Epilogue

The main goal of this lecture was to introduce to the techniques of impredicative ordinal analysis. The axiom system for noniterated inductive definitions served as an simple example for an impredicative theory. Of course, this is just a first step into the world of impredicativity. The most straightforward way to obtain more complicated axiom systems is to consider iterated inductive definitions. These theories are treated in [BFPS]. There it is also shown how these theories are connected to subsystems of classical analysis, i.e. second order number theory with comprehension. The real fascination of impredicative systems, however, becomes not visible till one considers subsystems of set theory. Pitily there are no text books in this area. The best references here are the papers [1979] and [1986] of *G.Jäger*. An impressing variety of subsystems of set theory is presentend by *M.Rathjen* [1989]. This paper is a good example for the interplay between recursion theoretical, set theoretical, model theoretical and proof theoretical methods in the ordinal analysis of subsystems of set theory. A survey of these methods is given in Pohlers [1990]. A text book with the title "Admissible Proof Theory" is in preparation and will appear in the Springer series "Ergebnisse der Mathematik und ihrer Grenzgebiete".

We will close this book by giving some comments on the 'constructive' meaning of ordinal analysis. In §14 we already indicated that it is sufficient for an ordinal analysis to regard only recursive proof trees of the semiformal system. This can be used to show

(1) $T \vdash F \quad \Leftrightarrow \quad Z_1 + TI(<|T|, X) \vdash F$

for all Π^1_1-sentences F. Here $TI(<|T|,X)$ means that we allow induction along all initial segments of the primitive recursive wellordering $\prec_T$ of ordertype $|T|$ which has been obtained from the notation system used in the ordinal analysis of T. A detailed proof is in Pohlers [BFPS]. The axiom system **PRA** for 'primitive recursive analysis' is essentially the system Σ^0_1 - **INDR** of exercise 3.15.6. (often considered as second order theory but without strong comprehensions). By the (formal) reflection principle (REF(T)) for an axiom system T one denotes the principle

(REF(T)) $Bew_T(\ulcorner F \urcorner) \rightarrow F$

where Bew_T is a provability predicate for T. (Π^0_2 - REF(T)) is the scheme (REF(T)) with F restricted to Π^0_2 - sentences, i.e. sentences of the form $\forall \vec{x}\, \exists y\, G(\vec{x},y)$ with G quantifier free. For a primitive recursive order relation $\prec$ on the natural

numbers we denote by PRWO($\prec$) that there are no primitive recursive infinitely $\prec$-descending sequences. Then we have

$$\text{(2)} \qquad \mathbf{PRA} \vdash \mathrm{PRWO}(\prec) \leftrightarrow (\Pi^0_2 - \mathrm{REF}(Z_1 + \mathrm{TI}(\prec, X))).$$

This is considered to be a folklore result of proof theory. Its proof needs a principle known as 'continuous cut elimination' originally developed by G.E.Mints. The most beautiful proof has been given by W.Buchholz in [1988a]. From (1) and (2) it already follows that **PRA** + PRWO($\prec_T$) proves the consistency of T. Moreover the theory **PRA** has a beautiful computational aspect. It has primitive recursive Π^0_2 – Skolem functions, i.e. if $\mathbf{PRA} \vdash \forall \vec{x}\, \exists y\, G(\vec{x}, y)$ for a quantifier free formula $G(\vec{x},y)$, then there is a primitive recursive function f such that $\mathbb{N} \models G(\vec{x}, f(\vec{x}))$. This result can be extended to **PRA** + PRWO($\prec$) in so far that this theory has Skolem functions which can be obtained from the basic functions C^n_k, P^n_k and S by substitution, primitive recursion (cf. 1.1.) and the $\prec$-descending μ-operator which for a given n+1-ary function f searches for the value

$$(\mu_\prec f)(\vec{x}) := \min\{y: (\neg f(\vec{x}, y+1) \prec f(\vec{x}, y))\}$$

i.e. $\mu_\prec f(\vec{x})$ computes the length of a $\prec$-descendent sequence $f(\vec{x}, 0) \succ f(\vec{x}, 1) \succ \succ f(\vec{x}, 2) \succ \ldots$. The class of these functions, the $\prec$-descendent functions, can also be obtained by $\prec$-recursion, i. e. using the scheme

$$f(\vec{x}, y) = \begin{cases} h(\vec{x}, f(\vec{x}, g(\vec{x}, y))) & \text{if } g(\vec{x}, y) \prec y \\ k(\vec{x}, y) & \text{otherwise} \end{cases}$$

in addition to substitution and primitive recursion. The functions can also be characterized using the Hardy hierarchy of computable functions which is given by

$$H_0(x) = x$$
$$H_{\alpha+1}(x) = H_\alpha(x+1)$$
$$H_\lambda(x) = H_{\lambda[x]}(x) \text{ for limit ordinals } \lambda$$

where $\{\lambda[n] : n < \omega\}$ is a fundamental sequence for λ, i.e. $\sup\{\lambda[n] : n < \omega\} = \lambda$ and $\lambda[n] < \lambda[n+1]$ for all $n < \omega$. It can be shown that the $\prec$-descending functions, where $\prec$ is an initial segment of $\prec_T$, are all majorizable by the function $H_{|T|}$. From (1) and (2) we obtain a characterization of the Π^0_2 – Skolem functions of the theory T. If $T \vdash \forall \vec{x}\, \exists y\, G(\vec{x}, y)$, then we obtain $Z_1 + \mathrm{TI}(\prec, X) \vdash \forall \vec{x}\, \exists y\, G(\vec{x}, y)$ for an initial segment $\prec$ of $\prec_T$ by (1). This entails

$$\mathbf{PRA} \vdash \mathrm{Bew}_{(Z_1 + \mathrm{TI}(\prec, X))}(\ulcorner \forall \vec{x}\, \exists y\, G(\vec{x}, y) \urcorner)$$

which by (2) implies $\mathbf{PRA} + \mathrm{PRWO}(\prec) \vdash \forall \vec{x}\, \exists y\, G(\vec{x}, y)$. Hence T has Π^0_2-Skolem functions which are $\prec$- descendent for initial segments of $\prec_T$. A recursive

function f with index e is provably recursive in T, if $T \vdash \forall \vec{x} \exists y\, T(e,\vec{x},y)$, where T denotes the Kleene predicate), i.e. if T proves f to be total. The provably recursive functions of T are thus Π_2^0 - Skolem functions and therefore majorizable by $H_{|T|}$.

Since (1) is a side result of the method of local predicativity (cf. [BFPS]) we obtain as as a corollary of the (proof of the) ordinal analysis for T a characterization of the Π_2^0 - Skolem functions, and thus also of the provable recursive functions of T. This characterization may be considered as a very constructive one since the wellorderings $\prec_T$ obtained from the ordinal analysis are so simple that it causes no problems to implement them on a computer. (For the system obtained in chapter III this has been done by K.Stroetmann in Münster). Therefore there is a program, implementable on a real computer, computing the provably recursive functions of T. As a matter of fact, however, these functions increase so incredibly fast that they only are computable for very small arguments.

The above stated facts are scattered in the literature. The best reference here is Takeuti's book [1987 CH.2 §12] where he proves similar results for the case of pure number theory.

A textbook treating this material systematically is still a challenge.

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Abbreviations:	
AMLG	Archiv für Mathematische Logik und Grundlagenforschung
APAL	Annals of Pure and Applied Logic (vorher:Annals of Mathematical Logic)
BFPS	Buchholz, Feferman, Pohlers, Sieg: Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies, LNM 897, 1981
HB	Handbook of Mathematical Logic
HF	L. Harrington, M.Morley, A. Scedrov and S.G. Simpson (eds.) Harvey Friedman's research on the foundations of mathematics, NH
	J. Barwise (ed.), North Holland, Amsterdam 1977
IPT	Intuitionism and Proof Theory, Proceedings of the summer conference at Buffalo, N.Y. 1968 A. Kino, J. Myhill, R.E. Vesley (eds.), Amsterdam-London 1970
JSL	Journal of Symbolic Logic
LMPS III	Logic, Methodology and Philosophy of Science III, B. van Rootselaar, J.F. Staal (eds.), Amsterdam 1968
LMPS VI	Logic, Methodology and Philosophy of Science VI L.J.Cohen, J.Los, H. Pfeiffer, K.-P. Podewski, Amsterdam 1982
LNM	Lecture Notes in Mathematics
MA	Mathematische Annalen
NH	North-Holland, Amsterdam
SDBA	Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse
SR	Stanford Report
ZML	Zeitschrift für mathematische Logik und Grundlagenforschung der Mathematik

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