APPENDIX

In this appendix we are going to describe briefly the connection between σ-positive functions on the one hand and projective cyclic unitary representations on the other hand. Suppose a projective continuous unitary representation $g \mapsto U_g$ with $U_{g_1} U_{g_2} = \sigma(g_1,g_2) U_{g_1 g_2}$ is given. Let x, with $\|x\|=1$, be a cyclic vector for this representation. Then we have

Lemma 1:

Let $f(g) := \langle U_g x, x\rangle$ be the expectation value of U_g with respect to x. Then f is σ-positive.

Proof:
The continuity of f is obvious, and we immediately obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \; \sigma(g_j^{-1}, g_i) f(g_j^{-1} g_i) =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \; \langle \sigma(g_j^{-1}, g_i) U_{g_j^{-1} g_i} x, x\rangle =$$

$$\| \sum_{i=1}^{n} \alpha_i U_{g_i} x \|^2 \geq 0 \; . \qquad \text{q.e.d.}$$

Remark:

If $\sigma \equiv 1$ then we obtain the well known fact about positive definite functions.

Rather more interesting is the fact that a converse to Lemma 1 exists. We recall that a central extension G_σ of G is defined by

$$G_\sigma : = G x S^1 \quad \text{as a set, with}$$

$$(g_1,t_1)\cdot(g_2,t_2) : = (g_1 g_2, \sigma(g_1,g_2) t_1 t_2)$$

where G_σ (cf. I.3) is furnished with the product topology (note that we assumed σ to be continuous!). This central extension allows us to consider "genuine" representations of G_σ instead of projective rep-

resentations of G .
Indeed, let $g \mapsto U_g$ be as above; we set

$$V_{(g,t)} := t\, U_g$$

and obtain as is readily verified a "genuine" representation for G_σ with cyclic vector x . Thus we can state

Lemma 2:

Let f be σ-positive on G . Then there exists a projective representation $g \mapsto U_g$ with $U_{g_1} U_{g_2} = \sigma(g_1,g_2)\, U_{g_1 g_2}$ and cyclic vector x satisfying

$$f(g) = \langle U_g x, x\rangle .$$

Moreover U_g and x are determined up to unitary equivalence.

Proof:
We first set

$$f_1(g,t) := t\, f(g) \; .$$

Then f_1 is positive definite on G_σ , since we have:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j\, f_1((g_j,t_j)^{-1} \cdot (g_i,t_i)) =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j\, f_1(g_j^{-1} g_i, t_j^{-1} t_i\; \sigma(g_j^{-1}, g_i)) =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i t_i)(\overline{\alpha_j t_j})\, \sigma(g_j^{-1}, g_i)\, f(g_j^{-1} g_i) \geq 0 \; .$$

(The last inequality follows since f is by assumption σ-positive.)

The well-known theorem concerning positive definite functions tells us now that there exists a representation $(g,t) \mapsto V_{(g,t)}$ of G_σ with cyclic vector x and

$$f_1(g,t) = \langle V_{(g,t)} x, x\rangle$$

where V and x are determined up to unitary equivalence. Since $f_1(g,t)=t\cdot f(g)$ we must have $V_{(g,t)}=t\cdot U_g$ for some unitary operator U_g . One verifies immediately that

$$U_{g_1}U_{g_2} = \sigma(g_1,g_2)U_{g_1 g_2} \quad \forall\, g_1,g_2 \,. \qquad \text{q.e.d.}$$

Remark:

The connection between σ-positive functions and projective representations was probably first recognized by Araki. Since it is of crucial importance in these notes we have given the description here again.

REFERENCES

[1] Araki, H.: Factorizable representation of Current Algebra-non commutative extension of Levy Kinchin formula and cohomology of a solvable group with values in a Hilbert space. RIMS, 41 (Revised Edition), Kyoto (1969), (Preprint)

[2] Bargmann, V.: Irreducible unitary representations of the Lorentz group. Annals of Mathematics 48, (1947), p. 568-640

[3] Delorme, P.: 1-cohomologie et produits tensoriel continus de representations. These, L'Université de Paris VI (1975)

[4] Erven, J.: Über Kozyklen erster Ordnung von $SL(2;\mathbb{R})$. Dissertation, TU München (1979)

[5] Erven, J., Falkowski, B.-J.: A note on the continuous second cohomology group for $SL(2;\mathbb{R})$. (Unpublished)

[6] Falkowski, B.-J.: Factorizable and infinitely divisible PUA representations of locally compact groups. J. of Math. Phys., 15, (1974), p. 1060-1066

[7] Falkowski, B.-J.: First order cocycles for $SL(2;\mathbb{C})$. J. of the Ind. Math. Soc., 41, (1977), p. 245-254

[8] Falkowski, B.-J.: A note on the first order cohomology for $SL(n;\mathbb{C})$. J. of the Ind. Math. Soc., 42, (1978), p. 105-107

[9] Falkowski, B.-J.: Infinitely divisible positive definite functions on $SO(3) \circledS \mathbb{R}^3$. p. 111-115 in: Probability measures on groups. Lecture Notes in Mathematics, Vol. 706, Springer-Verlag, Berlin (1979)

[10] Falkowski, B.-J.: Die Kohomologie gewisser Gruppenerweiterungen und faktorisierbare Darstellungen. Unpublished manuscript, Hochschule der Bundeswehr München (1979)

[11] Guichardet, A.: Symmetric Hilbert spaces and related topics. Lecture Notes in Mathematics, Vol. 261, Springer-Verlag, Berlin (1972)

[12] Helgason, S.: Differential geometry and symmetric spaces. Academic Press, New York and London (1962)

[13] Lang, S.: $SL_2(\mathbb{R})$. Addison-Wesley Publishing Company, Reading, Mass. (1975)

[14] Lipsman, R.L.: Group representations. Lecture Notes in Mathematics, Vol. 388, Springer-Verlag, Berlin (1974)

[15] Mackey, G.W.: Induced representations of groups and quantum mechanics. Benjamin, New York (1968)

[16] Mackey, G.W.: The theory of unitary group representations. Chicago Lecture Notes (unpublished manuscript)

[17] Newman, C.M.: Ultralocal quantum field theory in terms of currents. Comm. in Math. Phys., 26, (1972), p. 169-204

[18] Parry, W., Schmidt, K.: A note on cocycles of unitary representations. Proc. Americ. Math. Soc. 55 (1976), p. 185-190

[19] Parthasarathy, K.R.: Probability measures on metric spaces. Academic Press, New York (1967)

[20] Parthasarathy, K.R., Schmidt, K.: Positive definite kernels, continuous tensor products, and central limit theorems of probability theory. Lecture Notes in Mathematics, Vol. 272, Springer-Verlag, Berlin (1972)

[21] Parthasarathy, K.R., Schmidt, K.: Factorizable representations of current groups and the Araki-Woods imbedding theorem. Acta Math. 128, (1972), p. 53-71

[22] Parthasarathy, K.R., Schmidt, K.: A new method for constructing factorisable representations for current groups and current algebras. Comm. in Math. Phys., 50, (1976), p. 167-175

[23] Pinczon, G., Simon, E.: On the 1-cohomology of Lie groups. Letters in Math. Phys. 1, (1975), p. 83-91

[24] Schmidt, K.: Algebras with quasilocal structure and factorizable representations. p. 237-251 in: Mathematics of Contemporary Physics. Academic Press, New York (1972)

[25] Simms, D.J.: Lie groups and quantum mechanics. Lecture Notes in Mathematics, Vol. 52, Springer-Verlag, Berlin (1968)

[26] Streater, R.F.: Current commutation relations, continuous tensor products and infinitely divisible group representations. Rend. Sci. Int. Fisica E. Fermi, XI, (1969), p. 247-263

[27] Takahashi, R.: Sur les fonctions sphériques et la formule de Plancherel dans le group hyperbolique. Japan J. Math., 31, (1961), p. 55-90

[28] Vershik, A.M., Gelfand, J.M., Graev, M.J.: Representations of the group SL(2;R) , where R is a ring of functions. Russ. Math. Surv., 28, (1973), p. 87-132

[29] Vershik, A.M., Gelfand, J.M., Graev, M.J.: Irreducible representations of the group G^X and cohomologies. Funct. Anal. and Appl. (Translation), Vol. 8, No. 2, p. 67-69 (1974)

[30] Warner, G.: Harmonic analysis on semi-simple Lie groups I. Springer-Verlag, Berlin (1972)

[31] Stasheff, J.D.: Continuous cohomology of groups and classifying spaces. Bull. of the AMS, Vol. 84, No. 4, (1978), p. 513 - 530.

[32] Delorme, P.: 1-cohomologie des représentations unitaires des Groupes de Lie semi-simples et résolubles. Bull. Soc. math. France, 105, 1977, p. 281 - 336

[33] Dixmier, J.: C-Algebras, North-Holland, 1977

[34] Faraut/Harzallah: Distances hilbertiennes invariantes sur un espace homogéne. Ann. Inst. Fourier

[35] Gangolli: Positive Definite Kernels on Homogeneous Spaces and Certain Stochastic Processes related to Levy's Brownian Motion of Several Parameters. Ann. I.H.P.B. Vol. III, No. 2 (1967)

[36] Warner, G.: Harmonic analysis on semi-simple Lie groups II. Springer-Verlag, Berlin (1972)

[37] Wang, S.P.: The Dual of Semi-simple Lie Groups. Am. J. Math. (1969), p. 921 - 937