Addenda: Moufang Buildings

When the typing of these Notes was already completed, it occurred to the author that some results could be improved and some proofs simplified by a more systematic usage of §4 and the end of § 13. However, a revision of the text on that basis, which would imply the retyping of large sections of the Notes and a further delay in their publication, does not seem worthwhile, all the less as the observations made open a new approach, not yet completely worked out, to the classification problem (see below). The purpose of these addenda is to briefly outline what can be done.

Let Δ be a building of spherical type and, for every root a of an apartment of Δ, let U_a denote the group of all automorphisms of Δ fixing elementwise the stars of all elements of codimension 1 contained in $a - \partial a$. We say that Δ is Moufang if, for every a, the group U_a permutes transitively the apartments containing a. If the type of Δ has no direct factor of type A_1, this means that Δ is the building of a BN-pair with a system of subgroups (U_a) satisfying the conditions of 13.36 (cf. 13.37e)). A generalized polygon is said to be Moufang if its flag complex is a Moufang building; in the case of projective planes, this coincides with the usual notion, hence the terminology.

Our main observation is that the following statement is an easy corollary of proposition 4.16:

(1) Every building of spherical, irreducible type and rank $\geqslant 3$ is Moufang.

As a result:

(2) The stars of all elements of such a building are Moufang.

This generalizes proposition 7.11.

The assertion (1) suggests that, in order to extend our results on classification to the rank 2 case, a natural thing to do would be to impose the Moufang condition. One may conjecture that the only Moufang buildings of spherical, irreducible type and rank $\geqslant 2$ are the buildings associated with known groups: classical groups, algebraic simple groups and "mixed groups" of the type described in 10.3.2 (in the case of B_2, the construction of 10.3.2 must be slightly generalized still). This is known to be true for projective planes (cf. 5.12 and [58]). The progress recently made on that conjecture gives reasonable hope that it might be established soon. Among other things, the author has proved it for the types $G_2^{(5)}$ and $G_2^{(6)}$ (cf. 2.17). For $G_2^{(5)}$, this means that there exists no Moufang generalized pentagon; in view of (2), it follows that

(3) <u>There exists no building of type</u> H_3 <u>or</u> H_4 (cf. 2.17).

Thus, in view of the results of these Notes, the above conjecture is true for buildings of rank $\geqslant 3$. On the other hand, once all Moufang buildings of type B_2 are determined (which seems to be within reach), it will be possible to prove the results on rank $\geqslant 3$ in a uniform way, using 4.1.2 or rather 13.39, which is probably more efficient; this would considerably simplify our §§ 8 and 9.

So much for the classification problem. But the results of § 13 also throw an interesting light on the isomorphism problem for Moufang buildings. Indeed, let (G, B, N) and (G', B', N') be two BN-pairs having "the same" Weyl group, Δ and Δ' their buildings, Φ their common root systems, Φ^+ the set of "positive roots", $(U_a)_{a \in \Phi^+}$ (resp. $(U'_a)_{a \in \Phi^+}$) a system of subgroups of B (resp. B') satisfying the conditions of 13.36 and U (resp. U') the subgroup generated by all U_a (resp. U'_a). Suppose that G and G' operate faithfully on Δ and Δ'. Then, it readily follows from 13.37b) and 13.39 that:

<u>there exists a type-preserving isomorphism of</u> Δ <u>onto</u> Δ' <u>if and only if there exists an isomorphism</u> $\varphi : U \to U'$ <u>such that</u> $\varphi(U_a) = U'_a$ <u>for all</u> a ;

every automorphism of U _normalizing each_ U_a _induces_ (_by_ 13.39) _an automorphism of_ Δ , _and the automorphisms thus obtained, together with the automorphisms induced by the elements of_ G , _generate the full group_ Spe Δ (cf. 5.1).

This provides an elementary method for proving (case by case) the theorem 5.8 without using [10]. In fact, one can even, for groups of rank $\geqslant 2$ (including the Ree groups of type ${}^2F_4$ and the "mixed groups" of 10.3.2, which are not covered by [10]), deduce the automorphisms of the groups from the automorphisms of the buildings, determined as above. Finally notice that, applied to the polar spaces of hermitian and pseudo-quadratic forms, the present method gives new and simpler proofs of the theorem 8.6 (at least for non-degenerate forms) and the proposition 10.10.

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Index of notations

Index

Complements to the second printing (July 1985)

We shall briefly indicate a few recent developments in the area of the present Lecture Notes. For technical reasons, the length of these complements is restricted to four pages; a more or less complete survey is therefore out of the question.

The rank 2 case. The main result of these Notes was that buildings of irreducible spherical type and rank $\leqslant 3$ are essentially those - let us call them "algebraic" - that come from algebraic and classical groups. This is no longer true in rank 2 but, in the Addenda above, it was conjectured that the Moufang condition introduced there is precisely what is needed to characterize the algebraic buildings of rank 2. For a precise statement, i. e. a conjectural list of all Moufang generalized m-gons, see [127]. That conjecture is now proved for $m \neq 4$ (cf. [125], [137], [127], [114], [133]). For $m = 4$, only very partial results are published ([114], [126]) but a lot more is known; the finite case is completely settled ([115]). Alternative formulations of the Moufang condition, e. g. in terms of configurations, have been studied (see [119], chap. 9, for $m = 4$ and [120] for $m = 6$). It is not known whether there exist finite non-Moufang hexagons or octagons. On rank 2 buildings in general, which include projective planes, there is a huge literature; let us just mention the standard source [119] for what concerns generalized quadrangles.

Affine buildings. Using the classification of buildings of spherical type and rank $\leqslant 3$ established in this volume, one can also classify the buildings of affine type and rank $\leqslant 4$ (cf. [135]). They are essentially the affine buildings coming from algebraic or classical groups over local fields or skew fields (cf. [14], [108], [129]). As in the case of spherical buildings of rank 2, a complete classification of the affine buildings of rank 3 is out of the question (cf. [121] and the next section).

Other types. Let M be a Coxeter matrix. A "free construction" (unpublished), generalizing that of [128] and using the main result of [132], shows that if the Coxeter diagram of M has no subdiagram of type A_3, C_3 or H_3, there exist hosts of non-isomorphic buildings and BN-pairs of type M. For buildings, another construction, due to

M. Ronan [121], also based on [132] and applicable under the same restriction on M as above, gives a better insight into the variety of existing buildings. It is likely that a refinement of Ronan's method will provide a constructive approach to all buildings an show that buildings of type M exist if and only if the diagram of M has no subdiagram of type H_3 (this refers to joint work of M. Ronan and the author, in progress; for a group-theoretic counterpart, see [136], théorème 2). If all coefficients of M are 2, 3, 4, 6 or ∞, Kac-Moody groups provide many buildings and BN-pairs "over arbitrary fields" (cf. [134] and its bibliography).

Geometries of diagrams. These are at the origin of the notion of building (cf. [81], [130], [132]) and the source of fruitful generalizations. They have gained new interest with the discovery of sporadic simple groups and the classification of finite simple groups. See [109], [110], [123], [124], [107], and their lists of references.

Chamber transitive automorphism groups. Generators and relations. The definition of groups with BN-pairs by amalgamation of parabolic subgroups of rank 2 (cf. § 13 above and [131]) extends to arbitrary chamber transitive automorphism groups of buildings (cf. [136], proposition 3). Such groups have been much studied lately. In the finite "algebraic" case, they have all been classified ([122]), and for discrete automorphism groups of locally finite affine buildings, they will probably soon be (by work in progress of W. Kantor, P. Köhler, R. Liebler, Th. Meixner, F. Timmesfeld, M. Wester and the author; cf. for instance [118], [123], [124], [138] and references given there). For related results, see also [71], [113], [117].

The point-line approach. A building of rank n can be viewed as a collection of objects (the vertices) belonging to n types, with an incidence relation. In most cases, two types of objects, usually called points and lines, suffice to determine the whole structure. In the case of polar spaces (type C_n), an elegant set of axioms has been given by F. Buekenhout and E. Shult [111]. This line of thought has been pursued later on by several authors (cf. e. g. [112], [116], and their lists of references).

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