Appendix

ITERATED SOUSLIN FORCING AND $\mathcal{P}(\omega)$

In [Jn Jo], Jensen and Johnsbråten give a generic construction of a non-constructible Δ^1_3 set of integers by means of an ω-iteration of Souslin forcing. Their construction is a specialisation of an earlier construction which Jensen developed in order to show that iterated Souslin forcing over L must introduce new reals. For the convenience of the reader, and because the construction in [Jn Jo] is (necessarily) fairly cumbersome, we outline here a simplification which just gives the above result.

In L, we define a sequence $\langle T^n \mid n < \omega \rangle$ of ω_1-trees such that:

(i) T^0 is Souslin;

(ii) if b_0 is an L-generic branch of T^0, then we can canonically define a subtree $\hat{T}^1$ of T^1 in $L[b_0]$ such that $\hat{T}^1$ is Souslin in $L[b_0]$;

(iii) if b_1 is an $L[b_0]$-generic branch of $\hat{T}^1$, then we can canonically define a subtree $\hat{T}^2$ of T^2 in $L[b_0,b_1]$ such that $\hat{T}^2$ is Souslin in $L[b_0,b_1]$;

(iv)...$<(\omega)$ similar;

(ω) if $\langle b_0, b_1, \dots \rangle$ is any such sequence of branches of the respective trees $T^0, \hat{T}^1, \dots$, then $L[\langle b_0, b_1, \dots \rangle] = L[a]$ for some $a \subseteq \omega$.

This shows that, regardless of what we do at limit stages, if we try to obtain SH by forcing over L, iteratively destroying Souslin trees, if the method we use to destroy Souslin trees results in their being given a branch, then the iteration must introduce new reals.

As we proceed, given $x \in T^n$ we shall define a subtree t_x of T^{n+1}, such that:

(i) t_x is a $ht(x)$-tree;

(ii) $x \leq_n y \to t_x = t_y | ht(x)$;

(iii) if x, y are incomparable in T^n and z is the largest $z \leq_n x, y$ in T^n, then $t_x \cap t_y = t_z$ (unless $ht(z) = 0$, when $t_x \cap t_y = \{\langle n+1 \rangle\}$) ;

(iv) $T^{n+1} = \bigcup_{x \in T^n} t_x$.

Each T^n_α will consist of α-sequences of natural numbers, and the ordering, $\leq_n$, will be the usual one. We define $T^0|1$ first, then $T^1|1$, then $T^2|1, \ldots$, then $T^0|2$, then $T^1|2$, then $T^2|2, \ldots$, etc.

Fix some canonical, recursive bijection $\langle\!\langle -,- \rangle\!\rangle : \omega \times \omega \leftrightarrow \omega$.

<u>Stage 0</u>. For each $n \in \omega$, set $T^n_0 = \{\langle n \rangle\}$, $t_{\langle n \rangle} = \emptyset$.

<u>Stage 1</u>. For each $n \in \omega$, set $T^n_1 = \{\langle n, i \rangle \mid i \in \omega\}$, $t_{\langle n,i \rangle} = \{\langle n+1 \rangle\}$.

<u>Stage $\alpha + 2$</u>. For each $n \in \omega$, set $T^n_{\alpha+2} = \{x ^\frown \langle i \rangle \mid x \in T^n_{\alpha+1} \ \& \ i \in \omega\}$, and for $x \in T^n_{\alpha+1}$, $i \in \omega$, set $t_{x ^\frown \langle i \rangle} = t_x \cup \{y ^\frown \langle \langle\!\langle i, j \rangle\!\rangle \rangle \mid y \in t_x \ \& \ ht(y) = \alpha \ \& \ j \in \omega\}$.

<u>Stage α, $\lim(\alpha)$</u>. We first define T^0_α. Let $\eta = \eta_{0,\alpha}$ be least such that $T^0|\alpha \in L_\eta$, $L_\eta \models ZF^-$, and $|\alpha|^{L_\eta} = \omega$. For each $x \in T^0|\alpha$, let b_x be the $<_L$-least L_η-generic branch of $T^0|\alpha$ through x, and let $T^0_\alpha = \{\bigcup b_x \mid x \in T^0|\alpha\}$. Assume now that T^n_α is defined. Define t_x for $x \in T^n_\alpha$ by $t_x = \bigcup_{y <_n x} t_y$. Let $\eta = \eta_{n+1,\alpha}$ be least such that $T^n|\alpha+1$, $T^{n+1}|\alpha$, $\langle t_x \mid x \in T^n_\alpha \rangle \in L_\eta$, $L_\eta \models ZF^-$, and $|\alpha|^{L_\eta} = \omega$. For each $x \in T^n_\alpha$, define a poset $\mathbb{P} = \mathbb{P}_x$ in L_η as follows. The elements of $\mathbb{P}$ are finite maps f with $dom(f) \subseteq \omega$ and $ran(f) \subseteq t_x$, such that $i, j \in dom(f) \to ht(f(i)) = ht(f(j))$. The ordering is defined by $f' \leq f \leftrightarrow dom(f') \supseteq dom(f) \ \& \ (\forall i \in dom(f))(f'(i) \geq_{n+1} f(i))$. Let $G = G_x$ be the $<_L$-least L_η-generic subset of $\mathbb{P}$. Set $b^x_i = \{f(i) \mid f \in G\}$, each $i \in \omega$. Clearly, each b^x_i is a branch of t_x, $i \neq j \to b^x_i \neq b^x_j$, and $t_x \subseteq \bigcup_{i \in \omega} b^x_i$. Set $T^{n+1}_\alpha = \{\bigcup b^x_i \mid x \in T^n_\alpha \ \& \ i \in \omega\}$.

Stage $\alpha+1$, $\lim(\alpha)$. For each $n \in \omega$, set $T^n_{\alpha+1} = \{x^\frown\langle i\rangle \mid x \in T^n_\alpha \ \&\ i \in \omega\}$. Let $x \in T^n_\alpha$, $i \in \omega$. We define $t_{x^\frown\langle i\rangle}$ as follows: $t_{x^\frown\langle i\rangle} = t_x \cup \{\cup b^x_{\langle i,j\rangle} \mid j \in \omega\}$, where b^x_k are as above. By the genericity of the construction of T^{n+1}_α (above), we clearly have $t_x \subseteq \cup_{j\in\omega} b^x_{\langle i,j\rangle}$ for each fixed $i \in \omega$, and $i \neq j \to t_{x^\frown\langle i\rangle} \cap t_{x^\frown\langle j\rangle} = t_x$.

That completes the construction. Clearly, T^0 will be a Souslin tree in L. Suppose b_0 is any branch of T^0 (necessarily L-generic, of course). Define $\hat{T}^1 \subseteq T^1$ in $L[b_0]$ by $\hat{T}^1 = \cup_{x\in b_0} t_x$. Note that as T^0 is Souslin in L, $\omega_1^{L[b_0]} = \omega_1^L$, so $\hat{T}^1$ is clearly an ω_1-tree in $L[b_0]$. We show that, in fact, $\hat{T}^1$ is Souslin in $L[b_0]$. Work in $L[b_0]$. Write T for $\hat{T}^1$. Assume $A \subseteq T$ is a maximal antichain. Let $M \prec L_{\omega_2}[b_0]$ be the smallest M such that $A, T \in M$. Set $\alpha = \omega_1 \cap M$. Let $\pi : M \cong L_\beta[b_0 \restriction \alpha]$. Then $\pi(\omega_1) = \alpha$, $\pi(T) = T|\alpha$, $\pi(A) = A \cap (T|\alpha)$, and $A \cap (T|\alpha)$ is a maximal antichain of $T|\alpha$ in $L_\beta[b_0 \restriction \alpha]$. Now, α is uncountable in $L_\beta[b_0 \restriction \alpha]$, hence also in L_β. But α is countable in $L_{\eta_{1,\alpha}}$. Hence $\beta < \eta_{1,\alpha}$. But look, $T^0|\alpha+1 \in L_{\eta_{1,\alpha}}$, so $b_0 \restriction \alpha \in L_{\eta_{1,\alpha}}$. Thus $L_\beta[b_0 \restriction \alpha] \subseteq L_{\eta_{1,\alpha}}$. But then $A \cap (T|\alpha)$ lies in $L_{\eta_{1,\alpha}}$, whence $A \cap (T|\alpha)$ is maximal in $T|\alpha+1$, and hence in T. Thus $A = A \cap (T|\alpha)$, and we are done. Similarly, if b_1 is a branch of $\hat{T}^1$, then $\hat{T}^2 = \cup_{x \in b_1} t_x$ is a Souslin tree in $L[b_0, b_1]$, etc.

Suppose now that $b_0, b_1, \ldots$ is an arbitrary sequence of branches, b_0 a branch of T^0, b_1 a branch of $\hat{T}^1 = \cup_{x\in b_0} t_x$, etc. We show that $L[\langle b_0, b_1, \ldots\rangle] = L[a]$ for some $a \subseteq \omega$.

In L, define a function f as follows: Given $n \in \omega$, $x \in T^{n+1}$, $ht(x) \geq 1$, let $f(x) =$ the T^n-least $y \in T^n$ such that $x \in t_y$. It is easily seen that f is well-defined, and that $ht(f(x)) = ht(x) + 1$ for all x.

Let $a = \{\langle n,i\rangle \mid \langle n,i\rangle \in b_n \cap T^n_1\}$. Clearly, $L[a] \subseteq L[\langle b_0, b_1, \ldots\rangle]$. Conversely, we show that $L[\langle b_0, b_1, \ldots\rangle] \subseteq L[a]$.

Working in $L[a]$, define a sequence $\langle b'_n | n < \omega \rangle$ of sets as follows:

(i) Let $x^n_0 = \langle n \rangle$, each $n \in \omega$.

(ii) Let x^n_1 = that pair $\langle n,i \rangle$ such that $\langle n,i \rangle \in a$, each $n \in \omega$.

(iii) Let $x^n_{\alpha+1} = f(x^{n+1}_\alpha)$, each $n \in \omega$, whenever $\alpha < \omega_1$ and each x^{n+1}_α is defined.

(iv) Let x^n_α be the unique sucessor of $\{x^n_\beta \mid \beta < \alpha\}$ on T^n_α whenever $\{x^n_\beta \mid \beta < \alpha\}$ is a branch of $T^n | \alpha$ which extends on T^n_α, each $n \in \omega$, <u>when</u> $\alpha < \omega_1$, $\lim(\alpha)$, and each x^n_β, $\beta < \alpha$, is defined.

(v) Let $\gamma \leq \omega^L_1$ be the first point where the above definition breaks down, and set $b'_n = \{x^n_\alpha \mid \alpha < \gamma\}$.

But by a simple induction on α, $(\forall n \in \omega)(x^n_\alpha \in b_n)$. Hence $\gamma = \omega^L_1$, and $b'_n = b_n$ for each n. Thus $\langle b_0, b_1, \ldots \rangle \in L[a]$, and we are done.

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GLOSSARY OF NOTATION

SYMBOL	APPROXIMATE MEANING	PAGE
V	the universe	
V_α	the α'th level in the cumulative hierarchy	
$\mathrm{dom}(f)$	domain of the function f	
$f \upharpoonright A$	the restriction of f to A	
$\lvert X \rvert$	the cardinality of the set X	
$\mathcal{P}(X)$	the power set of X, i.e. $\{Y \mid Y \subseteq X\}$	
2^α	$\{f \mid f : \alpha \to 2\}$ or the cardinality of this set, according to context	
$2^{<\alpha}$	$\bigcup_{\beta<\alpha} 2^\beta$	
On	the set of all ordinals	
$\lim(\alpha)$	α is a limit ordinal	
$\mathbb{Q}$, $\mathbb{R}$	the rational and real numbers (as ordered sets)	
CH	the continuum hypothesis	
GCH	the generalised continuum hypothesis	
ZF^-	the theory ZF without the power set axiom	
$A \models \varphi$	A is a model of φ	
$f : A \Longleftrightarrow B$	f is an isomorphism between A and B	
$A \cong B$	A is isomorphic to B	
$A \prec B$	A is an elementary submodel of B	
LST	the language of set theory	1
$F''X$	the "F-image" of X	1
$\mathrm{Def}(X)$	the set of all "definable" subsets of X	1
L	the constructible universe	2
L_α	the α'th level in the constructible hierarchy	2
$V = L$	the axiom of constructibility	2
$<_L$	the canonical well-ordering of L	2

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