Appendix A
Electromagnetic Waves

A.1 Helmholtz's Equation

Taking the curl of $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ in Eq. (2.4):

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} \right) \\ \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t} \\ \nabla^2 \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t} &= 0 \end{aligned} \tag{A.1}$$

Similarly, the other part of the Helmholtz's equation can be solved by taking curl of $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0$ in Eq. (2.4).

A.2 Electromagnetic Waves *Are* Transverse

Let us consider a plane wave (in x-direction) that satisfies both Maxwell's and Helmholtz's equation

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \mathbf{E}_0 e^{i(kx-\omega t)} \\ \mathbf{B}(\mathbf{x}, t) &= \mathbf{B}_0 e^{i(kx-\omega t)} \end{aligned} \tag{A.2}$$

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Whereas every solution to Maxwell's equations (in empty space or nonconductors) must obey the wave equation, the converse is *not* true; it imposes special constraints on $\mathbf{E}_0$ and $\mathbf{B}_0$. Let us consider $\mathbf{E}(\mathbf{x}, t)$ with three orthogonal components as

$$\mathbf{E}(\mathbf{x}, t) = \left(\mathbf{E}_{0x} + \mathbf{E}_{0y} + \mathbf{E}_{0z}\right) e^{i(kx-\omega t)} \tag{A.3}$$

Since $\nabla \cdot \mathbf{E} = 0$,

$$\frac{\partial \mathbf{E}_{0x} e^{ikx}}{\partial x} + \frac{\partial \mathbf{E}_{0y} e^{ikx}}{\partial y} + \frac{\partial \mathbf{E}_{0z} e^{ikx}}{\partial z} = 0 \tag{A.4}$$

Equation (A.4) is satisfied only if

$$\mathbf{E}_{0x} = 0 \tag{A.5}$$

Similarly, it can be shown that

$$\mathbf{B}_{0x} = 0 \tag{A.6}$$

Moreover, it can be shown from Faraday's law, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ that

$$\mathbf{B}_0 = \frac{k}{\omega}\left(\hat{i} \times \mathbf{E}_0\right) \tag{A.7}$$

Thus, Eqs. (A.5)–(A.7) show that the EM plane wave needs to be *transverse* with the vectors $\mathbf{B}_0, \mathbf{E}_0, k$ forming a right-handed triplet.