

## Appendix

### ITERATED SOUSLIN FORCING AND $\mathcal{P}_{(\omega)}$

In [JnJo], Jensen and Johnsbråten give a generic construction of a non-constructible  $\Delta^1_3$  set of integers by means of an  $\omega$ -iteration of Souslin forcing. Their construction is a specialisation of an earlier construction which Jensen developed in order to show that iterated Souslin forcing over  $L$  must introduce new reals. For the convenience of the reader, and because the construction in [JnJo] is (necessarily) fairly cumbersome, we outline here a simplification which just gives the above result.

In  $L$ , we define a sequence  $\langle T^n \mid n < \omega \rangle$  of  $\omega_1$ -trees such that:

- (i)  $T^0$  is Souslin;
- (ii) if  $b_0$  is an  $L$ -generic branch of  $T^0$ , then we can canonically define a subtree  $\hat{T}^1$  of  $T^1$  in  $L[b_0]$  such that  $\hat{T}^1$  is Souslin in  $L[b_0]$ ;
- (iii) if  $b_1$  is an  $L[b_0]$ -generic branch of  $\hat{T}^1$ , then we can canonically define a subtree  $\hat{T}^2$  of  $T^2$  in  $L[b_0, b_1]$  such that  $\hat{T}^2$  is Souslin in  $L[b_0, b_1]$ ;
- (iv) ...  $< \omega$  similar;
- (v) if  $\langle b_0, b_1, \dots \rangle$  is any such sequence of branches of the respective trees  $T^0, \hat{T}^1, \dots$ , then  $L[\langle b_0, b_1, \dots \rangle] = L[a]$  for some  $a \subseteq \omega$ .

This shows that, regardless of what we do at limit stages, if we try to obtain SH by forcing over  $L$ , iteratively destroying Souslin trees, if the method we use to destroy Souslin trees results in their being given a branch, then the iteration must introduce new reals.

As we proceed, given  $x \in T^n$  we shall define a subtree  $t_x$  of  $T^{n+1}$ , such that:

- (i)  $t_x$  is a  $ht(x)$ -tree;

- (ii)  $x \leq_n y \rightarrow t_x = t_y \upharpoonright \text{ht}(x)$ ;
- (iii) if  $x, y$  are incomparable in  $T^n$  and  $z$  is the largest  $z \leq_n x, y$  in  $T^n$ , then  $t_x \cap t_y = t_z$  (unless  $\text{ht}(z) = 0$ , when  $t_x \cap t_y = \{\langle n+1 \rangle\}$ );
- (iv)  $T^{n+1} = \bigcup_{x \in T^n} t_x$ .

Each  $T_\alpha^n$  will consist of  $\alpha$ -sequences of natural numbers, and the ordering,  $\leq_n$ , will be the usual one. We define  $T^0 \upharpoonright 1$  first, then  $T^1 \upharpoonright 1$ , then  $T^2 \upharpoonright 1, \dots$ , then  $T^0 \upharpoonright 2$ , then  $T^1 \upharpoonright 2$ , then  $T^2 \upharpoonright 2, \dots$ , etc.

Fix some canonical, recursive bijection  $\{-, -\} : \omega \times \omega \rightarrow \omega$ .

Stage 0. For each  $n \in \omega$ , set  $T_0^n = \{\langle n \rangle\}$ ,  $t_{\langle n \rangle} = \emptyset$ .

Stage 1. For each  $n \in \omega$ , set  $T_1^n = \{\langle n, i \rangle \mid i \in \omega\}$ ,  $t_{\langle n, i \rangle} = \{\langle n+1 \rangle\}$ .

Stage  $\alpha+2$ . For each  $n \in \omega$ , set  $T_{\alpha+2}^n = \{x \smallfrown \langle i \rangle \mid x \in T_{\alpha+1}^n \text{ \& } i \in \omega\}$ , and for  $x \in T_{\alpha+1}^n$ ,  $i \in \omega$ , set  $t_{x \smallfrown \langle i \rangle} = t_x \cup \{y \smallfrown \langle \{i, j\} \rangle \mid y \in t_x \text{ \& } \text{ht}(y) = \alpha \text{ \& } j \in \omega\}$ .

Stage  $\alpha, \text{lim}(\alpha)$ . We first define  $T_\alpha^0$ . Let  $\eta = \eta_{0, \alpha}$  be least such that  $T^0 \upharpoonright \alpha \in L_\eta$ ,  $L_\eta \models \text{ZF}^-$ , and  $|\alpha|^{L_\eta} = \omega$ . For each  $x \in T^0 \upharpoonright \alpha$ , let  $b_x$  be the  $<_L$ -least  $L_\eta$ -generic branch of  $T^0 \upharpoonright \alpha$  through  $x$ , and let  $T_\alpha^0 = \{\cup b_x \mid x \in T^0 \upharpoonright \alpha\}$ . Assume now that  $T_\alpha^n$  is defined. Define  $t_x$  for  $x \in T_\alpha^n$  by  $t_x = \bigcup_{y <_n x} t_y$ . Let  $\eta = \eta_{n+1, \alpha}$  be least such that  $T^n \upharpoonright \alpha+1, T^{n+1} \upharpoonright \alpha, \langle t_x \mid x \in T_\alpha^n \rangle \in L_\eta$ ,  $L_\eta \models \text{ZF}^-$ , and  $|\alpha|^{L_\eta} = \omega$ . For each  $x \in T_\alpha^n$ , define a poset  $\mathbb{P} = \mathbb{P}_x$  in  $L_\eta$  as follows. The elements of  $\mathbb{P}$  are finite maps  $f$  with  $\text{dom}(f) \subseteq \omega$  and  $\text{ran}(f) \subseteq t_x$ , such that  $i, j \in \text{dom}(f) \rightarrow \text{ht}(f(i)) = \text{ht}(f(j))$ . The ordering is defined by  $f' \leq f \leftrightarrow \text{dom}(f') \supseteq \text{dom}(f) \text{ \& } (\forall i \in \text{dom}(f))(f'(i) \geq_{n+1} f(i))$ . Let  $G = G_x$  be the  $<_L$ -least  $L_\eta$ -generic subset of  $\mathbb{P}$ . Set  $b_i^x = \{f(i) \mid f \in G\}$ , each  $i \in \omega$ . Clearly, each  $b_i^x$  is a branch of  $t_x$ ,  $i \neq j \rightarrow b_i^x \neq b_j^x$ , and  $t_x \subseteq \bigcup_{i \in \omega} b_i^x$ . Set  $T_{\alpha}^{n+1} = \{\cup b_i^x \mid x \in T_\alpha^n \text{ \& } i \in \omega\}$ .

Stage  $\alpha+1, \text{lin}(\alpha)$ . For each  $n \in \omega$ , set  $T_{\alpha+1}^n = \{x \frown \langle i \rangle \mid x \in T_\alpha^n \text{ \& } i \in \omega\}$ . Let  $x \in T_\alpha^n$ ,  $i \in \omega$ . We define  $t_{x \frown \langle i \rangle}$  as follows:  $t_{x \frown \langle i \rangle} = t_x \cup \{U b_{\langle i, j \rangle}^x \mid j \in \omega\}$ , where  $b_k^x$  are as above. By the genericity of the construction of  $T_\alpha^{n+1}$  (above), we clearly have  $t_x \subseteq \bigcup_{j \in \omega} b_{\langle i, j \rangle}^x$  for each fixed  $i \in \omega$ , and  $i \neq j \rightarrow t_{x \frown \langle i \rangle} \cap t_{x \frown \langle j \rangle} = t_x$ .

That completes the construction. Clearly,  $T^0$  will be a Souslin tree in  $L$ . Suppose  $b_0$  is any branch of  $T^0$  (necessarily  $L$ -generic, of course). Define  $\hat{T}^1 \subseteq T^1$  in  $L[b_0]$  by  $\hat{T}^1 = \bigcup_{x \in b_0} t_x$ . Note that as  $T^0$  is Souslin in  $L$ ,  $\omega_1^{L[b_0]} = \omega_1^L$ , so  $\hat{T}^1$  is clearly an  $\omega_1$ -tree in  $L[b_0]$ . We show that, in fact,  $\hat{T}^1$  is Souslin in  $L[b_0]$ . Work in  $L[b_0]$ . Write  $T$  for  $\hat{T}^1$ . Assume  $A \subseteq T$  is a maximal antichain. Let  $M \prec_{\omega_2} L_{\omega_2}[b_0]$  be the smallest  $M$  such that  $A, T \in M$ . Set  $\alpha = \omega_1 \cap M$ . Let  $\pi: M \xrightarrow{\sim} L_\beta[b_0 \restriction \alpha]$ . Then  $\pi(\omega_1) = \alpha$ ,  $\pi(T) = T \restriction \alpha$ ,  $\pi(A) = A \cap (T \restriction \alpha)$ , and  $A \cap (T \restriction \alpha)$  is a maximal antichain of  $T \restriction \alpha$  in  $L_\beta[b_0 \restriction \alpha]$ . Now,  $\alpha$  is uncountable in  $L_\beta[b_0 \restriction \alpha]$ , hence also in  $L_\beta$ . But  $\alpha$  is countable in  $L_{\eta_1, \alpha}$ . Hence  $\beta < \eta_1, \alpha$ . But look,  $T^0 \restriction \alpha+1 \in L_{\eta_1, \alpha}$ , so  $b_0 \restriction \alpha \in L_{\eta_1, \alpha}$ . Thus  $L_\beta[b_0 \restriction \alpha] \subseteq L_{\eta_1, \alpha}$ . But then  $A \cap (T \restriction \alpha)$  lies in  $L_{\eta_1, \alpha}$ , whence  $A \cap (T \restriction \alpha)$  is maximal in  $T \restriction \alpha+1$ , and hence in  $T$ . Thus  $A = A \cap (T \restriction \alpha)$ , and we are done. Similarly, if  $b_1$  is a branch of  $\hat{T}^1$ , then  $\hat{T}^2 = \bigcup_{x \in b_1} t_x$  is a Souslin tree in  $L[b_0, b_1]$ , etc.

Suppose now that  $b_0, b_1, \dots$  is an arbitrary sequence of branches,  $b_0$  a branch of  $T^0$ ,  $b_1$  a branch of  $\hat{T}^1 = \bigcup_{x \in b_0} t_x$ , etc. We show that  $L[\langle b_0, b_1, \dots \rangle] = L[a]$  for some  $a \subseteq \omega$ .

In  $L$ , define a function  $f$  as follows: Given  $n \in \omega$ ,  $x \in T^{n+1}$ ,  $\text{ht}(x) \geq 1$ , let  $f(x) =$  the  $T^n$ -least  $y \in T^n$  such that  $x \in t_y$ . It is easily seen that  $f$  is well-defined, and that  $\text{ht}(f(x)) = \text{ht}(x) + 1$  for all  $x$ .

Let  $a = \{\langle n, i \rangle \mid \langle n, i \rangle \in b_n \cap T_1^n\}$ . Clearly,  $L[a] \subseteq L[\langle b_0, b_1, \dots \rangle]$ . Conversely, we show that  $L[\langle b_0, b_1, \dots \rangle] \subseteq L[a]$ .

Working in  $L[a]$ , define a sequence  $\langle b'_n \mid n < \omega \rangle$  of sets as follows:

- (i) Let  $x_0^n = \langle n \rangle$ , each  $n \in \omega$ .
- (ii) Let  $x_1^n =$  that pair  $\langle n, i \rangle$  such that  $\langle n, i \rangle \in a$ , each  $n \in \omega$ .
- (iii) Let  $x_{\alpha+1}^n = f(x_\alpha^{n+1})$ , each  $n \in \omega$ , whenever  $\alpha < \omega_1$  and each  $x_\alpha^{n+1}$  is defined.
- (iv) Let  $x_\alpha^n$  be the unique successor of  $\{x_\beta^n \mid \beta < \alpha\}$  on  $T_\alpha^n$  whenever  $\{x_\beta^n \mid \beta < \alpha\}$  is a branch of  $T^n \restriction \alpha$  which extends on  $T_\alpha^n$ , each  $n \in \omega$ , when  $\alpha < \omega_1$ ,  $\text{lim}(\alpha)$ , and each  $x_\beta^n$ ,  $\beta < \alpha$ , is defined.
- (v) Let  $\gamma \leq \omega_1^L$  be the first point where the above definition breaks down, and set  $b'_n = \{x_\alpha^n \mid \alpha < \gamma\}$ .

But by a simple induction on  $\alpha$ ,  $(\forall n \in \omega)(x_\alpha^n \in b'_n)$ . Hence  $\gamma = \omega_1^L$ , and  $b'_n = b_n$  for each  $n$ . Thus  $\langle b_0, b_1, \dots \rangle \in L[a]$ , and we are done.

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# GLOSSARY OF NOTATION

SYMBOL	APPROXIMATE MEANING	PAGE
$V$	the universe	
$V_\alpha$	the $\alpha$ 'th level in the cumulative hierarchy	
$\text{dom}(f)$	domain of the function $f$	
$f \upharpoonright A$	the restriction of $f$ to $A$	
$ X $	the cardinality of the set $X$	
$\mathcal{P}(X)$	the power set of $X$ , i.e. $\{Y \mid Y \subseteq X\}$	
$2^\alpha$	$\{f \mid f: \alpha \rightarrow 2\}$ or the cardinality of this set, according to context	
$2^{<\alpha}$	$\bigcup_{\beta < \alpha} 2^\beta$	
$\text{On}$	the set of all ordinals	
$\text{lim}(\alpha)$	$\alpha$ is a limit ordinal	
$\mathbb{Q}, \mathbb{R}$	the rational and real numbers (as ordered sets)	
CH	the continuum hypothesis	
GCH	the generalised continuum hypothesis	
$\text{ZF}^-$	the theory ZF without the power set axiom	
$A \models \varphi$	$A$ is a model of $\varphi$	
$f: A \xrightarrow{\sim} B$	$f$ is an isomorphism between $A$ and $B$	
$A \cong B$	$A$ is isomorphic to $B$	
$A \prec B$	$A$ is an elementary submodel of $B$	
LST	the language of set theory	1
$F''X$	the " $F$ -image" of $X$	1
$\text{Def}(X)$	the set of all "definable" subsets of $X$	1
$L$	the constructible universe	2
$L_\alpha$	the $\alpha$ 'th level in the constructible hierarchy	2
$V = L$	the axiom of constructibility	2
$<_L$	the canonical well-ordering of $L$	2

$L[A], L_{\alpha}[A], V = L[A]$	relative constructibility concepts	2
$p q, x y$	incompatibility, incomparability in posets	3, 11
c.c.c.	countable chain condition	3
c.t.m.	countable transitive model	4
$M[G]$	the generic extension of $M$ by $G$	4
$BA(\mathbb{P})$	the boolean algebra determined by the poset $\mathbb{P}$	4
$[p]$	$\{q \in \mathbb{P} \mid q \leq p\}$	4
$M^{\mathbb{B}}$	the $\mathbb{B}$ -valued extension of $M$	4
$\vee$	the canonical map of $M$ into $M^{\mathbb{B}}$	4
$\ \varphi\ $	the boolean value of $\varphi$	5
$\dot{a}$	a name for $a$ in the forcing language	5
$\Vdash$	the forcing relation	5, 6
$\text{lub}(A)$	the least upper bound of $A$	8
$\text{glb}(A)$	the greatest lower bound of $A$	8
SH	the Souslin hypothesis	9
$\hat{x}$	$\{y \in T \mid y \leq x\}$	11
$\text{ht}(x), \text{ht}(\underline{T})$	the height of $x, \underline{T}$	11
$T_{\alpha}$	the $\alpha$ 'th level of $\underline{T}$	11
$T _{\alpha}$	$\bigcup_{\beta < \alpha} T_{\beta}$	11
$\underline{T} _{\alpha}$ (or $T _{\alpha}$ )	$\langle T _{\alpha}, \leq_T \cap (T _{\alpha})^2 \rangle$	11
$T _C$	$\bigcup_{\alpha \in C} T_{\alpha}, \text{ for } C \subseteq \omega_1$	38
$\underline{T} _C$ (or $T _C$ )	$\langle T _C, \leq_T \cap (T _C)^2 \rangle$	38
$T^x$	$\{y \in T \mid x \leq y\}$	11
$\diamond$		23
$\diamond^*$		25
$\diamond^+$		25
BA	complete boolean algebra	56
MA	Martin's axiom	62
$\square$		75
$H_{\omega_1}, H_{\omega_1}(\alpha)$		80



$\mathbb{C}$

the "closed set forcing" poset

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$\mathcal{A}_\eta$

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