y-axis" direction so that $\tilde{h}(0) = 0$. Explicitly, let $s + t = r$ and $Y = h^i_c(r)$, then

$$h^{Ci}_{n-c}(t) = h^i_c(s) - Y.$$

Since $\tilde{h}$ is a reflection about a vertical line, the last maximum of h_c^i (at place $\bar{q}$), becomes the first maximum of h_{n-c}^{Ci} (at place $r - \bar{q}$). Furthermore, if $\bar{q} = r$ then the maximum of $\tilde{h}$ is zero. This shows that if $\tilde{e}_c(i) = \infty$ then $\tilde{f}_{n-c}(Ci) = \infty$.

Assume now that $\bar{q} < r$. By (A.3c) we know that $i_{\bar{q}+1} = c + 1$, and $\tilde{e}_c(i)$ is obtained from i by replacing $i_{\bar{q}+1} = c + 1$ by c. We get the word

$$C(\tilde{e}_c(i)) = \ \cdots (n - c + 1)(n - i_{\bar{q}} + 1) \cdots$$

where the letters shown are at places $r - \bar{q}$ and $r - \bar{q} + 1$.

Now consider $C(i) = \ \cdots (n - c)(n - i_{\bar{q}} + 1) \cdots$ where the letters shown are at places $r - \bar{q}$ and $r - \bar{q} + 1$. We know that $\tilde{h}$ assumes its maximum at place $r - \bar{q}$ for the first time. So $\tilde{f}_{n-c}(Ci)$ replaces the letter $n - c$ at place $r - \bar{q}$ by $n - c + 1$. Hence $\tilde{f}_{n-c}(Ci)$ is equal to $C(\tilde{e}_c(i))$.

From the definition of C we see immediately the following.

(D.3f) Lemma. *If a word $i \in I(n, r)$ has weight $\mu = (\mu_1, \ldots, \mu_n)$, then the word Ci has weight $(\mu_n, \ldots, \mu_1)$.*

We want to show now:

(D.3g) Lemma. *The operator C preserves the shape.*

Proof. Let $i \in I(n, r)$ and $Q(i) = Q$, and suppose Q has shape λ.

Assume first that $i = i^Q$, then $C(i) = i_R$ for some standard tableau R. By (D.2f) we can identify the shapes of the words i^Q and i_R from their weights. The weight of i^Q is λ, hence the weight of $C(i^Q)$ is $(n^{\lambda_1}, (n-1)^{\lambda_2}, \ldots)$. So i_R also has shape λ.

In general, by (D.2d)(iii), there are $c_1, \ldots, c_t \in \{1, 2, \ldots, n-1\}$ such that

$$i = \tilde{f}_{c_1} \cdots \tilde{f}_{c_t}(i^Q).$$

From (D.3e) it follows that $C(i) = \tilde{e}_{n-c_1} \cdots \tilde{e}_{n-c_t}(Ci^Q)$. But we have already seen that Ci^Q has shape λ; now Proposition B implies that $C(i)$ also has shape λ.

(D.3h) Remark. The operator C does not preserve the Q-symbol in general. But it gives a pairing on the set of standard tableaux of the same shape.

D.4 The Littelmann algebra $L(n, r)$

(D.4a) Let V be an n-dimensional vector space over a field F with basis $v_1, \ldots, v_n$. Then the r-fold tensor product $V^{\otimes r}$ has basis $\{v_i : i \in I(n, r)\}$. (In §§1–6, $V^{\otimes r}$ is called $E^{\otimes r}$.) For each $\alpha \in \Sigma$, where $\alpha = \alpha_{c,c+1}$ (see §A.3), we let $\tilde{f}_c$ and $\tilde{e}_c$ act on the tensor space by linear maps, defining

$$\tilde{f}_c v_i := v_{\tilde{f}_c i}, \qquad \tilde{e}_c v_i := v_{\tilde{e}_c i}$$

and using linear extension. We set $v_\infty := 0$.

Let $L = L(n,r)$ be the subalgebra of $\mathsf{End}_F(V^{\otimes r})$ generated by these linear maps $\tilde{f}_c$ and $\tilde{e}_c$, for $c \in \{1, 2, \ldots, n-1\}$. This algebra will be called the *Littelmann algebra*.

(D.4b) As an F-space, the Littelmann algebra L is spanned the set of all monomials $m = m_1 m_2 \ldots m_t$ of lengths $t \geq 1$, where each m_τ is either $\tilde{f}_c$ or $\tilde{e}_c$ (for some $c \in \{1, 2, \ldots, n-1\}$).

We do *not* include the monomial $m = 1_{\mathsf{End}_F V^{\otimes r}}$ of length zero. But it may happen that L does contain this element (see Proposition (D.4e), below).

(D.4c) An element $H \in \mathsf{End}_F(V^{\otimes r})$ will often be described by its matrix $(H_{i,j})_{i,j \in I(n,r)}$, whose entries $H_{i,j} \in F$ are defined by the equations

(D.4d) $Hv_j = \sum_{i \in I(n,r)} H_{i,j} v_i$, all $j \in I(n,r)$.

We often identify H with its matrix $(H_{i,j})_{i,j \in I(n,r)}$, and we often identify $\tilde{f}_c$, $\tilde{e}_c$ with the elements of $\mathsf{End}_F(V^{\otimes r})$ defined by (D.4a).

(D.4e) Proposition. *L has an identity element, viz. D_S, the diagonal matrix having $(D_S)_{i,i} = 1$, 0 according as $i \in S$ or not; here $S := I(n,r) \setminus (\Upsilon \cap \mathsf{T})$.*

Reminder: from §D.1 we have

$$\Upsilon = \left\{\, i \in I(n,r) \,:\, \tilde{f}_c(i) = \infty \text{ for all } c \in \{1, 2, \ldots, n-1\} \,\right\}$$

and

$$\mathsf{T} = \left\{\, i \in I(n,r) \,:\, \tilde{e}_c(i) = \infty \text{ for all } c \in \{1, 2, \ldots, n-1\} \,\right\}.$$

Proof of Proposition (D.4e). For any subset A of $I(n,r)$, define D_A to be the element of $\mathsf{End}_F(V^{\otimes r})$ whose matrix with respect to the basis $\{v_i : i \in I\}$ is diagonal, and $(D_A)_{ii} = 1$ or 0, according as $i \in A$ or $i \notin A$. The following facts are easily checked.

(i) $D_{I(n,r)}$ is the identity element of $\mathsf{End}_F(V^{\otimes r})$.

(ii) For any c, the matrix of $\tilde{f}_c \tilde{e}_c$ is equal to $D_{Z(c)}$, where $Z(c)$ is the set of all i such that $\tilde{e}_c(i) \neq \infty$. Similarly $\tilde{e}_c \tilde{f}_c = D_{Y(c)}$, where $Y(c)$ is the set of all i such that $\tilde{f}_c(i) \neq \infty$.

(iii) If $A, B \subseteq I(n,r)$, then $D_A D_B = D_{A \cap B}$ and $D_{A \cup B} = D_A + D_B - D_{A \cap B}$.

(iv) If $A_1, \ldots, A_w \subseteq I(n,r)$ such that $D_{A_t} \in L$ for all $t = 1, 2, \ldots, w$, then $D_A \in L$ where $A = A_1 \cup A_2 \cup \ldots \cup A_w$.

Now check that $I(n,r) \setminus \mathsf{T} = \bigcup_c Z(c)$ and $I(n,r) \setminus \Upsilon = \bigcup_c Y(c)$, hence

$$S = I(n,r) \setminus (\Upsilon \cap \mathsf{T}) = \bigcup_c Z(c) \cup \bigcup_c Y(c).$$

The partition $I(n,r) = S \cup (\Upsilon \cap \mathsf{T})$ of $I(n,r)$ allows us to decompose each linear operator $H \in \mathsf{End}_F(V^{\otimes r})$ in matrix form as

$$H = \begin{pmatrix} H^{(1,1)} & H^{(1,2)} \\ H^{(2,1)} & H^{(2,2)} \end{pmatrix},$$

with $H^{(1,1)} \in \mathsf{End}_F(S,S)$, $H^{(1,2)} \in \mathsf{Hom}_F(\Upsilon \cap \mathsf{T}, S)$, $H^{(2,1)} \in \mathsf{Hom}_F(S, \Upsilon \cap \mathsf{T})$ and $H^{(2,2)} \in \mathsf{End}_F(\Upsilon \cap \mathsf{T})$. For each $c \in \{1, 2, \ldots, n-1\}$, it is easy to verify the following facts.

(v) If $H = \tilde{e}_c$, or if $H = \tilde{f}_c$, then $H^{(1,2)}$, $H^{(2,1)}$, $H^{(2,2)}$ are all zero matrices; also $\tilde{f}_c$ is the transpose of $\tilde{e}_c$.

(vi) If $H \in L$, then $H^{(1,2)}$, $H^{(2,1)}$, $H^{(2,2)}$ are all zero and

$$H = \begin{pmatrix} H^{(1,1)} & 0 \\ 0 & 0 \end{pmatrix}.$$

(vii) D_S is the matrix shown, with $H^{(1,1)}$ the identity matrix.

Proposition (D.4e) follows from these facts.

(D.4f) Example. If $n = r = 2$, then $I(n,r) = \{11, 12, 21, 22\} = S \cup (\Upsilon \cap \mathsf{T})$, where $S = \{11, 12, 22\}$, and $\Upsilon \cap \mathsf{T} = \{21\}$. We have

$$\tilde{e}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{f}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

D.5 The modules M_Q

Let $\lambda \in \Lambda^+(n,r)$. For each standard tableau Q in $\mathcal{Q}(\lambda)$ we define M_Q to be the subspace of $V^{\otimes r}$ which has F-basis all v_i such that $Q = Q(i)$, that is, all i in $I_\lambda(Q, \approx)$. By Proposition B, this is an L-submodule of $V^{\otimes r}$.

We get therefore a direct sum decomposition of the tensor space $V^{\otimes r}$ into L-submodules

$$V^{\otimes r} = \bigoplus_{\lambda \in \Lambda^+(n,r)} \bigoplus_{Q \in \mathcal{Q}(\lambda)} M_Q.$$

(D.5a) $M_Q = Lv_{i^Q} = Lv_{i_Q}$.

This follows from (D.2d).

For $z = \sum_i \xi_i v_i \in V^{\otimes r}$, define the *support* of z to be

$$\mathsf{supp}(z) = \{\, i \in I(n,r) \,:\, \xi_i \neq 0 \,\}.$$

(D.5b) Lemma. *If* $z = \sum_i \xi_i v_i$ *and* $c \in \{1, 2, \ldots, n-1\}$ *such that* $\tilde{e}_c z = 0$, *then* $\mathsf{supp}(z)$ *lies in the set* $I(n,r) \setminus Z(c)$.

Proof. Let $U' := \{\, i : \tilde{e}_c(i) \neq \infty \,\} = Z(c)$ and $U'' := \{\, i : \tilde{e}_c(i) = \infty \,\}$. Then $z = z' + z''$, where $z' = \sum_{i \in U'} \xi_i v_i$ and $z'' = \sum_{i \in U''} \xi_i v_i$.

We have by assumption

$$(*) \quad \sum_{i \in U'} \xi_i \, v_{\tilde{e}_c(i)} = \tilde{e}_c z = 0.$$

But for each $i \in U'$, $\tilde{e}_c(i) \neq \infty$, hence $\tilde{f}_c \tilde{e}_c(i) = i$. Applying $\tilde{f}_c$ to $(*)$, we get $0 = \sum_{i \in U'} \xi_i v_i$. But the v_i are linearly independent, so all the ξ_i (for $i \in U'$) are zero. Therefore $z' = 0$ which shows that $z = z''$ has support in $U'' = I(n,r) \setminus Z(c)$.

(D.5c) Corollary. *If* $z \in V^{\otimes r}$ *is annihilated by all* $\tilde{e}_c$, $c \in \{1, 2, \ldots, n-1\}$, *then* $\mathsf{supp}(z)$ *lies in* $\bigcap_c I(n,r) \setminus Z(c) = I(n,r) \setminus \bigcup_c Z(c) = \mathsf{T}$.

Similarly, if $z \in V^{\otimes r}$ is annihilated by all $\tilde{f}_c$, $c \in \{1, 2, \ldots, n-1\}$, then $\mathsf{supp}(z)$ lies in $I(n,r) \setminus \bigcup_c Y(c) = \Upsilon$. There may be a word $i \in I(n,r)$ such that v_i is annihilated by the algebra L. According to Proposition (D.2d), we must have $i^Q = i = i_Q$ in this case, and the $\approx$-class of i consists of i alone. From (D.2f), the shape λ of Q has the property $(\lambda_1, \ldots, \lambda_n) = (\lambda_n, \ldots, \lambda_1)$, hence $\lambda = (k^n)$ (and $r = nk$). In this case $M_Q = F v_i$. There may be more than one such $i \in I_\lambda(n,r)$. For example, $I_{(2,2)}(2,4)$ contains $i = (2211)$ and $j = (2121)$ in $\Upsilon \cap \mathsf{T}$.

As a consequence, we have to allow L-modules which are not unital. An L-module M is then defined to be an F-space on which L acts by linear transformations so that $x(ym) = (xy)m$, for all $x, y \in L$ and $m \in M$. The L-module M is defined to be simple (= irreducible) if *either*

(1) $LM = 0$ and M is a simple F-space, that is, has F-dimension 1, *or*
(2) $M \neq 0$, the element D_S of L acts as the identity on M, and M has no L-submodules except M and $\{0\}$.

We aim to show that the modules M_Q are simple as L-modules. To do so, it is helpful to exploit a subalgebra of L.

(D.5d) The involutory anti-automorphism J of $\mathbf{L} = \mathbf{L}(\mathbf{n}, \mathbf{r})$. At this point we find a strong similarity with the involutory anti-automorphism J of $S(n,r)$ defined in §2.7.

Define a symmetric bilinear map $\langle \ , \ \rangle$ on $V^{\otimes r}$ by the rule $\langle v_i, v_j \rangle = \delta_{i,j}$ for all $i, j \in I(n,r)$. Given $H \in \mathsf{End}_F(V^{\otimes r})$, we defined in (D.4c), (D.4d) its matrix $(H_{i,j})$. Now define $J(H) \in \mathsf{End}_F(V^{\otimes r})$ by the rule: the matrix of $J(H)$ is the transpose of the matrix of H.

By (D.4d), $H_{i,j} = \langle Hv_j, v_i \rangle$ for all $i, j \in I(n, r)$. Replacing H by $J(H)$, we get $J(H)_{i,j} = \langle J(H)v_j, v_i \rangle$. But by definition, $J(H)_{i,j} = H_{j,i} = \langle Hv_i, v_j \rangle$. Therefore $\langle Hv_i, v_j \rangle = \langle J(H)v_j, v_i \rangle = \langle v_i, J(H)v_j \rangle$ for all $i, j \in I(n, r)$. Equivalently,

(D.5e) $\langle Hv, w \rangle = \langle v, J(H)w \rangle$ for all $v, w \in V^{\otimes r}$.

We may use (D.5e) as a definition of $J(H)$ when H is given.

From the elementary properties of transposed matrices we see that the linear map $J : \mathsf{End}_F V^{\otimes r} \to \mathsf{End}_F V^{\otimes r}$ is involutory (i.e. J^2 is the identity) and is an anti-automorphism (i.e. $J(H_1H_2) = J(H_2)J(H_1)$ for all H_1, H_2). But from our present viewpoint the important fact is

(D.5f) For all $c \in \{1, 2, \dots, n-1\}$ there holds $J(\tilde{f}_c) = \tilde{e}_c$ and $J(\tilde{e}_c) = \tilde{f}_c$.

These facts follow from the properties stated in (A.3g)(4). We leave the details as an exercise for the reader. But from (D.5f) we see that J maps L into itself, so it gives a map $J : L \to L$ which is an involutory anti-automorphism of the algebra $L = L(n, r)$.

(D.5g) Take a (total) order $\leq$ on the set $I(n, r)$ such that $\mathsf{sz}(i) \leq \mathsf{sz}(j)$ implies that $i \leq j$ for all words i, j in $I(n, r)$. Using such an order, the matrix of $\tilde{e}_c$ is upper triangular, and the matrix of $\tilde{f}_c$ is lower triangular.

We have therefore:

(D.5h) Corollary. *Let L^+ be the subalgebra of L, generated by the elements $\tilde{e}_c$, $c \in \{1, 2, \dots, n-1\}$. Then L^+ is nilpotent.*

(D.5i) Lemma. *The module M_Q is simple.*

Note that this holds for arbitrary fields F.

Proof. This is clear if M_Q is an 1-dimensional module which is annihilated by L, so we assume that this is not the case.

We fix $i = i^Q$, then v_i generates M_Q as an L-module, by (D.2a).

Let $0 \neq x \in M_Q$, it suffices to show that v_i lies in the L-submodule generated by x.

To do so, we consider the L^+-submodule of M_Q generated by x. By (D.5h), this submodule contains some non-zero element z such that $\tilde{e}_c z = 0$ for all c. By (D.5c), the support of z lies in T. But it also lies in $I(Q, \approx)$ since $z \in M_Q$. It follows that z is a scalar multiple of v_i, by (D.2d)(i). We assumed z is non-zero, hence v_{i^Q} lies in the submodule generated by x.

(D.5j) It follows by a well-known theorem on finite dimensional algebras (or on rings with minimum condition; see e.g. [11, Theorem (25.2), page 164]), that L is semisimple.

Furthermore, every unital simple L-module M occurs as a submodule (and hence as a summand) of $V^{\otimes r}$. Take any non-zero element $x \in M$. Then $M = Lx$, and the map $\theta : L \to M$ which takes $u \mapsto ux$ is an epimorphism of L-modules. But since L is semisimple, it follows that θ maps some simple submodule N of L isomorphically onto M. And the simple submodules of L are submodules of $V^{\otimes r} = \bigoplus_{\lambda \in \Lambda^+(n,r)} \bigoplus_{Q \in \mathcal{Q}(\lambda)} M_Q$ (see the displayed formula, above (D.5a)).

This shows that every unital simple L-module is isomorphic to M_Q for some $Q \in \mathcal{Q}(\lambda)$, for some $\lambda \in \Lambda^+(n,r)$.

To classify the simple L-modules we must find out when M_Q and M_R are isomorphic.

D.6 The $\boldsymbol{\lambda}$-rectangle

Fix $\lambda \in \Lambda^+(n,r)$ and use the following notation:

(D.6a) $\mathcal{P}(\lambda) = \{P_1, P_2, \dots, P_{d_\lambda}\}$ is the set of all standard λ-tableaux whose entries all lie in $\underline{n}$, and

(D.6b) $\mathcal{Q}(\lambda) = \{Q_1, Q_2, \dots, Q_{f_\lambda}\}$ is the set of all standard λ-tableaux whose entries are $\{1, 2, \dots, r\}$ in some order (see D.1).

Definition. If $P \in \mathcal{P}(\lambda)$ and $Q \in \mathcal{Q}(\lambda)$, let $P : Q$ denote[2] the word $i \in I(n,r)$ such that $P(i) = P$ and $Q(i) = Q$. In the notation of B.7,

(D.6c) $P : Q = \mathsf{M}(\lambda, P, Q) = \mathsf{Sch}^{-1}(\lambda, P, Q)$.

It is useful to display the set of words of shape λ in the following "λ-rectangle"

(D.6d)
$$\begin{array}{cccc}
P_1 : Q_1 & P_1 : Q_2 & \cdots & P_1 : Q_{f_\lambda} \\
P_2 : Q_1 & P_2 : Q_2 & \cdots & P_2 : Q_{f_\lambda} \\
\vdots & \vdots & & \vdots \\
P_{d_\lambda} : Q_1 & P_{d_\lambda} : Q_2 & \dots & P_{d_\lambda} : Q_{f_\lambda}
\end{array}$$

This rectangle has the following properties:

(D.6e) (i) Every element of $I_\lambda(n,r)$ appears once and only once in (D.6d) (see (B.6a)).

(ii) The h^{th} row $\{P_h : Q_1, \dots, P_h : Q_{f_\lambda}\}$ is the $\sim$-class $I_\lambda(P_h, \sim)$, for each $h \in \{1, \dots, d_\lambda\}$ (see (C.1d)).

(iii) The k^{th} column $\{P_1 : Q_k, \dots, P_{d_\lambda} : Q_k\}$ is the $\approx$-class $I_\lambda(Q_k, \approx)$, for each $k \in \{1, \dots, f_\lambda\}$ (see (C.1e)).

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[2]Not to be confused with the bideterminant $(T_i : T_j)$ defined in (4.3a).

(D.6f) From now on we shall arrange the notation in the λ-rectangle (D.6d) so that $P_1 = T_\lambda$ and $Q_1 = \mathsf{Q}^{(\lambda)}$. Recall from (C.2i) that $\mathsf{Q}^{(\lambda)} \in \mathcal{Q}(\lambda)$ has the property: an element $i \in I_\lambda(n,r)$ has $Q(i) = \mathsf{Q}^{(\lambda)}$ if and only if $i = KP(i)$.

D.7 Canonical maps

Fix $\lambda \in \Lambda^+(n,r)$ again. The entries $P : Q$ in (D.6d) are elements of $I(n,r)$. From now on we shall make the

(D.7a) Convention. When convenient, we shall regard each $i \in I(n,r)$ as the element v_i of $V^{\otimes r}$.

With this convention, the column of the rectangle (D.6d) corresponding to a given $Q \in \mathcal{Q}(\lambda)$ is a basis of the L-module M_Q (see §D.5).

(D.7b) Definition. If $Q, R \in \mathcal{Q}(\lambda)$, then the F-linear map $\gamma_{Q,R} : M_Q \to M_R$ which takes $P_h : Q \mapsto P_h : R$ for each $P_h \in \mathcal{P}(\lambda)$, is the *canonical map* from M_Q to M_R.

Since any two columns in (D.6d) have the same length, the canonical map is an F-linear isomorphism. It is clear that $\gamma_{Q,R}\gamma_{S,Q} = \gamma_{S,R}$ and $\gamma_{Q,R} = (\gamma_{R,Q})^{-1}$, for all $Q, R, S \in \mathcal{Q}(\lambda)$. Our ambition in this section is to prove that any canonical map is an isomorphism of L-modules (see (D.7f) and (D.7i)), and that any L-homomorphism $M_Q \to M_R$ is a scalar multiple of the canonical map $\gamma_{Q,R}$ (see (D.7h)).

(D.7c) Lemma. *Let $Q \in \mathcal{Q}(\lambda)$ and $P \in \mathcal{P}(\lambda)$, and let $i = P : Q$. Then*

$$KP(i) = P : \mathsf{Q}^{(\lambda)} = \gamma_{Q,\mathsf{Q}^{(\lambda)}}(i).$$

In other words, the operation $i \to KP(i)$ is achieved (for $i \in I_\lambda(n,r)$) by the canonical map $\gamma_{Q,\mathsf{Q}^{(\lambda)}}$.

Proof. By (C.3p) one may make a sequence of basic moves joining i to $KP(i)$. Since basic moves do not change P-symbols, we know that $KP(i) = P : Q'$ for some $Q' \in \mathcal{Q}(\lambda)$. But $KP(i)$ equals $KP(KP(i))$, hence its Q-symbol is $\mathsf{Q}^{(\lambda)}$ (see (C.2i)). Therefore $KP(i) = P : \mathsf{Q}^{(\lambda)}$.

Notice that this holds for any i in column Q of (D.6d), and in particular it holds for $\tilde{f}_c(i)$, for any $c \in \{1, \ldots, n-1\}$. By Proposition B (see C.6) we have $\tilde{f}_c(KP(i)) = KP(\tilde{f}_c(i))$, and in our case this gives

(D.7d) $\tilde{f}_c(P : \mathsf{Q}^{(\lambda)}) = KP(\tilde{f}_c(i))$

or, as a commutative diagram,

$$\text{(D.7e)}\qquad \begin{array}{ccc} KP(i) & \xleftarrow{\ \gamma\ } & i \\ \Big\downarrow & & \Big\downarrow \\ KP(\tilde{f}_{c(i)}) & \xleftarrow{\ \gamma\ } & \tilde{f}_{c(i)} \end{array}$$

where $\gamma = \gamma_{Q,\mathsf{Q}^{(\lambda)}}$ and the vertical arrows indicate action of $\tilde{f}_c$. Thus the action of $\tilde{f}_c$ commutes with γ, when applied to any i in the Q-column of (D.6d). In the same way, one has a diagram like (D.7e), with $\tilde{e}_c$ replacing $\tilde{f}_c$. Then we can replace $\tilde{f}_c$ in (D.7e) by any element of L and still have a commutative diagram, since the elements $\tilde{f}_c$ and $\tilde{e}_c$, $c \in \{1, \dots, n-1\}$ generate L as F-algebra. So we get a

(D.7f) Corollary to (D.7c). *For each pair Q, R of tableaux in $\mathcal{Q}(\lambda)$, the canonical map $\gamma_{Q,R} : M_Q \to M_R$ is an isomorphism of L-modules.*

Proof. The argument above shows that the corollary holds if $R = \mathsf{Q}^{(\lambda)}$, and this gives the general case, since $\gamma_{Q,R} = \gamma^{-1}_{R,\mathsf{Q}^{(\lambda)}} \gamma_{Q,\mathsf{Q}^{(\lambda)}}$.

(D.7g) Lemma. *Suppose Q, R are both tableaux whose entries are $1, 2, \dots, r$ in some order (possibly of different shapes). If $\psi : M_Q \to M_R$ is a homomorphism of L-modules, then $\psi(v_{i^Q}) = \alpha v_{i^R}$ for some $\alpha \in F$.*

Proof. Let $z = \psi(v_{i^Q})$, then $\tilde{e}_c(z) = \psi(\tilde{e}_c(v_{i^Q})) = 0$ for all $c \in \{1, 2, \dots, n-1\}$. Therefore $\mathsf{supp}(z) \subseteq \mathsf{T}$, by Corollary (D.5c). But also $\mathsf{supp}(z) \subseteq I(R, \approx)$ since $z \in M_R$, hence $\mathsf{supp}(z) \subseteq \mathsf{T} \cap I(R, \approx) = \{i^R\}$ (see (D.2d)); this means that $z = \alpha v_{i^R}$ for some $\alpha \in F$.

(D.7h) Corollary. *Suppose Q, R are tableaux whose entries are $1, 2, \dots, r$ in some order. If Q, R have the same shape then $\mathsf{Hom}_L(M_Q, M_R) = F\gamma_{Q,R}$.*

Proof. If $\psi \in \mathrm{Hom}_L(M_Q, M_R)$, i.e. if $\psi : M_Q \to M_R$ is an L-homomorphism, then $\psi(v_{i^Q}) = \alpha v_{i^R}$ for some $\alpha \in F$. But in the present case we know that $\gamma_{Q,R} : M_Q \to M_R$ also is an L-homomorphism, by (D.7f). Therefore we have L-homomorphisms ψ and $\alpha\gamma_{Q,R}$ which take v_{i^Q} to the same element αv_{i^R}. Since v_{i^Q} is an L-generator of M_Q, by (D.5a), the map ψ is equal to $\alpha\gamma_{Q,R}$.

The module M_Q is simple, and $M_Q \cong M_{\mathsf{Q}^{(\lambda)}}$. For each λ, let $M_\lambda = M_{\mathsf{Q}^{(\lambda)}}$.

(D.7i) Lemma. *Let $\lambda, \mu \in \Lambda^+(n, r)$, then $M_\lambda \cong M_\mu$ if and only if $\lambda = \mu$.*

Proof. Suppose that there is an isomorphism $\psi : M_\lambda \to M_\mu$ of L-modules. Then $\psi(v_{i^\lambda}) = \alpha v_{i^\mu}$ for some $\alpha \in F$, by (D.7g). (Note that $M_\lambda = M_{\mathsf{Q}^{(\lambda)}}$ and, by (D.1i), $i^{\mathsf{Q}^{(\lambda)}} = i^\lambda$ and $i^{\mathsf{Q}^{(\mu)}} = i^\mu$.)

If we apply repeatedly $\tilde{f}_1$'s to i^λ then we replace each time the last 1 by a 2, and we can do this $\lambda_1 - \lambda_2$ times, and the next time we get zero. Similarly

if we apply $\tilde{f}_1$ to i^μ repeatedly, then we can do this $\mu_1 - \mu_2$ times before we get zero. The isomorphism shows now that $\lambda_1 - \lambda_2 = \mu_1 - \mu_2$. The same argument with $\tilde{f}_2$ shows that $\lambda_2 - \lambda_3 = \mu_2 - \mu_3$, and so on. Both λ and μ have degree r which forces $\lambda = \mu$.

Exercise 1. Let $\lambda = (5, 4, 2)$. Find $\tilde{f}_1 v_{i^\lambda}$ and verify that $\tilde{f}_1^{\,2} v_{i^\lambda} = v_\infty = 0$. Find also $\tilde{f}_2^{\,t} v_{i^\lambda}$ for $t = 1, 2, 3$.

Exercise 2. If $\lambda = (k, \dots, k)$, where $kn = r$, we have

$$T_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & \cdots & 2 \\ \hline \vdots & \vdots & & \vdots \\ \hline k & k & \cdots & k \\ \hline \end{array}\,.$$

Check by direct calculation that $\tilde{f}_c(KT_\lambda) = \infty = \tilde{e}_c(KT_\lambda)$ for all c.

D.8 The algebra structure of $L(n, r)$

Each M_λ has endomorphism algebra F. It follows now that L is isomorphic to the direct sum of matrix algebras,

$$L \cong \bigoplus_\lambda M_{d_\lambda}(F),$$

where the sum is taken over all $\lambda \in \Lambda^+(n, r)$ with $\lambda \neq (k^n)$, and where d_λ denotes the dimension of M_λ. This follows from the Frobenius–Schur theorem, see [11, Theorem 27.8, page 183], but we shall give a direct proof that the representation $L \to \mathsf{End}_F(M_\lambda)$ afforded by the simple module M_λ is surjective, for all $\lambda \in \Lambda^+(n, r)$, $\lambda \neq (k^n)$.

The problem is to give elements of L which realize the "matrix units". Fix $\lambda \in \Lambda^+(n, r)$. The module M_λ is simple, it has F-basis $\{\, v_i \,:\, i \in I(\lambda) \,\}$ where we set

$$I(\lambda) := I(\mathsf{Q}^{(\lambda)}, \approx) = \{\, i \in I_\lambda \,:\, KP(i) = i \,\}$$

(see (C.2i)). An element Φ of $\mathsf{End}_F(M_\lambda)$ is regarded as a matrix $(\Phi_{ij})_{i,j \in I(\lambda)}$ in the usual way,

$$\Phi(v_j) = \sum_{i \in I(\lambda)} \Phi_{ij} v_i, \ \text{ all } j \in I(\lambda).$$

Monomials in $\tilde{f}_c, \tilde{e}_c$ are often identified with the matrices of the linear transformations which they determine on M_λ.

(D.8a) Lemma. *Suppose Φ is a monomial in the $\tilde{e}_c$ and $\tilde{f}_c$, $c \in \{1, \ldots, n-1\}$. Let $(\Phi_{ij})_{i,j \in I(\lambda)}$ be the matrix of Φ restricted to M_λ.*

(i) *Each row or column of the matrix has at most one non-zero entry (which is then equal to 1).*

(ii) *If $(\Phi_{ij})_{i,j \in I(\lambda)}$ has rank 1, then it is a matrix unit.*

Proof. For each $i \in I(\lambda)$, $\Phi(v_i)$ is either zero, or a basis element. So all but at most one entries of each column are zero, and if there is a non-zero entry then it is equal to 1.

This implies part (i), since if $\Phi = m_1 \cdots m_t$, then $J(\Phi) = J(m_t) \cdots J(m_1)$ is also a monomial. But $(J(\Phi))$ is the transpose of (Φ) (see (D.5d)).

Part (ii) follows.

(D.8b) We want to show that every element of $\mathsf{End}_F(M_\lambda)$ can be represented by some element of L, i.e. that the map $L \to \mathsf{End}_F(M_\lambda)$ is surjective. And we would like to do this by showing that each "matrix unit" $E_{i,j} \in M_{d_\lambda}(F)$ can be represented by some polynomial in the $\tilde{f}_c, \tilde{e}_c$.

It is enough to prove that E_{i_λ, i^λ} can be represented by a monomial in $\mathcal{L}$. Namely, if $s, t \in I(\lambda)$ then by (D.2d) there are monomials p and q in $\tilde{f}$'s and $\tilde{e}$'s with $p(v_{i_\lambda}) = v_s$ and $q(v_t) = v_{i^\lambda}$. The linear map $pE_{i_\lambda, i^\lambda}q$ of $V^{\otimes r}$ has rank at most 1 (the rank of E_{i_λ, i^λ}), so by (D.8a) it is equal to E_{st}, and this is then also represented by an element in L.

The L-module $M_\lambda = M_{\mathsf{Q}^{(\lambda)}}$ has basis $\{\, v_i \,:\, i \in I(\lambda) = I(\mathsf{Q}^{(\lambda)}, \approx) \,\}$. The d_λ elements of $I(\lambda)$ can be arranged (see (D.1g) and (D.6d)) as

$$i(1) = i^{\mathsf{Q}^{(\lambda)}} = i^\lambda, \quad i(2), \quad \ldots, \quad i(d_\lambda - 1), \quad i(d_\lambda) = i_{\mathsf{Q}^{(\lambda)}} = i_\lambda.$$

(D.8c) Proposition.

(i) *There are $c(1), \ldots, c(b) \in \{1, \ldots, n-1\}$ such that $\Phi := \tilde{f}_{c(b)} \ldots \tilde{f}_{c(1)}$ maps i^λ to i_λ.*

(ii) *The number b in (i) is given by $\mathsf{sz}(i^\lambda) + b = \mathsf{sz}(i_\lambda)$.*

(iii) *If $d(1), \ldots, d(s) \in \{1, \ldots, n-1\}$ are such that $\tilde{f}_{d(s)} \ldots \tilde{f}_{d(1)} i^\lambda \neq \infty$, then $s \leq b$.*

(iv) *$\Phi(v_{i(a)}) = 0$, for all $a \in \{2, 3, \ldots, d_\lambda\}$.*

This shows that the matrix of Φ on M_λ is the matrix unit E_{i_λ, i^λ}.

Proof. (i) is a direct application of (D.2d)(iii)(2). We recall the proof, because it brings up useful information.

The idea is to apply operators $\tilde{f}_{c(1)}, \tilde{f}_{c(2)}, \ldots$ in succession to the word i^λ, in such a way that, for each $t = 1, 2, \ldots$

$$\tilde{f}_{c(t)} \cdots \tilde{f}_{c(1)} i^\lambda \neq \infty.$$

The word $\tilde{f}_{c(t)} \cdots \tilde{f}_{c(1)} i^\lambda$ has size $\mathsf{sz}(i^\lambda) + t$, by (D.2c). The sizes of words in $I(n, r)$ are bounded by rn. Hence, however we choose $c(1), c(2), \ldots$, we

must reach b such that $\tilde{f}_{c(b)} \cdots \tilde{f}_{c(1)} i^\lambda \neq \infty$, but $z = \tilde{f}_{c(b)} \cdots \tilde{f}_{c(1)} i^\lambda$ has the property $\tilde{f}_c(z) = \infty$ for all $c \in \{1, \ldots, n-1\}$. This implies $z \in \Upsilon$; however $z \in I(\lambda) = I(\mathsf{Q}^{(\lambda)}, \approx)$, by (D.2b), hence $z = i_\lambda$ by (D.2d)(ii). This proves parts (i) and (ii) of (D.8c).

To prove part (iii), note that the argument above (replace t by s) shows that, applying further operators $\tilde{f}_{d(s+1)}, \tilde{f}_{d(s+2)}, \ldots, \tilde{f}_{d(b')}$ to $\tilde{f}_{d(s)} \ldots \tilde{f}_{d(1)} i^\lambda$ if necessary, we must reach b' such that

$$\tilde{f}_{d(b')} \cdots \tilde{f}_{d(s+1)} \tilde{f}_{d(s)} \cdots \tilde{f}_{d(1)} i^\lambda = i_\lambda.$$

Taking the size of each side of this equation, we get $\mathsf{sz}(i^\lambda) + b' = \mathsf{sz}(i_\lambda)$; this shows that $b' = b$. Therefore $s \leq b' = b$.

(iv) If $a \in \{2, 3, \ldots, d_\lambda\}$, then by (D.2d)(iii), $i(a) = \tilde{f}_{d(1)} \ldots \tilde{f}_{d(u)} i^\lambda$ for some $d(1), \ldots, d(u) \in \{1, \ldots, n-1\}$, and $u \geq 1$. But this implies that $\Phi(i(a)) = \tilde{f}_{c(b)} \cdots \tilde{f}_{c(1)} \tilde{f}_{d(u)} \cdots \tilde{f}_{d(1)} i^\lambda$. If this were $\neq \infty$, it would contradict (iii), since $b + u > b$. Therefore $\Phi(i(a)) = \infty$, hence $\Phi(v_{i(a)}) = 0$.

Remarks.

(i) In (D.8c)(i), there may be several ways of choosing $c(1), \ldots, c(b)$ so that $\Phi = \tilde{f}_{c(b)} \ldots \tilde{f}_{c(1)}$ maps i^λ to i_λ. But by (ii) the length b of any such sequence is always the same, namely $b = \mathsf{sz}(i_\lambda) - \mathsf{sz}(i^\lambda)$.

(ii) For any $\Phi \in L$, the matrix of $J(\Phi)$ is the transpose of the matrix of Φ. This is true by definition if the matrices are defined in terms of the natural basis $\{ v_i : i \in I(n,r) \}$ of $V^{\otimes r}$ (see (D.5d)), hence it is true also for the matrices defined in terms of the basis $\{ v_i : i \in I(\lambda) \}$ of M_λ. Therefore, if $\Phi = \tilde{f}_{c(b)} \ldots \tilde{f}_{c(1)}$ as in (D.8c), then the map $J(\Phi) = \tilde{e}_{c(1)} \ldots \tilde{e}_{c(b)}$ has matrix E_{i^λ, i_λ}.

Example (see chapter E). . Take $\lambda = (2,1,0)$ and $\mathsf{Q}^{(\lambda)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. We can take

$$i(1) = 211, \quad i(2) = 311, \quad i(3) = 312, \quad i(4) = 322, \quad i(5) = 323.$$

Another possibility is to take

$$i(1) = 211, \quad i(2) = 212, \quad i(3) = 213, \quad i(4) = 313, \quad i(5) = 323.$$

D.9 The character of M_λ

The basis $\{ v_i : i \in I(Q, \approx) \}$ for M_Q consists of eigenvectors for the diagonal matrices in the general linear group $\mathsf{GL}(n, F)$. Hence M_Q has a formal character (as defined in §3.4), also when the field F is finite.

Explicitly, let $(M_Q)^\alpha = \xi_\alpha M_Q$ (see §3), then the formal character of M_Q is by definition

$$\Phi_{M_Q}(X_1, \ldots, X_n) = \sum_{\alpha \in \Lambda(n,r)} \mathsf{dim}(M_Q)^\alpha \, X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

The P-symbol preserves weights, hence $\mathsf{dim}(M_Q)^\alpha = \mathsf{dim}(M_\lambda)^\alpha$, where λ is the shape of Q. Therefore $\Phi_{M_Q} = \Phi_{M_\lambda}$, that is, the formal character of M_Q depends only on the shape of Q.

Let V_λ be the "Weyl module" associated to λ; this is a module for the Schur algebra, see section 5.2. The following is due to P. Littelmann, in far more generality [35, Introduction].

(D.9a) Corollary. *The modules M_λ and V_λ have the same formal character.*

Proof. In (5.4a) we saw that $(V_\lambda)^\alpha$ has F-basis indexed by standard λ-tableaux of weight α. We also know from the characterisation of M_λ given above that $(M_\lambda)^\alpha$ has basis v_i labelled by standard λ-tableaux of weight α. Hence M_λ has the same formal character as V_λ.

Note that it follows that $\Phi_{M_\lambda} = \Phi_{M_\mu}$ if and only if $\lambda = \mu$. (This is also visible directly, by considering the "highest terms" of the formal characters.)

D.10 The Littlewood–Richardson Rule

Suppose λ and μ are partitions with $\lambda \in \Lambda^+(n,r)$, $\mu \in \Lambda^+(n,s)$. Then

$$\Phi_{V_\lambda} \cdot \Phi_{V_\mu} = \sum_\nu c^\nu_{\lambda,\mu} \Phi_{V_\nu}.$$

The coefficients $c^\nu_{\lambda,\mu}$ are non-negative integers, and the Littlewood–Richardson rule is a combinatorial rule for computing these integers. As we have seen, the L-module M_λ has the same formal character as the Schur algebra module V_λ. Then we have

$$\Phi_{M_\lambda} \cdot \Phi_{M_\mu} = \sum_\nu c^\nu_{\lambda,\mu} \Phi_{M_\nu}.$$

Here the sum is taken over all $\nu \in \Lambda^+(n, r+s)$. This leads to the following combinatorial description of the coefficients.

(D.10a) *Let $\mathcal{W}$ be the set of words $i \in I(n, r+s)$ of the form $i = jk$ with the following properties:*

(a) *$KP(j) = j$ and $P(j)$ has shape λ,*

(b) *$k = i^\mu$, and*

(c) *the reverse $B(i)$ of i is a lattice permutation of weight ν.*

Then $c^\nu_{\lambda,\mu}$ is equal to $\#\mathcal{W}$, the number of elements of $\mathcal{W}$.

A number of proofs of (D.10a) exist, and Littelmann gives a wide ranging generalization to cover any complex symmetrizable Kac-Moody Lie algebra [35, Introduction]. The proof we give is the special case which applies to $\mathsf{gl}(n)$ or to GL_n.

Proof. By definition M_λ is a direct summand of $V^{\otimes r}$, and M_μ is a direct summand of $V^{\otimes s}$. Then $M_\lambda \otimes M_\mu$ is a direct summand of $V^{\otimes(r+s)}$, as a vector space, since $v_i \otimes v_j = v_{ij}$. It is invariant under the linear maps $\tilde{f}_c$ and $\tilde{e}_c$. For example, $\tilde{f}_c(v_{ij})$ is either $v_{\tilde{f}_c(i)j}$ or $v_{i\tilde{f}_c(j)}$, or zero; and each of these belong again to $M_\lambda \otimes M_\mu$.

Furthermore, $M_\lambda \otimes M_\mu$ is the direct sum of L-modules M_R for some standard tableaux R with shapes in $\Lambda^+(n, r+s)$.

Therefore $c^\nu_{\lambda,\mu}$ is precisely the number of such R such that M_R occurs as a direct summand in $M_\lambda \otimes M_\mu$ and $M_R \cong M_\nu$. Each M_R contains a unique "highest weight vector" v_i such that $\tilde{e}_c(i) = \infty$ for all c, namely the basis vector for $i = i^R$.

Hence $c^\nu_{\lambda\mu}$ is equal to the number of words $i = jk$ where

(i) i belongs to T (see §D.1), and $P(i)$ has shape ν;

(ii) $k = KP(k)$, and $P(k)$ has shape μ;

(iii) $KP(j) = j$ and $P(j)$ has shape λ.

We know $i \in \mathsf{T}$ if and only if $B(i)$ is a lattice permutation (see (D.1b)). For $i \in \mathsf{T}$, the shape is the same as the weight (see (D.2f)). Furthermore, if $B(i)$ is a lattice permutation then so is $B(k)$, and since $k = KP(k)$, we have $k = i^\mu$. This completes the proof of (D.10a).

(D.10b) Very often, the Littlewood–Richardson rule is stated in a different form. It says that the coefficient $c^\nu_{\lambda,\mu}$ is equal to the size of the set $\mathcal{C}$ of standard (skew) tableaux T of shape $\nu \setminus \mu$ and of weight λ such that the word $w(T)$ is a lattice permutation. Here the word $w(T)$ is obtained by reading T from right to left and from rows $1, 2, 3, \ldots$.

We will now show directly that $\#\mathcal{C} = \#\mathcal{W}$, by means of a bijection from $\mathcal{W}$ onto $\mathcal{C}$.

Suppose i belongs to $\mathcal{W}$, where $i = jk$ as in (D.10a). Then always $k = i^\mu$, and we must consider the tableau $P(j)$. Let r_{ts} be the number of times the letter s occurs in row t of $P(j)$; since $P(j)$ is standard, row t of $P(j)$ starts with some letter $\geq t$, and it has the form

$$t^{r_{tt}}(t+1)^{r_{t,t+1}} \ldots\ n^{r_{tn}},\ t \geq 0$$

Write the multiplicities r_{st} as an upper triangular matrix:

$$U = \begin{pmatrix} r_{11} & r_{12} & & \cdots \\ & r_{22} & r_{23} & \cdots \\ & & r_{33} & \cdots \\ \vdots & & & \end{pmatrix}.$$

By transposing this matrix, we can define a skew tableaux $T = \psi(U)$, depending on $i = jk$, as follows: The t^{th} row of T starts at position $(t, \mu_t + 1)$ and has the multiplicities taken from the t^{th} column of U, that is, row t is

$$t^{r_{tt}}(t-1)^{r_{t-1,t}} \ldots 1^{r_{1t}}.$$

The associated word is then

$$w(T) = 1^{r_{11}}(2^{r_{22}}1^{r_{12}})(3^{r_{33}}2^{r_{23}}1^{r_{13}}) \cdots$$

We will show that T belongs to $\mathcal{C}$, and that the map $\psi : U \to T$ is a bijection between $\mathcal{W}$ and $\mathcal{C}$.

(1) The word j has weight $\nu \setminus \mu$ if and only if for each s, the sum of the entries in column s of the matrix U is equal to $\nu_s - \mu_s$. This means for the skew tableau T that the sum of the entries in row s is equal to $\nu_s - \mu_s$, for each s, that is T has shape $\nu \setminus \mu$.

(2) The tableau $P(j)$ has shape λ provided row t of $P(j)$ has λ_t entries, for each t, that is

$$\sum_{v \geq t} r_{tv} = \lambda_t\,.$$

This is equivalent with saying that the skew tableau $w(T)$ has weight λ.

(3) The tableau $P(j)$ is standard if and only if

$$\sum_{y=s+1}^{v+1} r_{s+1,y} \leq \sum_{y=s}^{v} r_{s,y}.$$

for all $s \geq 1$ and all $v \geq s$. This means for the word $w(T)$ that in each initial section the number of entries equal to s is $\geq$ the number of entries equal to $s+1$, for each $s \geq 1$. That is, $P(j)$ is standard if and only if $w(T)$ is a lattice permutation.

(4) The word $B(i)$ is of the form

$$(1^{\mu_1}2^{\mu_2}\cdots)(\cdots x^{r_{1x}}\cdots 2^{r_{12}}1^{r_{11}})(\cdots x^{r_{2x}}\cdots 2^{r_{22}})(\cdots x^{r_{3x}}\cdots 3^{r_{33}})\cdots$$

This is a lattice permutation if and only if for each $s \geq 1$ and each v

$$\mu_s + \sum_{y=1}^{v} r_{ys} \geq \mu_{s+1} + \sum_{y=1}^{v+1} r_{y,s+1}.$$

This is equivalent with T being standard.

Combining (1) to (4), we see that if $i = jk \in \mathcal{W}$ and if U is the matrix encoding j, then the skew tableau $T = \psi(U)$ belongs to $\mathcal{C}$. Conversely if we start with some $T \in \mathcal{C}$, then T is the transpose of a matrix U, and this encodes a word $i = jk$ in $\mathcal{W}$. So ψ is a bijection.

D.11 Lascoux, Leclerc and Thibon

This is a brief summary of Chapter 6 of the collective work "Algebraic combinatorics on words" [38]. This chapter is called "The plactic monoid", and its authors are A. Lascoux, B. Leclerc and J.-Y. Thibon. We refer to this chapter, and to its authors, as LLT. Our main purpose is to show that LLT prove facts which imply Theorem A and Proposition B (see (D.11h)).

Reference numbers for sections, propositions, etc. in LLT are enclosed in square brackets (so that, for example, [6.1] stands for [38, 6.1]).

(D.11a) The background of LLT is work of M. P. Schützenberger, which expresses the combinatoric background of work by A. Young, G. de B. Robinson, D. E. Littlewood, etc. on the representation theory of the finite symmetric group.

(D.11b) Words and tableaux. In LLT the set of all words on the alphabet $A = \{1, \dots, n\}$ is denoted A^*. So in our language, $A^* = \bigcup_{r \geq 0} I(n,r)$.

In LLT (page 3), a tableau[3] is a word i in A^* such that $i = KP$ for some standard tableau P in the sense of section B.1. For example, $i = 544135$ is a tableau, because $i = KP$ for $P = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array}$. If we know that i is a tableau, the corresponding tableau P (which LLT call its *planar representation*) is uniquely defined. The shape λ of i is, by definition, the shape of P. In the example above, the shape is $\lambda = (3,2,1,0,0)$.

(D.11c) In [6.1] the Schensted algorithm is described. It takes each word i to a tableau $KP(i)$. We can take $P(i)$ to be the tableau defined in (B.4b), (B.4c). The *equivalence* $\sim$ on A^* is defined in [6.2, bottom of page 4]: if i, j are words, then $i \sim j$ means $KP(i) = KP(j)$. The *equivalence* $\equiv$ on A^* is defined on page 5 to be the equivalence on A^* generated by *basic moves* (see (C.3c), (C.3d) and [6.2.3, 6.2.4]. LLT do not use the term "basic move".)

(D.11d) Knuth's theorem (C.3a), [6.2.5] says that $\equiv$ coincides with $\sim$. This is proved in [6.2], elegantly and economically, by a theorem of C. Greene [21]. Greene's theorem itself is also proved in [6.2]. Now the main

(D.11e) Definition (see [6.2.2]). The *plactid monoid* $\mathsf{Pl}(A) := A^*/\sim$ is the quotient of A^* by $\sim$. Elements of $\mathsf{Pl}(A)$ are the $\sim$-classes, or "plactic classes" in A^*.

[3] We write tableau (underlined) for a word which is a "tableau in the sense of LLT". A tableau (not underlined) is a standard tableau in the sense of section B.1 of this Appendix. Later in LLT a tableau KP and its planar representation P are often identified.

Knuth shows that $\sim$ is compatible with the product of words: if u, u', v, v' are words, then $u \sim u'$ and $v \sim v'$ implies $uu' \sim vv'$ (see [34, Corollary, page 724]). Product of words is by concatenation, so that $uu' = u \mid u'$; see (A.3g)(6).

Therefore $\mathsf{Pl}(A)$ is a monoid (i.e. a semigroup with identity): the product of the $\sim$-class of u with the $\sim$-class of u', is defined to be the $\sim$-class of uu'. If u is any word, then $u \sim KP(u)$ (see (C.3p), [6.2.3]). Every $\sim$-class contains exactly one <u>tableau</u>; see Theorem [6.2.5].

(D.11f) A main theme in LLT is that it is often useful to "lift" a symmetric polynomial to $\mathbb{Z}[\mathsf{Pl}(A)]$. Suppose that M is any monoid. Then $\mathbb{Z}[M]$, which is the free $\mathbb{Z}$-module with M as $\mathbb{Z}$-basis, is a ring. In case $M = A^*$ we can identify the ring $\mathbb{Z}[A^*]$ with the tensor ring $T(V) = \mathbb{Z} \oplus V \oplus (V \otimes V) \oplus \cdots$ over the free $\mathbb{Z}$-module $V = \mathbb{Z}\nu_1 \oplus \cdots \oplus \mathbb{Z}\nu_n$, by identifying each word $i = i_1 \cdots i_r \in A^*$ with the tensor product $\nu_i = \nu_{i_1} \otimes \cdots \otimes \nu_{i_r}$ (compare with (D.4a)). Yet another interpretation of $\mathbb{Z}[A^*]$ is as the ring of all polynomials (over $\mathbb{Z}$) in non-commuting variables $\nu_1, \ldots, \nu_n$; here one regards every tensor product $\nu_i = \nu_{i_1} \otimes \cdots \otimes \nu_{i_r}$ as the monomial $\nu_{i_1} \cdots \nu_{i_r}$.

Now suppose that $\xi_1, \ldots, \xi_n$ are commuting variables. Then there is an epimorphism of rings $\kappa : \mathbb{Z}[A^*] \to \mathbb{Z}[\xi_1, \ldots, \xi_n]$ which takes $\nu_\sigma \mapsto \xi_\sigma$ for all $\sigma \in \{1, \ldots, n\}$. And this map factors through the map $\pi : \mathbb{Z}[A^*] \to \mathbb{Z}[\mathsf{Pl}(A)]$ induced by the natural epimorphism $A^* \to \mathsf{Pl}(A)$; this means that $i \sim j$ implies $\kappa(i) = \kappa(j)$. (It is enough to check this in case i is connected to j by a basic move.) So there exists a ring epimorphism $\eta : \mathbb{Z}[\mathsf{Pl}(A)] \to \mathbb{Z}[\xi_1, \ldots, \xi_n]$ such that $\kappa = \eta\pi$. In section [6.4] LLT define a "plactic Schur function" S_λ in $\mathbb{Z}[\mathsf{Pl}(A)]$ which is mapped by η onto the classical Schur function in the variables $\xi_1, \ldots, \xi_n$ (see remark (iii) in section 3.5). Then they deduce the Littlewood–Richardson rule from an identity in $\mathbb{Z}[\mathsf{Pl}(A)]$ (see Theorem [6.4.5]).

(D.11g) Returning to section [6.3]; LLT define Schensted's Q-symbol. So for any $i \in A^*$, one defines the tableau $Q(i)$ (or more correctly the <u>tableau</u> $KQ(i)$) which is a byproduct of the sequence of tableaux $P(i_1), P(i_1 i_2), \ldots$ which is used to make $P(i)$; see the example (B.4c), or the example in LLT (page 7). By its construction, $Q(i)$ is what LLT call a "standard" tableau, i.e. if $i \in I(n,r)$, then the entries of $Q(i)$ are the numbers $1, 2, \ldots, r$ in some order. The shape of $Q(i)$ is the shape λ of $P(i)$. LLT prove the Robinson–Schensted theorem [6.3.1], which says that the map $\rho : i \mapsto (P(i), Q(i))$ induces a bijection from the set $I_\lambda(n,r)$ of all words i of given shape λ (see §C.1) to the set $\mathsf{Tab}(\lambda, A) \times \mathsf{STab}(\lambda)$. (In our notation, $\mathsf{Tab}(\lambda, A) = \mathcal{P}(\lambda)$ and $\mathsf{STab}(\lambda) = \mathcal{Q}(\lambda)$; see (D.6a) and (D.6b).) This is essentially the theorem (B.6a) which says the map Sch is bijective. It is proved in the same way, by constructing the inverse map ρ^{-1}.

The rest of section [6.3] is devoted to applications to representations of the symmetric group $S(n)$. A permutation σ of $\{1, \ldots, n\}$ is regarded as a word $\sigma = \sigma_1 \cdots \sigma_n$ of length n. Then G. de B. Robinson discovered and Schützenberger proved the theorem [6.3.3]: $Q(\sigma) = P(\sigma^{-1})$. LLT give a short

proof of this fact, and also generalize it to obtain, for any word i, a description of $Q(i)$ as $P(\sigma^{-1})$ for a certain permutation σ constructed from i (see [6.3.7]). Then a further generalization, gives them a generating function for the number d_λ of plactic classes of given weight λ (see [6.3.10]). Notice that d_λ appears in the "λ-rectangle" (D.6d).

(D.11h) In section [6.5], the set of all $i \in A^*$ for which $Q(i)$ is a given "standard" tableau Q is called a *coplactic class.* In our terminology (see section C.1) this is the $\approx$-class $I_\lambda(Q, \approx)$, where $\approx$ is the equivalence relation on A^* defined in (A.4b): $i \approx j$ means $Q(i) = Q(j)$. (LLT do not give a symbol for $\approx$.)

In order to give "structure" to the coplactic classes, LLT introduce three operations on words (which then induce linear operations on $\mathbb{Z}[A^*]$). For a given $c \in \{1, \ldots, n-1\}$, the LLT operators are called e_c, f_c, σ_c. We shall see in (D.11i) that e_c, f_c are just the Littelmann operators $\tilde{e}_c$, $\tilde{f}_c$ defined in section A.3. We do not have the operator σ_c in the Appendix, but it is used extensively in the latter part of LLT.

Proof of Theorem A. Theorem [6.5.1(i)] says that if θ is either e_c or f_c, then $Q(\theta i) = Q(i)$ for any word i such that $\theta i \neq 0$. This is Proposition (D.2b); it is the "if" part of Theorem A. The "only if" part follows from Proposition [6.5.2(i)]; one defines a graph Γ (called the *Littelmann graph* in section E.2) to have for vertices all words $i \in A^*$, with arrow $i \xrightarrow{c} j$ where $f_c i = j$. If i, j are such that $i \approx j$, i.e. if i, j are in the same coplactic class, then [6.5.2(i)] says that i, j are in the same connected component of Γ, which means that we connect i and j by a chain of links, each link being of the form either $\xrightarrow{c}$ or $\xleftarrow{c}$. But this is "only if" for Theorem A.

Proof of Proposition B. Theorem [6.5.1(ii)] says that LLT operators are compatible with the equivalence $\equiv$. For example, if $i, j \in A^*$ and if $i \equiv j$, then for any $c \in \{1, \ldots, n-1\}$ there holds $f_c(i) \equiv f_c(j)$. (This includes the statement $f_c(i) = 0$ if and only if $f_c(j) = 0$.) But this is essentially the Lemma (C.6c). To deduce Proposition B, we combine (C.6c) with (C.3p), which says that for any i there holds $i \equiv KP(i)$. However LLT have proved this in Proposition [6.2.3]. Therefore LLT have proved Proposition B.

(D.11i) We sketch the proof that the LLT operators e_c, f_c are the same as the Littelmann operators $\tilde{e}_c$, $\tilde{f}_c$, respectively. Let $c \in \{1, \ldots, n-1\}$, and keep this fixed. To calculate $\tilde{e}_c(i)$ and $\tilde{f}_c(i)$ for a given word i of length p, use the function $h_c^i(t)$ (see (A.3b)). This gives parameters M^i, q^i, $\overline{q}^i$, and these are sufficient to determine $\tilde{e}_c(i)$ and $\tilde{f}_c(i)$. Let us say that words i, j are *isologous* if M^i, q^i, $\overline{q}^i$ are equal to M^j, q^j, $\overline{q}^j$, respectively.

To calculate $h_c^i(t)$, one needs only the entries c, $c+1$ in i. We say that letters other than c, $c+1$ are *neutral.* In the example below $i = 235342233$ is a word in $I(5, 9)$, and $c = 2$. Our first move is to replace each neutral entry by the empty square, indicated by a "$\cdot$" in the third line of table D.1 below. Now we look for an *adjacent* $(c+1, c)$ *pair*, i.e. entries i_a, i_b of i

t	1	2	3	4	5	6	7	8	9
i_t	2	3	5	3	4	2	2	3	3
i_t	2	3	$\cdot$	3	$\cdot$	2	2	3	3
j_t	2	3	$\cdot$	$\cdot$	$\cdot$	$\cdot$	2	3	3
k_t	2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	3	3

Table D.1. Successive construction of isologous words.

such that $i_a = c + 1$, $i_b = c$, $a < b$, and i_z is neutral for all places z such that $a < z < b$, if there is any such place. In our example, (i_4, i_6) is an adjacent $(3, 2)$ pair. Now replace both entries i_a, i_b by neutral letters. It is (very) easy to see that the resulting word j is isologous to i.

We next look for adjacent $(c + 1, c)$ pairs in j; in the example, (j_2, j_7) is such a pair. Then "neutralize" this pair, etc. After a finite number of steps we reach a word k that contains no adjacent $(c + 1, c)$ pair. In this word, there may be r entries c, and they all occur before any of the s entries $c + 1$. (Either or both of r, s may be zero.) By construction k is isologous to i. But it is very simple to describe the function $h^k_c(t)$: starting from the left, it ascends by the r steps c, moves horizontally if there are some neutral entries between the last c and the first $c + 1$, then descends by the s entries $c + 1$. If $r = 0$ then $M^i = M^k = 0$, and if $s = 0$ we have $h^i_c(p) = h^k_c(p) = M^k$; if $r > 0$ then $q^i = q^k$ is the last place with entry c, and if $s > 0$ then $\overline{q}^i = \overline{q}^k$ is the place immediately before the first $c+1$. In the example given in table D.1, we have exactly one c at place 1 in k, and two $c + 1$'s at places 8, 9, respectively; hence $q^i = 1$ and $\overline{q}^i = 7$. We now have all that is needed to construct $\tilde{e}_c(i)$ and $\tilde{f}_c(i)$.

We leave it to the reader to compare our construction of $\tilde{e}_c$, $\tilde{f}_c$ with LLT's construction of their operators e_c, f_c, and to show that the two constructions are identical.

E

Tables

E.1 Schensted's decomposition of $I(3, 3)$

Let $n = 3$ and $r = 3$. For each $i \in I(3, 3)$, the Q-symbol $Q = Q(i)$ of i then contains each of the numbers 1, 2, 3 exactly once. We write $I(Q) = I(Q, \approx)$ for all these tableaux Q. Then

$$\begin{aligned} I(3,3) = \quad & I_{(300)} \quad \dot\cup \quad I_{(210)} \quad \dot\cup \quad I_{(111)} \\ = & I\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\right) \;\dot\cup\; I\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right) \cup I\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right) \;\dot\cup\; I\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right) . \end{aligned}$$

Table E.1 below contains, for each $\lambda \in \Lambda^+(3, 3)$, the tableaux $\psi^{(\lambda)}$, $\mathsf{Q}^{(\lambda)}$ (see (C.2g) and (C.2h)), the tableaux T_λ, Z_λ, and the words i^λ, i_λ obtained from these (see (D.1d) and (D.1e)).

λ	$\psi^{(\lambda)}$	$\mathsf{Q}^{(\lambda)}$	T_λ	i^λ	Z_λ	i_λ
$(3,0,0)$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 1 \\ \hline\end{array}$	1 1 1	$\begin{array}{\|c\|c\|c\|}\hline 3 & 3 & 3 \\ \hline\end{array}$	3 3 3
$(2,1,0)$	$\begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$	2 1 1	$\begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$	3 2 3
$(1,1,1)$	$\begin{array}{\|c\|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$	3 2 1	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$	3 2 1

Table E.1. Various data associated with $\lambda \in \Lambda^+(3, 3)$.

The elements of the sets $I(Q)$ with their P-symbols and Q-symbols are listed in table E.2.

λ = $(3,0,0)$

$P(i)$	i
$\begin{smallmatrix}1&1&1\end{smallmatrix}$	1 1 1
$\begin{smallmatrix}1&1&2\end{smallmatrix}$	1 1 2
$\begin{smallmatrix}1&2&2\end{smallmatrix}$	1 2 2
$\begin{smallmatrix}1&1&3\end{smallmatrix}$	1 1 3
$\begin{smallmatrix}2&2&2\end{smallmatrix}$	2 2 2
$\begin{smallmatrix}1&2&3\end{smallmatrix}$	1 2 3
$\begin{smallmatrix}2&2&3\end{smallmatrix}$	2 2 3
$\begin{smallmatrix}1&3&3\end{smallmatrix}$	1 3 3
$\begin{smallmatrix}2&3&3\end{smallmatrix}$	2 3 3
$\begin{smallmatrix}3&3&3\end{smallmatrix}$	3 3 3

$Q(i)$: $\begin{smallmatrix}1&2&3\end{smallmatrix}$

λ = $(2,1,0)$

$P(i)$	i	
$\begin{smallmatrix}1&1\\2&\end{smallmatrix}$	1 2 1	2 1 1
$\begin{smallmatrix}1&2\\2&\end{smallmatrix}$	2 2 1	2 1 2
$\begin{smallmatrix}1&1\\3&\end{smallmatrix}$	1 3 1	3 1 1
$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	2 3 1	2 1 3
$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	1 3 2	3 1 2
$\begin{smallmatrix}1&3\\3&\end{smallmatrix}$	3 3 1	3 1 3
$\begin{smallmatrix}2&2\\3&\end{smallmatrix}$	2 3 2	3 2 2
$\begin{smallmatrix}2&3\\3&\end{smallmatrix}$	3 3 2	3 2 3
$Q(i)$	$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$

λ = $(1,1,1)$

$P(i)$	i
$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	3 2 1

$Q(i)$: $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$

Table E.2. P-symbols and Q-symbols of the words $i \in I(3,3)$.

E.2 The Littelmann graph $I(3,3)$

Let n, r be positive integers. Following Littelmann [35, §2], we define the structure of a graph on $I(n,r)$ by saying that $i, j \in I(n,r)$ are connected

by an edge if there exists an element $c \in \{1, \ldots, n-1\}$ such that $\tilde{f}_c(i) = j$ or $\tilde{f}_c(j) = i$.

This graph is the *Littelmann graph* (it is the undirected form of the directed graph Γ in (D.11i)). The connected components of this graph are precisely the *coplactic*, or $\approx$*-classes* $I(Q, \approx)$, where Q is a standard tableau with entries $1, \ldots, r$ in some order. This follows from Theorem A (and (A.3g)(5), where we have seen that $\tilde{f}_c(i) = j$ if and only if $\tilde{e}_c(j) = i$). Therefore we can use these tableaux to label the connected components of the Littelmann graph.

For $n = r = 3$, the Littelmann graph is shown in table E.3. In this display, two words i, j are connected by a (directed) edge labelled by 1 or 2 according as $\tilde{f}_1(i) = j$ or $\tilde{f}_2(i) = j$.

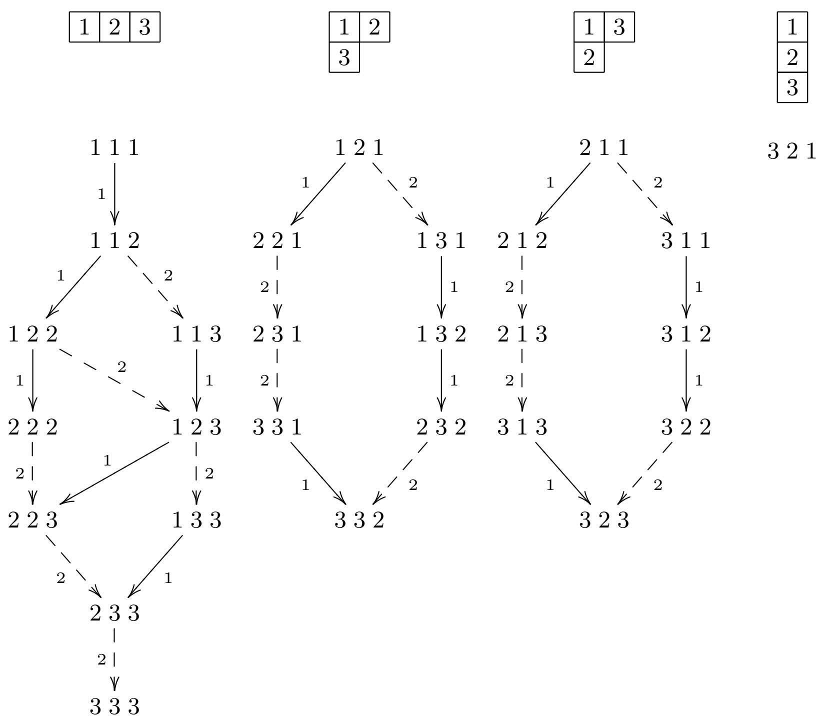

Table E.3. The four Littelmann graphs in $I(3,3)$.

Note that, if $\lambda \in \Lambda^+(n,r)$ and $Q = \mathsf{Q}^{(\lambda)}$, then i^λ is at the top and i_λ is at the bottom of the corresponding connected component of the Littelmann graph (see table E.1 and (D.2d), (C.2c)).

Index of symbols

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Index

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