APPENDIX: THE ARF INVARIANT

Our discussion of the Arf invariant is lifted directly from [**R-S**, Appendix]. Let V be a vector space over $Z/2$ equipped with an inner product, i.e. $x \cdot y \in Z/2$, $x \cdot y = y \cdot x$, and given a linear map $\lambda : V \to Z/2$, there exists $y \in V$ such that $\lambda(x) = x \cdot y$ for all $x \in V$. $V = H_1(F^2; Z/2)$ with the standard intersection form is the obvious example.

Let $q : V \to Z/2$ be a quadratic function, i.e. q must satisfy

$$q(x+y) = q(x) + q(y) + x \cdot y \quad (2) \quad \text{for all } x, y \in V. \tag{*}$$

Note that applying (*) to $0+0$ and to $x+x$ shows that $q(0) = 0$ and $x \cdot x = 0$. Choose x and a dual y to x so that $x \cdot y = 1$ (y corresponds to the linear map which sends x to 1 and all other elements of V to zero). Then the hyperbolic pair $H = \begin{array}{c} x \\ y \end{array}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ defines a unimodular subspace and hence an orthogonal complement $H^{\perp}$. We continue to split off hyperbolic pairs until nothing is left and we have shown that $V = H_1 \oplus H_2 \oplus \cdots \oplus H_n$.

It is easy to verify using (*) that on the three non-zero elements of H, x, y, and $x+y$, q is either always one (this case is called $H^{1,1}$) or q is zero on two of the elements and one on the third (this case is called $H^{0,0}$ because we can choose a basis of two elements on which q is zero). Thus $(V, q) = \overset{r}{\oplus} H^{0,0} \overset{s}{\oplus} H^{1,1}$, where $r + s = 2n$.

Given a basis x_1, y_1, x_2, y_2 for $H^{0,0} \oplus H^{0,0}$ we can choose a new basis $x_1 + y_1 + x_2$, $x_1 + y_1 + y_2$, $x_1 + x_2 + y_2$, $y_1 + x_2 + y_2$ on each element of which q is 1, so it follows that $H^{0,0} \oplus H^{0,0} \cong H^{1,1} \oplus H^{1,1}$. Thus (V, q) is isomorphic to either $\overset{n}{\oplus} H^{0,0}$ or $\overset{n-1}{\oplus} H^{0,0} \oplus H^{1,1}$ depending on whether s is even or odd.

Finally, $\overset{n}{\oplus} H^{0,0}$ is not isomorphic to $\overset{n-1}{\oplus} H^{0,0} \oplus H^{1,1}$ because out of the 2^{2n} elements in V, q is zero on $2^{2n-1} + 2^{n-1}$ of them in $\overset{n}{\oplus} H^{0,0}$ and is zero on $2^{2n-1} - 2^{n-1}$ of them in $\overset{n-1}{\oplus} H^{0,0} \oplus H^{1,1}$.

We define the Arf invariant of (V, q), $\mathrm{Arf}(V, q) \in Z/2$, to be zero if $(V, q) \cong \overset{n}{\oplus} H^{0,0}$ and one if $(V, q) \cong \overset{n-1}{\oplus} H^{0,0} \oplus H^{1,1}$. Note that the Arf invariant is additive under direct sum.

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