Appendix A
Infinite Dynamics

Abstract In this appendix we shortly review the principal results on Hamiltonian evolution of infinitely extended systems.

A.1 Time Evolution of Infinitely Many Particles Systems

It is well known that in classical mechanics the time evolution of a system of N particles is very well described by the Newton's law, founded in the seventeenth century, that gives rise to a system of ordinary differential equations. Only 150 years after, Cauchy proved an existence and uniqueness theorem, giving solid mathematical basis to the theory.

However, Statistical Mechanics deals with systems composed by a very large number of point particles ($N \sim 10^{23}$), and to catch their asymptotic behavior, due to the huge number of degrees of freedom, it is very often useful to consider these systems as infinitely extended (i.e., to perform the so-called thermodynamic limit). It is not obvious what happens to the existence and uniqueness theorem in this limit. Of course, the time evolution is well defined for any large but finite N, but it is not obvious the convergence of the dynamics for $N \to \infty$, which is the content of this appendix.

A phase point of the system is an infinite sequence $\{(\mathbf{r}_i, \mathbf{v}_i)\}_{i\in\mathbb{N}}$ of the position and the velocity of the particles and its time evolution is given by the Newton law,

$$m\ddot{\mathbf{r}}_i(t) = \sum_{j\in\mathbb{N};\, j\neq i} \mathbf{F}(\mathbf{r}_i(t) - \mathbf{r}_j(t)) , \qquad i \in \mathbb{N} , \tag{A.1}$$

where m is the mass of each particle and $\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$, with Φ is a two body potential. The system (A.1) is completed by assigning the initial data $\{(\mathbf{r}_i(0), \mathbf{v}_i(0))\}_{i\in\mathbb{N}}$.

In general, a necessary condition to give meaning to the right hand side of (A.1) is to have a finite number of particles in any bounded region of the space $\mathbb{R}^d$. We can assume that the initial states enjoy this property, but the time evolution could destroy it, as we can see in this simple example in dimension $d = 1$ (already discussed in Sect. 1.1). Consider a system of free particles of unitary mass moving on the real line with the initial condition $r_i(0) = i$, $\dot{r}_i(0) = -i$, $i \in \mathbb{N}$. It is evident that

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at time $t = 1$ all the particles are in the origin. We can forbid this "collapse" by restricting the allowed initial conditions, but we cannot be too drastic. For instance, these pathologies are removed by choosing the initial velocities uniformly bounded and the initial particle distribution locally finite. But this set of initial data is exceptional with respect of any Gibbs measure, as it can be easily seen observing that, at equilibrium, the probability to have particles with velocities smaller than a fixed value in a unitary interval is less than one. Therefore, the probability to have velocities smaller than any fixed value in infinitively many intervals is vanishing. Of course, the time evolution defined on a state with measure less than one is not able to produce a time evolution of all the functions of the phase space (the observables), i.e., the quantities with a physical meaning that can be compared with experiments. Obviously, a free particles system can be solved "ad hoc", but in general it is not so.

In fact, for the model to be meaningful, the initial conditions have to be chosen in a set which is typical for any reasonable thermodynamic (equilibrium or non-equilibrium) state. In conclusion, we have to construct the dynamics for initial data in a set sufficiently large to be the support of states of interest from a thermodynamical point of view. At the same time, this set cannot be so large to produce pathological collapses.

The difficulty of this problem increases with the dimension d of the space in which the particles move. For simplicity, we explain this in the following (not realistic) example, in which the initial particle velocities are uniformly bounded.

Let the potential $\Phi(|\mathbf{r}|)$ be twice differentiable and short-range, i.e., $\Phi(|\mathbf{r}|) = 0$ if $|\mathbf{r}| > r_0$, $r_0 > 0$, and assume that the velocities and the density are bounded, that is,

$$\sup_{i\in\mathbb{N}} |\mathbf{v}_i| < \infty \ , \qquad \sup_{\mu\in\mathbb{R}^d} \sup_{R>1} \frac{N(\mathbf{X};\mu,R)}{R^d} = \rho_0 < \infty \ , \tag{A.2}$$

where $\mathbf{X} = \{(\mathbf{r}_i, \mathbf{v}_i)\}_{i\in\mathbb{N}}$ is the particle configuration and $N(\mathbf{X};\mu,R)$ is the number of particles in a ball of radius R centered in μ.

The main difficulty in the control on the density of the evolved state. Let $V(t)$ be the modulus of the maximal velocity carried by the particles during the time $[0,t]$ and let $\mathbf{X}(t)$ be the evolved configuration. The conservation of the number of particles implies,

$$N(\mathbf{X}(t);\mu,R_0) \le N(\mathbf{X}(0);\mu,R(t)) \le \rho_0 R(t)^d \ , \qquad R_0 \ge 1 \ , \tag{A.3}$$

where

$$R(t) = R_0 + \int_0^t \mathrm{d}s\ V(s) \ . \tag{A.4}$$

On the other hand, $V(t)$ is controlled by the force, which turns out to be bounded by $\sup_{\mathbf{r}} |\nabla \Phi(|\mathbf{r}|)| \sup_{\mu} N(\mathbf{X}(s); \mu, r)$. From Eqs. (A.3) and (A.4) we arrive at the following integral inequality,

$$R(t) \leq R_0 + \text{Const.}\ t + \text{Const.} \int_0^t \mathrm{d}s\ (t-s)\ R(s)^d\ , \tag{A.5}$$

which is solvable globally in time only if $d = 1$. We next discuss separately the three cases $d = 1, 2, 3$.

Dimension $d = 1$ The previous inequality does not guarantee the existence of dynamics, because it applies to a set of initial data of null measure. However, it is possible to enlarge the set given by (A.2) to obtain a significant inequality like (A.5). A pioneering result was obtained by Lanford in 1968 [20, 21] for smooth, short-range interactions. Later, this was extended by Dobrushin and Fritz [15] to the case of potential with a hard core and by Marchioro et al. [25] to the case of very singular interactions. See also [23] for a one-dimensional Coulomb system.

In Chap. 1 the proof of the existence of dynamics in one dimension is explicitly given in the context of long time estimates. In conclusion, the problem has been solved in almost all cases.

Finally, we emphasize that by "one dimension" we do not mean only particles moving on a straight line, but particles moving in a region with an infinite extension in one direction only. In Chap. 1 we studied a system of particles moving in an infinite tube. For charged particles in a tube with a magnetic confinement see [4]. As an other example, we can consider a gas moving in the space under an external potential like $U(\mathbf{r}) = U(|x_1|)$, where $U(r) = -1$ if $r = 0$, is monotonically increasing, and $U(r) \to 0$ as $r \to \infty$. If the initial state is similar to a Gibbs state, i.e., mainly concentrated around the x_1-axis, it is possible to define the dynamics. We also refer to [9] for the time evolution of general system infinitely extended in some direction.

In conclusion, in one dimension infinite dynamics is more handy. However, it is difficult to find exactly solvable examples. An example quite simple, but nontrivial in the hydrodynamical limit, is the following one. A set of particle moving on a straight line and mutual interacting by a hard core of size L. At time $t = 0$ we fix the origin in a tagged particle labeled by $i = 0$, and denote by x_i, $i \in \mathbb{Z}$, the coordinate of the i-th particle, labeled in such a way that $x_i < x_j$ if $i < j$. On this system we change coordinates denoting by $\hat{x}_i = x_i - iL$ the position of the i-th particle. In these new coordinates, the motion is analogous to that of a free particles system.

There exist other one-dimensional exactly solvable systems (for instance, when the particles mutually interact via a potential like $(x_i - x_j)^{-2}$ or $[\sinh(x_i - x_j)]^{-2}$), but in the thermodynamical limit the time evolution becomes complicated.

Dimension $d = 2$ In this case the situation is more difficult. As already discussed, we need to control the increasing in time of the local density and it is hard to do it by using directly the Newton's law. A very nice result in this direction, based on the

energy conservation, was obtained by Dobrushin and Fritz in 1977 [17]. We give here only a short sketch of the proof in a particular case, to give some taste on the argument.

We assume the interaction $\Phi(r)$ to be non-negative and with short-range, i.e., $\Phi(|\mathbf{r}|) = 0$ if $|\mathbf{r}| > r_0 > 0$. We also assume $\Phi(0) > 0$, which guarantees Φ to be superstable, see Sect. 1.1. For the sake of simplicity, we also assume that each particle has unitary mass, i.e., $m = 1$. To characterize the set of initial data we proceed as in Chap. 1. For a locally finite configuration $\mathbf{X}$, we introduce a mollified version of the energy plus the number of particles contained in a circle of radius R centered in $\mu \in \mathbb{R}^2$,

$$W(\mathbf{X};\mu,R) = \sum_i f_i^{\mu,R}\left\{\frac{\mathbf{v}_i^2}{2} + \frac{1}{2}\sum_{j:j\neq i}\Phi_{i,j} + 1\right\}\,, \tag{A.6}$$

where

$$f_i^{\mu,R} = f\left(\frac{|\mathbf{x}_i - \mu|}{R}\right)\,, \qquad \Phi_{i,j} = \Phi(\mathbf{r}_i - \mathbf{r}_j)\,, \tag{A.7}$$

and the function $f \in C^\infty(\mathbb{R}_+)$ is not increasing and satisfies: $f(x) = 1$ for $x \in [0,1]$, $f(x) = 0$ for $x \geq 2$, and $|f'(x)| \leq 2$. Defining,

$$Q(\mathbf{X}) = \sup_{\mu}\ \sup_{R:R>\phi(|\mu|)} \frac{W(\mathbf{X};\mu,R)}{R^2}\,, \tag{A.8}$$

with

$$\phi(r) = \sqrt{\log(\mathrm{e}+r)}\,, \qquad r > 0\,, \tag{A.9}$$

the set $\mathscr{H} = \{\mathbf{X}\colon Q(\mathbf{X}) < \infty\}$ is a full measure set for any Gibbs states [17], see also Sect. 1.1.1.

We define "n-partial dynamics" (and we denote it by $\{\mathbf{r}_i^n(t), \mathbf{v}_i^n(t)\}$) the time evolved system in which only the particles initially contained in a circle of radius n centered in the origin are present.

Theorem A.1 *Let $\mathbf{X} \in \mathscr{H}$. There exists a unique flow $t \to \mathbf{X}(t) \in \mathscr{H}$, satisfying (A.1) with $\mathbf{X}(0) = \mathbf{X}$. Moreover, for all $t \in \mathbb{R}$ and $i \in \mathbb{N}$,*

$$\lim_{n\to\infty}\mathbf{r}_i^n(t) = \mathbf{r}_i(t)\,, \qquad \lim_{n\to\infty}\mathbf{v}_i^n(t) = \mathbf{v}_i(t)\,. \tag{A.10}$$

The main point in the proof is an a priori bound on the maximal velocity $V^n(t)$ in the partial dynamics, where $V^n(t) = \sup_{i,s}|\mathbf{v}_i^n(s)|\,,\ 0 \leq s \leq t$. We will obtain a bound of the form,

$$V^n(t) \leq C\,\sqrt{\log(\mathrm{e}+n)}\,. \tag{A.11}$$

From this bound it is quite easy to obtain (A.10) via standard methods. A short exposition is given in the sequel. By Gronwall's Lemma, the inequality (A.11) is a consequence of the following lemma.

Lemma A.2 *For each* $\mathbf{X} \in \mathscr{H}$ *there exists a positive constant* C *such that,*

$$V^n(t) \le C R_n(t) \qquad \forall\, t \ge 0\,, \tag{A.12}$$

where

$$R_n(t) = \phi(n) + \int_0^t \mathrm{d}s \; V^n(s)\,. \tag{A.13}$$

Proof For $0 \le s \le t$, let

$$R_n(t,s) = \phi(n) + \int_0^t \mathrm{d}\tau \; V^n(\tau) + \int_s^t \mathrm{d}\tau \; V^n(\tau) \tag{A.14}$$

(note that $R_n(t) = R_n(t,t)$). We compute the derivative with respect to s of the quantity

$$W(\mathbf{X}^n(s); \mu, R_n(t,s)) = \sum_i f^{\mu, R_n(t,s)} \left\{ \frac{\mathbf{v}_i^2}{2} + \frac{1}{2} \sum_{j:j\neq i} \Phi_{i,j} + 1 \right\}\,. \tag{A.15}$$

We have,

$$\begin{aligned} \partial_s W(\mathbf{X}^n(s); \mu, R_n(t,s)) = & \sum_{i,j:i\neq j} f_i^{\mu, R_n(t,s)} \left(\mathbf{v}_i \cdot \mathbf{F}_{i,j} - \frac{1}{2} \mathbf{F}_{i,j} \cdot (\mathbf{v}_i - \mathbf{v}_j) \right) \\ & + \sum_i f' \left(\frac{|\mathbf{r}_i - \mu|}{R_n(t,s)} \right) \left(\frac{\hat{\mathbf{r}}_i^{\mu} \cdot \mathbf{v}_i}{R_n(t,s)} - \frac{\partial_s R_n(t,s)}{R_n^2(t,s)} |\mathbf{r}_i - \mu| \right) \\ & \times \left(\frac{\mathbf{v}_i^2}{2} + \frac{1}{2} \sum_{j;j\neq i} \Phi_{i,j} + 1 \right)\,, \end{aligned} \tag{A.16}$$

where $\mathbf{F}_{i,j} = -\nabla \Phi(\mathbf{r}_i - \mathbf{r}_j)$. In the previous formula, we omit the explicit dependence on s and n of $\mathbf{r}_i$ and $\mathbf{v}_i$, and denote by $\hat{\mathbf{r}}_i^{\mu}$ the unitary vector in the direction of $\mathbf{r}_i - \mu$.

We note that the second sum in the right-hand side of (A.16) is not positive, as $f'\left(\frac{|\mathbf{r}_i - \mu|}{R_n(t,s)}\right) \le 0$ and vanishes if $|\mathbf{r}_i - \mu| < R_n(t,s)$ or $|\mathbf{r}_i - \mu| > 2R_n(t,s)$.

Moreover, $|\mathbf{v}_i| \le V^n(s)$ and $\partial_s R_n(t,s) = -V^n(s)$. Hence, for $R_n(t,s) \le |\mathbf{r}_i - \mu| \le 2R_n(t,s)$,

$$\frac{\hat{x}_i^{\mu} \cdot \mathbf{v}_i}{R_n(t,s)} - \frac{\partial_s R_n(t,s)}{R_n^2(t,s)}|\mathbf{r}_i - \mu| \ge -\frac{|\mathbf{v}_i|}{R_n(t,s)} - \frac{\partial_s R_n(t,s)}{R_n(t,s)} \ge 0 \,. \tag{A.17}$$

By using $\mathbf{F}_{i,j} = -\mathbf{F}_{j,i}$, the first sum in the right-hand side of (A.16) equals,

$$\frac{1}{2}\sum_{i,j;i\neq j} f_i^{\mu,R_n(t,s)}\, \mathbf{F}_{i,j}\cdot(\mathbf{v}_i + \mathbf{v}_j) = \frac{1}{2}\sum_{i,j;i\neq j}\left(f_i^{\mu,R_n(t,s)} - f_j^{\mu,R_n(t,s)}\right)\mathbf{F}_{i,j}\cdot \mathbf{v}_i \,. \tag{A.18}$$

Recalling that the force is bounded and it has a finite range r_0, and using the obvious inequality,

$$\left| f_i^{\mu,R} - f_j^{\mu,R}\right| \cdot \mathbf{v}_i \big| \le 2\frac{|\mathbf{r}_i - \mathbf{r}_j|}{R}[\chi_i(\mu, 2R) + \chi_j(\mu, 2R)] \,, \tag{A.19}$$

where $\chi_i(\mu, R) = \chi_i(|\mathbf{r}_i - \mu| \le R)$ ($\chi(A)$ denotes the characteristic function of the set A), the modulus of the quantity in the right-hand side of (A.18) is bounded by

$$\begin{aligned} &- \bar{C}_1 \frac{\partial_s R_n(t,s)}{R_n(t,s)} \sum_{i,j:i\neq j} \chi(|\mathbf{r}_i - \mathbf{r}_j| < r) \\ &\quad \times \chi(|\mathbf{r}_i - \mu| < 2R_n(t,s) + r)\, \chi(|\mathbf{r}_j - \mu| < 2R_n(t,s) + r) \,, \end{aligned} \tag{A.20}$$

where $\bar{C}_1$ is a positive constant depending only on Φ. Since Φ is superstable, by arguing as in proof of [7, Eq. (2.15)] and assuming n large enough that $\phi(n)$ is larger than the range r_0 of the potential, the double sum in the right-hand side the previous formula can be bounded by $\bar{C}_2 W(\mathbf{X}^n(s); \mu, 4R_n(t,s))$ for some $\bar{C}_2$ (depending only on Φ). Moreover, setting

$$W(\mathbf{X}; R) = \sup_{\mu\in\mathbb{R}^2} W(\mathbf{X}; \mu, R) \,, \tag{A.21}$$

by the form of $W(\mathbf{X}; \mu, R)$ and the superstability estimates on the interaction it can be proved [7] that there exists $\bar{C}_3 > 0$ (depending only on Φ) such that

$$W(\mathbf{X}; \mu, KR) \le \bar{C}_3 K^2 W(\mathbf{X}; R) \,. \tag{A.22}$$

Therefore, by (A.20),

$$\partial_s W(\mathbf{X}^n(s); \mu, R_n(t,s)) \le \bar{C}\, \frac{\partial_s R_n(t,s)}{R_n(t,s)} W(\mathbf{X}^n(s); R_n(t,s)) \,, \tag{A.23}$$

where $\bar{C}$ is a positive constant depending only on Φ. From the previous inequality we obtain

$$W(\mathbf{X}^n(s); R_n(t,s)) \leq W(\mathbf{X}^n(0); R_n(t,0)) \exp\left[-\bar{C}\int_0^s \mathrm{d}\tau \, \frac{\partial_s R_n(t,\tau)}{R_n(t,\tau)}\right] . \tag{A.24}$$

Hence, for $s \leq t$,

$$W(\mathbf{X}^n(s); R_n(t,s)) \leq W(\mathbf{X}^n(0); R_n(t,0)) \left(\frac{R_n(t,0)}{R_n(t,s)}\right)^{\bar{C}} . \tag{A.25}$$

By (A.18) and using $\frac{R_n(t,0)}{R_n(t,s)} < 2$ (by definition), we conclude that

$$W(\mathbf{X}^n(t); R_n(t)) \leq 2^{\bar{C}} W(\mathbf{X}^n(0); R_n(t,0)) \leq 2^{\bar{C}} Q(\mathbf{X}) R_n^2(t,0) =: C^2 R_n^2(t) . \tag{A.26}$$

Since $V^n(t)$ is bounded by $\sup_{s\in[0,t]} W(\mathbf{X}^n(s); R_n(s))^{1/2}$, the estimate (A.12) is thus proved. □

Proof of Theorem A.1 Let

$$\delta_i(n,t) = |\mathbf{r}_i^n(t) - \mathbf{r}_i^{n+1}(t)| , \qquad u_k(n,t) = \sup_{i\in I_k} \delta_i(n,t) ,$$

where I_k denotes the set of those particles which are initially contained in the sphere of radius k and centered in the origin, and define

$$d_n(t) = \sup_{s\in[0,t]} \sup_{i\in I_n} |\mathbf{r}_i^n(s) - \mathbf{r}_i^n(0)| .$$

By (A.11),

$$d_n(t) \leq C t \phi(n) . \tag{A.27}$$

Therefore, the maximal number of particles that can interact with a given particle i cannot be larger than the number of particles that initially are contained in the disk of radius $r_0 + Ct\phi(n)$ and centered in $\mathbf{r}_i^n(t)$. Hence, setting

$$N(\mathbf{X}; \mu, R) = \sum_i \chi(|\mathbf{r}_i - \mu| < R) , \tag{A.28}$$

we get, recalling the definition (A.8) of Q,

$$\begin{aligned} N(\mathbf{X}^n(t); \mathbf{r}_i(t), r_0) &\leq N(\mathbf{X}; \mathbf{r}_i, r_0 + Ct\phi(n)) \leq W(\mathbf{X}; \mathbf{r}_i, r_0 + Ct\phi(n)) \\ &\leq Q(\mathbf{X})[\phi(n + Ct\phi(n)) + r_0 + Ct\phi(n)]^2 \leq C_0 t^2 \phi^2(n) , \end{aligned} \tag{A.29}$$

for a suitable $C_0 > 0$ (depending on Φ and $\mathbf{X}$). By the equations of motion in integral form,

$$\mathbf{r}_i^n(t) = \mathbf{r}_i^n(0) + \mathbf{v}_i^n(0)t + \int_0^t \mathrm{d}s\ (t-s) \sum_j \mathbf{F}(\mathbf{r}_i^n(s) - \mathbf{r}_j^n(s)) \ , \tag{A.30}$$

we have, for $i \in I_k$ and n sufficiently large,

$$\delta_i(n,t) \le C \int_0^t \mathrm{d}s\ (t-s) {\sum_j}^* [\delta_i^n(s) + \delta_j^n(s)] \ , \tag{A.31}$$

where ${\sum_j}^*$ means the sum restricted to the particles which can fall in the interaction disk (radius r_0) with center $\mathbf{r}_i^n(s)$ or $\mathbf{r}_j^n(s)$ for $s \le t$. We observe that, since $k + r_0 + d_n(s) + d_{n+1}(s) < k + r_0 + C\phi(n+1) \ll n$ (provided n is sufficiently large), the particle i cannot interact with the particles j such that $n \le |\mathbf{r}_j^n(0)| \le n+1$. By (A.27) and (A.29) we then have, for $k < n$,

$$u_k(n,t) \le C_0\, t^2 \phi^2(n) \int_0^t \mathrm{d}s\ (t-s)\ u_{k_1}(n,s) \ , \tag{A.32}$$

where $k_1 = [k + C_3\phi(n)] + 1$ (here $[\cdot]$ denotes the integer part) and

$$C_3 = \sup_{n \ge 1} \frac{r_0 + 2Ct\phi(n+1)}{\phi(n)} \ . \tag{A.33}$$

Defining $k_r = [k_{r-1} + C_3\phi(n)] + 1$, with $k_0 = k$, we can iterate (A.32) ℓ times, where

$$\ell = \left[\frac{n}{10 C_3 \phi(n)} \right] . \tag{A.34}$$

Since $u_\ell(n,t) \le Ct\phi(n)$, we obtain

$$u_k(n,t) \le (Ct\phi(n))^{2\ell+1} \frac{t^{2\ell}}{(2\ell)!} \ . \tag{A.35}$$

We realize that $u_k(n,t)$ vanishes summably as $n \to \infty$.

The velocity can be studied in a similar way, as it is the integral of the force. Other steps in the proof are straightforward.

The proof of the uniqueness is similar. We observe that the assumption that the solutions belong to $\mathscr{H}$ (i.e., the velocities and the density do not increase too fast at infinity) seems to be essential. We have not an explicit counter-example with smooth interaction, but Lanford in [22] investigates a case of an infinitely extended hard core in which the uniqueness is violated allowing very large velocities at infinity.

We can show that the i-th particle can move at most by $C \log(e + |\mathbf{r}_i(0)|)$ and the evolved state belongs to the same space of the initial data $\mathbf{X}$. To obtain the first result, we use the fast convergence (A.35) to approximate the infinite dynamics by the partial dynamics with $n \approx |\mathbf{r}_i(0)|$. For the second result, we use a similar trick and (A.26). More precisely, given $\mu \in \mathbb{R}^2$ and $R > \phi(|\mu|)$ we choose n_0 such that, for a positive constant $C_4 < 1$,

$$R = C_4 \phi(n_0) . \tag{A.36}$$

Therefore,

$$\begin{aligned} W\left(\mathbf{X}^{n_0}(t); \mu, R\right) &\leq W\left(\mathbf{X}^{n_0}(t); \mu, R + R_{n_0}(t)\right) \\ &\leq C \left(\frac{R + R_{n_0}(t)}{R_{n_0}(t)}\right)^2 W\left(\mathbf{X}^{n_0}(t); R_{n_0}(t)\right) \leq C[R + R_{n_0}(t)]^2. \end{aligned}$$

On the other hand, by (A.12), (A.13), and (A.36) we have,

$$R_{n_0}(t) \leq \phi(n_0) + Ct\phi(n_0) = \frac{1 + Ct}{C_4} R .$$

Therefore,

$$W\left(\mathbf{X}^{n_0}(t); \mu, R\right) \leq \left[\frac{1 + C_4 + Ct}{C_4}\right]^2 R^2 . \tag{A.37}$$

For the infinite dynamics, we bound,

$$\begin{aligned} W\left(\mathbf{X}(t); \mu, R\right) &\leq W\left(\mathbf{X}^{n_0}(t); \mu, R\right) \\ &\quad + \sum_{n > n_0} \left| W\left(\mathbf{X}^n(t); \mu, R\right) - W\left(\mathbf{X}^{n-1}(t); \mu, R\right) \right| \end{aligned} \tag{A.38}$$

and, by the dependence of W on positions and velocities and the bound on $V^n(t)$, we get for the generic term of the sum in (A.38) an upper bound analogous to (A.35). By the choice (A.36) of n_0 (which in particular implies that $n_0 > |\mu|$), the sum in (A.38) converges uniformly with respect to $\mu \in \mathbb{R}^2$ and $R > \phi(|\mu|)$, so it is bounded by a constant independent of μ and R. Combining (A.37) and (A.38), dividing by R^2, and taking the supremum over $R > \phi(|\mu|)$ and $\mu \in \mathbb{R}^2$, we obtain that $\mathbf{X}(t) \in \mathscr{H}$.

The previous result has been obtained for bounded, positive, and short-range interaction. Actually, in the first original paper [17] the authors include power law interactions at short distances. In fact, in (A.16) and (A.19) the force is multiplied by the distance, and hence for power-like interaction is bounded by the interaction Φ and hence by W.

The proof has been extended to potentials with a negative part [16], to potentials with long range [2], and to very singular potentials [5].

We give just a sketch of the proof in the case of very singular interaction [5]. The aim is to prove that the assumption of power-like singularity for short distance is not essential and can be relaxed. We assume $\Phi(r)$ a non-negative twice differentiable function. Moreover, the interaction has a finite range and, close to the origin, of the form $\Phi(r) = g_1 \exp\{g_2 r^{-b}\}$, with $g_1, g_2, b > 0$. This interaction has the property that $|\Phi'| \leq Cr^{-1}(1 + \Phi)\log(\mathrm{e} + \Phi)$. Inserting this inequality in (A.18) and using (A.19) we obtain the equivalent of (A.20) times $\log(\mathrm{e} + \Phi)$. This term can be controlled by a change in the definition of $R_n(t,s)$. Precisely, we define $M(t) = \max\left\{V(t); \sup_{i,j,s} \sqrt{\Phi_{i,j}(s)}; \mathrm{e}\right\}$ and $R_n(t,s) = C\psi(n) + \int_0^t \mathrm{d}\tau\, M(\tau)\log M(\tau) + \int_s^t \mathrm{d}\tau\, M(\tau)\log M(\tau)$. Then, $|v_i| \leq -\partial_s R \log^{-1} M(s)$, which allows to control the extra term $\log(\mathrm{e} + \Phi)$. The other step in the proof are similar to those used before and we arrive to an estimate like $M(t) \leq (\log n)^C$ that is sufficient to achieve the result.

We spend few words on the case of singular but very weakly diverging interactions. In this case it is easy to control the growth of the energy, but it is difficult to control the Lipschitz constant of the force by the energy itself. In [17] the authors observe that the method works only for interactions not too weakly diverging at the origin.

A paradigmatic example of this case arises by considering point particles interacting via the Green function of the two dimensional Laplace operator. Physically, it is a system of charged wires mutually interacting via a Coulomb force. The potential energy is $\Phi(r) = g \log r$, $g \in \mathbb{R}$ ($g > 0$ for the attractive case, $g < 0$ for the repulsive case).

We discuss the repulsive case with short-range, i.e., $\Phi(r)$ non-negative, twice differentiable for $r > 0$, $\Phi(r) = 0$ if $r > 1$, with a behavior near the origin of the form

$$\Phi(r) = g \log r \ , \quad r \leq r_0 < 1 \ ,$$

with $g < 0$. In this case we are able to prove the existence of the infinite dynamics only for small initial data and short times (the shortness depending on the smallness of the initial state). The key observation is that by the form of the interaction it results

$$\left|\Phi''(r)\right| \leq |g| \exp \frac{2\Phi(r)}{|g|} \ ,$$

that allows to control the maximum Lipschitz constant $L^n(t)$ of the force in terms of the energy, showing it satisfies a bound of the form $L^n(t) \leq \mathrm{Const.}\, n^b$ with $b < 1$ (which allows the convergence of the series $\sum u_k(n,t)$ as the displacement is bounded by order $\log n$), only for small $Q(\mathbf{X})$ and short times. Hence we get a local existence of the infinite dynamics. The uniqueness and the property of remaining in the same space could be discussed. For a detailed analysis of this point see [5]. We

finally remark that the above interaction is a threshold: for interactions less singular than $C|\log r|$ near the origin the method does not work, whereas for interactions more singular than $C|\log r|$ we have a global existence of the dynamics.

Until now we have considered systems of particles with singular interaction at most in the origin. There is another kind of interaction largely used in physics (for instance in statistical mechanics): the hard core interaction, in which the particles cannot approach each other at a distance shorter than $2a$. For binary interactions, the impulsive motion can be fixed by the conservation law and for fixed number of particles N we can neglect the initial states that lead to a multiple shock. It would be interesting to study the problem in the thermodynamical limit, in particular proving as exceptional the initial data producing at some time a close packing situation. As far as we now this problem is still open.

To make the time evolution continuous we can add a smooth interaction potential diverging as the distance between the particles becomes $2a$. In this case the dynamics is well defined for fixed N and we can study the infinitely many particles case. In one dimension the problem is easy, because by the transformation previously explained it is related to the case without hard core. In two dimensions the problem is unsolved. In fact, in the proof of [17] the singularity of the interaction is compensated by the Lipschitz property of the function f appearing in the mollifier defined in (A.7). Of course, this is not possible here. However, in dimensions less than two the problem can be solved (see later).

Dimension $d = 3$ The energy, used as a sort of Liapunov function, is not enough to construct the time evolution in the three dimensional space and we must add some a technical device: an average on a short time interval. It is a trick used in other problems from mathematical physics, for instance in the analysis of the three dimensional Vlasov–Poisson equation, i.e., a gas of particles with Coulomb interaction in the mean field limit [28, 31, 38], or point charges and a Vlasov–Poisson gas [26]. In our problem, we want that the growth of the local energy and density needs some time to happens. This excludes the singular interactions and suggests to consider only the case of smooth interaction. A delicate point is to decide how small can be this time interval on which we average. The answer to this question and the whole proof can be found in the original papers [7], written assuming a nonnegative, smooth, and short-range interaction which is positive at the origin. We state the result. Let

$$Q(\mathbf{X};\mu,R) = \sum_i \chi(|\mathbf{r}_i - \mu| \le R) \left[\frac{|\mathbf{v}_i|^2}{2} + \frac{1}{2} \sum_{j;j\neq i} \Phi(\mathbf{r}_i - \mathbf{r}_j) + 1 \right] ,$$

which gives the energy and density contained in a ball centered at $\mu \in \mathbb{R}^3$ with radius R, and define

$$Q_\alpha(\mathbf{X}) = \sup_{\mu} \sup_{R: R > \phi_\alpha(\mu)} \frac{Q(\mathbf{X};\mu,R)}{R^3} , \quad \phi_\alpha(\mu) = \log^\alpha(\mathrm{e} + |\mu|) , \quad \alpha > 0 .$$

We denote by $\mathscr{H}_\alpha$ the set of the phase points $\mathbf{X}$ such that $Q_\alpha(\mathbf{X}) < \infty$. It is possible to prove that, for any $\alpha \geq \frac{1}{3}$, $\mathscr{H}_\alpha$ has full measure with respect to any Gibbs measure. As before, define the n-partial dynamics as the evolution in which only particles initially contained in a ball centered in the origin and radius n are present. Then the following theorem is proven in [7].

Theorem A.3 *If* $\mathbf{X} \in \mathscr{H}_\alpha$ *there exists a unique flow* $t \to \mathbf{X}(t) \in \mathscr{H}_{3\alpha/2}$ *satisfying the Newton's equations with* $\mathbf{X}(0) = \mathbf{X}$. *Moreover, the* n*-partial dynamics locally converges to* $\mathbf{X}(t)$ *as* $n \to \infty$.

The result has been extended in [14] to superstable smooth and long-range potentials. We remark that in two and in three dimensions the estimates on the behavior of the time evolution for very long time are very bad and do not allow an analysis as that of Chap. 1.

Other Dimensions Assume that the motion is confined to a region D. As evident from the proofs, the main point is not the real dimension of the space in which the motion takes place but the intersection of D with the infinity line, i.e., the direction in which D appears unbounded. Consider a sphere $B_R(\mathbf{r}, R)$ centered in a fixed point $\mathbf{r}$ and with radius R and the volume V of the region $D \cap B_R(\mathbf{r}, R)$. If $V \leq R^d$, d is a bound of the "dimension" appearing in the study of the time evolution. A related problem was discussed in [9]. Moreover, in a recent paper the present authors investigate the time evolution of a system of particles mutually interacting via a hard core plus a very singular potential, moving in a suitable region D [6].

The previous results hold for generic nonequilibrium states. If we restrict the initial data to belong to an equilibrium state the results can be stronger. We quote the first result by Sinai [36] in which he proves in one dimension a cluster structure of the evolved state, result extended in many dimension for a diluted gas [37]. In many dimensions, for smooth interactions the time evolution has been proved to exists in [22, 24], for singular interaction in [29] and for an hard sphere gas in [1]. For a generic (nonequilibrium) stationary state see [35]. Finally, on these topics see also [30].

A.2 Vlasov Equation with Infinite Mass

Here we study the initial value problem for the Vlasov equation when it describes the time evolution of a plasma distributed in the whole space $\mathbb{R}^d$ and with infinite total mass. As for point particle systems, this problem is nontrivial since it is not easy to exclude a priori the blow-up of the mass distribution in a finite time. There are many studies on the Vlasov equations, here we focus our attention on the difficulty related to the assumption of infinite total mass. In order to separate the difficulties, we assume the interaction is positive, smooth, and short-range. In analogy to the case of point particle systems, we believe that the positiveness and short-range assumptions can be relaxed by assuming that the interaction is superstable and satisfies some

decaying property at large distance. But this task needs a nontrivial effort and it has not been done. The case of singular interaction (the Coulomb interaction being the most interesting one) is discussed later on.

The difficulty of the problem grows with the dimension of the physical space. We start with an heuristic consideration similar to that discussed in the case of point particle systems, by ulteriorly showing the importance of the physical space dimension in this framework.

Consider the Vlasov equation (2.2) and assume $\Phi = \Phi(|\mathbf{x}|)$ to be a non-negative function such that

$$\Phi \in C^2(\mathbb{R}) , \quad \Phi(0) > 0 , \quad \Phi(|\mathbf{x}|) = 0 \quad \text{if} \quad |\mathbf{x}| > r \quad (r > 0) . \tag{A.39}$$

Moreover, we assume that the initial distribution f_0 satisfies

$$0 \leq f_0(\mathbf{x}, \mathbf{v}) \leq C_0 e^{-\lambda|\mathbf{v}|^2} \quad (C_0, \lambda > 0) . \tag{A.40}$$

We remark that we are really needed to postulate some decay in the velocity variable as shown by the following example. Consider the free evolution in one dimension of an initial datum $f_0(x, v)$ which is the characteristic function of the set $\{(x, v) \colon x > 0,\ -(x+1) < v < -x\}$. Therefore, the initial density of mass is equal to zero for $x \leq 0$ and to one for $x > 0$. It is clear that for $t = 1$ we have a blow-up of the density.

The main issue in proving the existence of solutions is to show that the force $\mathbf{F}(\mathbf{x}, t)$ acting on the element of fluid located in $\mathbf{x} \in \mathbb{R}^d$ is bounded. By (2.3) and (2.4) we have,

$$|\mathbf{F}(\mathbf{x}, t)| \leq \|\nabla\Phi\|_\infty \int_{B(\mathbf{x},r)} d\mathbf{y}\ \rho(\mathbf{y}, t) = \|\nabla\Phi\|_\infty\ m(B(\mathbf{x}, r), t) , \tag{A.41}$$

where $B(\mathbf{x}, r)$ in an open ball around $\mathbf{x}$ of radius r, $m(B(\mathbf{x}, r), t)$ is the mass contained in such a ball at time t and r is defined in (A.39). To simplify the situation we first assume that

$$f_0(\mathbf{x}, \mathbf{v}) \leq C_0\ \chi(|\mathbf{v}| < \hat{V}_0) . \tag{A.42}$$

Letting

$$\hat{V}(t) = \sup_{0 \leq s \leq t} \sup_{\mathbf{x},\mathbf{v}} |\mathbf{V}(\mathbf{x}, \mathbf{v}, s)| ,$$

we have, for any $\mathbf{a} \in \mathbb{R}^d$,

$$\begin{aligned} m(B(\mathbf{a}, r), t) &= \int d\mathbf{x}\, d\mathbf{v}\ f_0(\mathbf{x}, \mathbf{v})\ \chi(\mathbf{X}(\mathbf{x}, \mathbf{v}; t) \in B(\mathbf{a}, r)) \\ &\leq \|f_0\|_{L_\infty}\ \hat{V}_0^d\ [r + \hat{V}(t)t]^d . \end{aligned}$$

The last inequality follows from the fact that $\chi(\mathbf{X}(\mathbf{x},\mathbf{v};t) \in B(\mathbf{a},r)) = 0$ if $|\mathbf{x}-\mathbf{a}|$ is larger than $r + \hat{V}(t)t$. On the other hand,

$$\mathbf{V}(\mathbf{x},\mathbf{v};t) = \mathbf{v} + \int_0^t \mathrm{d}s\ \mathbf{F}(\mathbf{X}(\mathbf{x},\mathbf{v};s),s)\ ,$$

which gives

$$\hat{V}(t) \leq \hat{V}_0 + \|\nabla\Phi\|_\infty\ \|f_0\|_\infty\ \hat{V}_0^d \int_0^t \mathrm{d}s\ [r + \hat{V}(s)s]^d\ . \tag{A.43}$$

The above inequality is solvable globally in time only if $d = 1$. We remark that, as for the particle systems, a rigorous proof where the assumption (A.42) is relaxed requires some care. In dimension $d > 1$, other tools are needed besides the naive use of mass conservation. More precisely, to control the maximal velocity $\hat{V}$ a deep use of energy conservation is needed in $d = 2$, while for $d = 3$ suitable time averages have to been used. We discuss directly the more difficult case, i.e., the three dimensional one.

For a given function $f(\mathbf{x},\mathbf{v})$ and any couple $(\mu, R) \in (\mathbb{R}^3 \times \mathbb{R}^+)$ we introduce a sort of "smoothed energy" of a ball of center μ and radius R,

$$W(f;\mu,R) = \frac{1}{2}\int \mathrm{d}\mathbf{x}\ g^{\mu,R}(\mathbf{x}) \left[\int \mathrm{d}\mathbf{v}\ |\mathbf{v}|^2 f(\mathbf{x},\mathbf{v}) + \rho(\mathbf{x}) \int \mathrm{d}\mathbf{y}\ \rho(\mathbf{y})\ \Phi(|\mathbf{x}-\mathbf{y}|)\right]\ ,$$

where $g^{\mu,R}$ is a smoothing function defined as

$$g^{\mu,R}(\mathbf{x}) = g\left(\frac{|\mathbf{x}-\mu|}{R}\right)\ ,$$

with $g \in C^\infty(\mathbb{R}^+)$ such that

$$g(\eta) = 1 \quad \text{if} \quad \eta \in [0,1]\ , \quad g(\eta) = 0 \quad \text{if} \quad \eta \in [2,\infty)\ , \quad -2 \leq g'(\eta) \leq 0\ .$$

For the positivity of the potential Φ, W is a well-defined positive functional for any f satisfying (A.40). Moreover, it is straightforward to see that there exists a positive constant C_1 such that

$$\sup_{(\mu,R)\in\mathbb{R}^3\times\mathbb{R}^+} \frac{W(f;\mu,R)}{R^3} \leq C_1\ .$$

The following theorem is proved in [8].

Theorem A.4 *Let f_0 satisfy (A.40). Then, there exists a pair of functions*

$$(\mathbf{x},\mathbf{v}) \to (\mathbf{X}(\mathbf{x},\mathbf{v};t),\mathbf{V}(\mathbf{x},\mathbf{v};t)) \ , \quad f_0(\mathbf{x},\mathbf{v}) \to f(\mathbf{x},\mathbf{v};t) \ , \quad (\mathbf{x},\mathbf{v},t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+ \ ,$$

satisfying the Vlasov equations (2.2). *This is the unique solution in the class of functions $f(t) = f(\cdot,\cdot;t)$ such that*

$$\sup_{t\in[0,T]} \sup_{(\mu,R)\in\mathbb{R}^3\times\mathbb{R}+} \frac{W(f(t);\mu,R)}{R^3} < \infty \quad \forall\, T > 0 \ .$$

Moreover, for each $\lambda_1 < \lambda$ and $T > 0$ there exists $C_2 > 0$ such that

$$f(\mathbf{x},\mathbf{v};t) \le C_2 \mathrm{e}^{-\lambda_1|\mathbf{v}|^2} \quad \forall\, t \in [0,T] \ .$$

The proof is obtained in analogy with the case of point particle systems in three dimensions. First, we introduce a partial dynamics with a cut-of on the positions and the velocities, i.e., we introduce the sequence of problems,

$$\begin{aligned}
&\dot{\mathbf{X}}^{M,N}(\mathbf{x},\mathbf{v};t) = \mathbf{V}^{M,N}(\mathbf{x},\mathbf{v};t) \ , \quad \dot{\mathbf{V}}^{M,N}(\mathbf{x},\mathbf{v};t) = \mathbf{F}^{M,N}(\mathbf{X}(\mathbf{x},\mathbf{v},t),t) \ , \\
&\mathbf{X}^{M,N}(\mathbf{x},\mathbf{v},0) = \mathbf{x} \ , \quad \mathbf{V}^{M,N}(\mathbf{x},\mathbf{v},0) = \mathbf{v} \ , \quad |\mathbf{x}| \le M \ , \quad |\mathbf{v}| \le N \ ,
\end{aligned}$$

where M, N are positive integers,

$$\begin{aligned}
\mathbf{F}^{M,N}(\mathbf{x},t) &= \int \mathrm{d}\mathbf{y}\ \nabla\Phi(|\mathbf{x}-\mathbf{y}|) \int \mathrm{d}\mathbf{v}\ f^{M,N}(\mathbf{x},\mathbf{v};t) \ , \\
f^{M,N}(\mathbf{X}^{M,N}(\mathbf{x},\mathbf{v};t),\mathbf{V}^{M,N}(\mathbf{x},\mathbf{v},t),t) &= f_0^{M,N}(\mathbf{x},\mathbf{v}) \ ,
\end{aligned}$$

and

$$f_0^{M,N}(\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v})\ \chi(|\mathbf{x}| \le M)\ \chi(|\mathbf{v}| \le N) \ .$$

The above problem is a well posed Vlasov evolution with finite mass, which admits an unique positive solution $f^{M,N}(\mathbf{x},\mathbf{v};t)$ (see for instance [18] and the references quoted in).

We next investigate the limit $M, N \to \infty$. We introduce the quantity,

$$\hat{V}^{M,N}(t) = \sup_{0\le s\le t} \sup_{\mathbf{x},\mathbf{v}} |\mathbf{V}^{M,N}(\mathbf{x},\mathbf{v},s)| \ .$$

We can prove, after many efforts, that for each $T > 0$ there exists a positive constant C such that

$$V^{M,N}(T) \le CN.$$

This is the key of the proof, that develops through complicated steps. We forward to the original paper [8].

In the proof the smoothness of the interaction plays an essential role. In fact, in this case the only way to obtain a large growth of the velocity in a point is to crowd a lot of mass in a point. In this case, the superstability condition imposes a large energy, that in its turn controls the maximal velocity. This proof fails in three dimensions. In two dimensions indeed it is possible to do it if the interaction is not too singular [11], while in a three dimensional domain which is unbounded in one direction only the Vlasov equation can be studied for interactions with a singularity [13]. Another direction of (nontrivial) generalization is to consider also long range interactions.

Some results have been obtained in this direction in the physically relevant case of the so-called Vlasov–Helmholtz equation, where the interaction at short distance behaves as the Coulomb one and decays exponentially at large distances by a screening effect [11]. We remark that the analogous problem in the framework of point particle dynamics remains unsolved. This is possible because in the continuum case the energy controls the local density and the local density gives a good control to the average of the Lipschitz constant, that guarantees the convergence of the partial dynamics. Perhaps, the same equation in three dimension but with a cylindrical symmetry could be approachable, but it has not be done. Other problems for unbounded plasma have been studied [12, 19, 27, 32–34].

It is interesting to consider also situations where point particles coexists with a Vlasov fluid. Of course, the Coulomb interaction plays a privileged role because of its physical importance. Some results have been obtained for localized Vlasov fluid that we do not quote here, but only one: a two-dimensional system composed by a point charge particle that interacts with an unbounded Vlasov fluid with charges of the same sign. The interaction behaves at short distance as the Coulomb one and it is exponentially decreasing at large distances [10].

We mention that in this direction it would be interesting the study of the following case: a Vlasov gas in three dimensions with a cylindrical symmetry and a point particle moving along the symmetry axis. Of course, it would be a model of viscous friction. A study of the long time behavior is too hard, but at least the existence of the infinite dynamics seems to be an approachable issue.

Finally, we remark that the relation between the infinite point particle system and the corresponding infinite Vlasov equation have been studied, as far as we know, only in one dimension [3].

References

1. Alexander, R.: Time evolution for infinitely many hard spheres. Commun. Math. Phys. **49**, 217–232 (2076)
2. Bahn, C., Park, Y.M., Yoo, H.J.: Non equilibrium dynamics of infinite particle systems with infinite range interaction. J. Math. Phys. **40**, 4337–4358 (1999)

3. Buttà, P., Caglioti, E., Marchioro, C.: On the long time behavior of infinitely extended systems of particles interacting via Kac Potentials. J. Stat. Phys. **108**, 317–339 (2002)
4. Buttà, P., Caprino, S., Cavallaro, G., Marchioro, C.: On the dynamics of infinitely many particles with magnetic confinement. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **8**, 371–395 (2006)
5. Buttà, P., Cavallaro, G., Marchioro, C.: Time evolution of two dimensional systems with infinitely many particles mutually interacting via very singular forces. J. Stat. Phys. **108**, 317–339 (2012)
6. Buttà, P., Cavallaro, G., Marchioro, C.: Dynamics of infinitely extended hard core systems. Rep. Math. Phys. **72**, 369–377 (2013)
7. Caglioti, E., Marchioro, C., Pulvirenti, M.: Non-equilibrium dynamics of three-dimensional infinite particle systems. Commun. Math. Phys. **215**, 25–43 (2000)
8. Caglioti, E., Caprino, S., Marchioro, C., Pulvirenti, M.: The Vlasov equation with infinite Mass. Arch. Ration. Mech. Anal. **159**, 85–108 (2001)
9. Calderoni, P., Caprino, S.: Time evolution of infinitely many particles: an existence theorem. J. Stat. Phys. **28**, 815–833 (1982)
10. Caprino, S., Marchioro, C.: On a charge interacting with a plasma of unbounded mass. Kinetic Relat. Models **4**, 215–226 (2011)
11. Caprino, S., Marchioro, C., Pulvirenti, M.: On the Vlasov–Helmholtz equations with infinite mass. Commun. Partial Differ. Equ. **27**, 791–808 (2002)
12. Caprino, S., Cavallaro, G., Marchioro, C.: On a magnetically confined plasma with infinite charge. SIAM J. Math. Anal. **46**, 133–164 (2014)
13. Caprino, S., Cavallaro, G., Marchioro, C.: Time evolution of an infinitely extended Vlasov system with singular mutual interaction (in preparation)
14. Cavallaro, G., Marchioro, C., Spitoni, C.: Dynamics of infinitely many particles mutually interacting in three dimensions via a bounded superstable long-range potential. J. Stat. Phys. **120**, 367–416 (2005)
15. Dobrushin, R.L., Fritz, J.: Non equilibrium dynamics of one-dimensional infinite particle system with hard-core interaction. Commun. Math. Phys. **55**, 275–292 (1977)
16. Fritz, F.: Some remark on non equilibrium dynamics of infinite particle systems. J. Stat. Phys. **34**, 539–556 (1985)
17. Fritz, F., Dobrushin, R.L.: Non-equilibrium dynamics of two-dimensional infinite particle systems with a singular interaction. Commun. Math. Phys. **57**, 67–81 (1977)
18. Glassey, R.: The Cauchy Problem in Kinetic Theory. Society for Industrial and Applied Mathematics, Philadelphia, PA (1996)
19. Jabin, P.E.: The Vlasov–Poisson system with infinite mass and energy. J. Stat. Phys. **103**, 1107–1123 (2001)
20. Lanford, O.E.: Classical mechanics of one-dimensional systems with infinitely many particles. I An existence theorem. Commun. Math. Phys. **9**, 176–191 (1968)
21. Lanford, O.E.: Classical mechanics of one-dimensional systems with infinitely many particles. II Kinetic theory. Commun. Math. Phys. **11**, 257–292 (1969)
22. Lanford, O.E., III: Time Evolution of Large Classical Systems. Dynamical systems, theory and applications (Recontres, Battelle Research Institute, Seattle, Washington, 1974). Lecture Notes in Physics, vol. 38, pp. 1–111. Springer, Berlin (1975)
23. Marchioro, C., Pulvirenti, M.: Time evolution of infinite one-dimensional Coulomb systems. J. Stat. Phys. **27**, 809–822 (1982)
24. Marchioro, C., Pellegrinotti, S., Presutti, E.: Existence of time evolution in ν-dimensional statistical mechanics. Commun. Math. Phys. **40**, 175–185 (1975)
25. Marchioro, C., Pellegrinotti, S., Pulvirenti, M.: Remarks on the existence of non-equilibrium dynamics. Proceedings Esztergom Summer School. Coll. Math. Soc. Janos Bolyai **27**, 733–746 (1978)
26. Marchioro, C., Miot, E., Pulvirenti, M.: The Cauchy problem for the 3-D Vlasov–Poisson system with point charges. Arch. Ration. Mech. Anal. **201**, 1–26 (2011)

27. Pankavich, S.: Global existence for a three dimensional Vlasov–Poisson system with steady spatial asymptotics. Commun. Partial Differ. Equ. **31**, 349–370 (2006)
28. Pfaffelmoser, K.: Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data. J. Differ. Equ. **95**, 281–303 (1992)
29. Presutti, E., Pulvirenti, M., Tirozzi, B.: Time evolution of infinite classical systems with singular, long-range, two-body interaction. Commun. Math. Phys. **47**, 81–95 (1976)
30. Pulvirenti, M.: On the time evolution of states of infinitely extended particle systems. J. Stat. Phys. **27**, 693–713 (1982)
31. Schaeffer, J.: Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions. Commun. Partial Differ. Equ. **16**, 1313–1335 (1991)
32. Schaeffer, J: Steady spatial asymptotics for the Vlasov–Poisson system. Math. Methods Appl. Sci. **26**, 273–296 (2003)
33. Schaeffer, J: The Vlasov Poisson system with steady spatial asymptotics. Commun. Partial Differ. Equ. **28**, 1057–1084 (2003)
34. Schaeffer, J.: Global existence for the Vlasov–Poisson system with steady spatial asymptotic behavior. Kinet. Relat. Models **5**, 129–153 (2012)
35. Sigmund-Shultze, R: On non-equilibrium dynamics of multidimensional infinite particle systems in the translational invariant case. Commun. Math. Phys. **100**, 487–501 (1985)
36. Sinai, Ya: Construction of the dynamics for one-dimensional systems of statistical mechanics. Sov. Theor. Math. Phys. **12**, 487–501 (1973)
37. Sinai, Ya: The construction of cluster dynamics of dynamics systems in statistical mechanics. Vest. Moskow Univ. Sez I Math. Mech. **29**, 152–176 (1974)
38. Wollman S: Global in time solution to the three-dimensional Vlasov–Poisson system. J. Math. Anal. Appl. **176**, 76–91 (1993)

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