Appendix A: Units, Constants and Formulas, Vector Relations

Units

The formulas in this book are written in the mks units of the International System (SI). In much of the research literature, however, the cgs-Gaussian system is still used. The following table compares the vacuum Maxwell equations, the fluid equation of motion, and the idealized Ohm's law in the two systems:

mks-SI	cgs-Gaussian
$\nabla \cdot \mathbf{D} = e(n_i - n_e)$	$\nabla \cdot \mathbf{E} = 4\pi e(n_i - n_e)$
$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$	$c\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{H} = \mathbf{j} + \dot{\mathbf{D}}$	$c\nabla \times \mathbf{B} = 4\pi \mathbf{j} + \dot{\mathbf{E}}$
$\mathbf{D} = \epsilon_0 \mathbf{E} \quad \mathbf{B} = \mu_0 \mathbf{H}$	$\epsilon = \mu = 1$
$mn\frac{d\mathbf{v}}{dt} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla p$	$mn\frac{d\mathbf{v}}{dt} = qn\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) - \nabla p$
$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$	$\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B} = 0$

The equation of continuity is the same in both systems.

In the Gaussian system, all electrical quantities are in electrostatic units (esu) except **B**, which is in gauss (emu); the factors of c are written explicitly to accommodate this exception. In the mks system, **B** is measured in tesla (Wb/m^2), each of which is worth 10^4 gauss. Electric fields **E** are in esu/cm in cgs and V/m in mks. Since one esu of potential is 300 V, one esu/cm is the same as 3×10^4 V/m. The ratio of E to B is dimensionless in the Gaussian system, so that $v_E = cE/B$. In the mks system, E/B has the dimensions of a velocity, so that $v_E = E/B$. This fact is useful to keep in mind when checking the dimensions of various terms in an equation in looking for algebraic errors.

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The current density $\mathbf{j} = ne\mathbf{v}$ has the same form in both systems. In cgs, n and $\mathbf{v}$ are in cm^{-3} and cm/s, and e has the value $e = 4.8 \times 10^{-10}$ esu; then $\mathbf{j}$ comes out in esu/cm^2, where 1 esu of current equals c^{-1} emu or $10/c = 1/(3 \times 10^9)$ A. In mks, n and $\mathbf{v}$ are in m^{-3} and m/s, and e has the value $e = 1.6 \times 10^{-19}$ C; then $\mathbf{j}$ comes out in A/m^2.

Most cgs formulas can be converted to mks by replacing B/c by B and 4π by ϵ_0^{-1}, where $1/4\pi\epsilon_0 = 9 \times 10^9$. For instance, electric field energy density is $E^2/8\pi$ in cgs and $\epsilon_0 E^2/2$ in mks, and magnetic field energy density is $B^2/8\pi$ in cgs and $B^2/2\mu_0$ in mks. Here we have used the fact that $(\epsilon_0\mu_0)^{-1/2} = c = 3 \times 10^8$ m/s.

The energy KT is usually given in electron volts. In cgs, one must convert T_{eV} to ergs by multiplying by 1.6×10^{-12} erg/eV. In mks, one converts T_{eV} to joules by multiplying by 1.6×10^{-19} J/eV. This last number is, of course, just the charge e in mks, since that is how the electron volt is defined.

Useful Constants and Formulas

Constants			
		mks	cgs
c	Velocity of light	3×10^8 m/s	3×10^{10} cm/s
e	Electron charge	1.6×10^{-19} C	4.8×10^{-10} esu
m	Electron mass	0.91×10^{-30} kg	0.91×10^{-27} g
M	Proton mass	1.67×10^{-27} kg	1.67×10^{-24} g
M/m		1837	1837
$(M/m)^{1/2}$		43	43
K	Boltzmann's constant	1.38×10^{-23} J/K	1.38×10^{-16} erg/K
eV	Electron volt	1.6×10^{-19} J	1.6×10^{-12} erg
1 eV	Of temperature KT	11,600 K	11,600 K
ϵ_0	Permittivity of free space	8.854×10^{-12} F/m	
μ_0	Permeability of free space	$4\pi \times 10^{-7}$ H/m	
πa_0^2	Cross section of H atom	0.88×10^{-20} m^2	0.88×10^{-16} cm^2
Density of neutral atoms at room temperature and 1 mTorr pressure		3.3×10^{19} m^{-3}	3.3×10^{13} cm^{-3}

Formulas (H) for hydrogen				
		mks	cgs-Gaussian	Handy formula (n in cm^{-3})
ω_p	Plasma frequency	$\left(\frac{ne^2}{\epsilon_0 m}\right)^{1/2}$	$\left(\frac{4\pi ne^2}{m}\right)^{1/2}$	$f_p = 9000\sqrt{n}\,s^{-1}$
ω_c	Electron cyclotron frequency	$\frac{eB}{m}$	$\frac{eB}{mc}$	$f_c = 2.8$ GHz/kG
λ_D	Debye length	$\left(\frac{\epsilon_0 KT_e}{ne^2}\right)^{1/2}$	$\left(\frac{KT_e}{4\pi ne^2}\right)^{1/2}$	$740(T_{\text{eV}}/n)^{1/2}$ cm
r_L	Larmor radius	$\frac{mv_\perp}{eB}$	$\frac{mv_\perp c}{eB}$	$\frac{1.4T_{\text{ev}}^{1/2}}{B_{\text{kG}}}$ mm(H)

(continued)

Formulas (H) for hydrogen				
		mks	cgs-Gaussian	Handy formula (n in cm^{-3})
v_A	Alfvén speed	$\frac{B}{(\mu_0\rho)^{1/2}}$	$\frac{B}{(4\pi\rho)^{1/2}}$	$2.2\times10^{11}\frac{B}{\sqrt{n}}\frac{\text{cm}}{s}$ (H)
v_s	Acoustic speed ($T_i = 0$)	$\left(\frac{KT_e}{M}\right)^{1/2}$	$\left(\frac{KT_e}{M}\right)^{1/2}$	$10^6\,T_{\text{eV}}^{1/2}\frac{\text{cm}}{s}$ (H)
v_E	$E\times B$ drift speed	$\frac{E}{B}$	$\frac{cE}{B}$	$10^8\frac{E(\text{V/cm})}{B(\text{G})}\frac{\text{cm}}{s}$
v_D	Diamagnetic drift speed	$\frac{KT}{eB}\frac{n'}{n}$	$\frac{cKT}{eB}\frac{n'}{n}$	$10^8\frac{T_{\text{eV}}}{B}\frac{1}{R}\frac{\text{cm}}{s}$
β	Magnetic/plasma pressure	$\frac{nKT}{B^2/2\mu_0}$	$\frac{nKT}{B^2/8\pi}$	
v_{the}	Electron thermal speed	$\left(\frac{2KT_e}{m}\right)^{1/2}$	$\left(\frac{2KT_e}{m}\right)^{1/2}$	$5.9\times10^7\,T_{\text{eV}}^{1/2}\frac{\text{cm}}{s}$
ν_{ei}	Electron–ion collision frequency		$\approx\frac{\omega_p}{N_D}$	$\simeq 2\times10^{-6}\frac{Zn_e\ln\Lambda}{T_{\text{eV}}^{3/2}}s^{-1}$
ν_{ee}	Electron–electron collision frequency			$\simeq 5\times10^{-6}\frac{n\ln\Lambda}{T_{\text{eV}}^{3/2}}s^{-1}$
ν_{ii}	Ion–ion collision frequency		$Z^4\left(\frac{m}{M}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\nu_{ee}$	
λ_{ei}	Collision mean free path		$\approx\lambda_{ee}\approx\lambda_{ii}$	$\simeq 3.4\times10^{13}\frac{T_{\text{eV}}^2}{n\ln\Lambda}$ cm(H)
v_{osc}	Peak electron quiver velocity	$\frac{eE_0}{m\omega_0}$	$\frac{eE_0}{m\omega_0}$	$\frac{v_{\text{osc}}^2}{c^2}=7.3I_{19}\lambda_\mu^2$
				$\frac{v_{\text{osc}}^2}{v_e^2}=3.7\frac{I_{13}\lambda_\mu^2}{T_{\text{eV}}}$

Useful Vector Relations

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\equiv(\mathbf{ABC})$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$
$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=(\mathbf{ABD})\mathbf{C}-(\mathbf{ABC})\mathbf{D}=(\mathbf{ACD})\mathbf{B}-(\mathbf{BCD})\mathbf{A}$$
$$\nabla\cdot(\phi\mathbf{A})=\mathbf{A}\cdot\nabla\phi+\phi\nabla\cdot\mathbf{A}$$
$$\nabla\times(\phi\mathbf{A})=\nabla\phi\times\mathbf{A}+\phi\nabla\times\mathbf{A}$$

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = \nabla(\mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$(\mathbf{A} \cdot \nabla)\mathbf{A} = \nabla\left(\frac{1}{2}A^2\right) - \mathbf{A} \times (\nabla \times \mathbf{A})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \times [(\mathbf{A} \cdot \nabla)\mathbf{A}] = (\mathbf{A} \cdot \nabla)(\nabla \times \mathbf{A}) + (\nabla \cdot \mathbf{A})(\nabla \times \mathbf{A}) - [(\nabla \times \mathbf{A}) \cdot \nabla]\mathbf{A}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}$$

$$\nabla \times \nabla\phi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Cylindrical Coordinates (*r*, *θ*, *z*)

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial\theta}A_\theta + \frac{\partial}{\partial z}A_z$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\hat{\mathbf{z}}$$

$$\nabla^2\mathbf{A} = (\nabla \cdot \nabla)\mathbf{A} = \left[\nabla^2 A_r - \frac{1}{r^2}\left(A_r + 2\frac{\partial A_\theta}{\partial\theta}\right)\right]\hat{\mathbf{r}}$$
$$+ \left[\nabla^2 A_\theta - \frac{1}{r^2}\left(A_\theta + 2\frac{\partial A_r}{\partial\theta}\right)\right]\hat{\boldsymbol{\theta}} + \nabla^2 A_z\hat{\mathbf{z}}$$

$$(\mathbf{A} \cdot \nabla)\mathbf{B} = \hat{\mathbf{r}}\left(A_r\frac{\partial B_r}{\partial r} + A_\theta\frac{1}{r}\frac{\partial B_r}{\partial\theta} + A_z\frac{\partial B_r}{\partial z} - \frac{1}{r}A_\theta B_\theta\right)$$
$$+ \hat{\boldsymbol{\theta}}\left(A_r\frac{\partial B_\theta}{\partial r} + A_\theta\frac{1}{r}\frac{\partial B_\theta}{\partial\theta} + A_z\frac{\partial B_\theta}{\partial z} - \frac{1}{r}A_\theta B_r\right)$$
$$+ \hat{\mathbf{z}}\left(A_r\frac{\partial B_z}{\partial r} + A_\theta\frac{1}{r}\frac{\partial B_z}{\partial\theta} + A_z\frac{\partial B_z}{\partial z}\right)$$

Appendix B: Theory of Waves in a Cold Uniform Plasma

As long as $T_e = T_i = 0$, the waves described in Chap. 4 can easily be generalized to an arbitrary number of charged particle species and an arbitrary angle of propagation θ relative to the magnetic field. Waves that depend on finite T, such as ion acoustic waves, are not included in this treatment.

First, we define the dielectric tensor of a plasma as follows. The fourth Maxwell equation is

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \dot{\mathbf{E}} \right) \tag{B.1}$$

where $\mathbf{j}$ is the plasma current due to the motion of the various charged particle species s, with density n_s, charge q_s, and velocity $\mathbf{v}_s$:

$$\mathbf{j} = \sum_s n_s q_s \mathbf{v}_s \tag{B.2}$$

Considering the plasma to be a dielectric with internal currents $\mathbf{j}$, we may write Eq. (B.1) as

$$\nabla \times \mathbf{B} = \mu_0 \dot{\mathbf{D}} \tag{B.3}$$

where

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \frac{i}{\omega} \mathbf{j} \tag{B.4}$$

Here we have assumed an exp $(-i\omega t)$ dependence for all plasma motions. Let the current $\mathbf{j}$ be proportional to $\mathbf{E}$ but not necessarily in the same direction (because of the magnetic field $B_0\hat{\mathbf{z}}$); we may then define a conductivity tensor $\boldsymbol{\sigma}$ by the relation

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$$\mathbf{j} = \boldsymbol{\sigma} \cdot \mathbf{E} \tag{B.5}$$

Equation (B.4) becomes

$$\mathbf{D} = \epsilon_0 \left(\mathbf{I} + \frac{i}{\epsilon_0 \omega} \boldsymbol{\sigma} \right) \cdot \mathbf{E} = \boldsymbol{\epsilon} \cdot \mathbf{E} \tag{B.6}$$

Thus the effective dielectric constant of the plasma is the tensor

$$\boldsymbol{\epsilon} = \epsilon_0 (\mathbf{I} + i\boldsymbol{\sigma}/\epsilon_0 \omega) \tag{B.7}$$

where $\mathbf{I}$ is the unit tensor.

To evaluate $\boldsymbol{\sigma}$, we use the linearized fluid equation of motion for species s, neglecting the collision and pressure terms:

$$m_s \frac{\partial \mathbf{v}_s}{\partial t} = q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}_0) \tag{B.8}$$

Defining the cyclotron and plasma frequencies for each species as

$$\omega_{cs} \equiv \left| \frac{q_s B_0}{m_s} \right| \quad \omega_{ps}^2 \equiv \left| \frac{n_0 q_s^2}{\epsilon_0 m_s} \right|, \tag{B.9}$$

we can separate Eq. (B.8) into x, y, and z components and solve for $\mathbf{v}_s$, obtaining

$$v_{xs} = \frac{iq_s}{m_s \omega} \frac{\left[E_x \pm i(\omega_{cs}/\omega) E_y \right]}{1 - (\omega_{cs}/\omega)^2} \tag{B.10a}$$

$$v_{ys} = \frac{iq_s}{m_s \omega} \frac{\left[E_y \mp i(\omega_{cs}/\omega) E_x \right]}{1 - (\omega_{cs}/\omega)^2} \tag{B.10b}$$

$$v_{zs} = \frac{iq_s}{m_s \omega} E_z \tag{B.10c}$$

where $\pm$ stands for the sign of q_s. The plasma current is

$$\mathbf{j} = \sum_s n_{0s} q_s \mathbf{v}_s \tag{B.11}$$

so that

$$\begin{aligned} \frac{i}{\epsilon_0 \omega} j_x &= \sum_s \frac{in_{0s}}{\epsilon_0 \omega} \frac{iq_s^2}{m_s \omega} \frac{E_x \pm i(\omega_{cs}/\omega) E_y}{1 - (\omega_{cs}/\omega)} \\ &= \sum_s - \frac{\omega_{ps}^2}{\omega^2} \frac{E_x \pm i(\omega_{cs}/\omega) E_y}{1 - (\omega_{cs}/\omega)} \end{aligned} \tag{B.12}$$

Using the identities

$$\frac{1}{1-(\omega_{cs}/\omega)^2} = \frac{1}{2}\left[\frac{\omega}{\omega \mp \omega_{cs}} + \frac{\omega}{\omega \pm \omega_{cs}}\right]$$
$$\pm\frac{\omega_{cs}/\omega}{1-(\omega_{cs}/\omega)^2} = \frac{1}{2}\left[\frac{\omega}{\omega \mp \omega_{cs}} + \frac{\omega}{\omega \pm \omega_{cs}}\right], \tag{B.13}$$

we can write Eq. (B.12) as follows:

$$\frac{1}{\epsilon_0\omega} j_x = -\frac{1}{2}\sum_s \frac{\omega_{ps}^2}{\omega^2}\left[\left(\frac{\omega}{\omega \mp \omega_{cs}} + \frac{\omega}{\omega \pm \omega_{cs}}\right)E_x + \left(\frac{\omega}{\omega \mp \omega_{cs}} + \frac{\omega}{\omega \pm \omega_{cs}}\right) iE_{\mathrm{y}}\right]. \tag{B.14}$$

Similarly, the y and z components are

$$\frac{1}{\epsilon_0\omega} j_y = -\frac{1}{2}\sum_s \frac{\omega_{ps}^2}{\omega^2}\left[\left(\frac{\omega}{\omega \pm \omega_{cs}} + \frac{\omega}{\omega \mp \omega_{cs}}\right) iE_x + \left(\frac{\omega}{\omega \mp \omega_{cs}} + \frac{\omega}{\omega \pm \omega_{cs}}\right)E_y\right] \tag{B.15}$$

$$\frac{i}{\epsilon_0\omega} j_z = -\sum_s \frac{\omega_{ps}^2}{\omega^2} E_z \tag{B.16}$$

Use of Eq. (B.14) in Eq. (B.4) gives

$$\frac{1}{\epsilon_0} D_x = E_x - \frac{1}{2}\sum_s \left[\frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \mp \omega_{cs}} + \frac{\omega}{\omega \pm \omega_{cs}}\right)E_x + \frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \mp \omega_{cs}} - \frac{\omega}{\omega \pm \omega_{cs}}\right) iE_y\right]. \tag{B.17}$$

We define the convenient abbreviations

$$\begin{aligned}
R &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \pm \omega_{cs}}\right) \\
L &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\left(\frac{\omega}{\omega \mp \omega_{cs}}\right) \\
S &\equiv \frac{1}{2}(R+L) \qquad D \equiv \frac{1}{2}(R-L)^{*} \\
P &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}
\end{aligned} \tag{B.18}$$

Using these in Eq. (B.17) and proceeding similarly with the y and z components, we obtain

$$\begin{aligned}
\epsilon_0^{-1} D_x &= SE_x - iDE_y \\
\epsilon_0^{-1} D_y &= iDE_x + SE_y \\
\epsilon_0^{-1} D_z &= PE_z
\end{aligned} \tag{B.19}$$

Comparing with Eq. (B.6), we see that

$$\boldsymbol{\epsilon} = \epsilon_0 \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \equiv \epsilon_0 \boldsymbol{\epsilon}_R \tag{B.20}$$

We next derive the wave equation by taking the curl of the equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ and substituting $\nabla \times \mathbf{B} = \mu_0 \boldsymbol{\epsilon} \cdot \dot{\mathbf{E}}$, obtaining

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \epsilon_0 \left(\boldsymbol{\epsilon}_R \cdot \ddot{\mathbf{E}}\right) = -\frac{1}{c^2} \boldsymbol{\epsilon}_R \cdot \ddot{\mathbf{E}} \tag{B.21}$$

Assuming an exp $(i\mathbf{k} \cdot \mathbf{r})$ spatial dependence of $\mathbf{E}$ and defining a vector index of refraction

$$\boldsymbol{\mu} = \frac{c}{\omega} \mathbf{k}, \tag{B.22}$$

*Note that D here stands for "difference." It is not the displacement vector $\mathbf{D}$.

we can write Eq. (B.21) as

$$\boldsymbol{\mu} \times (\boldsymbol{\mu} \times \mathbf{E}) + \boldsymbol{\epsilon}_R \cdot \mathbf{E} = 0. \tag{B.23}$$

The uniform plasma is isotropic in the $x - y$ plane, so we may choose the y axis so that $k_y = 0$, without loss of generality. If θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$, we then have

$$\mu_x = \mu \sin\theta \qquad \mu_z = \mu \cos\theta \qquad \mu_y = 0 \tag{B.24}$$

The next step is to separate Eq. (B.23) into components, using the elements of $\boldsymbol{\epsilon}_R$ given in Eq. (B.20). This procedure readily yields

$$\mathbf{R} \cdot \mathbf{E} \equiv \begin{pmatrix} S - \mu^2 \cos^2\theta & -iD & \mu^2 \sin\theta \cos\theta \\ iD & S - \mu^2 & 0 \\ \mu^2 \sin\theta \cos\theta & 0 & P - \mu^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0. \tag{B.25}$$

From this it is clear that the E_x, E_y components are coupled to E_z only if one deviates from the principal angles $\theta = 0, 90°$.

Equation (B.25) is a set of three simultaneous, homogeneous equations; the condition for the existence of a solution is that the determinant of $\mathbf{R}$ vanish: $||\mathbf{R}|| = 0$. Expanding in minors of the second column, we then obtain

$$\begin{aligned} &(iD)^2 (P - \mu^2 \sin^2\theta) + (S - \mu^2) \\ &\quad \times [(S - \mu^2 \cos^2\theta)(P - \mu^2 \sin^2\theta) - \mu^4 \sin^2\theta \cos^2\theta] = 0. \end{aligned} \tag{B.26}$$

By replacing $\cos^2\theta$ by $1 - \sin^2\theta$, we can solve for $\sin^2\theta$, obtaining

$$\sin^2\theta = \frac{-P(\mu^4 - 2S\mu^2 + RL)}{\mu^4(S - P) + \mu^2(PS - RL)}. \tag{B.27}$$

We have used the identity $S^2 - D^2 = RL$. Similarly,

$$\cos^2\theta = \frac{S\mu^4 - (PS + RL)\mu^2 + PRL}{\mu^4(S - P) + \mu^2(PS - RL)}. \tag{B.28}$$

Dividing the last two equations, we obtain

$$\tan^2\theta = \frac{P(\mu^4 - 2S\mu^2 + RL)}{S\mu^4 - (PS + RL)\mu^2 + PRL}.$$

Since $2S = R + L$, the numerator and denominator can be factored to give the cold-plasma dispersion relation

$$\boxed{\tan^2\theta = \frac{P(\mu^2 - R)(\mu^2 - L)}{(S\mu^2 - RL)(\mu^2 - P)}} \tag{B.29}$$

The principal modes of Chap. 4 can be recovered by setting $\theta = 0°$ and $90°$. When $\theta = 0°$, there are three roots: $P = 0$ (Langmuir wave), $\mu^2 = R$ (R wave), and $\mu^2 = L$ (L wave). When $\theta = 90°$, there are two roots: $\mu^2 = RL/S$ (extraordinary wave) and $\mu^2 = P$ (ordinary wave). By inserting the definitions of Eq. (B.18), one can verify that these are identical to the dispersion relations given in Chap. 4, with the addition of corrections due to ion motions.

The resonances can be found by letting μ go to ∞. We then have

$$\tan^2\theta_{\text{res}} = -P/S \tag{B.30}$$

This shows that the resonance frequencies depend on angle θ. If $\theta = 0°$, the possible solutions are $P = 0$ and $S = \infty$. The former is the plasma resonance $\omega = \omega_p$, while the latter occurs when either $R = \infty$ (electron cyclotron resonance) or $L = \infty$ (ion cyclotron resonance). If $\theta = 90°$, the possible solutions are $P = \infty$ or $S = 0$. The former cannot occur for finite ω_p and ω, and the latter yields the upper and lower hybrid frequencies, as well as the two-ion hybrid frequency when there is more than one ion species.

The cutoffs can be found by setting $\mu = 0$ in Eq. (B.26). Again using $S^2 - D^2 = RL$, we find that the condition for cutoff is independent of θ:

$$PRL = 0 \tag{B.31}$$

The conditions $R = 0$ and $L = 0$ yield the ω_R and ω_L cutoff frequencies of Chap. 4, with the addition of ion corrections. The condition $P = 0$ is seen to correspond to cutoff as well as to resonance. This degeneracy is due to our neglect of thermal motions. Actually, $P = 0$ (or $\omega = \omega_p$) is a resonance for longitudinal waves and a cutoff for transverse waves.

The information contained in Eq. (B.29) is summarized in the Clemmow–Mullaly–Allis diagram. One further result, not in the diagram, can be obtained easily from this formulation. The middle line of Eq. (B.25) reads

$$iDE_x + \left(S - \mu^2\right)E_y = 0 \tag{B.32}$$

Thus the polarization in the plane perpendicular to B_0 is given by

$$\frac{iE_x}{E_y} = \frac{\mu^2 - S}{D}. \tag{B.33}$$

From this it is easily seen that waves are linearly polarized at resonance ($\mu^2 = \infty$) and circularly polarized at cutoff ($\mu^2 = 0$, $R = 0$ or $L = 0$; thus $S = \pm D$).

Appendix C: Sample Three-Hour Final Exam

Part A (One Hour, Closed Book)

1. The number of electrons in a Debye sphere for $n = 10^{17}$ m^{-3}, $KT_e = 10$ eV is approximately

   (A) 135
   (B) 0.14
   (C) 7.4×10^3
   (D) 1.7×10^5
   (E) 3.5×10^{10}

2. The electron plasma frequency in a plasma of density $n = 10^{20}$ m^{-3} is

   (A) 90 MHz
   (B) 900 MHz
   (C) 9 GHz
   (D) 90 GHz
   (E) None of the above to within 10 %

3. A doubly charged helium nucleus of energy 3.5 MeV in a magnetic field of 8 T has a maximum Larmor radius of approximately

   (A) 2 mm
   (B) 2 cm
   (C) 20 cm
   (D) 2 m
   (E) 2 ft

4. A laboratory plasma with $n = 10^{16}$ m^{-3}, $KT_e = 2$ eV, $KT_i = 0.1$ eV, and $B = 0.3$ T has a beta (plasma pressure/magnetic field pressure) of approximately

   (A) 10^{-7}
   (B) 10^{-6}

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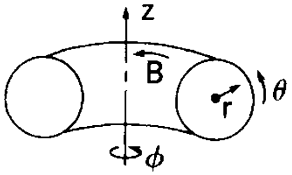
(C) 10^{-4}
(D) 10^{-2}
(E) 10^{-1}

5. The grad-B drift $\mathbf{v}_{\nabla B}$ is

(A) always in the same direction as $\mathbf{v}_E$
(B) always opposite to $\mathbf{v}_E$
(C) sometimes parallel to $\mathbf{B}$
(D) always opposite to the curvature drift $\mathbf{v}_R$
(E) sometimes parallel to the diamagnetic drift $\mathbf{v}_D$

6. In the toroidal plasma shown, the diamagnetic current flows mainly in the direction

(A) $+\hat{\boldsymbol{\phi}}$
(B) $-\hat{\boldsymbol{\phi}}$
(C) $+\hat{\boldsymbol{\theta}}$
(D) $-\hat{\boldsymbol{\theta}}$
(E) $+\hat{\mathbf{z}}$



7. In the torus shown above, torsional Alfvén waves can propagate in the directions

(A) $\pm\hat{\mathbf{r}}$
(B) $\pm\hat{\boldsymbol{\theta}}$
(C) $\pm\hat{\boldsymbol{\phi}}$
(D) $+\hat{\boldsymbol{\theta}}$ only
(E) $-\hat{\boldsymbol{\theta}}$ only

8. Plasma A is ten times denser than plasma B but has the same temperature and composition. The resistivity of A relative to that of B is

(A) 100 times smaller
(B) 10 times smaller
(C) approximately the same
(D) 10 times larger
(E) 100 times larger

9. The average electron velocity $\overline{|\mathbf{v}|}$ in a 10-keV Maxwellian plasma is

(A) 7×10^2 m/s
(B) 7×10^4 m/s
(C) 7×10^5 m/s
(D) 7×10^6 m/s
(E) 7×10^7 m/s

10. Which of the following waves cannot propagate when $B_0 = 0$?

(A) electron plasma wave
(B) the ordinary wave

(C) Alfvén wave
(D) ion acoustic wave
(E) Bohm–Gross wave

11. A “backward wave” is one which has

(A) $\mathbf{k}$ opposite to $\mathbf{B}_0$
(B) $\omega/k < 0$
(C) $d\omega/dk < 0$
(D) $\mathbf{v}_i = -\mathbf{v}_e$
(E) $\mathbf{v}_\phi$ opposite to $\mathbf{v}_g$

12. “Cutoff” and “resonance,” respectively, refer to conditions when the dielectric constant is

(A) 0 and ∞
(B) ∞ and 0
(C) 0 and 1
(D) 1 and 0
(E) not calculable from the plasma approximation

13. The lower and upper hybrid frequencies are, respectively,

(A) $(\Omega_p\Omega_c)^{1/2}$ and $(\omega_p\omega_c)^{1/2}$
(B) $\left(\Omega_p^2 + \Omega_c^2\right)^{1/2}$ and $\left(\omega_p^2 + \omega_c^2\right)^{1/2}$
(C) $(\omega_c\Omega_c)^{1/2}$ and $\left(\omega_p^2 + \omega_c^2\right)^{1/2}$
(D) $\left(\omega_p^2 - \omega_c^2\right)^{1/2}$ and $\left(\omega_p^2 + \omega_c^2\right)^{1/2}$
(E) $(\omega_R\omega_L)^{1/2}$ and $(\omega_p\omega_c)^{1/2}$

14. In a fully ionized plasma, diffusion across **B** is mainly due to

(A) ion–ion collisions
(B) electron–electron collisions
(C) electron–ion collisions
(D) three-body collisions
(E) plasma diamagnetism

15. An exponential density decay with time is characteristic of

(A) fully ionized plasmas under classical diffusion
(B) fully ionized plasmas under recombination
(C) weakly ionized plasmas under recombination
(D) weakly ionized plasmas under classical diffusion
(E) fully ionized plasmas with both diffusion and recombination

16. The whistler mode has a circular polarization which is

    (A) clockwise looking in the $+B_0$ direction
    (B) clockwise looking in the $-B_0$ direction
    (C) counterclockwise looking in the $+k$ direction
    (D) counterclockwise looking in the $-k$ direction
    (E) both, since the wave is plane polarized

17. The phase velocity of electromagnetic waves in a plasma

    (A) is always $>c$
    (B) is never $>c$
    (C) is sometimes $>c$
    (D) is always $<c$
    (E) is never $<c$

18. The following is *not* a possible way to heat a plasma:

    (A) Cyclotron resonance heating
    (B) Adiabatic compression
    (C) Ohmic heating
    (D) Transit time magnetic pumping
    (E) Neoclassical transport

19. The following is *not* a plasma confinement device:

    (A) Baseball coil
    (B) Diamagnetic loop
    (C) Figure-8 stellarator
    (D) Levitated octopole
    (E) Theta pinch

20. Landau damping

    (A) is caused by "resonant" particles
    (B) always occurs in a collisionless plasma
    (C) never occurs in a collisionless plasma
    (D) is a mathematical result which does not occur in experiment
    (E) is the residue of imaginary singularities lying on a semicircle

Part B (Two Hours, Open Book; Do 4 Out of 5)

1. Consider a cold plasma composed of n_0 hydrogen ions, $\frac{1}{2}n_0$ doubly ionized He ions, and $2n_0$ electrons. Show that there are two lower-hybrid frequencies and give an approximate expression for each. [Hint: You may use the plasma approximation, the assumption $m/M \ll 1$, and the formulas for $\mathbf{v}_1$ given in the

text. (You need not solve the equations of motion again; just use the known solution.)]

2. Intelligent beings on a distant planet try to communicate with the earth by sending powerful radio waves swept in frequency from 10 to 50 MHz every minute. The linearly polarized emissions must pass through a radiation belt plasma in such a way that $\mathbf{E}$ and $\mathbf{k}$ are perpendicular to $\mathbf{B}_0$. It is found that during solar flares (on their sun), frequencies between 24.25 and 28 MHz do not get through their radiation belt. From this deduce the plasma density and magnetic field there. (Hint: Do not round off numbers too early.)
3. When β is larger than m/M, there is a possibility of coupling between a drift wave and an Alfvén wave to produce an instability. A necessary condition for this to happen is that there be synchronism between the parallel wave velocities of the two waves (along B_0).

   (a) Show that the condition $\beta > m/M$ is equivalent to $v_A < v_{\mathrm{th}}$.
   (b) If $KT_e = 10$ eV, $B = 0.2$ T, $k_y = 1\ \mathrm{cm}^{-1}$, and $n = 10^{21}\ \mathrm{m}^{-3}$ find the required value of k_z for this interaction in a hydrogen plasma. You may assume $n_0'/n_0 = 1\,\mathrm{cm}^{-1}$, where $n_0' = dn_0/dr$.

4. When anomalous diffusion is caused by unstable oscillations, Fick's law of diffusion does not necessarily hold. For instance, the growth rate of drift waves depends on $\nabla n/n$, so that the diffusion coefficient $D_\perp$ can itself depend on ∇n. Taking a general form for $D_\perp$ in cylindrical geometry, namely,

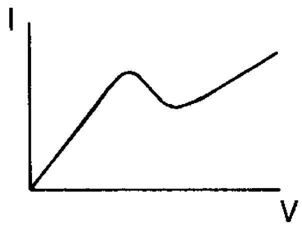
$$D_\perp = Ar^s n^p \left(\frac{\partial n}{\partial r}\right)^q.$$

   Show that the time behavior of a plasma decaying under diffusion follows the equation

$$\frac{\partial n}{\partial t} = f(r) n^{p+q+1}$$

   Show also that the behavior of weakly and fully ionized plasmas is recovered in the proper limits.

5. In some semiconductors such as gallium arsenide, the current–voltage relation looks like this:



There is a region of negative resistance or mobility. Suppose you had a substance with negative mobility for all values of current. Using the equation of motion for weakly ionized plasmas with $KT = B = 0$, plus the electron continuity equation and Poisson's equation, perform the usual linearized wave analysis to show that there is instability for $\mu_e < 0$.

Appendix D: Answers to Some Problems

1.1 (a) At standard temperature and pressure, a mole of an ideal gas contains 6.022×10^{23} molecules (Avogadro's number) and occupies 22.4 L. Hence, the number per m^3 is $6.022 \times 10^{23}/2.24 \times 10^{-2} = 2.69 \times 10^{25}\ m^{-3}$.

(b) Since $PV = NRT$, $n = N/V = P/RT$. Hence $n_1/n_0 = P_1T_0/P_0T_1$. Taking n_0 to be the density in part (a) and n_1 to be that in part (b), we have

$$n_1 = \left(2.69 \times 10^{25}\right)\frac{10^{-3}}{760}\frac{273}{(273+20)} = 3.30 \times 10^{19}\mathrm{m}^{-3}$$

Note that a diatomic gas such as H_2 will have twice as many *atoms* per torr as, say, He.

1.2

$$1 = \int_{-\infty}^{\infty}\hat{f}(u)du = Ah\int_{-\infty}^{\infty}e^{-u^2/h^2}d[u/h]$$
$$= Ah\int_{-\infty}^{\infty}e^{-x^2}dx = Ah\sqrt{\pi}$$
$$A = (2\pi KT/m)^{-1/2}$$

1.2a

$$\iint\hat{f}(u,v)dudv = 1 = A\iint e^{-\left(u^2+v^2\right)/h^2}dudv$$
$$1 = A\int_{-\infty}^{\infty}e^{-(u/h)^2}du\int_{-\infty}^{\infty}e^{-(v/h)^2}dv = Ah^2\pi$$
$$A = (2\pi KT/m)^{-1}$$

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In cylindrical coordinates,

$$1 = A\iint_{-\infty}^{\infty} e^{-\left(u^2+v^2\right)/h^2} du\,dv = A\int_0^{\infty}\int_0^{2\pi} e^{-r^2/h^2} r dr d\phi = 2\pi A\int_0^{\infty} e^{-(r/h)^2} r dr$$

$$1 = 2\pi A h^2\int_0^{\infty} e^{-x^2} x dx = \pi A h^2\int_0^{\infty} e^{-y} dy = -\pi A h^2 e^{-y}\Big|_0^{\infty} = \pi A h^2$$

$$A = (2\pi KT/m)^{-1}$$

1.4

$$p = n(KT_e + KT_i) = 10^{21}\left(4\times 10^4\right)\left(1.6\times 10^{-19}\right) = 6.4\times 10^6\,\mathrm{N/m^2}$$

$$1\,\mathrm{atm} \simeq 10^5\,\mathrm{N/m^2} \qquad \therefore \quad p = 64\,\mathrm{atm}$$

$$1\,\mathrm{atm} \simeq 14.7\ \mathrm{lb/in^2}. = (14.7)(144)/(2000) = 1.06\,\mathrm{tons/ft^2}$$

$$p \simeq 68\,\mathrm{tons/ft^2}$$

1.5

$$\frac{d^2\phi}{dx^2} = -\frac{e(n_i - n_e)}{\varepsilon_0} = -\frac{1}{\varepsilon_0} n_\infty e\left(e^{-e\phi/KT_i} - e^{e\phi/KT_e}\right) \simeq \frac{n_\infty e}{\varepsilon_0}\left(\frac{e\phi}{KT_i} + \frac{e\phi}{KT_e}\right)$$

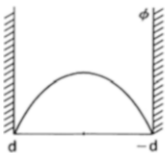
$$\phi = \phi_0 e^{-|x|/\lambda_D},\ \text{where} \frac{1}{\lambda_D^2} = \frac{n_\infty e^2}{\varepsilon_0}\left(\frac{1}{KT_e} + \frac{1}{KT_i}\right)$$

$$\text{If } T_i \ll T_e \quad \lambda_D \simeq \left(\varepsilon_0 KT_i/n_\infty e^2\right)^{1/2}$$

$$\text{If } T_e \ll T_i \quad \lambda_D \simeq \left(\varepsilon_0 KT_e/n_\infty e^2\right)^{1/2}$$

However, this result is deceptive because in most experiments the ions move too slowly to shield charges. Electrons do the shielding, so λ_D depends on T_e, even when $T_e \gg T_i$, which is the usual case.

1.6 (a)



$$\frac{d^2\phi}{dx^2} = -\frac{nq}{\varepsilon_0}$$

Let $\phi = Ax^2 + Bx + C$; $\phi' = 2Ax + B$; $\phi'' = 2A$. At $x=0$, $\phi' = 0$ by symmetry $\therefore B = 0$. At $x = \pm d$, $\phi = 0$; therefore, $0 = Ad^2 + C$ and $C = -Ad^2$.

$$\phi'' = 2A = -\frac{nq}{\varepsilon_0} \;\therefore\; A = -\frac{1}{2\varepsilon_0}nq,$$

$$\phi = Ax^2 - Ad^2 = \frac{1}{2\varepsilon_0}nq\left(d^2 - x^2\right)$$

(b) Energy E to move a charge q from x_1 to x_2 is the change in potential energy $\Delta(q\phi) = q(\phi_2 - \phi_1)$. Let $\phi_1 = 0$ at $x = \pm d$ and $\phi_2 = (\frac{1}{2}\varepsilon_0)nqd^2$ at $x = 0$. Then

$$E = \frac{1}{2\varepsilon_0}nq^2d^2.$$

Let $d = \lambda_D$; then

$$E = \frac{1}{2\varepsilon_0}nq^2\frac{KT\varepsilon_0}{nq^2} = ½KT = E_{av}$$

for a one-dimensional Maxwellian distribution. Hence, if $d > \lambda_D$, $E > E_{av}$. If the velocities are distributed in three dimensions, we have $E_{av} = \frac{3}{2}KT$ and $E > \frac{1}{3}E_{av}$. The factor 3 is not important here. The point is that a thermal particle would not have enough energy to go very far in a plasma ($d >> \lambda_D$) if the charge of one species is not neutralized by another species.

1.7 (a) $\lambda_D = 7400(2/10^{16})^{1/2} = 10^{-4}$ m, $N_D = 4.8 \times 10^4$.

(b) $\lambda_D = 7400(0.1/10^{12})^{1/2} = 2.3 \times 10^{-3}$ m, $N_D = 5.4 \times 10^4$.

(c) $\lambda_D = 7400(800/10^{23})^{1/2} = 6.6 \times 10^{-7}$ m, $N_D = 1.2 \times 10^5$.

1.8

$$\begin{aligned} N_D &= 1.38 \times 10^6 T^{3/2}/n^{1/2} \\ &= \left(1.38 \times 10^6\right)\left(5 \times 10^7\right)^{3/2}/\left(10^{33}\right)^{1/2} \\ &= 15.4 \end{aligned}$$

1.9 From Eq. (1.18), $\lambda_D = 69(T/n)^{1/2}$m, T in °K

From Problem 1.5, $\lambda_D{}^{-2} = \frac{1}{69^2}\frac{n}{T} = \frac{10^6}{4760}\left(\frac{1}{100} + \frac{1}{100}\right) = 4.20$

Hence, $\lambda_D =$ 0.49 m. The particle masses do not matter.

1.10 $\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{e}{\varepsilon_0}(n_e - n_i) = \frac{e}{\varepsilon_0}n_0\left(e^{e\phi/KT_e} - 1\right) \approx \frac{e^2 n_0}{\varepsilon_0 K T_e}\phi = \phi/\lambda_D^2 = \kappa^2\phi$ where $\kappa \equiv 1/\lambda_D$.

Let $\phi = \Phi\frac{e^{-kr}}{r}$, where Φ is a constant in units of V-m.

$$\frac{d\phi}{dr} = -\Phi\left(\frac{1}{r^2}e^{-kr} + \frac{k}{r}e^{-kr}\right) = -e^{-kr}\frac{\Phi}{r^2}(1+kr) \qquad r^2\frac{d\phi}{dr} = \Phi(1+kr)e^{-kr}$$

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\Phi\left[ke^{-kr} - k(1+kr)e^{-kr}\right] = -k\Phi e^{-kr}[1-(1+kr)] = k^2r^2\phi$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = k^2\phi \quad \therefore \quad k^2 = \kappa^2$$

$$\phi(r) = \Phi\frac{e^{-\kappa r}}{r} = \Phi\frac{e^{-r/\lambda_D}}{r}, \quad \phi_0 = \Phi\frac{e^{-\kappa a}}{a}, \quad \Phi = \frac{a\phi_0}{e^{-\kappa a}}$$

$$\phi = a\phi_0 e^{\kappa a}\frac{e^{-\kappa r}}{r} = \phi_0\frac{a}{r}e^{-\kappa(r-a)} = \phi_0\frac{a}{r}e^{-(r-a)/\lambda_D}$$

1.11 Let $T_e = 300$ K. Then $\lambda_D = 69(T/n)^{1/2} = 69\left(300/\left(10^{22}\right)^{1/2} = 1.20 \times 10^{-8} =$ 12 nm

1.12 From Eq. (1.13), $f(u) = A\exp\left(-mu^2/2KT_e\right)$. From Eq. (1.6) and Problem 1.2,

$$v_{th} \equiv (2KT_e/m)^{1/2} \text{ and } A = n(2\pi KT/m)^{-1/2} = n/v_{th}\sqrt{\pi}$$

So $f(u) = (n/v_{th}\sqrt{\pi})\exp\left(-u^2/v_{th}^2\right)$. We wish to integrate $f(u)$ from u_{crit} to ∞ and from $-u_{\text{crit}}$ to $-\infty$. $\int_{u_{crit}}^{\infty} f(u)du = \frac{n}{v_{th}\sqrt{\pi}}\int_{u_{crit}}^{\infty} e^{-u^2/v_{th}^2}du = \frac{n}{\sqrt{\pi}}\int_{y_{crit}}^{\infty} e^{-y^2}dy$, where $y \equiv u/v_{th}$.

Let $½mu_{crit}^2 = \mathrm{e}V_{ioniz}$, $u_{crit} = (2\mathrm{e}V_{ioniz}/m)^{1/2}$, where $V_{ioniz} = 15.8$ eV.

The error function erf(x) is defined as $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt$

The integral from u_{crit} to ∞ is then $\frac{n}{\sqrt{\pi}}\int_{y_{crit}}^{\infty} e^{-y^2}dy = \frac{n}{2}\mathrm{erfc}(y_{crit})$, where $y_{crit} = u_{crit}/v_{th} = \left(\frac{2eV_{ioniz}}{m}\right)^{1/2}\left(\frac{m}{2KT_e}\right)^{1/2} = \left(\frac{eV_{ioniz}}{KT_e}\right)^{1/2}$. This density has to be doubled to account for negative velocities u. Finally, the fraction of electrons that can ionize is

$$\frac{\Delta n}{n} = \operatorname{erfc}\left(\frac{eV_{ioniz}}{KT_e}\right)^{1/2}.$$

2.1 (a) $E = \frac{1}{2}mv_\perp^2 \quad \therefore v_\perp = (2E/m)^{1/2}, r_L = mv_\perp/eB.$

$$v_\perp = \left[\frac{(2)(10^4)(1.6\times10^{-19})}{9.11\times10^{-31}}\right]^{1/2} = 5.93\times10^7\,\text{m}/s$$

$$r_L = \frac{(9.11\times10^{-31})(5.93\times10^7)}{(1.6\times10^{-19})(0.5\times10^{-4})} = 6.75\,\text{m}$$

(b)

$$v_\perp = (300)(1000) = 3\times10^5\,\text{m}/s$$

$$r_L = \frac{(1.67\times10^{-27})(3\times10^5)}{(1.6\times10^{-19})(5\times10^{-9})} = 6.26\times10^5\,\text{m} = 626\,\text{km}$$

(c)

$$v_\perp = \left[\frac{(2)(10^3)(1.6\times10^{-19})}{(4)(1.67\times10^{-27})}\right]^{1/2} = 2.19\times10^5\,\text{m}/s$$

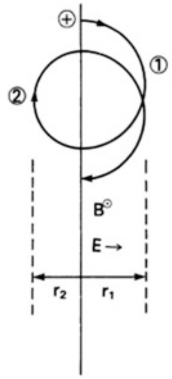
$$r_L = \frac{(4)(1.67\times10^{-27})(2.19\times10^5)}{(1.6\times10^{-19})(5.00\times10^{-2})} = 0.183\,\text{m}$$

(d)

$$r_L = \frac{2ME}{qB} = \frac{[(2)(4)(1.67\times10^{-27})(3.5\times10^6)(1.6\times10^{-19})]^{1/2}}{(2)(1.6\times10^{-19})(8)}$$

$$= 3.38\times10^{-2}\,\text{m}$$

2.4 Let initial energy be $\mathcal{E}_0$, and Larmor radii r_1 and r_2, as shown. Energy at ① is $\mathcal{E}_1 = \mathcal{E}_0 + eEr_1$; energy at ② is $\mathcal{E}_2 = \mathcal{E}_0 - eEr_2$. (It would be acceptable to say: $\mathcal{E}_{1,2} = \mathcal{E}_0 \pm eE\bar{r}_L$ here.) Also $v_{\perp 1,2}^2 = 2\mathcal{E}_{1,2}/M$. We are asked to make the approximation



$$r_{1,2} = \frac{Mv_{\perp 1,2}}{eB} = \frac{M}{eB}\left(\frac{2\mathcal{E}_{1,2}}{M}\right)^{1/2}$$
$$= \frac{1}{\Omega_c}\left(\frac{2\mathcal{E}_0}{M}\right)^{1/2}\left(1 + \frac{eE}{\mathcal{E}_0} r_{1,2}\right)^{1/2}$$

For small E, expand the square root in a Taylor series:

$$r_{1,2} \simeq \frac{1}{\Omega_c}\left(\frac{2\mathcal{E}_0}{M}\right)^{1/2}\left(1 \pm \frac{1}{2}\frac{eE}{\mathcal{E}_0} r_{1,2}\right)$$

$$r_{1,2} = \frac{1}{\Omega_c}\left(\frac{2\mathcal{E}_0}{M}\right)^{1/2}\left[1 \pm \frac{1}{2}\frac{eE}{\mathcal{E}_0}\frac{1}{\Omega_c}\left(\frac{2\mathcal{E}_0}{M}\right)^{1/2}\right]^{-1}$$

$$\simeq \frac{1}{\Omega_c}\left(\frac{2\mathcal{E}_0}{M}\right)^{1/2}\left[1 \pm \frac{1}{2}\frac{eE}{\mathcal{E}_0}\frac{1}{\Omega_c}\left(\frac{2\mathcal{E}_0}{M}\right)^{1/2}\right]$$

Thus

$$r_1 - r_2 = \frac{eE}{\mathcal{E}_0}\frac{1}{\Omega_c^2}\left(\frac{2\mathcal{E}_0}{M}\right) = \frac{2eE}{M\Omega_c^2}$$

independent of $\mathcal{E}_0$. The guiding center moves a distance $2(r_1 - r_2)$ in a time $2\pi/\Omega_c$, so

$$v_{gc} = 2(r_1 - r_2)(\Omega_c/2\pi) = \frac{4eE}{M\Omega_c}\frac{1}{2\pi} = \frac{2}{\pi}\frac{E}{B} \simeq \frac{E}{B}$$

Thus the guiding center drift is independent of the ion energy $\mathcal{E}_0$. The factor $2/\pi$ would be 1 if we did not make the crude approximation.

2.5

(a)
$$n = n_0 e^{e\phi/KT_e} \quad \therefore \quad \phi = (KT_e/e)\ln(n/n_0)$$
$$\mathbf{E} = -\frac{\partial \phi}{\partial r}\hat{\mathbf{r}} = -\frac{KT_e}{e}\frac{1}{n}\frac{\partial n}{\partial r}\hat{\mathbf{r}} = \frac{KT_e}{e\lambda}\hat{\mathbf{r}}$$

(b) $\mathbf{v}_E = -\dfrac{E_r}{B}\hat{\boldsymbol{\theta}} = -\dfrac{KT_e}{eB\lambda}\hat{\boldsymbol{\theta}}$

Consider electrons:

$$v_{\text{th}} = \left(\frac{2KT_e}{m}\right)^{1/2} \therefore \ \left|v_E\right| = \frac{KT_e}{m}\frac{m}{eB}\frac{1}{\lambda} = \frac{1}{2}\frac{v_{\text{th}}^2}{\omega_c}\frac{1}{\lambda}$$

Now, $r_L = mv_\perp/eB$, so for a distribution of velocities we must find an average r_L. Since $v_\perp$ contains two degrees of freedom, we have

$$\tfrac{1}{2}m\overline{v_\perp^2} = 2 \times \tfrac{1}{2}KT_e$$

The most convenient average is

$$\langle v_\perp \rangle_{\text{rms}} = (2KT_e/m)^{1/2} = v_{\text{th}}$$

Using this for $v_\perp$ in r_L, we have $\left|v_E\right| = \dfrac{1}{2}\dfrac{v_{\text{th}}}{\lambda}\dfrac{v_\perp}{\omega_c} = \dfrac{1}{2}\dfrac{v_{\text{th}} r_L}{\lambda}$

so that $|v_E| = v_{\text{th}}$ implies $r_L = 2\lambda$.

(c) If we take ions instead of electrons, we have $v_{\text{th}i} = (2KT_i/M)^{1/2} = v_{\perp i}$, $r_{Li} = v_{\perp i}/\omega_{ci}$, and $\left|v_E\right| = \dfrac{1}{2\lambda}\left(\dfrac{2KT_e}{M}\right)\left(\dfrac{M}{eB}\right) = \dfrac{1}{2\lambda}\dfrac{T_e}{T_i}\dfrac{v_{\text{th}i} v_{\perp i}}{\omega_{ci}} = \dfrac{1}{2\lambda}\dfrac{T_e}{T_i} v_{\text{th}i} r_{Li}$.

If $|v_E| = v_{\text{th}i}$, it is still true that $r_{Li} = 2\lambda$ *provided* that $T_i = T_e$.

2.6 (a) $n = n_0 \exp\left(e^{-r^2/a^2} - 1\right) = n_0 \ e^{e\phi/KT_e} \ \therefore \ \dfrac{e}{KT_e}\phi(r) = e^{-r^2/a^2} - 1$

$$\mathbf{E} = -\nabla\phi = \frac{\partial \phi}{\partial r}\hat{\mathbf{r}} \qquad E_r(r) = -\frac{\partial \phi}{\partial r} = \frac{KT_e}{e}\frac{2r}{a^2}e^{-r^2/a^2}$$
$$\frac{dE_r}{dr} = \frac{2KT_e}{ea^2}\left(1 - \frac{2r^2}{a^2}\right)e^{-r^2/a^2} = 0 \qquad \frac{r^2}{a^2} = \frac{1}{2}$$
$$E_{\max} = \frac{KT_e}{ea}\frac{2}{\sqrt{2}}e^{-1/2} = \frac{(0.2)\left(1.6 \times 10^{-19}\right)}{\left(1.6 \times 10^{-19}\right)(.01)}\sqrt{2}e^{-1/2} = 17\,\text{V/m}$$

$$\mathbf{v}_E = -\frac{E_r}{B}\hat{\boldsymbol{\theta}} \qquad V_{E\max} = \frac{E_{\max}}{B} = \frac{17}{0.2} = 8500 \;\; \mathrm{m}/s$$

(b) Compare the force Mg with the force eE for an ion. (mg for an electron would be 1836 times smaller.) $g = 9.80$ m/s^2. $Mg = (39)(1.67 \times 10^{-27})(9.80) = 6.38 \times 10^{-25}$ N. $eE_{\max} = (1.6 \times 10^{-19})(17) = 2.75 \times 10^{-18}$ N $= 4 \times 10^6 Mg$. Hence gravitational drift is 4 million times smaller.

(c)

$$r_L = \frac{Mv_\perp}{eB} = 10^{-2}\mathrm{m}$$

$$v_\perp = (2KT/M)^{1/2} = \left[\frac{(2)(0.2)\left(1.6 \times 10^{-19}\right)}{(39)\left(1.67 \times 10^{-27}\right)}\right]^{1/2}$$

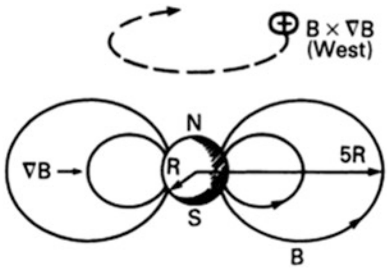
$$= 9.9 \times 10^2 \mathrm{m/s}$$

$$B = \frac{(39)\left(1.67 \times 10^{-27}\right)\left(9.9 \times 10^2\right)}{\left(10^{-2}\right)\left(1.6 \times 10^{-19}\right)} = 4.00 \times 10^{-2}\mathrm{T}$$

2.8

$$B = \frac{c}{r^3} = \frac{0.3 \times 10^{-4}}{(r/R)^3}\mathrm{T}$$

$$v_{\nabla B} = \frac{1}{2}v_\perp r_L \left|\frac{\mathbf{B} \times \nabla\mathbf{B}}{B^2}\right| = \frac{1}{2}v r_L \left|\frac{\nabla B}{B}\right|$$



(a)

$$\nabla B = \frac{\partial B}{\partial r}\hat{\mathbf{r}} = -3\frac{c}{r^4}\hat{\mathbf{r}} = \frac{3}{r}B(-\hat{\mathbf{r}}) \qquad \left|\frac{\nabla B}{B}\right| = \frac{3}{r}$$

$$\frac{1}{2}v_\perp r_L = \frac{1}{2}\frac{v_\perp^2}{\omega_c} = \frac{1}{2}\frac{2KT/m}{eB/m} = \frac{KT}{eB} = \frac{\left(1.6 \times 10^{-19}\right)(KT)_{eV}}{1.6 \times 10^{-19}}\frac{1}{B} = \frac{(KT)_{eV}}{B}$$

$$B(r = 5R) = \frac{0.3 \times 10^{-4}}{5^3} = 2.4 \times 10^{-7}\mathrm{T}$$

$5R = (5)(4000 \text{ miles})(1.6 \text{ km/mile})(10^3 \text{ m/km}) = 3.2 \times 10^7$ m

$$v_{\nabla B} = 10^8 \frac{(KT)_{\text{eV}}}{2.4 \times 10^{-7}} = 0.39(KT)_{\text{eV}} \text{m}/s$$

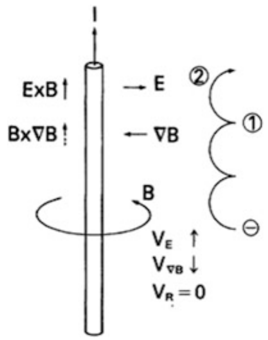
$$\text{Ions}: KT = 1\,\text{eV} \qquad v_{\nabla B} = \underline{0.39\text{m}/s}$$

$$\text{Electrons}: KT = 3 \times 10^4 \text{eV} \qquad v_{\nabla B} = \underline{1.17 \times 10^4 \text{m}/s}$$

(b) Ions: westward; electrons: eastward.
(c) $2\pi r = (6.28)(3.2 \times 10^7) = 2.0 \times 10^8$ m

$$t = \frac{2\pi r}{v_{\nabla B}} = \frac{(2.0 \times 10^8)}{(1.17 \times 10^4)} = 1.7 \times 10^4 s = 4.8\,\text{h}$$

(d)
$$j = nev_{\nabla B} \qquad \text{neglect ions}$$
$$= (10^7)(1.6 \times 10^{-19})(1.17 \times 10^4) = 1.87 \times 10^{-8} \text{A}/\text{m}^2$$



2.9 (a) $v_R = 0$, since the electron gains no energy in the parallel $(\hat{\boldsymbol{\theta}})$ direction. Since the electron starts at rest with no thermal energy, it will come back to rest after one cycle. Hence, the orbit has sharp cusps instead of loops. It is clear that the v_E drift must dominate, since the electron starts to the left, and the Lorentz force makes it move upwards.
(b) In cylindrical geometry, $\phi = A \ln r + B$. Since

$$\begin{aligned}
\phi\left(10^{-3}\right) &= 460\text{V} \quad \text{and} \quad \phi(0.1\ \text{m}) = 0,\\
460 &= A\ \ln\ \left(10^{-3}\right) + B\\
0 &= A\ \ln\ (0.1) + B \qquad B = -A\ \ln\ (0.1)\\
460 &= A\ \ln\ \left(10^{-3}\right) - A\ \ln\ (0.1)\\
&= A\ \ln\ (0.01) \qquad A = 460/\ln\ (0.01)\\
\phi(r) &= \frac{460}{\ln(0.01)}[\ln r - \ln(0.1)] = 460\frac{\ln(0.1r)}{\ln 100}V\\
E &= \frac{-\partial\phi}{\partial r} = \frac{-460}{\ln 100}\left(\frac{r}{0.1}\right)\left(\frac{-0.1}{r^2}\right) = \frac{460/r}{\ln 100}\frac{V}{m}\\
&= \frac{460}{(4.6)(1)} = 10^4\frac{V}{m}\text{at } r = 10^{-2}\text{m}\\
B &= \frac{I(A)10^{-4}}{5r} = \frac{500 \times 10^{-4}}{(5)(1)} = 0.01\,\text{T}\\
|v_E| &= |E/B| = 10^8\frac{10^4\text{V/cm}}{0.01\,\text{T}} = 10^6\text{m}/s
\end{aligned}$$

To estimate the ∇B drift, we must find $v_\perp$ in the frame moving with the guiding center. Remember that in deriving $v_{\nabla B}$, $v_\perp$ was taken as the velocity in the undisturbed circular orbit. Here, the latter is moving with velocity v_E, so that it does not look circular in the lab frame. Nonetheless, it can still be decomposed into a circular motion with velocity $v_\perp$ plus an $E \times B$ drift of the guiding center. Consider the z component of velocity (along the wire). At point ① on the orbit, $v_z = v_E + v\ \cos\ \omega_c t = 0$, where $\cos\ \omega_c t = -1$, its maximum negative value; hence, $v_E = v_\perp$. The same result can be obtained by considering that at point ② $v_z = v_E + v_\perp$ ($\cos\ \omega_c t = 1$). The energy there, $\frac{1}{2}\left(mv_z^2\right)$, must equal the energy gained in falling a distance $2r_L$ in an electric field. Thus

$$\begin{aligned}
&\frac{1}{2}m(v_E + v_\perp)^2 2r_L eE = 2eE\frac{mv_\perp}{eB} = 2mv_\perp\frac{E}{B} = 2mv_\perp v_E\\
&v_E^2 + 2v_\perp v_E + v_\perp^2 = 4v_\perp v_E \quad (v_E - v_\perp)^2 = 0 \quad v_E = v_\perp
\end{aligned}$$

Now we can calculate $v_{\nabla B}$:

$$\begin{aligned}
v_{\nabla B} &= \frac{1}{2}\frac{v_\perp^2}{\omega_c}\left|\frac{\nabla B}{B}\right| \quad \omega_c = \frac{eB}{m} = \frac{\left(1.6 \times 10^{-19}\right)\left(10^{-2}\right)}{\left(9.11 \times 10^{-31}\right)} = 1.76 \times 10^9 s^{-1}\\
\frac{dB}{dr} &= \frac{I(-1)10^{-4}}{r^2} = -\frac{B}{r} \qquad \left|\frac{\nabla B}{B}\right| = 10^2\text{m}^{-1}\\
v_{\nabla B} &= \frac{1}{2}\frac{v_E^2}{\omega_c} = \frac{1}{2 10^{16} 1.8 \times 10^9} = \underline{2.8 \times 10^4\text{m}/s}
\end{aligned}$$

This amounts to a slowing down of the v_E drift due to a distortion of the orbit into a hairpin shape ⋲ because of the change in Larmor radius.

The *undisturbed* orbit is the path taken by the valve on a bicycle wheel as it rolls along:

Finally, we note that the finite Larmor radius correction to v_E is negligible:

$$\frac{1}{4}r_L^2\nabla^2\frac{E}{B} \simeq \frac{1}{4}\frac{r_L^2}{r^2}\frac{E}{B}$$

$$r_L = \frac{\left(9.11\times10^{-31}\right)\left(10^6\right)}{\left(1.6\times10^{-19}\right)(0.01)} = 5.7\times10^{-4}\text{m}$$

$$r \simeq 10^{-2}\text{m} \quad \therefore \frac{1}{4}\frac{r_L^2}{r^2} = 0.08\%$$

2.12 Let all velocities refer to the midplane, and let subscripts i and f refer to initial and final states (before and after acceleration).

(a) Given: $\boldsymbol{R}_m = 5$, $v_{\perp i} = v_{\parallel i}$ since μ is conserved, $v_{\perp f} = v_{\perp i}$, and only $v_\parallel$ will increase. It will increase until the pitch angle θ reaches the loss cone:

$$\sin^2\theta_m = \frac{v_{\perp f}^2}{v_{\perp f}^2 + v_{\parallel f}^2} = \frac{1}{1 + v_{\parallel f}^2/v_{\perp i}^2} = \frac{1}{\boldsymbol{R}_m} = \frac{1}{5}$$

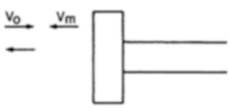
Hence $v_{\parallel f}^2/v_{\perp i}^2 = 4$, $v_{\parallel f} = 2v_{\perp i}$. Energy is

$$E_f = \frac{1}{2}M\left(v_{\parallel f}^2 + v_{\perp f}^2\right) = \frac{1}{2}M(4+1)v_{\perp i}^2 = \frac{5}{2}mv_{\perp i}^2$$

$$E_i = \frac{1}{2}M\left(v_{\parallel i}^2 + v_{\perp i}^2\right) = \frac{1}{2}M(1+1)v_{\perp i}^2 = Mv_{\perp i}^2$$

$$\therefore E_f = 2.5E_i = (2.5)(1) = \underline{2.5\text{keV}}$$

(b) (1) Let particle have $v_0 > 0$ and hit piston moving at velocity $v_\text{m} < 0$. *In the frame of the piston*, the particle bounces elastically and comes off with its initial velocity, but in the opposite direction. Let $'$ refer to the frame of the piston. Initial and final velocities in this frame are



$$v_i' = v_0 - v_m \quad v_f' = -(v_0 - v_m)$$

(Note: v_m is negative.) Transforming back to lab frame,

$$v_f = v_f' + v_m = -v_0 + 2v_m$$

Since v_m is negative, the change in velocity is $2|v_m|$. QED

(2) At each bounce, the change in momentum is $\Delta p_{\|} = 2\,m|v_m|$. If N is the number of bounces, $p_{\|f} = p_{\|i} + N\Delta p$. Thus

$$N = \frac{p_{\|f} - p_{\|i}}{\Delta p} = \frac{v_{\|f} - v_{\|i}}{2v_m} = \frac{2v_{\perp i} - v_{\perp i}}{2v_m} = \frac{1v_{\perp i}}{2vm}$$

$$E_i = Mv_{\perp i}^2 = 1\ \text{keV} = \left(10^3\right)\left(1.6 \times 10^{-19}\right) = 1.6 \times 10^{-16}\text{J}$$

$$\therefore v_{\perp i} = \left(\frac{1.6 \times 10^{-16}}{1.67 \times 10^{-27}}\right)^{1/2} = 3.1 \times 10^5 \text{m}/s$$

$$v_m = 10^4 \text{m}/s$$

$$\therefore N = \frac{1}{2}\frac{\left(3 \times 10^5\right)}{10^4} = 15\,\text{bounces}$$

(3) Average $v_{\|}$ is

$$\overline{v} = \frac{1}{2}\left(v_{\|i} + v_{\|f}\right) = \frac{1}{2}\left(v_{\perp i} + 2v_{\perp i}\right)$$

$$= \frac{3}{2}v_{\perp i} = 4.6 \times 10^5$$

$$L = 10^{13}\text{m}$$

$$\therefore t = \frac{NL}{\overline{v}} = \frac{(15)\left(10^{13}\right)}{4.6 \times 10^5} = 3.2 \times 10^8 s$$

$$(= 10y)$$

However, L changes during this time by a distance

$$\Delta L = 2v_m t = (2)\left(10^4\right)\left(3.2 \times 10^8\right) = 6.4 \times 10^{12}\text{m}$$

so that actual time is more like 2.5×10^8 s. Since only factor-of-two accuracy is required, it is not necessary to sum the series—the above answer of 3.2×10^8 s will do.

2.13 (a)

$$\int v_{\|} ds \simeq v_{\|} L = \text{constant} \ \therefore \dot{v}_{\|} L + v_{\|} \dot{L} = 0$$

(b)

$$\frac{\dot{v}_{\|}}{v_{\|}} = -\frac{\dot{L}}{L} \qquad \dot{v}_{\|} \simeq \frac{\Delta v_{\|}}{T} = \frac{v_{\|}}{L}\left(-\dot{L}\right)$$

$$T \simeq \frac{\Delta v_{\|}}{\overline{v}_{\|}} \frac{L}{-\dot{L}} = \frac{2v_{\perp i} - v_{\perp i}}{\frac{1}{2}(2v_{\perp i} + v_{\perp i})} \frac{L}{2v_m} = \frac{2}{3} \frac{10^{13}}{2 \times 10^4}$$
$$= \underline{3.3 \times 10^8 s}$$

2.14 As B increases, Maxwell's equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ predicts an E-field. This induced E-field has a component along $\mathbf{v}$ and accelerates the particle. If B increases slowly and adiabatically, E will be small; but the integrated effect over many Larmor periods will be finite. The invariance of μ allows us to calculate the energy increases without doing this integration.

3.1 $\partial\sigma/\partial t + \nabla \cdot \mathbf{j} = 0$, where $\mathbf{j} = \mathbf{j}_P = \left(\rho/B^2\right)\dot{\mathbf{E}}$. Hence, $\dot{\sigma} = -\nabla \cdot \left[\left(\rho/B^2\right)\dot{\mathbf{E}}\right]$. The time derivative of Poisson's equation is $\nabla \cdot \dot{\mathbf{E}} = \dot{\sigma}/\epsilon_0$

$$\therefore \nabla \cdot \dot{\mathbf{E}} = -\left(\frac{1}{\epsilon_0}\right)\nabla \cdot \left(\frac{\rho}{B^2}\right)\dot{\mathbf{E}} \quad \nabla \cdot \left(1 + \frac{\rho}{\epsilon_0 B^2}\right)\dot{\mathbf{E}} = 0$$

Assuming the dielectric constant ϵ to be constant in time, we have $\nabla \cdot \dot{\mathbf{D}} = \nabla \cdot \left(\epsilon\dot{\mathbf{E}}\right) = 0$. By comparison, $\epsilon = 1 + \rho/\epsilon_0 B^2$.

3.2

$$\epsilon \simeq 1 + \frac{nM}{\epsilon_0 B^2} \simeq \frac{\Omega_p^2}{\Omega_c^2} = \frac{ne^2}{\epsilon_0 M} \frac{M^2}{e^2 B^2} = \frac{nM}{\epsilon_0 B^2}$$

True if $\epsilon \gg 1$.

3.3 Take divergence of Eqs. (3.56) and (3.58):

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot \dot{\mathbf{B}} = 0 \ \therefore \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$$

$\therefore$ $\nabla \cdot \mathbf{B} = 0$ if it is initially zero. This is Eq. (3.57),

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0[q_i \nabla \cdot (n_i \mathbf{v}_i) + q_e \nabla \cdot (n_e \mathbf{v}_e)] + \frac{\nabla \cdot \dot{\boldsymbol{E}}}{c^2}$$

from Eq. (3.60), $\nabla \cdot (n_i \mathbf{v}_i) = -\dot{n}_i, \nabla \cdot (n_e \mathbf{v}_e) = -\dot{n}_e$

$$\therefore \mu_0(-q_i \dot{n}_i - q_e \dot{n}_e) + \frac{\nabla \cdot \dot{\mathbf{E}}}{c^2} = 0$$
$$\frac{\partial}{\partial t}\left[\nabla \cdot \boldsymbol{E} - \frac{1}{\epsilon_0}(n_i q_i + n_e q_e)\right] = 0$$

If [] = 0 initially, $\nabla \cdot \mathbf{E} = (1/\epsilon_0)(n_i q_i + n_e q_e)$. This is Eq. (3.55).

3.4

$$\mathbf{j}_D = (KT_i + KT_e)\frac{\mathbf{B}\times\nabla n}{B^2} \propto \frac{KT}{e}\frac{ne}{BL}$$

Since $KT \propto e\phi$ and $E \propto -\phi/L$, $KT/eL \propto E$ $\therefore$ $j_D \propto neE/B \propto nev$, since $E/B = v_E$.

3.5 Let j_D be constant in the box of width L. $\mathbf{\Delta}n = n'L$, $|J_D| = |\mathbf{\Delta}nev_y| = |n'Lev_y|$: from the difference between the currents on the two walls. This current J_D is over a box of width L, so the equivalent current density is

$$|j_D| = |J_D|/L = |n'ev_y|$$

Equation [3.69] gives $|j_D| \simeq |KT\nabla n/B| = |KTn'/B|$; hence, once v_y is chosen so the two formulas agree for one value of L, they agree for all L, since L cancels out.

3.6 (a)

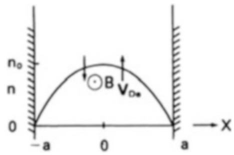
$$\mathbf{v}_{De} = -\frac{\gamma KT_e}{eB}\frac{\hat{z}\times\nabla n}{n}$$

Isothermal means $\gamma = 1$.

$$\nabla n = \hat{\mathbf{x}}\frac{\partial n}{\partial x} = -\frac{n_0 2x}{a^2}\hat{\mathbf{x}}$$

$$\mathbf{v}_{De} = \hat{\mathbf{y}}\frac{KT_e}{eB}\frac{2n_0}{a^2}\frac{x}{n_0}\left(1-\frac{x^2}{a^2}\right)^{-1} = \hat{\mathbf{y}}\frac{KT_e}{eB}\frac{2x}{a^2}\left(1-\frac{x^2}{a^2}\right)^{-1}$$

(b)



(c) $v_{De} = (2)/(0.2)\mathbf{\Lambda}$

$$\Lambda^{-1} = \left|\frac{n'}{n}\right| = \frac{(2n_0/a^2)(a/2)}{n_0(1-a^2/4a^2)} = \frac{1/0.04}{3/4} = 33.3\,\mathrm{m}^{-1}$$

$$\therefore v_{De} = (10)(33.3) = 333\,\mathrm{m/s}$$

3.7 $n = n_0 e^{-r^2/r_0^2} = n_0 e^{e\phi/KT_e}$

$$\phi = \frac{KT_e}{e}\ln\frac{n}{n_0} = \frac{KT_e}{e}\left(-\frac{r^2}{r_0^2}\right)$$

(a)

$$\mathbf{E} = -\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} = \frac{KT_e}{e}\frac{2r}{r_0^2}\hat{\mathbf{r}}$$

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = -\frac{E_r}{B_z}\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{2r}{r_0^2}$$

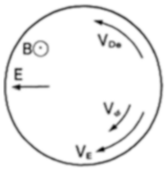
$$\mathbf{v}_{De} = -\frac{\mathbf{B}\times\nabla p}{enB^2} = -\frac{KT_e}{eB}\frac{\partial n/\partial r}{n}\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{\partial}{\partial r}(\ln n)$$

$$= -\hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{\partial}{\partial r}\left(\frac{-r^2}{r_0^2}\right) = \hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{2r}{r_0^2} = -\mathbf{v}_E \quad \text{QED}$$

(b) From (a), the rotation frequency is constant whether we take $\mathbf{v}_E$, $\mathbf{v}_{De}$, $\mathbf{v}_{Di}$, or any combination thereof, since $\omega = v_\theta/r$ and $v_\theta \propto r$.

(c) In lab frame,

$$\mathbf{v} = \mathbf{v}_\phi + \mathbf{v}_E = 0.5\mathbf{v}_{De} + (-\mathbf{v}_{De})$$
$$= -\frac{1}{2}\mathbf{v}_{De}$$



3.8 (a)

$$j_D = ne(v_{Di} - v_{De}) = -\hat{\theta}\,\frac{n_0(KT_e + KT_i)}{B}\cdot\frac{2r}{r_0^2}e^{-r^2/r_0^2}$$

(b)

$$j_D = \frac{(10^{16})(0.5)(1.6\times 10^{-19})}{0.4(r_0^2/2r)(2.718)} = 0.147\,\text{A/m}^2$$

or:

$$j_D = ne(|v_{De}| + |v_{Di}|)$$

$$|v_{De}| = |v_{Di}| = \frac{(KT)_{e\text{V}}}{B}\frac{2r}{r_0^2} = \frac{(0.25)2r}{0.4r_0^2} = 1.25\frac{r}{r_0^2}\text{m/s}$$

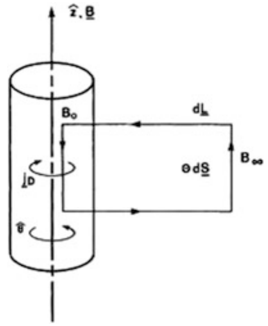
Using $e = 1.6\times 10^{-19}$ C, $\in = 2.718$,

$$j_D = (10^{16})(1.6\times 10^{-19})(2)(1.25)\frac{r\in^{-1}}{r_0^2} = 0.147\frac{\text{A}}{\text{m}^2}$$

(c)

Since $\mathbf{v}_e = \mathbf{v}_E + \mathbf{v}_{De} = \mathbf{v}_E - \mathbf{v}_E = 0$ in the lab frame, the current is carried entirely by ions.

3.9



$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_D$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int \mathbf{j}_D \cdot d\mathbf{S}$$

$$\oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 \int \mathbf{j}_\mathrm{D} \cdot d\mathbf{S}$$

Choose a loop with one leg along the axis ($B = B_0$) and one leg far away, where $B = B_\infty$. Since $\mathbf{j}_D$ is in the $-\hat{\boldsymbol{\theta}}$ direction, we can choose the direction of integration $d\mathbf{L}$ as shown, so that $\mathbf{j}_D \cdot d\mathbf{S}$ is positive. There is no B_r ∴

$$\oint \mathbf{B} \cdot d\mathbf{L} = (B_\infty - B_0)L$$

$$\mathbf{j}_D = -\hat{\boldsymbol{\theta}}\, \frac{n(KT_i + KT_e)}{B} \frac{2r}{r_0^2}$$

$$\int \mathbf{j}_D \cdot d\mathbf{S} = \frac{n_0(KT_i + KT_e)}{B_\infty r_0^2} \int_0^L \int_0^\infty e^{-r^2/r_0^2} 2r\, dr\, dz$$

$$= \frac{Ln_0(KT_i + KT_e)}{B_\infty} \left[-e^{-r^2/r_0}\right]_0^\infty = \frac{2Ln_0KT}{B_\infty}$$

where $T_e = T_i$. In this integral, we have approximated $B(r)$ by B_∞, since B is not greatly changed by such a small j_D. Thus,

$$\boldsymbol{\Delta} B = B_\infty - B_0 = \mu_0 \frac{2n_0KT}{B_\infty}$$

$$= \frac{2\left(4\pi \times 10^{-7}\right)\left(10^{16}\right)(0.25)\left(1.6 \times 10^{-19}\right)}{0.4}$$

$$= 2.5 \times 10^{-9}\mathrm{T}$$

4.1 (a) Solve for ϕ_1:

$$\phi_1 = \frac{KT_e}{e}\frac{n_1}{n_0}\frac{\omega + ia}{\omega^* + ia} \times \frac{\omega^* - ia}{\omega^* - ia}$$
$$= \frac{KT_e}{e}\frac{\omega\omega^* + a^2 + ia(\omega^* - \omega)}{\omega^{*2} + a^2}\frac{n_1}{n_0}$$

If n_1 is real,

$$\frac{Im(\phi_1)}{Re(\phi_1)} = \frac{a(\omega^* - \omega)}{\omega\omega^* + a^2} = \tan\delta$$

Hence,

$$\delta = \tan^{-1}\left[\frac{a(\omega^* - \omega)}{\omega\omega^* + a^2}\right]$$

(b) $n_1 = \overline{n}_1 e^{i(kx-\omega t)}$, while $\phi_1 = An_1 e^{i(kx-\omega t+\delta)}$, where A is a positive constant. For $\omega < \omega^*$, we have $\delta > 0$. Let the phase of n_1 be 0 at (x_0, t_0): $kx_0 - \omega t_0 = 0$. If ω and k are positive and x_0 is fixed, then the phase of ϕ_1 is 0 at $kx_0 - \omega t + \delta = 0$ or $t > t_0$. Hence ϕ_1 *lags* n_1 in time. If t_0 is fixed, $kx - \omega t_0 + \delta = 0$ at $x < x_0$, so ϕ_1 *lags* n_1 in space also (since $\omega/k > 0$ and the wave moves to the right, the leading wave is at larger x). If $k < 0$ and $\omega > 0$, the phase of ϕ_1 would be 0 at $x > x_0$; but since the wave now moves to the left, ϕ_1 still lags n_1.

4.3

$$ikE_1 = \frac{1}{\epsilon_0}e(n_{i1} - n_{e1})$$
$$-i\omega m v_{e1} = -eE_1 \quad \text{(electrons)}$$
$$-i\omega M v_{i1} = eE_1 \quad \text{(ions)}$$
$$-i\omega n_{e1} = -ikn_0 v_{e1} \quad \text{(electrons)}$$
$$-i\omega n_{i1} = -ikn_0 v_{i1} \quad \text{(ions)}$$
$$n_{e1} = \frac{k}{\omega}n_0\left(\frac{-ie}{m\omega}\right)E_1 \quad n_{i1} = \frac{k}{\omega}n_0\left(\frac{ie}{M\omega}\right)E_1$$
$$ikE_1 = \frac{1}{\epsilon_0}\frac{k}{\omega}n_0\frac{ie}{\omega}\left(\frac{1}{M} + \frac{1}{m}\right)E_1 = \frac{ikE_1}{\omega^2}\left(\Omega_p^2 + \omega_p^2\right)$$
$$\underline{\omega^2 = \left(\omega_p^2 + \Omega_p^2\right)}$$

4.4 Find ϕ_1, E_1, and v_1 in terms of n_1:

$$\text{Eq.[4-22]}: \quad v_1 = \frac{\omega}{k}\frac{n_1}{n_0}$$
$$\text{Eq.[4-23]}: \quad E_1 = \frac{ie}{\epsilon_0 k}n_1$$

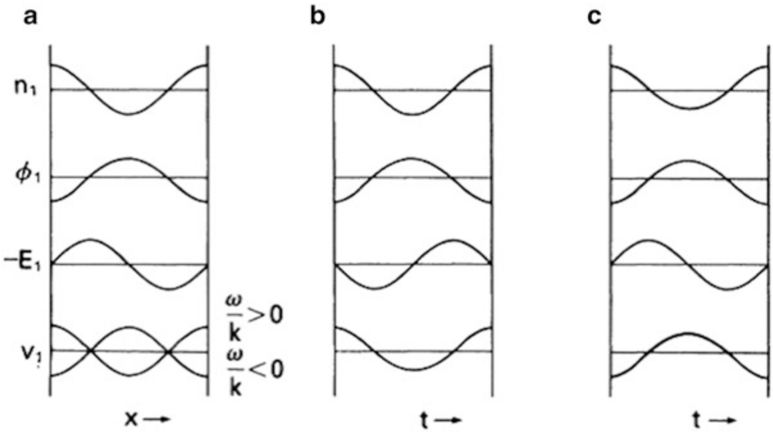
But $E_1 = -ik\phi_1$,

$$\therefore \phi_1 = -\frac{e}{\epsilon_0 k^2}n_1$$

Hence, E_1 is 90° out of phase with n_1; ϕ_1 is 180° out of phase; and v_1 is either in phase or 180° out of phase, depending on the sign of ω/k. In (a), E_1 is found from the slope of the ϕ_1 curve, since $E_1 = -\partial\phi_1/\partial x$. In (b), $E_1/n_1 \propto i \times \text{sgn}(k)$ $\therefore\ \delta = \pm\ \pi/2$. If $\omega/k > 0$,

$$E_1 \propto \exp i(kx \pm |\omega|t \pm \pi/2)$$

the $\pm$ standing for the sign of k. Hence, E_1 leads n_1 by 90°. Opposite if $\omega/k < 0$.



4.5

$$ikE_1 = -\frac{1}{\epsilon_0}en_1 = -\frac{1}{\epsilon_0}en_0\frac{k}{\omega}v_1 = -\frac{1}{\epsilon_0}en_0\frac{k}{\omega}\left(\frac{-ie}{m\omega}\right)E_1$$
$$ik\left(1 - \frac{n_0e^2}{\epsilon_0 m\omega^2}\right)E_1 = 0 \quad \text{or} \quad \nabla\cdot\left(1 - \frac{\omega_p^2}{\omega^2}\right)E_1 = 0$$
$$\therefore \epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

4.7 (a)

$$mn_0(-i\omega)v_1 = -en_0E_1 - mn_0\nu v_1$$
$$v_1\left(1+\frac{i\nu}{\omega}\right) = \frac{ieE_1}{m\omega}$$
$$ikE_1 = -\frac{1}{\epsilon_0}en_1 \quad n_1 = \frac{k}{\omega}n_0v_1 \quad \text{(continuity)}$$
$$ikE_1 = -\frac{1}{\epsilon_0}e\frac{k}{\omega}n_0\frac{ieE_1}{m\omega}\left(1+\frac{i\nu}{\omega}\right)^{-1}$$
$$\omega^2\left(1+\frac{i\nu}{\omega}\right) - \omega_p^2 \quad \underline{\omega^2 + i\nu\omega = \omega_p^2}$$

(b) Let $\omega = x + iy$. Then the dispersion relation is $x^2 - y^2 + 2ixy + i\nu x - \nu y = \omega_p^2$. We need the imaginary part: $2xy + \nu x = 0$, $y = (-1/2)\nu$ ∴ Im $(\omega) = -\nu/2$. Since $x = \text{Re}\ (\omega)$, $\nu > 0$, and

$$E_1 \propto e^{-i\omega t} = e^{-i\omega t}e^{yt} = e^{-ixt}e^{-(1/2)\nu t}$$

the oscillation is damped in time.

4.8 $mn_0(-i\omega)\mathbf{v}_1 = en_0\mathbf{E}_1 - en_0(\mathbf{v}_1 \times \mathbf{B}_0)$. Take $\mathbf{B}_0$ in the $\hat{\mathbf{z}}$ direction and $\mathbf{E}_1$ and $\mathbf{k}$ in the $\hat{\mathbf{x}}$ direction. Then the y-component is

$$-i\omega m v_y = ev_xB_0 \quad \frac{v_x}{v_y} = -i\frac{\omega}{\omega_c}$$

Since $\omega = \omega_h > \omega_c$, $|v_x/v_y| > 1$; and the orbit is elongated in the $\hat{\mathbf{x}}$ direction, which is the direction of $\mathbf{k}$.

4.9 (a)

$$\nabla\cdot\mathbf{E}_1 = -\frac{1}{\epsilon_0}en_1 \quad \mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}} \quad E_y = k_y = 0$$
$$i(k_xE_x + k_zE_z) = -\frac{1}{\epsilon_0}en_1$$

We need n_1:

$$\frac{\partial n_1}{\partial t} + n_0\nabla\cdot\mathbf{v}_1 = 0 \qquad -i\omega n_1 + n_0 i(k_xv_x + k_zv_z) = 0$$

We need v_x, v_z:

$$Mn_0(-i\omega)\mathbf{v}_1 = -en_0\mathbf{E}_1 - en_0(\mathbf{v}_1 \times \mathbf{B}_0)$$

$$x\text{-component}: \quad v_x = -\frac{ie}{m\omega}E_x - \frac{i\omega_c}{\omega}v_y$$

$$y\text{-component}: \quad v_y = 0 + \frac{i\omega_c}{\omega}v_x$$

$$v_x = -\frac{ie}{m\omega}E_x + \frac{\omega_c^2}{\omega^2}v_x = \frac{-ie}{m\omega}E_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$z\text{-component}: \quad v_z = -\frac{ie}{m\omega}E_z$$

$$\text{Continuity}: \quad n_1 = \frac{n_0}{\omega}\left(\frac{-ie}{m\omega}\right)\left[k_x E_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + k_z E_z\right]$$

$$k_x E_x + k_z E_z = i\frac{en_0}{e_0\omega}\left(\frac{-ie}{m\omega}\right)\left[k_x E_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + k_z E_z\right]$$

$$k_x = k\ \sin\ \theta \qquad k_z = k\ \cos\ \theta$$

$$\therefore E_1 \sin^2\theta + kE_1 \cos^2\theta = \frac{\omega_{\mathrm{p}}^2}{\omega^2}\left[kE_1 \sin^2\theta\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + kE_1 \cos^2\theta\right]$$

$$1 = \frac{\omega_{\mathrm{p}}^2}{\omega^2}\left[\sin^2\theta\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + \cos^2\theta\right]$$

$$1 - \frac{\omega_c^2}{\omega^2} = \frac{\omega_p^2}{\omega^2}\left[1 - \cos^2\theta + \left(1 - \frac{\omega_c^2}{\omega^2}\right)\cos^2\theta\right]$$

$$\omega^2 - \omega_c^2 - \omega_p^2 = -\frac{\omega_p^2\omega_c^2}{\omega^2}\cos^2\theta$$

$$\omega^2\left(\omega^2 - \omega_h^2\right) + \omega_p^2\omega_c^2\cos^2\theta = 0 \quad \text{QED}$$

(b)

$$\omega^4 - \omega_h^2\omega^2 + \omega_p^2\omega_c^2\cos^2\theta = 0$$

$$2\omega^2 = \omega_h^2 \pm \left(\omega_h^4 - 4\omega_p^2\omega_c^2\cos^2\theta\right)^{1/2}$$

For $\theta \to 0$, $\cos^2\theta \to 1$,

$$2\omega^2 = \omega_h^2 \pm \left[\left(\omega_p^2 + \omega_c^2\right)^2 - 4\omega_p^2\omega_c^2\right]^{1/2}$$
$$= \omega_p^2 + \omega_c^2 \pm \left(\omega_p^2 - \omega_c^2\right)$$
$$\omega^2 = \omega_p^2, \omega_c^2$$

The $\omega = \omega_p$ root is the usual Langmuir oscillation. The $\omega = \omega_c$ root is spurious because at $\theta \to 0$, B_0 does not enter the problem. For $\theta \to \pi/2$, $\cos^2\theta \to 0$, $2\omega^2$

$= \omega_h^2 \pm \omega_h^2$, $\omega = 0, \omega_h$. The $\omega = \omega_h$ root is the usual upper hybrid oscillation. The $\omega = 0$ root has no physical meaning, since on oscillating perturbation was assumed.

(c)

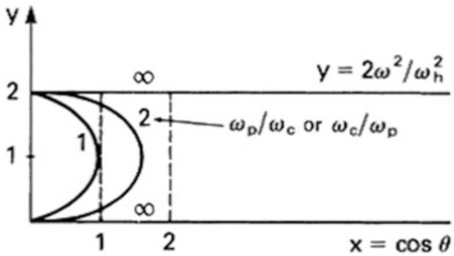
$$\omega^4 - \omega_h^2\omega^2 + \frac{1}{4}\omega_h^4 = \frac{1}{4}\omega_h^4 - \omega_p^2\omega_c^2 \cos^2\theta$$

$$\left(\omega^2 - \frac{1}{2}\omega_h^2\right) + \left(\omega_p\omega_c \cos\theta\right)^2 = \left(\frac{1}{2}\omega_h\right)^2$$

$$(y-1)^2 + \frac{x^2}{a^2} = 1 \quad \text{QED}$$

(d)

ω_p/ω_c	$a = 1/2(\omega_c/\omega_p + \omega_p/\omega_c)$
1	1
2	5/4
∞	∞



(e)

$$\omega^2 = \frac{1}{2}\left(\omega_p^2 + \omega_c^2\right) \pm \left[\left(\omega_p^2 + \omega_c^2\right)^2 - 4\omega_p^2\omega_c^2 \cos^2\theta\right)\right]^{1/2}$$

Lower root: Take (−) sign; ω is maximum when $\cos^2\theta$ is maximum (=1). Thus

$$\begin{aligned}\omega_-^2 &< \frac{1}{2}\left[\left(\omega_p^2 + \omega_c^2\right) - \left|\omega_p^2 - \omega_c^2\right|\right]\\ &= \omega_c^2 \quad \text{if } \omega_p > \omega_c\\ &= \omega_p^2 \quad \text{if } \omega_c > \omega_p\end{aligned}$$

Upper root: Take (+) sign; ω is maximum when $\cos^2\theta = 0, \omega^2 = \omega_h^2$. Thus $\omega_+^2 < \omega_h^2$. This root is minimum when $\cos^2\theta = 1$; thus

$$\begin{aligned}\omega_+^2 &> \frac{1}{2}\left[\left(\omega_p^2 + \omega_c^2\right) + \left|\omega_p^2 - \omega_c^2\right|\right]\\ &= \omega_p^2 \quad \text{if } \omega_p > \omega_c\\ &= \omega_c^2 \quad \text{if } \omega_c > \omega_p\end{aligned}$$

4.10 Use V_+, N_+ for proton velocity and density
V_-, N_- for antiprotons
v_-, n_- for electrons
v_+, n_+ for positrons

(a)

$$\nabla\times\mathbf{E} = -\dot{\mathbf{B}} \quad \nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{\dot{\mathbf{E}}}{c^2} \; \nabla\times\nabla\times\mathbf{E} = -\left(\mu_0\dot{\mathbf{j}} + \frac{\ddot{\mathbf{E}}}{c^2}\right)$$

$$-(\mathbf{k}\times\mathbf{k}\times\mathbf{E}) = -\left[\mu_0 n_0 e(\dot{\mathbf{v}}_+ - \dot{\mathbf{v}}_-) - \frac{\omega^2}{c^2}\mathbf{E}\right]$$
$$= k^2\mathbf{E} - \mathbf{k}(\mathbf{k}\cdot\overset{o}{\mathbf{E}})$$

$$(\omega^2 - c^2k^2)\mathbf{E} = \frac{1}{\epsilon_0} n_0 e(\dot{\mathbf{v}}_+ - \dot{\mathbf{v}}_-)$$

$$mn_0\mathbf{v}_\pm = \pm en_0\mathbf{E} \quad \dot{v}_\pm = \pm\frac{e}{m}\mathbf{E}$$

$$\omega^2 - c^2k^2 = \frac{1}{\epsilon_0} n_0 e\frac{e}{m}(1+1) = 2\omega_p^2$$

$$\omega_p^2 = \frac{n_0e^2}{\epsilon_0 m} \quad \underline{\omega^2 = 2\omega_p^2 + c^2k^2}$$

(Or the 2 can be incorporated into the definition of ω_p.)

(b) $\nabla\cdot\mathbf{E}_1 = (1/\epsilon_0)(N_+ - N_- + n_+ - n_-)_1$, where $n_+ = n_0 e^{-e\phi/KT+}$, $n_- = n_0 e^{e\phi/KT-}$.
Let $T_+ = T_- = T_e$ $n_{1\pm} = \mp n_0 e\phi/KT_e$. Note: $N_{0\pm} = n_{0\pm} \equiv n_0$.

$$\frac{\partial N_\pm}{\partial t} + N_{0\pm}\nabla\cdot V_\pm = 0 \quad N_{1\pm} = N_{0\pm}\frac{k}{\omega}V_\pm = n_0\frac{k}{\omega}V_\pm$$

$$M(-i\omega)V_\pm = \pm e\mathbf{E}_1 = \pm i\mathbf{k}e\phi \quad (M_+ = M_- = M)$$

$$V_\pm = \pm\frac{k}{\omega}\frac{e\phi}{M} \quad N_{1\pm} = \pm\frac{k^2}{\omega^2}\frac{n_0 e\phi}{M}$$

$$\nabla\cdot\mathbf{E}_1 = k^2\phi = \frac{e}{\epsilon_0}\left(\frac{k^2}{\omega^2} + \frac{k^2}{\omega^2}\right)\frac{n_0 e\phi}{M} + \frac{e}{\epsilon_0}(-n_0 - n_0)\frac{e\phi}{KT_e}$$

$$= \frac{n_0e^2}{\epsilon_0 M}\frac{2k^2}{\omega^2}\phi - \frac{n_0e^2}{\epsilon_0 kT_e}2\phi = 2\phi\left(\Omega_p^2\frac{k^2}{\omega^2} - \frac{1}{\lambda_D^2}\right)$$

$$k^2\lambda_D^2 + 2 = \frac{2k^2}{\omega^2}\Omega_p^2\lambda_D^2 = \frac{2k^2}{\omega^2}v_s^2 \quad v_s^2 \equiv \frac{kT_e}{M}$$

$$\frac{\omega^2}{k^2} = \frac{2v_s^2}{2 + k^2\lambda_D^2} = \frac{v_s^2}{1 + (1/2)k^2\lambda_D^2} \; \lambda_D \equiv \left(\frac{kT_e \epsilon_0}{n_0 e^2}\right)^{1/2}$$

4.11 $\tilde{n} = \dfrac{ck}{\omega} \quad \omega^2 = \omega_p^2 + c^2k^2 \quad \dfrac{c^2k^2}{\omega^2} = 1 - \dfrac{\omega_p^2}{\omega^2} = \epsilon$

$$\therefore \; \tilde{n} = \sqrt{\epsilon}$$

4.12 In $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_1$, $\mathbf{j}_1$ is the current carried by electrons only, since Cl^- ions are too heavy to move appreciably in response to a signal at microwave frequencies. Hence,

$$j_1 = -n_{0e} e v_e = -(1 - \kappa) n_0 e v_{e1}$$

If ω_p is defined with n_0 (i.e., $\omega_p^2 = n_0 e^2 / \epsilon_0 m$), the dispersion relation becomes

$$\frac{c^2k^2}{\omega^2} = 1 - (1 - \kappa)\frac{\omega_p^2}{\omega^2}$$

Cutoff occurs for $f = (1-\kappa)^{1/2} f_p = (0.4)^{1/2}(9)(n_0)^{1/2}$, where

$$f = \frac{c}{\lambda} = \frac{3 \times 10^{10}}{3} = 10^{10}$$

Thus

$$n_0 = \left[\frac{10^{10}}{(0.63)(9)}\right]^2 = 3.1 \times 10^{18} \text{m}^{-3}$$

4.13 (a) Method 1: Let N = No. of wavelengths in length $L = 0.08$ m, N_0 = No. of wavelengths in absence of plasma.

$$N = \frac{L}{\lambda} \quad N_0 = \frac{L}{\lambda_0} \quad \lambda = \frac{2\pi}{k} \quad \frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

$$\mathbf{\Delta} N = N_0 - N = \frac{L}{\lambda_0} - \frac{Lk}{2\pi} = \frac{L}{\lambda_0} - \frac{L}{2\pi}\frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

$$\frac{\omega}{2\pi c} = \frac{1}{\lambda_0} \; \therefore \mathbf{\Delta} N = \frac{L}{\lambda_0}\left[1 - \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}\right] = 0.1$$

$$\frac{L}{\lambda_0} = \frac{0.08}{0.008} = 10$$

$$\therefore \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = 1 - 10^{-2} \quad 1 - \frac{f_p^2}{f^2} = 1 - \left(2 \times 10^{-2}\right)$$

$$f_p^2 = f^2 \times 2 \times 10^{-2} = \left(\frac{c}{\lambda_0}\right)^2 2 \times 10^{-2} = 2.8 \times 10^{19}$$

$$n = \frac{2.8 \times 10^{19}}{(9)^2} = \underline{3.5 \times 10^{17} \mathrm{m}^{-3}}$$

Method 2: Let k_0 = free-space k. The phase shift is

$$\mathbf{\Delta}\phi = \int_0^L \mathbf{\Delta}k\, dx = (k_0 - k)L = (0.1)2\pi$$

This leads to the same answer.

(b) From above, $\mathbf{\Delta}N$ is small if ω_p^2/ω^2 is small; hence expand square root:

$$\mathbf{\Delta}N \approx \frac{L}{\lambda_0}\left[1 - \left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)\right] = \frac{L}{\lambda_0}\frac{1}{2}\frac{\omega_p^2}{\omega^2} \propto n \quad \text{QED}$$

4.14 From Eq. (4.101a), we have for the X-wave

$$\left(\omega^2 - \omega_h^2\right)E_x + i\frac{\omega_p^2\omega_c}{\omega}E_y = 0$$

At resonance, $\omega = \omega_h$) $E_y = 0, \mathbf{E} = E_x\hat{\mathbf{x}}$. Since $\mathbf{k} = k_x\hat{\mathbf{x}}$, $\mathbf{E}||\mathbf{k}$, and the wave is longitudinal and electrostatic.

4.15 Since $\omega_h^2 = \omega_c^2 + \omega_p^2$, clearly $\omega_p < \omega_h$. Further,

$$\begin{aligned}\omega_L &= \frac{1}{2}\left[-\omega_c + \left(\omega_c^2 + 4\omega_p^2\right)^{1/2}\right] \\ &< \frac{1}{2}\left[-\omega_c + \left(\omega_c^2 + 4\omega_c\omega_p + 4\omega_p^2\right)^{1/2}\right] \\ &= \frac{1}{2}\left[-\omega_c + \left(\omega_c + 2\omega_p\right)\right] = \omega_p \quad \therefore \quad \omega_L < \omega_p\end{aligned}$$

Also,

$$\omega_R = \frac{1}{2}\left[\omega_c + \left(\omega_c^2 + 4\omega_p^2\right)^{1/2}\right] > \omega_c$$

and

$$\omega_R^2 - \omega_R\omega_c - \omega_p^2 = 0 \quad (\text{Eq. [4-107]})$$
$$\therefore \quad \omega_R^2 = \omega_R\omega_c + \omega_p^2 > \omega_c^2 + \omega_p^2 = \omega_h^2$$

4.17 (a) Multiply Eq. (4.112b) by i and add to Eq. (4.112a):

$$\left(\omega^2 - c^2k^2 - \alpha\right)\left(E_x + iE_y\right) + \alpha\frac{\omega_c}{\omega}\left(E_x + iE_y\right) = 0$$

Now subtract from Eq. (4.112a):

$$\left(\omega^2 - c^2k^2 - \alpha\right)\left(E_x - iE_y\right) - \alpha\frac{\omega_c}{\omega}\left(E_x - iE_y\right) = 0$$

Thus

$$F(\omega) = \omega^2 - c^2k^2 - \alpha(1 + \omega_c/\omega)$$
$$G(\omega) = \omega^2 - c^2k^2 - \alpha(1 - \omega_c/\omega)$$

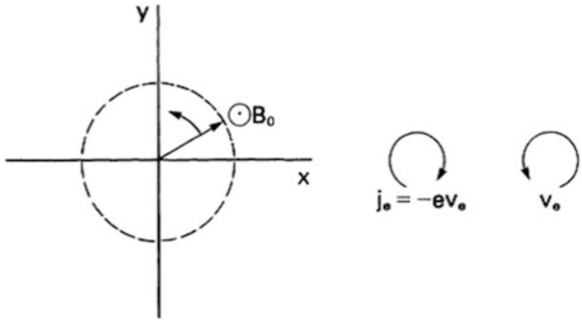
Since

$$\alpha \equiv \frac{\omega_p^2}{\left(1 - \omega_c^2/\omega^2\right)}$$
$$F(\omega) = \omega^2\left(1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega} - \frac{c^2k^2}{\omega^2}\right)$$
$$G(\omega) = \omega^2\left(1 - \frac{\omega_p^2/\omega^2}{1 + \omega_c/\omega} - \frac{c^2k^2}{\omega^2}\right)$$

From Eqs. (4.116) and (4.117),

$$F(\omega) = 0 \text{ for the } R \text{ wave} \quad \text{and}$$
$$G(\omega) = 0 \text{ for the } L \text{ wave}$$

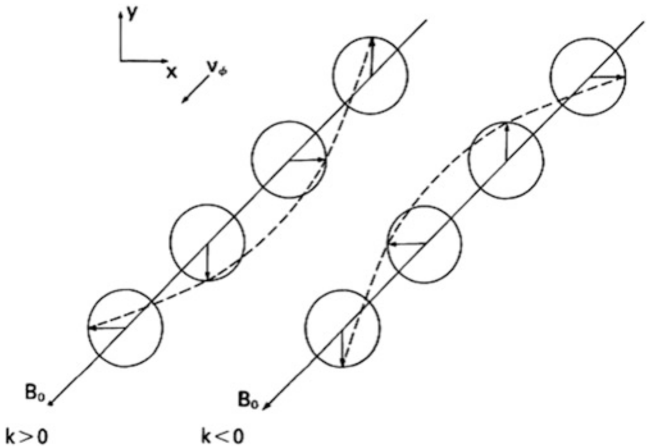
(b) $E_x = -iE_y)$ $E_y = iE_x$. Let $E_x = f(z)\, e^{-i\omega t}$. Then
$E_y = f(z) i\, e^{-i\omega t} = f(z)\, e^{-i\omega t + i(\pi/2)} = f(z)\, e^{-i[\omega t - (\pi/2)]}$



E_y lags E_x by 90°. Hence E rotates counterclockwise on this diagram. This is the same way electrons gyrate in order to create a clockwise current and generate a B-field opposite to $\mathbf{B}_0$. For the L wave, $E_y = -iE_x$ so that

$E_y = f(z)\, e^{-i(\omega t + \pi/2)}$ and E_y leads E_x by 90°.

(c) For an R-wave, $E_y = iE_x$. The space dependence is $E_x = f(t)\, e^{ikz}$, $E_y = f(t) i\, e^{ikz} = f(t)\, e^{i(kz+\pi/2)}$ For $k > 0$, E_y leads E_x (has the same phase at smaller z). For $k < 0$, E_y lags E_x (has the same phase at larger z).



4.19

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega} \quad c^2 v_\phi^{-2} = 1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega}$$

$$c^2(-2)v_\phi^{-3}\frac{dv_\phi}{d\omega} = -\omega_p^2 \frac{-1}{(\omega^2 - \omega\omega_c)^2}(2\omega - \omega_c) = 0$$

$$\therefore 2\omega - \omega_c = 0 \quad \omega = \tfrac{1}{2}\omega_c$$

At $\omega = 1/2\omega_c$,

$$\frac{c^2}{v_\phi^2} = 1 - \frac{\omega_p^2}{\frac{1}{4}\omega_c^2 - \frac{1}{2}\omega_c^2} = 1 + \frac{4\omega_p^2}{\omega_c^2} > 1$$
$$\therefore \quad v_\phi < c.$$

4.20 $$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega} \quad c^2k^2 = \omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}$$

$$c^2 2k\,dk = 2\omega\,d\omega - \frac{(\omega - \omega_c) - \omega}{(\omega - \omega_c)^2}\omega_p^2 d\omega$$
$$= \left[2\omega + \frac{\omega_c\omega_p^2}{(\omega - \omega_c)^2}\right] d\omega$$
$$\frac{d\omega}{dk} = \frac{kc^2}{\omega + \omega_c\omega_p^2/2(\omega - \omega_c)^2} \approx \frac{kc^2}{\omega + \omega_p^2/2\omega_c} \quad \text{if}\, \omega \ll \omega_c$$

But

$$ck = \left(\omega^2 - \frac{\omega_p^2}{1 - \omega_c/\omega}\right)^{1/2} \approx \left(\omega^2 + \frac{\omega\omega_p^2}{\omega_c}\right)^{1/2} \quad \text{if}\, \omega \ll \omega_c$$
$$\therefore \frac{d\omega}{dk} = c\frac{\left(\omega^2 + \omega\omega_p^2/\omega_c\right)^{1/2}}{\omega + \omega_p^2/2\omega_c} = c\frac{\left(1 + \omega_p^2/\omega\omega_c\right)^{1/2}}{1 + \omega_p^2/2\omega\omega_c}$$

To prove the required result, one must also assume $v_\phi^2 \ll c^2$, as is true for whistlers, so that $\omega_p^2/\omega\omega_c \ll 1$ (from line 1). Hence

$$\frac{d\omega}{dk} \approx 2c\left(\frac{\omega\omega_c}{\omega_p^2}\right)^{1/2} \propto \omega^{1/2}$$

4.21 $$\left(\omega^2 - c^2k^2\right)\mathbf{E}_1 = \frac{1}{\epsilon_0} i\omega\mathbf{j}_1 \quad \text{(Eq.[4-81])}$$

$$\mathbf{j}_1 = n_0 e\left(\mathbf{v}_p - \mathbf{v}_e\right) \quad \left(v_p \text{ is the positron velocity}\right)$$

From the equation of motion,

$$v_x = \frac{\pm ie}{m\omega}\left(E_x \pm \frac{i\omega_c}{\omega}E_y\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$v_y = \frac{\pm ie}{m\omega}\left(E_y \mp \frac{i\omega_c}{\omega}E_x\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$\therefore \left(\omega^2 - c^2k^2\right)E_x = \left(-\frac{1}{\epsilon_0}i\omega\right)(n_0e)\left(\frac{ie}{m\omega}\right)(1+1)E_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$
$$= \frac{2\omega_p^2}{1 - \omega_c^2/\omega^2}E_x$$

the E_y term canceling out. Similarly,

$$\left(\omega^2 - c^2k^2\right)E_y = \frac{2\omega_p^2}{1 - \omega_c^2/\omega^2}E_y$$

the E_x term cancelling out. Both equations give

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{2\omega_p^2}{\omega^2 - \omega_c^2}$$

The R and L waves are degenerate and have the same phase velocities—hence, no Faraday rotation.

4.22 Since the phase difference between the R and L waves is twice the angle of rotation,

$$\int_0^L \left(k_L - k_R\right)\ dz = \pi$$

$$k_{R,L} = k_0\left(1 - \frac{\omega_p^2/\omega^2}{1 \pm \omega_c/\omega}\right)^{1/2}$$

To get a simple expression for $k_L - k_R$, we wish to expand the square root. Let us assume we can, and then check later for consistency:

$$k_{R,L} \approx k_0\left(1 - \frac{1}{2}\frac{\omega_p^2/\omega^2}{1 \pm \omega_c/\omega}\right)$$

$$\begin{aligned}
k_L - k_R &= \frac{1}{2}k_0\frac{\omega_p^2}{\omega^2}\left(\frac{1}{1-\omega_c/\omega} - \frac{1}{1+\omega_c/\omega}\right) \\
&= \frac{1}{2}k_0\frac{\omega_p^2}{\omega^2}\frac{2\omega_c/\omega}{1-\omega_c^2/\omega^2} \\
\pi &= L(k_L - k_R) = k_0 L\frac{\omega_p^2\omega_c}{\omega}\frac{1}{\omega^2-\omega_c^2} \quad k_0 = \frac{\omega}{c} \\
\omega_p^2 &= \frac{\pi c}{L\omega_c}\left(\omega^2-\omega_c^2\right) \quad f_p^2 = \frac{c}{2L}\frac{f^2-f_c^2}{f_c} \\
f_c &= 2.8\times 10^{10}(0.1)\mathrm{Hz} \\
f &= \frac{c}{\lambda_0} = \frac{3\times 10^8}{8\times 10^{-3}} = 3.75\times 10^{10}\mathrm{Hz} \\
f_p^2 &= \frac{\left(3\times 10^8\right)}{(2)(1)}\frac{\left(1.41\times 10^{21} - 7.8\times 10^{18}\right)}{2.8\times 10^9} \\
&= 7.5\times 10^{19} = 9^2 n \\
n &= 9.3\times 10^{17}\mathrm{m}^{-3}
\end{aligned}$$

To justify expansion, note that $f_c \ll f$, so that

$$\frac{\omega_p^2/\omega^2}{1\pm\omega_c/\omega} \approx \frac{f_p^2}{f^2} = \frac{7.5\times 10^9}{\left(3.75\times 10^{10}\right)^2} = 0.05 \ll 1$$

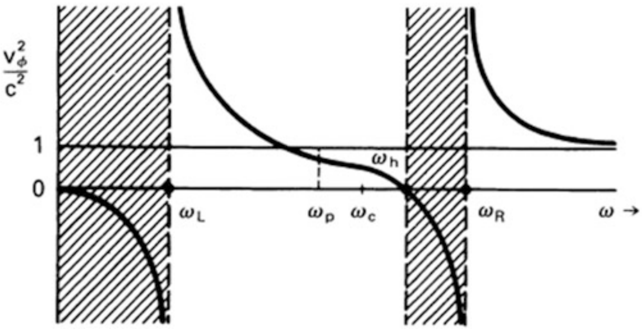
4.24 12.7°.

4.25 (a) The X-wave cutoff frequencies are given by Eq. (4.107). Thus,

$$\begin{aligned}
\omega_p^2 &= \omega(\omega\pm\omega_c) = \frac{4\pi n e^2}{m} \\
n_{cx} &= \frac{m\omega}{4\pi e^2}(\omega+\omega_c)
\end{aligned}$$

We choose the (+) sign, corresponding to the L cutoff, because that gives the higher density.

(b)

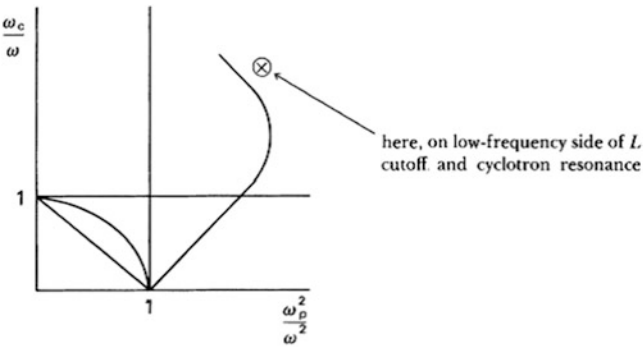


The left branch is the one that has a cutoff at $\omega = \omega_L$. One might worry that this branch is inaccessible if the wave is sent in from outside the plasma. However, if ω is kept less than ω_c, the stopband between ω_h and ω_R is avoided completely.

4.28 (a)

$$f_p = 9\sqrt{n} = (9)\left(10^{15}\right)^{1/2} = 2.85 \times 10^8\text{Hz}$$
$$f_c = 28\text{GHz/T} = \left(2.8 \times 10^{10}\right) \times \left(10^{-2}\right) = 2.8 \times 10^8\text{Hz}$$
$$f = 1.6 \times 10^8\text{Hz} \therefore \omega_p/\omega > 1 \quad \omega_c/\omega > 1$$
$$\omega_L = \frac{1}{2}\left[-\omega_c \pm \left(\omega_c^2 + 4\omega_p^2\right)^{1/2}\right] \approx \frac{1}{2}\left(-\omega_c + \sqrt{5}\omega_c\right)$$
$$= 0.62\omega_c \quad \text{for}\, \omega_c \approx \omega_p$$
$$f_L = (0.62)\left(2.8 \times 10^8\right) = 1.73 \times 10^8 > f$$

Also, $f >$ all ion frequencies.



(b) The R-wave (whistler mode) is the only wave that propagates here.

4.29 (a)

$$v_A = \frac{B}{(\mu_0 nM)} = \frac{1}{\left[\left(1.26\times10^{-6}\right)\left(10^{19}\right)\left(1.67\times10^{-27}\right)\right]^{1/2}}$$
$$= 6.9\times10^6 \mathrm{m/s}$$
$$\Omega_c = \frac{eB}{M} = \frac{\left(1.6\times10^{-19}\right)(1)}{\left(1.67\times10^{-27}\right)} = 9.58\times10^7 \mathrm{rad}/s$$
$$\omega = 0.1\Omega_c = 9.58\times10^6 \mathrm{rad}/s$$
$$\omega = kv_A = 2\pi v_A/\lambda$$

If $\lambda = 2$ L,

$$L = \frac{\pi v_A}{\omega} = \frac{\pi\left(6.9\times10^6\right)}{9.58\times10^6} = 2.26\,\mathrm{m}$$

(b)

$$L \propto v_A/\omega \propto v_A/\Omega_c \propto B(nM)^{-1/2}B^{-1}M \propto (M/n)^{1/2}$$
$$\therefore L = (2.26)\left(\frac{133}{1}\right)^{1/2}\left(\frac{10^{19}}{10^{18}}\right)^{1/2} = 82\,\mathrm{m}$$

This is why Alfvén waves cannot be studied in Q-machines, regardless of B.

4.30 (a)

$$\omega^2 = \omega_p^2 + c^2k^2 \quad 2\omega\, d\omega = c^2 2k\, dk$$

$$v_g = d\omega/dk = c^2k/\omega$$
$$\frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega}\right)^{1/2}$$
$$\therefore v_g = c\left(1 - \frac{\omega_p^2}{\omega}\right)^{1/2} \approx c\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega}\right) \quad \text{for}\, \omega^2 \gg \omega_p^2$$
$$v_g t = x \therefore t = x/v_g$$
$$\frac{dt}{d\omega} = \frac{x}{c}\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)^{-2}\left(-\frac{\omega_p^2}{\omega^2}\right) \approx -\frac{x}{c}\frac{\omega_p^2}{\omega^3}$$
$$\therefore \frac{df}{dt} \approx -\frac{c}{x}\frac{f^3}{f_p^2}$$

(b)

$$x = \frac{cf^3}{f_p^2}\left(-\frac{df}{dt}\right)^{-1} = \frac{\left(3\times10^8\right)\left(8\times10^7\right)^3}{(9)^2\left(2\times10^5\right)\left(5\times10^6\right)} = 1.9\times10^{18}\mathrm{m}$$
$$= \left(1.9\times10^{18}\right)\left(3\times10^{16}\right)^{-1} = 63\,\mathrm{parsec}$$

4.31 (a) Let $n_0^{(1)} = (1-\in)n_0,\ \ n_0^{(2)} = \in n_0, \quad n_e = n_0 e\phi/kT_e$

Poisson: $ikE_1 = k^2\phi = \frac{1}{\in_0} e\left(n_i^{(1)} + n_i^{(2)} - n_e\right)$

(Assume $z_{1,2} = 1$, since the ion charge is not explicitly specified.)

Continuity: $n_1^{(1)} = (1-\in)n_0 \dfrac{k}{\omega} v_1^{(1)},\ \ n_1^{(2)} = \in n_0 \dfrac{k}{\omega} v_1^{(2)}$

Equation of motion:

$$v_1^{(j)} = \frac{e}{M_j}\frac{k}{\omega}\phi\left(1 - \frac{\Omega_{cj}^2}{\omega^2}\right)^{-1} \tag{4.68}$$

$$\therefore k^2\phi = \frac{e}{\in_0}\left[(1-\in)n_0\frac{k^2}{\omega^2}\frac{e}{M_1}\left(1-\frac{\Omega_{c1}^2}{\omega^2}\right)^{-1}\right.$$
$$\left. + \in n_0 \frac{k^2}{\omega^2}\frac{e}{M_2}\left(1 - \frac{\Omega_{c2}^2}{\omega^2}\right)^{-1} - n_0\frac{e}{kT_e}\right]\phi \approx 0\,(\text{plasma approximation})$$
$$1 = (1-\in)\frac{k^2 v_{s1}^2}{\omega^2 - \Omega_{c1}^2} + \in \frac{k^2 v_{s2}^2}{\omega^2 - \Omega_{c2}^2} \quad \Leftarrow$$

(b) There are two roots, one near $\omega = \Omega c1$ and one near $\omega = \Omega c2$. If $\epsilon \to 0$, the root near $\Omega c2$ approaches $\Omega c2$ to keep the last term finite. The usual root, near $\Omega c1$, is shifted by the presence of the M_2 species:

$$\omega^2 - \Omega_{c1}^2 = k^2 v_{s1}^2 - \in\left[k^2 v_{s1}^2 - k^2 v_{s2}^2 \frac{\omega^2 - \Omega_{c1}^2}{\omega^2 - \Omega_{c2}^2}\right]$$

In the last term, we may approximate ω^2 by $\Omega_{c1}^2 + k^2 v_{s1}^2$. Thus,

$$\omega^2 \approx \Omega_{c1}^2 + k^2 v_{s1}^2 + \in\left[\frac{k^2 v_{s2}^2}{\Omega_{c1}^2 - \Omega_{c2}^2} - 1\right]k^2 v_{s1}^2$$

(c)

$$1 = \frac{1}{2}\frac{k^2 v_{sD}^2}{\omega^2 - \omega_{cD}^2} + \frac{1}{2}\frac{k^2 v_{sT}}{\omega^2 - \Omega_{cT}^2}$$

$$v_{sD}^2 = KT_e/M_D = \left(10^4\right)\left(1.6 \times 10^{-19}\right)/(2)\left(1.67 \times 10^{-27}\right) = 4.79 \times 10^{11}$$

$$v_{sT}^2 = \frac{2}{3}v_{sD}^2 = 3.19 \times 10^{11}$$

$$\Omega_{cD} = eB/M_D = \left(1.6 \times 10^{-19}\right)(5)/(2)\left(1.67 \times 10^{-27}\right) = 2.40 \times 10^8$$

$$\Omega_{cT} = \frac{2}{3}\Omega_{cD} = 1.60 \times 10^8 \quad k = 100\,\mathrm{m}^{-1}$$

$$\left(\omega^2 - \Omega_{cD}^2\right)\left(\omega^2 - \Omega_{cT}^2\right) = \frac{1}{2}k^2\left[v_{sD}^2\left(\omega^2 - \Omega_{cT}^2\right) + v_{sT}^2\left(\omega^2 - \Omega_{cD}^2\right)\right]$$

$$\omega^4 - \omega^2\left[\Omega_{cD}^2 + \Omega_{cT}^2 + \frac{1}{2}k^2\left(v_{sD}^2 + v_{sT}^2\right)\right]$$

$$+ \Omega_{cD}^2\Omega_{cT}^2 + \frac{1}{2}k^2\left(v_{sD}^2\Omega_{cT}^2 + v_{sT}^2\Omega_{cD}^2\right) = 0$$

$$\omega^4 - \omega^2\left[8.32 \times 10^{16} + 3.99 \times 10^{15}\right] + 1.47 \times 10^{33} + 1.53 \times 10^{32} = 0$$

$$\omega^4 - 8.72 \times 10^{16}\omega^2 + 1.63 \times 10^{33} = 0$$

$$\omega^2 = \frac{1}{2}\left[8.72 \times 10^{16} \pm \left(7.60 \times 10^{33} - 6.52 \times 10^{33}\right)^{1/2}\right]$$

$$= 6.0 \times 10^{16}, \quad 2.72 \times 10^{16}$$

$$\omega = 2.45,\ 1.65 \times 10^8\,\mathrm{sec}^{-1} \quad f = 39\,\mathrm{and}\,26.3\,\mathrm{MHz}$$

4.32

$$E = n_0\left\langle\frac{1}{2}mv_e^2\right\rangle \quad v_e = \frac{e}{im\omega}E$$

$$\therefore\left\langle v_e^2\right\rangle = \frac{e^2}{m^2\omega^2}\left\langle E^2\right\rangle$$

$$E = n_0\frac{1}{2}m\frac{e^2}{m^2\omega^2}\left\langle E^2\right\rangle = \frac{\epsilon_0\omega_p^2}{\omega^2}\frac{\left\langle E^2\right\rangle}{2}$$

But $\omega^2 = \omega_p^2\ \therefore E = \frac{1}{2}\epsilon_0\left\langle E^2\right\rangle$.

4.33

$$E = n_0\left\langle\frac{1}{2}Mv_i^2\right\rangle \quad v_i \approx E_1/B_0$$

$$\therefore E = \frac{1}{2}Mn_0\left\langle E_1^2\right\rangle/B_0.\ \mathrm{But}\,\nabla \times \mathbf{E}_1 = -\dot{B}_1 \therefore \left\langle E_1^2\right\rangle = \left(\omega^2/k^2\right)\left\langle B_1^2\right\rangle$$

$$E = \frac{Mn_0}{2B_0^2}\frac{\omega^2}{k^2}\left\langle B_1^2\right\rangle.$$

For Alfvén wave,

$$\frac{\omega^2}{k^2} = \frac{B_0^2}{\mu_0 n_0 M} = \frac{\left\langle B_1^2\right\rangle}{2\mu_0}$$

4.34 (a) With the L-wave, the cutoff occurs at $\omega = \omega_L$, so that one requires $\omega_L^2 < \in\omega^2$. Since $\omega_L < \omega_p$ if n_0 is fixed (Problem 4.15), one can go to higher values of n_0 (for constant $\in\omega^2$) with the L-wave than with the O-wave.

(b) For the L-cutoff,

$$\frac{\omega_p^2}{\omega^2} = 1 + \frac{\omega_c}{\omega} \therefore n_c = \frac{\in_0 m\omega^2}{e^2}\left(1 + \frac{\omega_c}{\omega}\right)$$

Thus, to double the usual cutoff density of $\in_0 m\omega^2/e^2$, one must have $f_c = f$

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{337 \times 10^{-6}} = 8.9 \times 10^{11}\,\mathrm{Hz}$$
$$f_c = 28 \times 10^9\,\mathrm{Hz/T} \therefore B_0 = \frac{8.9 \times 10^{11}}{28 \times 10^9} = 31.8\,\mathrm{T}$$

This would be unreasonably expensive.

(c) The plasma has a density maximum at the center, so it behaves like a convex lens. Such a lens focuses if $\tilde{n} > 1$ and defocuses if $\tilde{n} < 1$. The whistler wave always travels with $v_\phi < c$ (Problem 4.19), so $\tilde{n} = c/v_\phi > 1$, and the plasma focuses this wave.

(d) The question is one of accessibility. If $\omega < \omega_c$ everywhere, the whistler wave will propagate regardless of n_0. However, if $\omega > \omega_c$, the wave will be cut off in regions of low density. From (b) above, we see that a field of 31.8 T is required; this seems too large for the scheme to be practical.

4.35 The answer should come out the same as for cold plasma.

4.36 The linearized equation of motion for either species is

$$-i\omega m n_0 \mathbf{v}_1 = q n_0(\mathbf{E} + \mathbf{v}_1 \times \mathbf{B}_0) - \gamma k T_i \mathbf{k} n_1$$

Thus

$$-i\omega m n_0 \mathbf{k}.\mathbf{v}_1 = q n_0(\mathbf{k}\cdot\mathbf{E} + \mathbf{k}\cdot\mathbf{v}_1 \times \mathbf{B}_0) - \gamma k T_i k^2 n_1.$$

But $\mathbf{k}\cdot\mathbf{E} = 0$ for transverse wave, and $\mathbf{k}\cdot(\mathbf{v}_1 \times \mathbf{B_0}) = -\mathbf{v}_1\cdot(\mathbf{k}\times\mathbf{B}_0) = 0$ by assumption. The linearized equation of continuity is

$$-i\omega n_1 + n_0 i\mathbf{k}\cdot\mathbf{v}_1 = 0$$

Substituting for $\mathbf{k}\cdot\mathbf{v}_1$, we have

$$i\omega^2 m n_1 = i\gamma k T k^2 n_1$$

Thus n_1 is arbitrary, and we may take it to be 0. Then the ∇_p term vanishes for both ions and electrons.

4.44 For a given density, the highest cutoff frequency is ω_R. Thus the lowest bound for n is given by $\omega = \omega_R$.

$$\frac{\omega_p^2}{\omega^2} = \frac{f_p^2}{f^2} = 1 - \frac{\omega_c}{\omega} = 1 - \frac{(1.6 \times 10^{-19})(36 \times 10^{-4})}{(0.91 \times 10^{-30})(2\pi)(1.2 \times 10^8)} = 0.16$$

$$n = f_p^2/q^2 = (0.16)(1.2 \times 10^8)^2 q^{-2} = 2.8 \times 10^{13}\text{m}^{-3}$$

4.46 Let $\omega = \omega_R$ at r_1 where $n = n_1$, $\omega_p = \omega_{p1}$; and $\omega = \omega_h$ at r_2, where $n = n_2$, $\omega_p = \omega_{p2}$ Then

$$\omega_{p2}^2 = \omega^2 - \omega_c^2 \qquad (4.105)$$

$$\omega_{p1}^2 = \omega^2 - \omega\omega_c \qquad (4.107)$$

Thus

$$\omega_{p2}^2 - \omega_{p1}^2 = \omega_c(\omega - \omega_c) = (n_2 - n_1)e^2/\epsilon_0 m$$

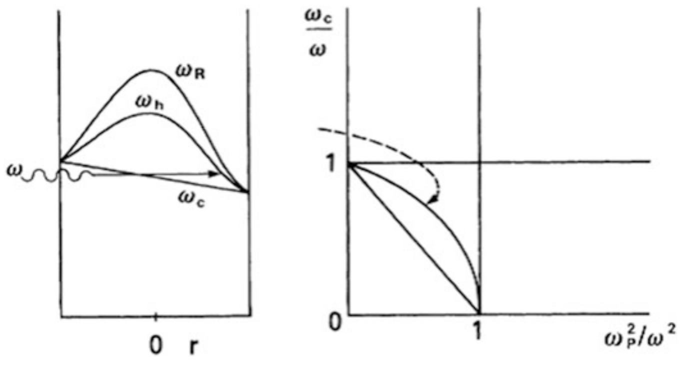
But

$$n_2 - n_1 \approx d|\partial n/\partial r| \approx n_1 d/r_0 = (\epsilon_0 m/e^2)(\omega)(\omega - \omega_c)(d/r_0)$$

So

$$d \approx (\omega_c/\omega)r_0$$

4.47 (a) The accessible resonance is on the far side, past the density maximum.



(b) Let ω_{c0} be ω_c at the left boundary, and ω_c be the value at the resonance layer, where $\omega = \omega_p$. Then we require

$$\omega_{c0} > \omega, \quad \text{where}\, \omega^2 = \omega_c^2 + \omega_p^2$$

Thus

$$\omega_{c0}^2 > \omega_c^2 + \omega_p^2 \qquad \omega_{c0}^2 - \omega_c^2 > \omega_p^2$$
$$(\omega_{c0} + \omega_c)(\omega_{c0} - \omega_c) \approx 2\omega_c \boldsymbol{\Delta}\omega_c > \omega_p^2$$
$$\frac{\boldsymbol{\Delta}\omega_c}{\omega_c} = \frac{\boldsymbol{\Delta}B_0}{B_0} > \frac{\omega_p^2}{2\omega_c^2}$$

4.48 These are the upper and lower hybrid frequencies and right- and left-hand cutoff frequencies with ion motions included. Note that $\omega_p^2/\omega_c = \Omega_p^2/\Omega_c$.

Resonance:

$$\omega^4 - \left(\omega_p^2 + \omega_c^2 + \Omega_p^2 + \Omega_c^2\right) + \omega_p^2\Omega_c^2 + \omega_c^2\Omega_p^2 + \omega_c^2\Omega_c^2 = 0$$
$$\omega_+^2 \approx \omega_h^2 + \Omega_p^2\left(1 - \omega_c^2/\omega_h^2\right) \quad \text{(upper hybrid)}$$
$$\omega_-^2 \approx \omega_c^2\Omega_p^2/\omega_h^2 \quad \text{or} \quad \frac{1}{\omega_-^2} = \frac{1}{\omega_c\Omega_c} + \frac{1}{\Omega_p^2} \quad \text{(lower hybrid)}$$

Cutoff:

$$\frac{\overline{\omega}_p^2}{\omega^2} = \left(1 \mp \frac{\omega_c}{\omega}\right)\left(1 \pm \frac{\Omega_c}{\omega}\right) \qquad \left(\begin{matrix} R \\ L \end{matrix}\ \text{cutoff}\right)$$

This is more easily obtained, without approximation, from the form given in Problem 4.50.

5.1 (a) $D_e = KT_e/mv$

$$\sigma = (6\pi)\left(0.53 \times 10^{-10}\right)^2 = 5.29 \times 10^{-20}\text{m}^2$$
$$v = \left(\frac{2E}{m}\right)^{1/2} = \left[\frac{(2)(2)\left(1.6 \times 10^{-19}\right)}{\left(9.11 \times 10^{-31}\right)}\right]$$
$$= 8.39 \times 10^5\,\text{m}/s$$

From Problem 1.1b,

$$n_0 = \left(3.3 \times 10^{19}\right)\left(10^3\right) = 3.3 \times 10^{22}\text{m}^{-3}$$
$$v = n_0\overline{\sigma v} = n_0\sigma v = \left(3.3 \times 10^{22}\right)\left(5.29 \times 10^{-20}\right)\left(8.39 \times 10^5\right)$$
$$= 1.46 \times 10^9 s^{-1}$$

$$D_e = \frac{(2)\left(1.6 \times 10^{-19}\right)}{\left(9.11 \times 10^{-31}\right)\left(1.46 \times 10^{9}\right)} = 2.4 \times 10^2 \mathrm{m}^2/\mathrm{s}$$

(b) $j = \mu neE$

$$\mu_e = eD_e/KT_e = \frac{\left(1.6 \times 10^{-19}\right)\left(2.4 \times 10^{2}\right)}{(2)\left(1.6 \times 10^{-19}\right)}$$
$$= 1.2 \times 10^2 \mathrm{m}^2/\mathrm{V\,s}$$

$$E = \frac{j}{\mu ne} = \frac{2 \times 10^3}{\left(1.2 \times 10^2\right)\left(10^{16}\right)\left(1.6 \times 10^{-19}\right)} = 1.04 \times 10^4 \mathrm{V/m}$$

5.2

$$\frac{\partial n}{\partial t} = D\nabla^2 n - \alpha n^2$$

$$D\nabla^2 n = D\frac{\partial^2 n}{\partial x^2} = -Dn_0\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L} = -D\left(\frac{\pi}{2L}\right)^2 n = -\alpha n^2$$

$$\therefore n = \frac{D}{\alpha}\left(\frac{\pi}{2L}\right)^2 = \frac{0.4}{10^{-15}}\left(\frac{\pi}{0.06}\right)^2 = 1.1 \times 10^{18} \mathrm{m}^{-3}$$

5.4 (a) From Problem 5.1a, $\mathrm{v}_{\mathrm{en}} = 1.46 \times 10^9\ \mathrm{s}^{-1}$. We need to find whether $\mu_{e\perp}/\mu_{i\perp}$ is large or small:

$$\frac{\mu_e}{\mu_i} = \frac{M v_{in}}{m v_{en}} \qquad v_{jn} = n_n \sigma v_j \propto v_{\mathrm{th}\,j} \propto m_j^{-1/2}$$

since σ is approximately the same for ion–neutral and electron–neutral collisions. Thus

$$\frac{\mu_e}{\mu_i} \approx \left(\frac{M}{m}\right)^{1/2} = (4 \times 1{,}836)^{1/2} = 85.7$$

$$\omega_c = \frac{eB}{m} = \frac{\left(1.6 \times 10^{-19}\right)(0.2)}{9.11 \times 10^{-31}} = 3.52 \times 10^{10}$$

$$\omega_c \tau_{en} = \frac{3.52 \times 10^{10}}{1.46 \times 10^9} \times 24 \qquad 1 + \omega_c^2 \tau_{en}^2 = 580$$

$$\Omega_c \tau_{in} = \omega_c \tau_{en}\left(\frac{m}{M}\right)\left(\frac{M}{m}\right)^{1/2} = (24)(85.7)^{-1} = 0.28$$

$$\frac{\mu_{e\perp}}{\mu_{i\perp}} = \frac{\mu_e}{\mu_i}\frac{1 + \Omega_c^2 \tau_{in}^2}{1 + \omega_c^2 \tau_{en}^2} = (85.7)\frac{1.08}{580} = 0.16 \ll 1$$

$$\therefore D_{a\perp} = \frac{\mu_{i\perp} D_{e\perp} + \mu_{e\perp} D_{i\perp}}{\mu_{i\perp} + \mu_{e\perp}} \approx D_{e\perp} + \frac{\mu_{e\perp}}{m_{i\perp}} D_{i\perp}$$
$$= D_{e\perp} + 0.16 D_{i\perp}$$

But

$$D = \frac{KT}{e}\mu$$
$$\therefore \frac{D_{i\perp}}{D_{e\perp}} = \frac{\mu_{i\perp} T_i}{\mu_{e\perp} T_e} = \frac{1 \quad 0.1}{0.16 \;\; 2} = 0.3$$
$$\therefore D_{a\perp} = D_{e\perp}[1 + (0.16)(0.3)] = 1.05 D_{e\perp} \approx D_{e\perp}$$

(b)

$$\frac{a}{(D\tau)^{1/2}} = 2.4 \therefore \tau = \left(\frac{a}{2.4}\right)^2 \frac{1}{D_{a\perp}}$$
$$\tau = \frac{1 \qquad\qquad 1}{\left(2.4 \times 10^{-2}\right)^2 D_{e\perp}}$$
$$D_{e\perp} = \frac{2.4 \times 10^2}{580} = 0.4140\,(\text{from Problem 5.1})$$
$$\therefore \tau = 42\,\mu s$$

5.5

$$\Gamma = -D\,dn/dx \quad n = n_0(1 - x/L)$$
$$\Gamma = Dn_0/L \quad (x > 0)$$
$$Q = 2\Gamma = 2Dn_0/L \therefore n_0 = QL/2D$$

5.7

$$\lambda_{ei} \approx v_{\text{th}e}\tau_{ei} = v_{\text{th}e}/\nu_{ei}$$

But $v_{\text{th}e} \propto T_e^{1/2}$ and $\nu_{ei} \propto T_e^{-3/2}$

$$\therefore \lambda_{ei} \propto T_e^{1/2}/T_e^{-3/2} \propto T_e^2$$

5.8

$$\eta_{\parallel} = 5.2 \times 10^{-5} \frac{\ln\boldsymbol{\Lambda}}{T_{ev}^{3/2}} \Omega\text{-m} \quad (\text{assume } Z = 1)$$
$$= \frac{\left(5.2 \times 10^{-5}\right)(10)}{(500)^{3/2}} = 4.65 \times 10^{-8}\Omega\text{-m}$$
$$j = I/A = \left(2 \times 10^5\right)/\left(7.5 \times 10^{-3}\right) = 2.67 \times 10^7 \text{A/m}^2$$
$$E = \eta_{\parallel j} = \left(4.65 \times 10^{-8}\right)\left(2.67 \times 10^7\right) = 1.2\text{V/m}$$

5.9 (a)

$$KT_i = 20\,\text{keV} \quad KT_e = 10\,\text{keV} \quad n = 10^{12}\,\text{m}^{-3}$$

$$B = 5T \quad D_\perp = \frac{\eta n(KT_i + KT_e)}{B^2}$$

$$\eta_\perp = (2.0)\left(5.2\times10^{-5}\right)\frac{\ln\Lambda}{T_{ev}^{3/2}} = \frac{\left(10^{-3}\right)(10)}{\left(10^4\right)^{3/2}}$$

$$= 1.0\times10^{-9}\Omega\text{-m}$$

$$\text{D}_\perp = \frac{\left(1.0\times10^{-9}\right)\left(10^{21}\right)\left(3\times10^4\right)\left(1.6\times10^{-19}\right)}{5^2}$$

$$= 3.0\times10^{-4}\text{m}^2/s$$

(b)

$$\frac{dN}{dt} = 2\pi rL\Gamma_r \quad \Gamma_r = -D_\perp\frac{\partial n}{\partial r}$$

$$\frac{\partial n}{\partial r} = \frac{n}{0.1} \quad r = 0.50\text{m} \;\; L = 100\,\text{m}$$

$$-\frac{dN}{dt} = (2\pi)(0.50)\left(10^2\right)\left(2.0\times10^{-4}\right)\left(10^{21}/0.10\right) = 6\times10^{20}s^{-1}$$

(c)

$$\tau = \frac{N}{-dN/dt} = \frac{n\pi r^2 L}{-dN/dt} \quad r_{\text{effective}} = 0.55\,\text{m}$$

$$\tau = \frac{\left(10^{21}\right)(\pi)(0.55)^2\left(10^2\right)}{6\times10^{20}} = 150s$$

5.13

$$\eta_\parallel = 5.2\times10^{-5}\frac{\ln\mathbf{\Lambda}}{T_{ev}^{3/2}}\Omega\text{-m} = \left(5.2\times10^{-5}\right)\frac{10}{10^{3/2}}$$

$$= 1.6\times10^{-5}\Omega\text{-m}$$

$$\eta j^2 = \left(1.6\times10^{-5}\right)\left(10^5\right)^2 = 1.6\times10^5\,\text{W/m}^3$$

$$= 1.6\times10^5\,\text{J/(m}^3\text{-s)}$$

$$= \left(1.6\times10^5\right)/\left(1.6\times10^{-19}\right) = 10^{24}\text{eV/m}^3\text{-}s$$

$$= \frac{dE_{\widehat{ev}}}{dt}$$

$$E = \frac{3}{2}nKT_e \therefore \frac{dE_{ev}}{dt} = \frac{3}{2}n\frac{dT_{ev}}{dt}$$

$$\frac{dT_{ev}}{dt} = \frac{2}{3}\frac{1}{10^{19}}10^{24} = 0.67 \times 10^5 \text{eV}/s = 0.067\text{eV}/\mu\text{s}$$

5.15 (a)

$$en(E_\theta^{\nearrow 0} - v_{ir}B) - \nabla_\theta^{\nearrow 0} p_i - e^2n^2\eta(v_{i\theta} - v_{e\theta}) = 0$$
$$-en(E_\theta^{\nearrow 0} - v_{er}B) - \nabla_\theta^{\nearrow 0} p_e + e^2n^2\eta(v_{i\theta} - v_{e\theta}) = 0$$

add:

$$-v_{ir}B + v_{er}B = 0 \therefore v_{ir} = v_{er}$$

(This shows ambipolar diffusion.)

(b)

$$en(E_r + v_{i\theta}B) - \frac{\partial p_i}{\partial r} - e^2n^2\eta(v_{ir}^{\nearrow 0} - v_{er}) = 0$$
$$-en(E_r + v_{e\theta}B) - \frac{\partial p_e}{\partial r} + e^2n^2\eta(v_{ir} - v_{er}) = 0$$
$$v_{i\theta} = -\frac{E_r}{B} + \frac{1}{enB}\frac{\partial p_i}{\partial r} = v_E + v_{Di}$$
$$v_{e\theta} = -\frac{E_r}{B} - \frac{1}{enB}\frac{\partial p_e}{\partial r} = v_E + v_{De}$$

(c) From the first equation in (a),

$$v_{ir} = -\frac{e^2n^2\eta}{enB}(v_{i\theta} - v_{e\theta})$$
$$= \frac{en\eta}{B}\frac{1}{enB}\left(\frac{\partial p_i}{\partial r} + \frac{\partial p_e}{\partial r}\right) = -\frac{\eta}{B^2}\frac{\partial p}{\partial r} = v_{er}$$

(This shows the absence of cross-field mobility.)

5.17 (a)

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{j}_1 \times \mathbf{B}_0 \tag{1}$$

$$\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = \eta \mathbf{j}_1 \tag{2}$$

$$\nabla \times \mathbf{E}_1 = -\dot{\mathbf{B}}_1 \quad \nabla \times \mathbf{B}_1 = \mu_0 \mathbf{j}_1$$
$$\nabla \times \nabla \times \mathbf{E}_1 = -\nabla \times \dot{\mathbf{B}}_1 = -\mu_0 \dot{\mathbf{j}}_1$$

$$-\mathbf{k}(\mathbf{k}^{\nearrow 0} \cdot \mathbf{E}) + k^2\mathbf{E}_1 = i\omega\mu_0\mathbf{j}_1 \tag{3}$$

$$\mathbf{k} \cdot \mathbf{E} = 0 \quad \text{(transverse wave)}$$

Solve for $\mathbf{v}_1$ in Eq. (2):

$$\mathbf{E}_1 \times \mathbf{B}_0 + \underbrace{(\mathbf{v}_1 \times \mathbf{B}_0) \times \mathbf{B}_0}_{-\mathbf{v}_{1\perp} B_0^2} = \eta \mathbf{j}_1 \times \mathbf{B}_0$$

$$v_{1\perp} = \frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2} - \frac{\eta \mathbf{j}_1 \times \mathbf{B}_0}{B_0^2}$$

Substitute in Eq. (1), which has no parallel component anyway:

$$-i\omega\rho_0 \left(\frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2} - \frac{\eta \mathbf{j}_1 \times \mathbf{B}_0}{B_0^2} \right) = \mathbf{j}_1 \times \mathbf{B}_0$$

Since, by Eq. (3), $\mathbf{E}$ and $\mathbf{j}_1$ are in the same direction, take them both to be in the $\hat{\mathbf{x}}$-direction. Then the y-component is

$$\frac{E_1}{B_0} = \left(\frac{iB_0}{\omega\rho_0} + \frac{\eta}{B_0} \right) j_1$$

Equation (3) becomes

$$\begin{aligned} k^2 E_1 &= \mu_0 i\omega \frac{E_1}{B_0} \left(\frac{iB_0}{\omega\rho_0} + \frac{\eta}{B_0} \right)^{-1} \\ &= \mu_0 \omega^2 \left(\frac{B_0^2}{\rho_0} - i\eta\omega \right)^{-1} E_1 \\ \frac{\omega^2}{k^2} &= \mu_0 \left(\frac{B_0^2}{\rho_0} - i\omega\eta \right) \end{aligned}$$

(b)

$$\begin{aligned} k &= \left(\mu_0 \omega^2\right)^{1/2} \left(\frac{B_0^2}{\rho_0} - i\omega\eta \right)^{-1/2} \\ &= \omega \left(\frac{\mu_0 \rho_0}{B_0^2} \right)^{1/2} \left(1 - \frac{i\omega\eta\rho_0}{B_0^2} \right)^{-1/2} \\ \mathrm{Im}(k) &= \omega \frac{\omega\eta\rho_0}{2B_0^2} \left(\frac{\mu_0\rho_0}{B_0^2} \right)^{1/2} = \frac{\omega^2 \eta}{2} \frac{1}{v_A^3} \end{aligned}$$

But for small η, $\omega \approx k v_A$, where $k = \mathrm{Re}\ (k)$

$$\therefore \mathrm{Im}(k) \approx \frac{(\eta)\left(k^2\right)}{2 v_A}$$

6.4 (a)

$$\mathbf{j} \times \mathbf{B} = \nabla p = KT\nabla n \quad (KT = KT_e + KT_i \text{ here})$$
$$(\mathbf{j} \times \mathbf{B}) \times \mathbf{B} = KT\nabla n \times \mathbf{B} = \mathbf{B}(\mathbf{j} \cdot \mathbf{B}) - \mathbf{j}B^2$$

The parallel component is $0 = \boldsymbol{j}_{\|}\boldsymbol{B}^2 - \boldsymbol{j}_{\|}\boldsymbol{B}^2 \therefore \boldsymbol{j}_{\|}$ is arbitrary. The perpendicular component is

$$\mathbf{j}_{\perp} = \frac{KT}{B^2}\mathbf{B} \times \nabla n = \frac{KT}{B}\frac{\partial n}{\partial r}\hat{\theta}$$

(b)

$$\int \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int \mathbf{j} \cdot d\mathbf{S}$$
$$\oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 \int \mathbf{j} \cdot d\mathbf{S} = \mu_0 \boldsymbol{L} \int_0^{\infty} j_\theta dr$$

since $\mathbf{j}$ and $d\mathbf{S}$ are both in the $\hat{\theta}$ direction, and L is the width of the loop in the $\hat{\mathbf{z}}$ direction. By symmetry, there can be no B_r, so only the two z-legs of the loop contribute to the line integral. Substituting for j_θ, we have

$$(B_{ax} - B_0)L = \mu_0 LKT \int_0^{\infty} \frac{\partial n/\partial r}{B(r)} dr$$

(c) $\partial n/\partial r = -n_0 \delta(r-a)$, since $\partial n/\partial r$ is a function that is zero everywhere except at $r = a$, is $-\infty$ there, and has an integral equal to $-n_0$. Thus

$$B_{ax} - B_0 = \mu_0 KT \int_0^{\infty} -n_0 \frac{\delta(r-a)}{B(r)} dr$$

Since all the diamagnetic current is concentrated at $r = a$, B takes a jump from a constant value B_{ax} inside the plasma to another constant value B_0 outside. (Remember that the field inside an infinite solenoid is uniform.) Upon integrating across the jump, one obtains the average value of B on the two sides, i.e., $B(a) = \frac{1}{2}(B_{ax} + B_0)$. Thus

$$B_{ax} - B_0 = \mu_0 KT n_0 \frac{-1}{\frac{1}{2}(B_{ax} + B_0)}$$
$$B_{ax}^2 - B_0^2 = -2\mu_0 n_0 KT$$
$$1 - \frac{B_{ax}^2}{B_0^2} = \frac{2\mu_0 n_0 KT}{B^2} \equiv \beta = 1 \therefore B_{ax} = 0$$

6.5 (a) By Faraday's law, $V = -d\Phi/dt$

$$\therefore \int V\,dt = -N\int \frac{d\Phi}{dt}dt = -N\mathbf{\Delta}\Phi$$

Since $\Delta\Phi$ is the flux change due to the diamagnetic decrease in B,

$$-N\mathbf{\Delta}\Phi = -N\int (\mathbf{B} - \mathbf{B}_0)\cdot d\mathbf{S}$$

The sign depends on which side of V is considered positive. In practice, this is of no consequence because the oscilloscope trace can easily be inverted by using the polarity switch.

(b) In Problem 6.4b, we can draw the loop so that its inner leg lies at an arbitrary radius r rather than on the axis. We then have

$$B(r) - B_0 = \mu_0 KT\int_r^\infty \frac{\partial n/\partial r}{B(r')}dr' \approx \mu_0 KT\int_r^\infty \frac{\partial n/\partial r'}{B_0}dr'$$

where again KT is short for $\Sigma\, KT$

$$\frac{\partial n}{\partial r} = n_0\left(\frac{-2r}{r_0^2}\right)e^{-r^2/r_0^2}$$

$$B(r) - B_0 = \frac{\mu_0 KT}{B_0}\frac{n_0}{r_0^2}\int_\infty^r e^{-r'^2/r_0^2}2r'\,dr'$$

$$= \frac{\mu_0 n_0 KT}{B_0}\left[e^{-r'^2/r_0^2}\right]_r^\infty = \frac{-\mu_0 n_0 KT}{B_0}e^{-r^2/r_0^2}$$

This is the diamagnetic change in B at any r. To get the loop signal, we must integrate over the plasma cross section.

$$\int V\,dt = -N\int (\mathbf{B} - \mathbf{B}_0)\cdot d\mathbf{S} = -N\int\int [B(r) - B_0]r\,dr\,d\theta$$

where both $\mathbf{B}$ and $d\mathbf{S}$ are in the $\hat{\mathbf{z}}$ direction. Substituting for $B(r) - B_0$ and assuming the coil lies well outside the plasma, we have

$$\int V\,dt = N\frac{\mu_0 n_0 KT}{B_0}2\pi\int_0^\infty e^{-r^2/r_0^2}r\,dr$$

$$= N\pi\frac{\mu_0 n_0 KT}{B_0}r_0^2\left[e^{-r^2/r_0^2}\right]_\infty^0 = \frac{1}{2}N\pi r_0^2\left(\frac{2\mu_0 n_0 KT}{B_0^2}\right)B_0$$

(c) The quantity in parentheses is β by definition; hence,

$$\int V\,dt = \frac{1}{2}N\pi r_0^2\beta B_0$$

Both sides of this equation have units of flux,

6.6 (a) For each stream, we have

$$m\left(\frac{\partial \mathbf{v}_1}{\partial \mathrm{t}} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1\right) = -e\mathbf{E}_1 = (-i\omega + ikv_0)\mathbf{v}_1$$

$$\mathbf{v}_1 = \frac{-ie\mathbf{E}_1}{m(\omega - kv_0)}$$

$$\frac{\partial n_1}{\partial t} + n_0(\nabla \cdot \mathbf{v}_1) + (\mathbf{v}_0 \cdot \nabla)n_1 = 0$$

$$(-i\omega + ikv_0)n_1 + ikn_0v_1 = 0 \quad n_1 = n_0\frac{kv_1}{\omega - kv_0}$$

$$\therefore n_{1j} = n_{0j}\frac{-ikE_1e}{m\left(\omega - kv_{0j}\right)^2}$$

Poisson: $ikE_1 = (e/\epsilon_0)(n_{1\mathrm{a}} + n_{1b})$, where stream a has $v_{0a} = v_0\hat{\mathbf{x}}$, $n_{0a} = \frac{1}{2}n_0$; stream b has $v_{0b} = -v_0\hat{\mathbf{x}}$, $n_{0b} = \frac{1}{2}n_0$. Thus

$$ikE_1 = -\left(\frac{e}{\epsilon_0}\right)\left(\frac{-ikeE_1}{m}\right)\left[\frac{\frac{1}{2}n_0}{(\omega - kv_0)^2} + \frac{\frac{1}{2}n_0}{(\omega + kv_0)^2}\right]$$

$$1 = \frac{n_0e^2}{\epsilon_0 m}\cdot\frac{1}{2}\left[\frac{1}{(\omega - kv_0)^2} + \frac{1}{\left(\omega + kv_0^2\right)}\right]$$

$$1 = \frac{1}{2}\omega_p^2\left[\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right]$$

(b)

$$1 = \omega_p^2\frac{\omega^2 + k^2v_0^2}{\left(\omega^2 - k^2v_0^2\right)^2}$$

$$\omega^4 - \left(\omega_p^2 + 2k^2v_0^2\right)\omega^2 + k^2v_0^2\left(k^2v_0^2 - \omega_p^2\right) = 0$$

$$\omega^2 = \frac{1}{2}\left(\omega_p^2 + 2k^2v_0^2\right) \pm \frac{1}{2}\left(\omega_p^4 + 8\omega_p^2k^2v_0^2\right)^{1/2}$$

Let

$$x = \frac{2k^2v_0^2}{\omega_p^2} \quad y^2 = \frac{2\omega^2}{\omega_p^2}$$

Then

$$y^2 = 1 + x \pm (1 + 4x)^{1/2}$$

y can be complex only if the (−) sign is taken. This y is pure imaginary, and we can let $y = i\gamma$:

$$\gamma^2 = (1 + 4x)^{1/2} - (1 + x)$$
$$\frac{d}{dx}(\gamma^2) = 2(1 + 4x)^{-1/2} - 1 = 0 \quad x = \frac{3}{4}$$

Thus

$$\gamma^2 = (1 + 3)^{1/2} - \frac{7}{4} = \frac{1}{4}$$
$$\gamma = \frac{1}{2} = \frac{\sqrt{2}\,\mathrm{Im}(\omega)}{\omega_p} \quad \mathrm{Im}(\omega) = \frac{\omega_p}{2^{3/2}}$$

6.8 (a)

$$1 = \omega_p^2 \left[\frac{1}{\omega^2} + \frac{\delta}{(\omega - ku)^2} \right]$$

where $\omega_p^2 \equiv n_0 e^2/\epsilon_0 m$.

(b) This equation is the same as Eq. (6.30) except that m/M is replaced by δ, which is also small, and that the rest frame has changed to one moving with velocity u. The maximum growth rate does not depend on frame, as can be seen from Fig. 6.11 by imagining γ to be plotted in the z direction vs. x and y; a shift in the origin of x will not affect the peak. Analogy with Eq. (6.35) then gives

$$\gamma_{\max} \approx \delta^{1/3} \omega_p$$

(The exact constant that should appear here is $3^{1/2}2^{-4/3} = 0.69$. The derivation of $\gamma_{\max}$, which is difficult because the dispersion relation is cubic, and the proof that it is independent of frame for real k are left as exercises for the advanced student.)

6.9 (a) Since only the y component of $\mathbf{v}_j$ and $\mathbf{E}$ are involved, the given relation is easily found from Eqs. (4.98b) and (6.23), plus continuity and Poisson's equation. Note that Ω_p is defined with n_0, not $(1/2)n_0$.

(b) Let $\alpha \equiv \frac{1}{2}\Omega_p^2 \left(1 + \omega_p^2/\omega_c^2\right)^{-1}, \beta \equiv k^2 v_0^2$. Then the dispersion relation reduces to

$$\omega^4 - 2(\alpha+\beta)\omega^2 + \beta^2 - 2\alpha\beta = 0$$

The dispersion $\omega(k)$ is given by

$$\omega^2 = \alpha + \beta \pm \left(\alpha^2 + 4\alpha\beta\right)^{1/2}$$

Instability occurs if $(\alpha^2 + 4\alpha\beta)^{1/2} > \alpha + \beta$, or $\beta < 2\alpha$, i.e.,

$$k^2 < \left(\Omega_p^2/v_0^2\right)\left(1 + \omega_p^2/\omega_c^2\right)^{-1}$$

Where this is satisfied, the growth rate is given by

$$\gamma = \left[\left(\alpha^2 + 4\alpha\beta\right)^{1/2} - (\alpha+\beta)\right]^{1/2}$$

7.3 (a)

$$f_p(v) = \frac{n_p}{a\pi^{1/2}} e^{-v^2/a^2}$$
$$f_b(v) = \frac{n_b}{b\pi^{1/2}} e^{-(v-V)^2/b^2}$$

(b)

$$f_b'(v) = \frac{n_b}{b\pi^{1/2}} \frac{-2(v-V)}{b^2} e^{-(v-V)^2/b^2}$$
$$f_b''(v) = \frac{-2n_b}{b^3\pi^{1/2}}\left[1 - \frac{2(v-V)^2}{b^2}\right] e^{-(v-V)^2/b^2} = 0$$
$$v - V = \pm b/\sqrt{2} \quad v_\phi = V - b/\sqrt{2}$$
$$f_b'(v_\phi) = \left(\tfrac{2}{\pi}\right)^{1/2} \frac{n_b}{b^2} e^{-1/2}$$

(c)

$$f_b'(v_\phi) = \frac{n_p}{a\pi^{1/2}}\left(\frac{-2}{a^2}\right)\left(V - \frac{b}{2^{1/2}}\right) e^{-\left(V-b/\sqrt{2}\right)^2/a^2}$$
$$\approx -\frac{2n_pV}{a^3\pi^{1/2}} e^{-V^2/a^2} \quad V \gg b$$

(d)

$$\left(\frac{2}{\pi}\right)^{1/2}\frac{n_b}{b^2}e^{-1/2} = \frac{2n_pV}{a^3\pi^{1/2}}e^{-V^2/a^2}$$
$$\frac{n_b}{n_p} = (2e)^{1/2}\frac{b^2}{a^3}Ve^{-V^2/a^2} \quad \frac{b^2}{a^2} = \frac{T_b}{T_p}$$
$$\therefore \frac{n_b}{n} = \left(2e^{1/2}\right)\frac{T_bV}{T_pa}e^{-V^2/a^2}$$

7.8 From Eq. (7.127), we obtain $\sum\alpha_j Z'(\zeta_j) = 2T_i/T_e$, where $\alpha_j = n_{0j}/n_{0e}$, $\zeta_j = \omega/kv_{thj}$. Assume at first that α_H is small, so that $\alpha_A \approx 1$, $\alpha_H = \alpha$; furthermore, small α means that v_ϕ will be nearly unchanged from v_s of argon. Then doubling the Landau damping rate means Im $Z'(\zeta_H) =$ Im $Z'(\zeta_A)$, where Im $Z'\left(\zeta_j\right) = -2i\sqrt{\pi}\zeta_j e^{-\zeta_j^2}. =$ Thus

$$\zeta_A e^{-\zeta_A^2} = \alpha\zeta_H e^{-\zeta_H^2} \quad \alpha = \frac{\zeta_A}{\zeta_H}e^{-\left(\zeta_A^2-\zeta_H^2\right)}$$

$$\frac{\zeta_A}{\zeta_H} = \left(\frac{M_A}{M_H}\right)^{1/2} \quad \alpha = (40)^{1/2}e^{-\zeta_A^2(1-1/40)}$$

$$\zeta_A^2 = \frac{KT_e + 3KT_i}{M_A}\cdot\frac{M_A}{2KT_i} = \frac{13}{2}$$
$$\alpha = \sqrt{40}e^{-6.5(0.975)} = 1.12 \times 10^{-2} \approx 1\%$$

Thus α is so small that our initial assumptions are justified.

7.9 (a)

$$\frac{2k^2}{k_{Di}^2} = Z'(\zeta_i) + \frac{1-\alpha}{\theta_e}Z'(\zeta_e) + \frac{\alpha}{\theta_h}Z'(\zeta_h)$$

(b)

$$Z'(\zeta_i) \approx -2 - 2i\sqrt{\pi}\zeta e^{-\zeta^2}$$

Since $\zeta_h \ll \zeta_e \ll 1$,

$$\left|\text{Im}Z'(\zeta_h)\right| \ll \left|\text{Im}Z'(\zeta_e)\right|$$

(c) Since $Z'(\zeta_h) \approx Z'(\zeta_e) \approx -2)$, the ζ_h term in (a) is negligible compared with the ζ_e term if $\theta_h \gg \theta_e$ and $\alpha < 1/2$. Now the dispersion relation is

$$Z'(\zeta_i)\big| = \frac{2k^2}{k_{Di}^2} + \frac{2(1-\alpha)}{\theta_e} = \frac{2T_i}{T_e}\left(1 \mp \alpha + \frac{T_ek^2}{T_ik_{Di}^2}\right)$$

The last term is $\approx k^2\lambda_D^2$ and is negligible when quasineutrality holds. Thus the ion wave dispersion relation is the same as usual, except that T_i/T_e has been replaced by $(1-\alpha)\ T_i/T_e$. Since small T_i/T_e means less Landau damping, the hot electrons have decreased ion Landau damping.

8.3 Refer to Fig. 8.4. Take a number of ions with $v = u_0$ and split them into two groups, one with $v = u_0 + \mathbf{\Delta}$ and one with $v = u_0 - \mathbf{\Delta}$. After acceleration in a potential ϕ, the faster half will have less fractional energy gain (because it started with more energy) and, hence, will have less fractional density decrease. The opposite is true for the slower half, and to first order the total density decrease is the same as if all ions had $v = u_0$. However, there is a second-order effect which makes the slower group dominate. This can be seen by making $\mathbf{\Delta}$ so large that $v \approx 0$ for the slower half, which clearly must then suffer a huge density decrease. To compensate for this, u_0 must be *increased* to higher than the Bohm value.

8.4 The maximum current occurs when the space charge of decelerated ions near grid 3 decreases the electric field to zero. Thus we can apply the Child-Langmuir law to the region between grids 2 and 3.

$$J = \frac{4}{9}\left[\frac{(2)\left(1.6\times10^{-19}\right)}{(4)\left(1.67\times10^{-27}\right)}\right]^{1/2}\frac{\left(8.85\times10^{-12}\right)(100)^{3/2}}{\left(10^{-3}\right)^2} = 27.2\frac{\text{A}}{\text{m}^2}$$

$$A = \frac{\pi}{4}\left(4\times10^{-3}\right)^2 = 1.26\times10^{-5}\,\text{m}^2$$

$$I = JA = \underline{0.34\,\text{mA}}$$

8.6 (a) At $\omega_p = \omega$,

$$F_{\text{NL}} = -\frac{\epsilon_0\left\langle E^2\right\rangle}{2L} = -\nabla p_{\text{eff}} = \frac{p_{\text{eff}}}{L}$$

$\therefore p_{\text{eff}} = \frac{1}{2}\epsilon_0\left\langle E^2\right\rangle$. But $I_0 = c\epsilon_0\left\langle E^2\right\rangle = P/A$, where $P = 10^{12}$ and $A = (\pi/4)\ (50\times10^{-6})^2 = 1.96\times10^{-9}\ \text{m}^2$

$$p_{\text{eff}} = \frac{P}{2cA} = \frac{10^{12}}{(2)\left(3\times10^8\right)\left(1.96\times10^{-9}\right)} = 8.50\times10^{11}\,\frac{\text{N}}{\text{m}^2}$$

$$= \frac{\left(8.50\times10^{11}\right)(0.2248)}{(39.37)^2} = 1.23\times10^8\frac{\text{lb}}{\text{in.}^2}$$

(b)

$$F = pA \quad P/2c = 10^{12}/(2)\left(3\times10^8\right) = 1667\,\text{N}$$

$$F = Mg \quad M = F/g = 1667/9.8 = 170\,\text{kg} = 0.17\,\text{tonnes}$$

(c)

$$2nKT = p_{\text{eff}}$$
$$\therefore n = \frac{8.5\times10^{11}}{(2)(10^3)(1.6\times10^{-19})} = 2.66\times10^{27}\text{m}^{-3}$$

8.7

$$F_{\text{NL}} = \nabla p.\therefore \frac{\partial}{\partial r}(n\boldsymbol{KT}) = -\frac{n}{n_c}\frac{\partial}{\partial r}\left(\frac{\epsilon_0\langle E^2\rangle}{2}\right)$$
$$\frac{1}{n}\frac{\partial n}{\partial r} - \frac{\epsilon_0}{2n_c\boldsymbol{KT}}\frac{\partial}{\partial r}\langle E^2\rangle \quad \ln n = -\frac{\epsilon_0\langle E^2\rangle}{2n_c\boldsymbol{KT}} + \ln n_0$$
$$n = n_0 e^{-\epsilon_0\langle E^2\rangle/2n_cKT}$$

At r = 0,

$$n_{\min} = n_0 e^{-\epsilon_0\langle E^2\rangle \max/2n_cKT} = n_0 e^{-\alpha}$$
$$\therefore \alpha = \frac{\epsilon_0\langle E^2\rangle_{\max}}{2n_cKT}$$

8.9

$$k_0 = 2\pi/\lambda_0 = 2\pi/1.06\times10^{-6} = 5.93\times10^6\,\text{m}^{-1}$$
$$k_i \approx 2k_0 = 1.19\times10^7\text{m}^{-1}$$
$$v_s = \left(\frac{KT_e + 3KT_i}{M}\right)^{1/2} = \left[\frac{(10^3)(1.6\times10^{-19})}{(2)(1.67\times10^{-27})}\right]^{1/2}\left(1+\frac{3}{\theta}\right)^{1/2}$$
$$\omega_i = \boldsymbol{\Delta}\omega = k_i v_s = (1.19\times10^7)(2.19\times10^5)\left(1+\frac{3}{\theta}\right)^{1/2}$$
$$= 2.61\times10^{12}\left(1+\frac{3}{\theta}\right)^{1/2}$$
$$\frac{\boldsymbol{\Delta}\omega}{\omega_0} = -\frac{\boldsymbol{\Delta}\lambda}{\lambda_0}.\therefore \boldsymbol{\Delta}\omega = -\frac{\omega_0}{\lambda_0}\boldsymbol{\Delta}\lambda = -\frac{2\pi c}{\lambda_0^2}\boldsymbol{\Delta}\lambda$$
$$= -\frac{(2\pi)(3\times10^8)}{(1.06\times10^{-6})^2}(21.9\times10^{-10})$$
$$= 3.67\times10^{12}$$
$$1+\frac{3}{\theta} = \left(\frac{3.67\times10^{12}}{2.61\times10^{12}}\right)^2 = 2 \quad \theta = \frac{T_e}{T_i} = 3 \;\therefore T_i = \frac{1}{3}\text{keV}$$

8.10 (a)

$$\left\langle E_0^2 \right\rangle = \frac{1}{2}\overline{E}^2 = \frac{8\omega_1\omega_2\Gamma_1\Gamma_2}{c_1c_2}$$

$$c_1c_2 = \frac{\epsilon_0 k_1^2\omega_p^4}{n_0\omega_0^2 M} \quad \Gamma_2 = \frac{\omega_p^2}{\omega_2^2}\frac{\nu}{2}$$

$$\left\langle E_0^2 \right\rangle = \frac{4\omega_1\Gamma_1\omega_0^2\nu}{\omega_2 k_1^2}\frac{n_0 M}{\epsilon_0\omega_p^2} = \frac{4\omega_1\Gamma_1\omega_0^2\nu Mm}{\omega_2 k_1^2 e^2}$$

$$\left\langle v_0^2 \right\rangle = \frac{e^2\left\langle E_0^2 \right\rangle}{m^2\omega_0^2} = \frac{4\omega_1\Gamma_1\nu M}{\omega_2 k_1^2 m}$$

$$k_1^2 = \frac{\omega_1^2}{v_s^2} = \frac{\omega_1^2 M}{KT_e} = \frac{\omega_1^2 v_e^2 M}{M} \therefore \frac{\left\langle v_0^2 \right\rangle}{v_e^2} = \frac{4\Gamma_1\nu}{\omega_1\omega_2}$$

(b)

$$\frac{\left\langle v_0^2 \right\rangle}{v_e^2} = \frac{4\Gamma_1\nu_{ei}}{\omega_1\omega_0}$$

since $\omega_2 \approx \boldsymbol{\omega_0}$ when $n \ll n_c$.

$$\omega_0 = \frac{2\pi c}{\lambda_0} = \frac{(2\pi)\left(3\times 10^8\right)}{10.6\times 10^{-6}} = 1.78\times 10^{14}s^{-1}$$

$$v_e^2 = \frac{KT_e}{m} = \frac{\left(10^2\right)\left(1.6\times 10^{-19}\right)}{\left(0.91\times 10^{-30}\right)} = 1.76\times 10^{13}\,\frac{\text{m}^2}{s^2}$$

$$\frac{\Gamma_1}{\omega_1} = \left(\frac{\pi}{8}\right)^{1/2}\theta(3+\theta)^{1/2}e^{-(3+\theta)/2} \quad \theta = \frac{T_e}{T_i} = 10$$

$$= 3.40\times 10^{-2}$$

$$\eta = 5.2\times 10^{-5}\frac{\ln\boldsymbol{\Lambda}}{T_{eV}^{3/2}} = \frac{\left(5.2\times 10^{-5}\right)(10)}{(100)^{3/2}} = 5.2\times 10^{-7}\Omega\text{-m}$$

$$v_{ei} = \frac{ne^2\eta}{m} = \frac{\left(10^{23}\right)\left(1.6\times 10^{-19}\right)^2\left(5.2\times 10^{-7}\right)}{\left(0.91\times 10^{-30}\right)} = 1.46\times 10^9 s^{-1}$$

$$\left\langle v_0^2 \right\rangle = \frac{(4)\left(3.4\times 10^{-2}\right)\left(1.46\times 10^9\right)}{1.78\times 10^{14}}\left(1.76\times 10^{13}\right) = 1.96\times 10^7\,\frac{\text{m}^2}{\text{s}^2}$$

From Problem 8.6(a):

$$I_0 = c\epsilon_0\left\langle E^2 \right\rangle = c\epsilon_0\frac{m^2\omega_0^2}{e^2}\left\langle v_0^2 \right\rangle$$

$$I_0 = \left(3 \times 10^8\right)\left(8.854 \times 10^{-12}\right) \frac{\left(0.91 \times 10^{-30}\right)^2\left(1.78 \times 10^{14}\right)^2\left(1.96 \times 10^7\right)}{\left(1.6 \times 10^{-19}\right)^2}$$

$$= 5.34 \times 10^{10}\frac{\mathrm{W}}{\mathrm{m}^2} = 5.34 \times 10^6\frac{\mathrm{W}}{\mathrm{cm}^2}$$

8.11 $$\left(\omega_s^2 + 2i\gamma\omega_s - \omega_1^2\right)\left[(\omega_s + i\gamma - \omega_0)^2 - \omega_2^2\right] = \tfrac{1}{4}c_1c_2\overline{E}_0^2.$$

If $\omega_{\mathrm{s}}^2 = \omega_1^2, (\omega_s - \omega_0)^2 = \omega_2^2$, and $\gamma/\omega_{\mathrm{s}} \ll 1$, then

$$(2i\gamma\omega_{\mathrm{s}})[2i\gamma(\omega_{\mathrm{s}} - \omega_0)] = \frac{1}{4}c_1c_2\overline{E}_0^2 = 4\gamma^2\omega_s\omega_2$$

From Problem 8.10,

$$c_1c_2 = \frac{\epsilon_0 k_1^2\omega_p^4}{n_0\omega_0^2 M} = \frac{k_1^2\omega_p^2 e^2}{\omega_0^2 mM}$$

$$\gamma^2 = \frac{k_1^2\omega_p^2 e^2\overline{E}_0^2}{16\omega_s\omega_2\omega_0^2 mM} = \frac{k_1^2\omega_p^2\overline{v}_0^2 m}{16\omega_s\omega_2 M} \approx \frac{(2k_0)^2\Omega_p^2\overline{v}_0^2}{16\omega_0\omega_s}$$

$$= \frac{\omega_0^2\Omega_p^2\overline{v}_0^2}{4c^2\omega_0\omega_s} \therefore \gamma = \frac{\overline{v}_0}{2}\left(\frac{\omega_0}{\omega_s}\right)^{1/2}\Omega_p$$

8.13 (a)

$$Mn_0\frac{\partial v}{\partial t} = en_0E - \gamma_{\mathrm{i}}KT_i\nabla_n - Mn_0\nu v + F_{NL}$$

$$Mn_0(-i\omega + \nu)v = en_0(-ik\phi) - \gamma_{\mathrm{i}}KT_i ikn_1 + F_{NL}$$

with $e\phi/KT_e = n_1/n_0$, this becomes

$$(\omega + i\nu)v = kv_s^2\frac{n_1}{n_0} + \frac{iF_{NL}}{Mn_0}$$

Continuity:

$$-i\omega n_1 + ikn_0v = -i\omega n_1 + ikn_0(\omega + i\nu)^{-1}\left[kv_s^2\frac{n_1}{n_0} + \frac{iF_{NL}}{Mn_0}\right] = 0$$

$$\left(\omega^2 + i\nu\omega - k^2v_s^2\right)n_1 = ikF_{NL}/M$$

When $F_{\mathrm{NL}} = 0$,

$$\omega^2\left(1+i\frac{\nu}{\omega}\right)=k^2v_s^2\therefore\omega\approx kv_s\left(1-\frac{1}{2}i\frac{\nu}{\omega}\right)=kv_s-\frac{i}{2}\nu$$

Hence $-\,\mathrm{Im}\,\omega\equiv\ \Gamma=\nu/2$. So $\left(\omega^2+2i\Gamma\omega-k^2\mathrm{v}_\mathrm{s}^2\right)n_1=ikF_{NL}/M$

(b)

$$F_{NL}=-\frac{\omega_p^2}{\omega_0\omega_2}\nabla\epsilon_0\langle E_0E_2\rangle=-\frac{\omega_p^2}{\omega_0\omega_2}ik\epsilon_0\langle E_0E_2\rangle$$

Thus,

$$c_1=\frac{ikF_{NL}}{M}\frac{1}{\langle E_0E_2\rangle}=\frac{ik}{M}\left(\frac{-\omega_p^2}{\omega_0\omega_2}ik\,\epsilon_0\right)=\frac{\omega_p^2}{\omega_0\omega_2}\frac{k^2\epsilon_0}{M}$$

8.14 The upper sideband has $\hbar\omega_2=\hbar\omega_0+\hbar\omega_1$, so that the outgoing photon has *more* energy than the original photon $\hbar\omega_0$. The lower sideband would be expected to be more favorable energetically, since it is an exothermic reaction, with $\hbar\omega_2=\hbar\omega_0-\hbar\omega_1$.

8.18 $\boldsymbol{U}(\xi-c\tau)=3c\ \ \mathrm{sech}^2\ \ [(c/2)^{1/2}(\xi-c\tau)]$, where $\xi=\delta^{1/2}(x'-t')$, $\tau=\delta^{3/2}t'$, $x'=x/\lambda_D$, $t'=\Omega_pt$, $\delta=\mathrm{M}-1$

$$\zeta=\xi-ct=\delta^{1/2}\left(\frac{x-v_st}{\lambda_D}-\delta c\frac{v_s}{\lambda_D}t\right)$$

since $\lambda_D\Omega_p=v_s$

$$\zeta=\frac{\delta^{1/2}}{\lambda_D}\left[x-(1+\delta c)v_st\right]$$

The soliton has a peak at $\zeta=0$. The velocity of the peak is $dx/dt=(1+\delta c)v_s$. By definition,

$$\frac{dx}{dt}=Mv_s=(1+\delta)v_s$$

$$\therefore c=1\qquad\therefore U_{\max}=3c=3$$

From Eq. (8.111),

$$x_{\max} \equiv \frac{e\phi_{\max}}{KT_e} \approx \delta x_{1\max} = \delta U_{\max}$$
$$\therefore \delta \frac{e \quad \phi_{\max}}{KT_e U_{\max}} = \frac{121}{103} = 0.4$$
$$v_\phi = (1+\delta) v_s = 1.4 v_s$$
$$v_s = \left(\frac{KT_e}{M}\right)^{1/2} = \left[\frac{(10)\left(1.6 \times 10^{-19}\right)}{1.67 \times 10^{-27}}\right] = 3.10 \times 10^4$$
$$v_\phi = \underline{4.33 \times 10^4 \mathrm{m/s}}$$

At half maximum, $\mathrm{sech}^2\, a = \frac{1}{2} \therefore a = 0.8814 = \sqrt{\frac{1}{2}}\zeta \therefore \zeta = 1.25 = \delta^{1/2} x/\lambda_{\mathrm{D}}$ at $t = 0$, say.

$$\delta^{1/2} = \sqrt{0.4} = 0.632$$
$$\lambda_{\mathrm{D}} = \left(\frac{\in_0 KT_e}{n_0 e^2}\right)^{1/2} = 2.35 \times 10^{-4}\mathrm{m} = 0.235\,\mathrm{mm}$$
$$x = \frac{1.25\lambda_{\mathrm{D}}}{0.632} = 0.46\,\mathrm{mm} \quad \mathrm{FWHM} = 2x = \underline{0.93\mathrm{mm}}$$

8.21
$$|u| = 4A^{1/2}|\sec\mathrm{h}\, x| \therefore |u|^2 = 16A|\sec\mathrm{h}\, x|^2$$

$$\delta n = \frac{1}{4}|u|^2\left(\frac{V^2}{\in^2} - 1\right)^{-1} \approx -\frac{1}{4}|u|^2 = -4A|\sec\mathrm{h}\, x|^2$$
$$\overline{\delta n} = -4A\overline{|sech\, x|^2} \approx -2A$$
$$\frac{\delta\omega_p}{\omega_p} = \frac{1}{2}\frac{\delta n}{n} = -\frac{1}{2}(2A) = -A$$

$\therefore$ A is frequency shifted due to δn.

8.22 In real units,

$$v = \frac{v}{v_e} = 4A^{1/2}\mathrm{sech}\left[\left(\frac{2A}{3}\right)^{1/2}\left(\frac{x}{\lambda_{\mathrm{D}}} - \frac{V}{v_e}\omega_p t\right)\right]\exp\left\{-i\left[\left(\frac{\omega_0}{\omega_p} + \frac{1}{6}\frac{V^2}{v_e} - A\right)\right]\omega_p t - \frac{V \quad x}{3v_e\lambda_{\mathrm{D}}}\right\}$$

$$v_e = \left(\frac{KT_e}{m}\right)^{1/2} = 5.93 \times 10^5 \text{m/s} \quad \omega_p = \left(\frac{ne^2}{\in_0 m}\right)^{1/2} = 1.78 \times 10^9 \frac{\text{rad}}{\text{s}}$$

$$\lambda_D = \frac{v_e}{\omega_p} = 3.33 \times 10^{-4}\text{m} \quad k = \frac{(k\lambda_D)}{\lambda_D} = \frac{0.3}{\lambda_D} = 9.02 \times 10^2 \text{m}^{-1}$$

$$u_{p-p} = 4A^{1/2} \qquad -i\omega mv = -eE = -e(-ik\phi) \therefore \phi = -\frac{m\omega v}{ek}$$

$$\phi_{p-p} \approx \frac{m\omega}{ek} 4A^{1/2} v_e \quad \omega \approx \left(\omega_p^2 + 3k^2 v_e^2\right)^{1/2} 2.01 \times 10^9$$

$$A^{1/2} = \frac{ke\phi_{p-p}}{4m\omega v_e} = \frac{k}{4\omega}\frac{e\phi_{p-p}}{KT_e}\frac{1}{v_e} = \frac{kv_e}{4\omega}\frac{e\phi_{p-p}}{KT_e}$$

$$= \frac{kv_e}{4\omega}\frac{3.2}{2} = 0.106$$

$$A = 1.13 \times 10^{-2}$$

(a)

$$\sec\text{h} X = \frac{1}{2} \qquad X = 1.315 = \left(\frac{2A}{3}\right)^{1/2} \frac{x}{\lambda_D}$$

$$x = \left(\frac{3}{2}\right)^{1/2} \frac{(1.315)\left(3.33 \times 10^{-4}\right)}{0.106} = 5.04 \times 10^{-3}$$

$$\text{FWHM} = 2x = 1.01 \times 10^{-2} = 10.1 \text{ mm}$$

(b)

$$N = \frac{1.01 \times 10^{-2}}{2\pi/k} = 1.45$$

(c)

$$\delta\omega = A\omega_p = \left(1.13 \times 10^{-2}\right)\left(1.78 \times 10^9\right) = 2 \times 10^7 \text{rad/s}$$

$$\delta f = \delta\omega/2\pi = 3.2 \times 10^6 = 3.2\,\text{MHz}$$

8.23

$$3v_e^2 = \frac{(3)(3)\left(1.6 \times 10^{-19}\right)}{0.91 \times 10^{-30}} = 1.58 \times 10^{12} \text{m}^2/s^2$$

$$\omega_p^2(\text{out}) = \frac{\left(10^{16}\right)\left(1.6 \times 10^{-19}\right)^2}{\left(8.824 \times 10^{-12}\right)\left(0.91 \times 10^{-30}\right)} = 3.18 \times 10^{19} \frac{\text{rad}^2}{\text{s}^2}$$

$$\omega_p^2(\text{in}) = 0.4\omega_p^2(\text{out})$$

$$k_{\max}^2 = \frac{\omega_p^2(\text{out}) - \omega_p^2(\text{in})}{3v_e^2} = \frac{3.18 \times 10^{19}}{1.58 \times 10^{12}}(1 - 0.4)$$

$$= 1.21 \times 10^7 \text{m}^{-2}$$

$$\lambda_{\min} = \frac{2\pi}{k_{\max}} = 1.81 \times 10^{-3}\text{m} = \underline{1.81 \text{ mm}}$$

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