Appendix A
Basic Principles and Supplementary Material

Abstract This appendix reviews elementary concepts of module theory, homological algebra, and differential algebra and presents additional basic material that is relevant for the previous chapters. Fundamental notions of homological algebra, which are used for the study of systems of linear functional equations, are to be found in the first section. In the second section the chain rule for higher derivatives is deduced, which is employed in the earlier section on linear differential elimination. The description of Thomas' algorithm and the development of nonlinear differential elimination rely on the principles of differential algebra that are recalled in the third section. Two notions of homogeneity for differential polynomials are discussed, and the analogs of Hilbert's Basis Theorem and Hilbert's Nullstellensatz in differential algebra are outlined.

A.1 Module Theory and Homological Algebra

In this section some basic notions of module theory and homological algebra are collected. General references for this material are, e.g., [Eis95], [Lam99], [Rot09].

Let R be a (not necessarily commutative) ring with multiplicative identity element 1. In this section all modules are understood to be left R-modules, and 1 is assumed to act as identity. For every statement about left R-modules in this section an analogous statement about right R-modules holds, which will not be mentioned in what follows.

A.1.1 Free Modules

Definition A.1.1. An R-module M is said to be *free*, if it is a direct sum of copies of R. If $M = \bigoplus_{i \in I} R\, b_i$ and $R\, b_i \cong R$ for all $i \in I$, where I is an index set, then $(b_i)_{i \in I}$ is a *basis* for M. If n denotes the cardinality of I, then M is said to be *free of rank n*.

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Example A.1.2. Let R be a field (or a skew-field). Then every R-module is free because, as a vector space, it has a basis (by the axiom of choice).

Remark A.1.3 (Universal property of a free module). A homomorphism from a free R-module M to another R-module N is uniquely determined by specifying the images of the elements of a basis $B = (b_i)_{i \in I}$ of M. In other words, every map $B \to N$ defines a unique homomorphism $M \to N$ by R-linear extension.

Remark A.1.4. Every R-module M is a factor module of a free module. Indeed, let $G \subseteq M$ be any generating set for M. We define the free (left) R-module

$$F = \bigoplus_{g \in G} R\,\hat{g},$$

where the $\hat{g}$ for $g \in G$ are pairwise different new symbols, and we define the homomorphism $\varphi\colon F \to M$ by $\varphi(\hat{g}) = g$. By the universal property of a free module, φ exists and is uniquely determined. Since G is a generating set for M, the homomorphism φ is surjective. Defining $N := \ker(\varphi)$, the homomorphism theorem for R-modules states that

$$M = \operatorname{im}(\varphi) \cong F/N. \tag{A.1}$$

Whereas in an arbitrary representation of M it may be difficult to address elements $m \in M$ or to tell such elements apart, the isomorphism in (A.1) is advantageous in the sense that the free module F allows a very explicit representation of its elements (e.g., coefficient vectors with respect to a chosen basis) and the ambiguity of addressing an element in M in that way is captured by N.

Definition A.1.5. By definition, an equivalent formulation for the statement in (A.1) is:

$$0 \longleftarrow M \xleftarrow{\;\varphi\;} F \longleftarrow N \longleftarrow 0 \quad \text{is a short exact sequence,} \tag{A.2}$$

where the homomorphism $N \to F$ is the inclusion map.

More generally, a *chain complex* of R-modules is defined to be a family $(M_i)_{i \in \mathbb{Z}}$ of R-modules with homomorphisms $d_i\colon M_i \to M_{i-1}$, $i \in \mathbb{Z}$, denoted by

$$M_\bullet\colon \qquad \ldots \xleftarrow{d_0} M_0 \xleftarrow{d_1} M_1 \xleftarrow{d_2} \ldots \xleftarrow{d_{n-1}} M_{n-1} \xleftarrow{d_n} M_n \xleftarrow{d_{n+1}} \ldots$$

such that the composition of each two consecutive homomorphisms is the zero map, i.e.,

$$\operatorname{im}(d_{i+1}) \subseteq \ker(d_i) \qquad \text{for all } i \in \mathbb{Z}.$$

If $\operatorname{im}(d_{i+1}) = \ker(d_i)$ holds, then $M_\bullet$ is said to be *exact at* M_i. The factor module

$$H_i(M_\bullet) := \ker(d_i)/\operatorname{im}(d_{i+1}), \qquad i \in \mathbb{Z},$$

is the *homology at* M_i or the *defect of exactness at* M_i.

A *cochain complex* of R-modules is defined to be a family $(N^i)_{i \in \mathbb{Z}}$ of R-modules with homomorphisms $d^i\colon N^i \to N^{i+1}$, $i \in \mathbb{Z}$, denoted by

$$N^\bullet: \qquad \ldots \longrightarrow N^0 \xrightarrow{d^0} N^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-2}} N^{n-1} \xrightarrow{d^{n-1}} N^n \xrightarrow{d^n} \ldots$$

such that

$$\operatorname{im}(d^{i-1}) \subseteq \ker(d^i) \qquad \text{for all } i \in \mathbb{Z}.$$

If $\operatorname{im}(d^{i-1}) = \ker(d^i)$ holds, then $N^\bullet$ is said to be *exact at* N^i. The factor module

$$H^i(N^\bullet) := \ker(d^i)/\operatorname{im}(d^{i-1}), \qquad i \in \mathbb{Z},$$

is the *cohomology at* N^i or the *defect of exactness at* N^i.

The term *exact sequence* is a synonym for a (co)chain complex with trivial (co)homology groups.

A short exact sequence as in (A.2), where F is a free module, is also called a *presentation* of M. If F and N are finitely generated, then (A.1) is called a *finite presentation* of M. If there exists a finite presentation of an R-module M, then M is said to be *finitely presented*.

A.1.2 Projective Modules and Injective Modules

Definition A.1.6. An R-module M is said to be *projective*, if for every epimorphism $\beta: B \to C$ and every homomorphism $\alpha: M \to C$ there exists a homomorphism $\gamma: M \to B$ satisfying $\alpha = \beta \circ \gamma$, i.e., such that the following diagram is commutative:

$$\begin{tikzcd} & M \arrow[dl, dashed, "\gamma"'] \arrow[d, "\alpha"] & \\ B \arrow[r, "\beta"'] & C \arrow[r] & 0 \end{tikzcd}$$

Remark A.1.7. Let M be a free R-module. Then M is projective. Indeed, knowledge of the images $\alpha(m_i)$ in C of elements in a basis $(m_i)_{i \in I}$ of M allows to choose preimages $b_i \in B$ of the $\alpha(m_i)$ under β, and by the universal property of free modules, there exists a unique homomorphism $\gamma: M \to B$ such that $\gamma(m_i) = b_i$ for all $i \in I$.

Remark A.1.8. The condition in Definition A.1.6 is equivalent to the right exactness of the (covariant) functor $\hom_R(M, -)$, i.e., the condition that for every exact sequence of R-modules

$$A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{A.3}$$

the complex of abelian groups

$$\hom_R(M,A) \longrightarrow \hom_R(M,B) \longrightarrow \hom_R(M,C) \longrightarrow 0$$

(with homomorphisms which compose with the corresponding homomorphisms in (A.3)) is exact. (The functor $\hom_R(M, -)$ is left exact for every R-module M, i.e.,

for every exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \tag{A.4}$$

the complex of abelian groups

$$0 \longrightarrow \hom_R(M,A) \longrightarrow \hom_R(M,B) \longrightarrow \hom_R(M,C)$$

(with homomorphisms which compose with the corresponding homomorphisms in (A.4)) is exact.)

Remark A.1.9. Let M be a projective R-module and let

$$0 \longleftarrow M \xleftarrow{\pi} F \xleftarrow{\varepsilon} N \longleftarrow 0$$

be a presentation of M with a free R-module F. By choosing the modules $B = F$, $C = M$ and the homomorphisms $\alpha = \mathrm{id}_M$, $\beta = \pi$, we conclude that there exists a homomorphism $\sigma\colon M \to F$ (viz. $\sigma = \gamma$) satisfying $\pi \circ \sigma = \mathrm{id}_M$. The short exact sequence is said to be *split* in this case (and the following discussion is essentially the Splitting Lemma in homological algebra).

Let $\varphi\colon F \to F$ be the homomorphism defined by

$$\varphi = \mathrm{id}_F - \sigma \circ \pi.$$

For $f \in F$ we have

$$\varphi(f) = f \quad \Longleftrightarrow \quad f \in \ker(\pi) = \mathrm{im}(\varepsilon)$$

because σ is a monomorphism. Since the homomorphism ε is injective, we get a homomorphism $\rho\colon F \to N$ such that $\rho \circ \varepsilon = \mathrm{id}_N$, and we have (cf., e.g., [BK00, Thm. 2.4.5])

$$\mathrm{id}_F = \varepsilon \circ \rho + \sigma \circ \pi.$$

This shows that

$$F = \varepsilon(N) \oplus \sigma(M)$$

and ρ, π are the projections $F \to N$ and $F \to M$ onto the respective summands (up to isomorphism) of this direct sum.

Conversely, let M be a direct summand of a free R-module F, i.e., there exists an R-module N such that $M \oplus N = F$. Then every element of F has a unique representation as sum of an element of M and an element of the complement N. In particular, for the elements of a basis $(f_i)_{i\in I}$ of F we have $f_i = m_i + n_i$, where $m_i \in M$ and $n_i \in N$ are uniquely determined by f_i. In the situation of Definition A.1.6 we may choose preimages $b_i \in B$ of the $\alpha(m_i)$ under β and use the universal property of free modules to define a homomorphism $F \to B$ by $f_i \mapsto b_i$, $i \in I$, whose restriction to M is a homomorphism γ as in Definition A.1.6.

Therefore, an R-module is projective if and only if it is (isomorphic to) a direct summand of a free R-module.

Examples A.1.10. a) Let R be a (commutative) principal ideal domain. Then every projective R-module is free, because every submodule of a free R-module is free.
b) (Quillen-Suslin Theorem, the resolution of Serre's Problem)
Every finitely generated projective module over $K[x_1,\ldots,x_n]$, where K is a field or a (commutative) principal ideal domain, is free. (We also refer to [Lam06] for details on Serre's Problem and generalizations, and to [FQ07] for constructive aspects and applications.)
c) Let R be a commutative local ring (with 1). Then, by using Nakayama's Lemma, every finitely generated projective R-module is free.

We refer to Example A.1.17 below for an example of a projective module which is not free.

Definition A.1.11. An R-module M is said to be *injective*, if for every monomorphism $\beta\colon A \to B$ and every homomorphism $\alpha\colon A \to M$ there exists a homomorphism $\gamma\colon B \to M$ satisfying $\alpha = \gamma \circ \beta$, i.e., such that the following diagram is commutative:

$$\begin{tikzcd} & M & \\ B \arrow[ur, dashed, "\gamma"] & A \arrow[l, "\beta"] \arrow[u, "\alpha"'] & 0 \arrow[l] \end{tikzcd}$$

The notion of an injective module is clearly dual to the notion of projective module in the sense that all arrows are reversed.

Remark A.1.12. The condition in Definition A.1.11 is equivalent to the right exactness of the contravariant functor $\hom_R(-,M)$, i.e., the condition that for every exact sequence of R-modules

$$C \longleftarrow B \longleftarrow A \longleftarrow 0 \tag{A.5}$$

the complex of abelian groups

$$\hom_R(C,M) \longrightarrow \hom_R(B,M) \longrightarrow \hom_R(A,M) \longrightarrow 0$$

(with homomorphisms which pre-compose with the corresponding homomorphisms in (A.5)) is exact. (The contravariant functor $\hom_R(-,M)$ is left exact for every R-module M, i.e., for every exact sequence of R-modules

$$0 \longleftarrow C \longleftarrow B \longleftarrow A \tag{A.6}$$

the complex of abelian groups

$$0 \longrightarrow \hom_R(C,M) \longrightarrow \hom_R(B,M) \longrightarrow \hom_R(A,M)$$

(with homomorphisms which pre-compose with the corresponding homomorphisms in (A.6)) is exact.)

If the monomorphism $\beta: A \to B$ in Definition A.1.11 is actually an inclusion of modules, then injectivity can be understood as the possibility to extend every homomorphism $A \to M$ to a homomorphism $B \to M$. Moreover, choosing $A = M$ and $\alpha = \mathrm{id}_M$, we conclude that an injective R-module is a direct summand of every R-module which contains it. Another particular case of the above condition is that $A = I$ is a (left) ideal of $B = R$. It turns out that the restriction of the condition to arbitrary (left) ideals of R is sufficient for injectivity.

Theorem A.1.13 (Baer's criterion). *A (left) R-module M is injective if and only if for every (left) ideal I of R every homomorphism $I \to M$ can be extended to a homomorphism $R \to M$.*

$$\begin{tikzcd}
 & M & \\
R \arrow[ur, dashed, "\gamma"] & I \arrow[l, "\beta"] \arrow[u, "\alpha"'] & 0 \arrow[l]
\end{tikzcd}$$

Example A.1.14. Let A be an abelian group (which is the same as a $\mathbb{Z}$-module). Then A is injective if and only if it is *divisible*, i.e., for every $a \in A$ and every integer $n \in \mathbb{Z} - \{0\}$ there exists $b \in A$ such that $a = n\,b$.

Definition A.1.15. An R-module M is said to be *stably free of rank $n \in \mathbb{Z}_{\geq 0}$*, if there exist $m \in \mathbb{Z}_{\geq 0}$ and a free R-module F of rank m such that $M \oplus F$ is isomorphic to a free R-module of rank $m + n$.

Remark A.1.16. Every free R-module of rank $n \in \mathbb{Z}_{\geq 0}$ is stably free of the same rank. Every stably free R-module is projective because it is a direct summand of a free module (cf. Rem. A.1.9). Hence, the following chain of implications holds for finitely generated R-modules M:

$$M \text{ free} \Rightarrow M \text{ stably free} \Rightarrow M \text{ projective.}$$

Example A.1.17. Let $R = \mathbb{R}[x_1, \ldots, x_n]/\langle 1 - x_1^2 - \ldots - x_n^2 \rangle$ and define the left R-module $M = R^{1 \times n}/R\,(x_1, \ldots, x_n)$. Then M is stably free of rank $n-1$, but not free if $n \notin \{1,2,4,8\}$ (because the tangent bundle to the $(n-1)$-sphere is trivial only if $n \in \{1,2,4,8\}$, cf. [Eis95, Ex. 19.17] and the references therein).

For an example of a projective, but not stably free module, cf. Example A.1.32.

A.1.3 Syzygies and Resolutions

As mentioned earlier, a presentation of a module M as $M \cong F/N$ with a free module F furnishes a very concrete description of M. We may treat N in the same way as M, in order to get an ever clearer picture of M.

For more details about the following material, we refer to, e.g., [Eis95], [Rot09]. A proof of the next proposition can be found, e.g., in [Lam99, Chap. 2, § 5A].

Proposition A.1.18 (Schanuel's Lemma). *Let P and Q be projective R-modules and*

$$0 \longleftarrow M \longleftarrow P \longleftarrow K \longleftarrow 0,$$

$$0 \longleftarrow M \longleftarrow Q \longleftarrow L \longleftarrow 0$$

short exact sequences of R-modules. Then we have $K \oplus Q \cong L \oplus P$ as R-modules.

The conclusion of Schanuel's Lemma suggests to define the following equivalence relation on the class of all R-modules.

Definition A.1.19. Two R-modules K and L are said to be *projectively equivalent* if there exist projective R-modules P and Q such that $K \oplus Q \cong L \oplus P$.

Remark A.1.20. Let M be an R-module with generating sets G_1 and G_2. Let

$$F_i := \bigoplus_{g \in G_i} R\hat{g} \quad \text{and} \quad \varphi_i \colon F_i \longrightarrow M \colon \hat{g} \longmapsto g, \qquad i = 1,2,$$

be the homomorphisms of R-modules discussed in Remark A.1.4, where all $\hat{g}$ for $g \in G_1 \cup G_2$ are pairwise different new symbols. The *syzygy module* of G_i is defined to be the kernel of φ_i, $i = 1, 2$. Every element of $\ker(\varphi_i)$ is also called a *syzygy* of G_i. By Schanuel's Lemma, $\ker(\varphi_1)$ and $\ker(\varphi_2)$ are projectively equivalent. Therefore, we may associate with M the equivalence class of $\ker(\varphi_1)$ (or of $\ker(\varphi_2)$, whose equivalence class is the same) under projective equivalence.

By iterating the syzygy module construction we get a *free resolution*

$$F_0 \longleftarrow F_1 \longleftarrow \ldots \longleftarrow F_{n-1} \longleftarrow F_n \longleftarrow \ldots \tag{A.7}$$

of M, i.e., a chain complex of free R-modules that is exact at each F_i except at F_0, where the homology (i.e., the cokernel of the leftmost map) is isomorphic to M.

If the modules F_i are assumed to be projective rather than free, then (A.7) is called a *projective resolution* of M.

Similarly, a cochain complex of injective modules

$$I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^{n-1} \longrightarrow I^n \longrightarrow \ldots$$

which is exact at each I^i except at I^0, where the cohomology (i.e., the kernel of the leftmost map) is isomorphic to M, is called an *injective resolution* of M.

If a resolution is *finite*, i.e., if there exists a non-negative integer n such that the i-th module is zero for all $i \geq n$, then the number of non-zero homomorphisms in the resolution is called the *length* of the resolution.

Theorem A.1.21. *Every R-module has a free resolution and an injective resolution.*

For examples of free resolutions, cf. Examples A.1.31 and A.1.32 below, or Subsect. 3.1.5, e.g., p. 152.

Example A.1.22. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}$. Then

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is a short exact sequence of $\mathbb{Z}$-modules, and

$$\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is an injective resolution of $\mathbb{Z}$. Note that $\mathbb{Q}$ is not finitely generated as $\mathbb{Z}$-module.

For particular kinds of rings it is known that finite free resolutions exist for every finitely generated module. For instance, every finitely generated module over a commutative polynomial algebra in n variables over a field has a free resolution of length at most n with finitely generated free modules, as ensured by Hilbert's Syzygy Theorem, cf. Corollary 3.1.47, p. 151, or, e.g., [Eis95, Cor. 19.7].

Definition A.1.23. The *left projective dimension* of a left R-module M is defined to be the smallest length of a projective resolution of M if such a finite resolution exists and ∞ otherwise. The *left global dimension* of a ring R is defined to be the supremum of the left projective dimensions of its left modules. In an analogous way the notions of right projective dimension and right global dimension are defined.

Remark A.1.24. The left projective dimension of a left R-module M is zero if and only if M is projective.

Proposition A.1.25 (cf. [MR01], Subsect. 7.1.11). *If R is left and right Noetherian, then its left and right global dimensions are equal.*

Since we only deal with Noetherian rings, we denote the left and right global dimension of R by $\mathrm{gld}(R)$. For notation concerning skew polynomial rings $R[\partial;\sigma,\delta]$, cf. Subsect. 2.1.2.

Theorem A.1.26 ([MR01], Thm. 7.5.3). *If* $\mathrm{gld}(R) < \infty$*, σ an automorphism of R and δ a σ-derivation of R, then we have* $\mathrm{gld}(R) \leq \mathrm{gld}(R[\partial;\sigma,\delta]) \leq \mathrm{gld}(R)+1$.

Remark A.1.27 (cf. [QR07], Cor. 21). Let D be a ring and

$$D^{1\times r_0} \xleftarrow{d_1} D^{1\times r_1} \xleftarrow{d_2} \ldots \xleftarrow{d_{m-1}} D^{1\times r_{m-1}} \xleftarrow{d_m} D^{1\times r_m} \longleftarrow 0 \tag{A.8}$$

be a finite free resolution of a left D-module M. If $m \geq 3$ and there exists a homomorphism $s\colon D^{1\times r_{m-1}} \to D^{1\times r_m}$ such that $s \circ d_m$ is the identity on $D^{1\times r_m}$, then a free resolution of M of length $m-1$ is given by

$$D^{1\times r_0} \xleftarrow{d_1} \ldots \xleftarrow{d_{m-3}} D^{1\times r_{m-3}} \xleftarrow{\widetilde{d_{m-2}}} D^{1\times r_{m-2}} \oplus D^{1\times r_m} \xleftarrow{\widetilde{d_{m-1}}} D^{1\times r_{m-1}} \longleftarrow 0,$$

where $\widetilde{d_{m-1}}$ combines the values of d_{m-1} and s to a pair and $\widetilde{d_{m-2}}$ applies d_{m-2} to the first component of such a pair. In case $m = 2$ the same reduction is possible, where now the canonical projection onto M is to be modified instead of d_{m-2}.

Theorem A.1.28. *Let $D = R[\partial_1;\sigma_1,\delta_1][\partial_2;\sigma_2,\delta_2]\dots[\partial_l;\sigma_l,\delta_l]$ be an Ore algebra (cf. Def. 2.1.14, p. 17), where R is either a field, or $\mathbb{Z}$, or a commutative polynomial algebra over a field or $\mathbb{Z}$ with finitely many indeterminates, and where every σ_i is an automorphism. Moreover, let M be a finitely generated (left or right) D-module. Then there exists a finite free resolution (A.8) of M, where either $m = 1$ and there exists a homomorphism $s\colon D^{1\times r_{m-1}} \to D^{1\times r_m}$ such that $s \circ d_m$ is the identity on $D^{1\times r_m}$, or where $m \geq 1$ and there exists no such s. In the first case, M is stably free; in the second case, M is not projective.*

An iteration of Theorem A.1.26 shows that a finite free resolution of M exists. A free resolution of M as claimed in Theorem A.1.28 is obtained by a repeated use of Remark A.1.27. The assertion that M is stably free in the first case is immediate, whereas an application of Schanuel's Lemma (Prop. A.1.18) proves that M is not projective in the second case. We refer to the proof of [Rob14, Thm. 3.23] for more details. The relevant techniques can also be found in [Lam99, Chap. 2, § 5A].

Remark A.1.27 allows to compute the projective dimension of a finitely generated D-module.

Theorem A.1.28 yields the following corollary (cf. also [MR01, Cor. 12.3.3]).

Corollary A.1.29. *Let D be an Ore algebra as in Theorem A.1.28. Then every finitely generated projective D-module is stably free.*

By applying Corollary A.1.29 and induction on the projective dimension one can prove the following corollary (cf. also [Rot09, Lem. 8.42], [CQR05, Prop. 8], [Rob14, Cor. 3.25]).

Corollary A.1.30. *Let D be an Ore algebra as in Theorem A.1.28. Then every finitely generated D-module has a free resolution with finitely generated free modules of length at most* $\mathrm{gld}(D)+1$.

Example A.1.31 ([QR07], Ex. 50). Let K be a field of characteristic zero and either $D = A_1(K) = K[z][\partial;\sigma,\delta]$ be the Weyl algebra or $D = B_1(K) = K(z)[\partial;\sigma,\delta]$ be the algebra of differential operators with rational function coefficients (cf. Ex. 2.1.18 b), p. 19). Moreover, let $R = (\partial \quad -z^k) \in D^{1\times 2}$ for some $k \in \mathbb{Z}_{\geq 0}$ and $M = D^{1\times 2}/DR$. Then the left D-module M is stably free of rank 1 because the short exact sequence

$$0 \longleftarrow M \xleftarrow{\ \pi\ } D^{1\times 2} \xleftarrow{\ \varepsilon\ } D \longleftarrow 0,$$

where $\pi\colon D^{1\times 2} \to M$ is the canonical projection and $\varepsilon\colon D \to D^{1\times 2}$ is induced by R, is split (cf. Rem. A.1.9), the homomorphism $\rho\colon D^{1\times 2} \to D$ represented with respect to the standard bases by the matrix

$$\begin{pmatrix} \displaystyle\sum_{j=1}^{k} (-1)^{j-1}\frac{z^j}{j!}\,\partial^{j-1} \\ (-1)^{k+1}\dfrac{1}{k!}\,\partial^k \end{pmatrix} \in D^{2\times 1}$$

satisfying $\rho \circ \varepsilon = \mathrm{id}_D$. In order to show that the left D-module M is not free, we consider the exact sequence of left D-modules

$$D \xleftarrow{\psi} D^{1\times 2} \xleftarrow{\varepsilon} D \longleftarrow 0,$$

where $\psi\colon D^{1\times 2} \to D$ is induced by

$$Q = \begin{pmatrix} z^{k+1} \\ z\partial + k + 1 \end{pmatrix} \in D^{2\times 1}.$$

The homomorphism of left D-modules

$$\phi\colon M \longrightarrow D\colon r + DR \longmapsto rQ, \qquad r \in D^{1\times 2},$$

is well-defined due to $RQ = 0$ and injective because of $\ker(\psi) = \mathrm{im}(\varepsilon)$. Hence, we have $M \cong \mathrm{im}(\phi) = D^{1\times 2}Q$, which is a left ideal of D. A Janet basis or Gröbner basis computation proves that this left ideal is not principal. Therefore, M is not free.

(The matrix Q is referred to as a *minimal parametrization* of M in [CQR05, Thm. 8], cf. also [PQ99].)

In contrast to the previous example we note that non-principal ideals of Dedekind domains are projective modules which are not stably free (cf. also [MR01, Ex. 11.1.4 (i)] for the general statement or [Rot09, Ex. 4.92 (iii)] for a different example).

Example A.1.32. Let $D = \mathbb{Z}[\sqrt{-5}]$. The left D-modules $M_i = D^{1\times 2}R_i$, $i = 1, 2$, where

$$R_1 = \begin{pmatrix} -2 & 1-\sqrt{-5} \\ -1-\sqrt{-5} & 3 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 3 & -1+\sqrt{-5} \\ 1+\sqrt{-5} & -2 \end{pmatrix},$$

are projective because the D-module homomorphisms $\pi_i\colon D^{1\times 2} \to D^{1\times 2}$ induced by R_i, $i = 1, 2$, satisfy

$$\pi_1 \circ \pi_1 = \pi_1, \quad \pi_2 \circ \pi_2 = \pi_2, \quad \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0, \quad \pi_1 + \pi_2 = \mathrm{id}_{D^{1\times 2}},$$

which implies $M_1 \oplus M_2 = D^{1\times 2}$. However, M_1 (and M_2) is not free, because the ideal of D which is generated by 2 and $1+\sqrt{-5}$ (by 3 and $1+\sqrt{-5}$, respectively) is isomorphic to M_1 (to M_2, respectively) and is not principal. The left D-modules M_1 and M_2 are not stably free either, because an isomorphism $M_i \oplus D^{1\times m} \cong D^{1\times n}$ of finitely generated D-modules for some $m, n \in \mathbb{N}$ would allow cancelation of free D-modules, i.e., would imply that M_i is free (as a consequence of Steinitz' Theorem, cf., e.g., [BK00, Cor. 6.1.8]). A free resolution of M_1 is given by the periodic resolution

$$0 \longleftarrow M_1 \xleftarrow{\pi_1} D^{1\times 2} \xleftarrow{\pi_2} D^{1\times 2} \xleftarrow{\pi_1} D^{1\times 2} \xleftarrow{\pi_2} \ldots,$$

and by exchanging the roles played by π_1 and π_2, we obtain one for M_2.

A.2 The Chain Rule of Differentiation

In this section we deduce the chain rule for higher derivatives, which is applied in Subsect. 3.2.2. For more details on the basic definitions, we refer to, e.g., [Die69], and for an equivariant viewpoint on the tensors discussed below, cf. [KMS93].

In what follows let E, E_1, ..., E_n, and F be Banach spaces over either $\mathbb{R}$ or $\mathbb{C}$, and let $A, A_1, \ldots, A_n$ be non-empty open subsets of $E, E_1, \ldots, E_n$, respectively.

If $f\colon A \to F$ is a continuously differentiable map, then the *derivative* Df of f associates to each $x \in A$ a certain continuous linear map $E \to F$, the linearization of f at x. We also write Df for the continuous map

$$A \times E \longrightarrow F\colon\ (x, v) \longmapsto (Df)(x)(v),$$

which is linear in its second argument.

Let $f\colon A_1 \times \ldots \times A_n \to F$ be a map, $j \in \{1, \ldots, n\}$, and define

$$A_j^{(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)} := \{x_1\} \times \ldots \times \{x_{j-1}\} \times A_j \times \{x_{j+1}\} \times \ldots \times \{x_n\}$$

for $x_i \in A_i$, $1 \le i \le n$, $i \ne j$. If the restriction of f to $A_j^{(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)}$ is continuously differentiable, then the derivative of this restriction associates to each $(x_1, \ldots, x_n) \in A_j^{(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)}$ a certain continuous linear map $E_j \to F$ as above. If all restrictions of f to $A_j^{(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)}$, $x_i \in A_i$, $1 \le i \le n$, $i \ne j$, are continuously differentiable, then the *j-th partial derivative* $D_j f$ of f associates to each $(x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n$ the continuous linear map $E_j \to F$ given by the derivative of the corresponding restriction of f. We also write $D_j f$ for the continuous map

$$A_1 \times \ldots \times A_n \times E_j \longrightarrow F\colon\ (x_1, \ldots, x_n, v) \longmapsto (D_j f)(x_1, \ldots, x_n)(v),$$

which is linear in its last argument.

Let $f\colon A \to F$ be a continuously differentiable map and assume that the first partial derivative of $Df\colon A \times E \to F$ exists and is continuous. Then

$$D^2 f\colon A \times E^2 \longrightarrow F\colon\ (x, v_1, v_2) \longmapsto D_1(Df)(x, v_1, v_2) = D_1(Df)(x, v_1)(v_2)$$

is a continuous map, which is bilinear and symmetric in its last two arguments. It is called the *second derivative* of f. More generally, if the $(k-1)$-st derivative of $f\colon A \to F$ is defined and if its first partial derivative exists and is continuous, then

$$D^k f\colon A \times E^k \longrightarrow F\colon\ (x, v_1, \ldots, v_k) \longmapsto D_1(D^{k-1} f)(x, v_1, \ldots, v_k)$$

is a continuous map, which is multilinear and symmetric in its last k arguments. It is called the *k-th derivative* of f.

Remark A.2.1. If all partial derivatives $D_j f$ of $f\colon A_1 \times \ldots \times A_n \to F$ exist and if $\alpha_i\colon A \to A_i$ is a continuously differentiable map, $i = 1, \ldots, n$, then

$$D(f \circ (\alpha_1, \ldots, \alpha_n))\colon A \times E \longrightarrow F$$

is given by

$$(D(f \circ \alpha))(x,v) = \sum_{i=1}^{n} (D_i f)(\alpha(x), (D\alpha_i)(x,v)), \qquad x \in A, \quad v \in E,$$

where $\alpha := (\alpha_1, \ldots, \alpha_n)$.

Remark A.2.2. Let $j \in \{1, \ldots, n\}$. If

$$g\colon A_1 \times \ldots \times A_{j-1} \times E_j \times A_{j+1} \times \ldots \times A_n \longrightarrow F$$

is a map such that

$$E_j \longrightarrow F\colon x_j \longmapsto g(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_n)$$

is a continuous linear map for all $a_i \in A_i$, $1 \leq i \leq n$, $i \neq j$, then the j-th partial derivative of g exists and we have

$$(D_j g)(x_1, \ldots, x_n, v) = g(x_1, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_n),$$
$$(x_1, \ldots, x_n) \in A_1 \times \ldots \times A_{j-1} \times E_j \times A_{j+1} \times \ldots \times A_n, \qquad v \in E_j$$

(i.e., the j-th partial derivative of g associates to each

$$(x_1, \ldots, x_n) \in A_1 \times \ldots \times A_{j-1} \times E_j \times A_{j+1} \times \ldots \times A_n$$

the continuous linear map

$$E_j \longrightarrow F\colon v \longmapsto g(x_1, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_n),$$

which does not depend on x_j).

Remark A.2.3. For every $(i_1, \ldots, i_j) \in \{1, \ldots, k\}^j$ we define

$$\varepsilon^k_{i_1,\ldots,i_j}\colon E^k \longrightarrow E^j\colon (v_1, \ldots, v_k) \longmapsto (v_{i_1}, \ldots, v_{i_j}).$$

In what follows, $(i_1, \ldots, i_j)$ will be chosen to have pairwise distinct entries. We have $\varepsilon^k_{1,\ldots,k} = \mathrm{id}_{E^k}$.

Let us assume that the k-th derivatives of the maps $f\colon A_1 \times \ldots \times A_n \to F$ and $\alpha_i\colon A \to A_i$, $i = 1, \ldots, n$, exist, and write $\alpha := (\alpha_1, \ldots, \alpha_n)$. Applying Remark A.2.1 repeatedly, we obtain the *chain rule* for higher derivatives

$$D^k(f\circ\alpha)=\sum_{i=1}^{k}\sum_{(I_1,\dots,I_i)} D^i f\left(D^{|I_1|}\alpha\circ\varepsilon_{I_1},\dots,D^{|I_i|}\alpha\circ\varepsilon_{I_i}\right), \tag{A.9}$$

where the dependencies of the derivatives $D^{|I_j|}\alpha$ on $x\in A$ and of the derivatives $D^i f$ on $\alpha(x)=(\alpha_1(x),\dots,\alpha_n(x))$ are suppressed, and where the inner sum is taken over all partitions

$$I_1\uplus\dots\uplus I_i=\{1,\dots,k\}$$

with $\min I_l<\min I_m$ for all $l<m$, and

$$\varepsilon_{I_j}:=\varepsilon^k_{i_1,\dots,i_r},\qquad I_j=\{i_1,\dots,i_r\},\qquad i_1<\dots<i_r$$

(where r depends on j). Formula (A.9) is a generalization of Faà di Bruno's formula, where $n=1$ and $A,A_1\subseteq\mathbb{R}$ (cf., e.g., [KP02, Sect. 1.3]).

Example A.2.4. For $k=2$, the chain rule (A.9) specializes to

$$D^2(f\circ\alpha)=D^2f(D\alpha\circ\varepsilon^2_1,D\alpha\circ\varepsilon^2_2)+Df(D^2\alpha\circ\varepsilon^2_{1,2}).$$

Remark A.2.5. Note that each (inner) summand on the right hand side of (A.9) is given by composing the j-th argument of D^if with a certain derivative of α, evaluated at the appropriate arguments of $D^k(f\circ\alpha)$. Moreover, the summand for $i=k$ involves the highest derivative of f which occurs in (A.9), and the arguments of this derivative are composed with the first derivative of α.

For use in Subsect. 3.2.2, we investigate more closely the summand for $i=k$ in the chain rule (A.9) in case $D\alpha$ is represented by a Jacobian matrix of square shape. First a general remark is made about the determinant of the k-th symmetric tensor power of a square matrix.

Remark A.2.6. Let V and W be vector spaces over $\mathbb{R}$ or $\mathbb{C}$ as above. Every linear map $\varphi\colon V\to W$ induces a linear map

$$\bigotimes^k\varphi\colon\bigotimes^k V\longrightarrow\bigotimes^k W\colon v_1\otimes\dots\otimes v_k\longmapsto\varphi(v_1)\otimes\dots\otimes\varphi(v_k)$$

between the k-th tensor powers of V and W and via symmetrization a linear map

$$S^k\varphi\colon S^kV\longrightarrow S^kW$$

between the k-th symmetric tensor powers of V and W.

Let V and W be of finite dimension with bases $b_1,\dots,b_s$ and $b'_1,\dots,b'_t$, respectively. We recall that if the $s\times t$ matrix M represents φ with respect to these bases, then the Kronecker product $M\otimes M$ represents $\varphi\otimes\varphi$ with respect to the bases $b_1\otimes b_1, b_1\otimes b_2,\dots,b_s\otimes b_{s-1}, b_s\times b_s$ and $b'_1\otimes b'_1, b'_1\otimes b'_2,\dots,b'_t\otimes b'_{t-1}, b'_t\otimes b'_t$ of $V\otimes V$ and $W\otimes W$.

Moreover, let us assume that $s=t$. Then $S^k\varphi$ is of shape $\binom{s+k-1}{k}\times\binom{s+k-1}{k}$ and we have

$$\det(S^k\varphi) = \det(\varphi)^N, \qquad N := \binom{s+k-1}{s}. \tag{A.10}$$

On the Zariski dense subset of diagonalizable matrices with pairwise distinct eigenvalues this may be checked by writing the determinant as product of the eigenvalues. Symmetry implies that the degree of $\det(S^k\varphi)$ is the same in each of the s eigenvalues, namely

$$k\binom{s+k-1}{k}\Big/ s = \frac{k}{s}\,\frac{(s+k-1)!}{k!\,(s-1)!} = \frac{(s+k-1)!}{(k-1)!\,s!} = \binom{s+k-1}{s}.$$

By continuity, formula (A.10) for the determinant of a k-th symmetric tensor power holds for all matrices.

Remark A.2.7. Let us assume that the k-th derivatives of $f\colon A_1 \times \ldots \times A_n \to F$ and $\alpha_i\colon A \to A_i$, $i = 1, \ldots, n$, exist and are continuous. We consider the map $D^k f$, which is k-linear and symmetric on $(E_1 \oplus \ldots \oplus E_n)^k$, as a map which is linear on the k-th symmetric tensor power $S^k(E_1 \oplus \ldots \oplus E_n)$.

If the vector spaces $E, E_1, \ldots, E_n$ are finite dimensional and if we have

$$s := \dim E_1 + \ldots + \dim E_n = n \cdot \dim E,$$

then the Jacobian matrix of α, which represents $D\alpha$, is of shape $s \times s$. If F is of finite dimension r, then the summand for $i = k$ in (A.9) may be represented as the product of two matrices of shape $r \times \binom{s+k-1}{k}$ and $\binom{s+k-1}{k} \times \binom{s+k-1}{k}$, respectively, the latter matrix representing the symmetrization of the k-th tensor power of the Jacobian matrix of α. By Remark A.2.6, the determinant of this symmetrization is the $\binom{s+k-1}{s}$-th power of the functional determinant of α.

Example A.2.8. Let $K \in \{\mathbb{R}, \mathbb{C}\}$ and let A be an open subset of $E := K^2$. Moreover, let $\alpha_i\colon A \to A_i$, $i = 1, 2$, and $f\colon A_1 \times A_2 \to F$ be maps, where A_i is an open subset of $E_i := K$, $i = 1, 2$, and $F := K$. Assume that the k-th derivatives of f and α_1, α_2 exist and are continuous. Then the summand for $i = k$ in the chain rule (A.9) may be represented as the product of two matrices of shape $1 \times (k+1)$ and $(k+1) \times (k+1)$, respectively. The determinant of the latter matrix equals the $\binom{k+1}{2}$-th power of the functional determinant of $\alpha = (\alpha_1, \alpha_2)$. Denoting the partial derivative $D_i D_j f$ by $D_{i,j} f$, the summand in question for $k = 2$ may be written as

$$\begin{pmatrix} D_{1,1}f & D_{1,2}f & D_{2,2}f \end{pmatrix} \begin{pmatrix} (D_1\alpha_1)^2 & (D_1\alpha_1)(D_2\alpha_1) & (D_2\alpha_1)^2 \\ 2(D_1\alpha_1)(D_1\alpha_2) & (D_1\alpha_1)(D_2\alpha_2) + (D_2\alpha_1)(D_1\alpha_2) & 2(D_2\alpha_1)(D_2\alpha_2) \\ (D_1\alpha_2)^2 & (D_1\alpha_2)(D_2\alpha_2) & (D_2\alpha_2)^2 \end{pmatrix}.$$

A.3 Differential Algebra

A.3.1 Basic Notions of Differential Algebra

In this subsection we recall a few basic definitions from differential algebra. Standard references are [Rit34], [Rit50], [Kol73], [Kol99], [Kap76].

A *differential ring* R is a ring together with a certain number of *derivations* acting on R, i.e., maps $\partial\colon R \to R$ satisfying

$$\partial(r_1+r_2) = \partial(r_1)+\partial(r_2), \quad \partial(r_1 \cdot r_2) = r_1 \cdot \partial(r_2)+\partial(r_1)\cdot r_2, \qquad r_1, r_2 \in R.$$

Let $\partial_1, \ldots, \partial_n$ be the derivations of the differential ring R. We usually assume that each two of $\partial_1, \ldots, \partial_n$ commute, i.e., $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for all i, $j = 1, \ldots, n$. For $r \in R$ and $i \in \{1, \ldots, n\}$, the element $\partial_i(r)$ of R is called a *derivative* of r. An element of R all of whose derivatives are zero is called a *constant*.

An ideal of R that is closed under the action of $\partial_1, \ldots, \partial_n$ is called a *differential ideal*. The homomorphism theorem for rings holds in an analogous way for differential rings. A differential ring which is a field is called a *differential field*.

Examples of differential fields are fields of rational functions or fields of (formal or convergent) Laurent series in n variables, or fields of meromorphic functions on connected open subsets of $\mathbb{C}^n$, where in each case the n commuting derivations are given by partial differentiation with respect to the n variables. (Choosing all derivations to be zero yields a trivial differential structure for any field.)

Let R be a differential ring with derivations $\partial_1, \ldots, \partial_n$ and S a differential ring with derivations $\delta_1, \ldots, \delta_n$. Then S is called a *differential algebra over R* if S is an algebra over R and

$$\delta_i(r \cdot s) = \partial_i(r) \cdot s + r \cdot \delta_i(s), \qquad r \in R, \quad s \in S, \quad i = 1, \ldots, n.$$

Let S_1 and S_2 be differential algebras over R with derivations $\partial_1, \ldots, \partial_n$ on S_1 and $\delta_1, \ldots, \delta_n$ on S_2. A *homomorphism* $\varphi\colon S_1 \to S_2$ *of differential algebras over R* is a homomorphism of algebras over R which satisfies

$$\varphi \circ \partial_i = \delta_i \circ \varphi, \qquad i = 1, \ldots, n.$$

All algebras are assumed to be associative and unital; algebra homomorphisms map the multiplicative identity element to the multiplicative identity element.

We assume that K is a differential field of characteristic zero with commuting derivations $\partial_1, \ldots, \partial_n$. The *differential polynomial ring* $K\{u_1, \ldots, u_m\}$ over K in the *differential indeterminates* $u_1, \ldots, u_m$ is by definition the commutative polynomial algebra $K[(u_k)_J \mid 1 \le k \le m, J \in (\mathbb{Z}_{\ge 0})^n]$ with infinitely many, algebraically independent indeterminates $(u_k)_J$, also called *jet variables*, which represent the partial derivatives

$$\frac{\partial^{J_1+\ldots+J_n} U_k}{\partial z_1^{J_1} \ldots \partial z_n^{J_n}}, \qquad k = 1, \ldots, m, \quad J \in (\mathbb{Z}_{\geq 0})^n,$$

of smooth functions $U_1, \ldots, U_m$ of $z_1, \ldots, z_n$. We use u_k as a synonym for $(u_k)_{(0,\ldots,0)}$, $k = 1, \ldots, m$. The ring $K\{u_1, \ldots, u_m\}$ is considered as differential ring with commuting derivations $\delta_1, \ldots, \delta_n$ defined by extending

$$\delta_i \, u_k := (u_k)_{1_i}, \qquad i = 1, \ldots, n, \quad k = 1, \ldots, m,$$

additively, respecting the product rule of differentiation, and restricting to the derivation ∂_i on K. (Here 1_i denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$ of length n with 1 at position i.) More generally, the differential polynomial ring may be constructed with coefficients in a differential ring rather than in a differential field in the same way.

A jet variable $(u_k)_J$ has *(differential) order*

$$\mathrm{ord}((u_k)_J) := |J| := J_1 + \ldots + J_n, \qquad k = 1, \ldots, m, \quad J \in (\mathbb{Z}_{\geq 0})^n.$$

The *(differential) order* of a non-constant differential polynomial p is defined to be the maximum of the orders of jet variables occurring in p.

We also use an alternative notation for jet variables. Denoting, for simplicity, by x, y, z the first three coordinates and by u one differential indeterminate,

$$u_{\underbrace{x, \ldots, x}_{i}, \underbrace{y, \ldots, y}_{j}, \underbrace{z, \ldots, z}_{k}} \quad \text{and} \quad u_{x^i, y^j, z^k}$$

are used as synonyms for the jet variable $u_{(i,j,k)}$.

Let F and E be differential fields such that E is a differential algebra over F and $F \subseteq E$. Then $F \subseteq E$ is called a *differential field extension*. A family of elements e_i, $i \in I$, of E, where I is an index set, is said to be *differentially algebraically independent over F* if the family of all derivatives of e_i, $i \in I$, is algebraically independent over F (which is a condition that does not involve the differential structure of F).

The differential polynomial ring $K\{u_1, \ldots, u_m\}$ is the free differential algebra over K generated by $u_1, \ldots, u_m$, in the sense that $u_1, \ldots, u_m$ are differentially algebraically independent over K, which gives rise to the following universal property of $K\{u_1, \ldots, u_m\}$. Let A be any differential algebra over K. For any $a_1, \ldots, a_m \in A$ there exists a unique homomorphism $\varphi \colon K\{u_1, \ldots, u_m\} \to A$ of differential algebras over K satisfying $\varphi(u_k) = a_k$, $k = 1, \ldots, m$.

Let R be a differential ring of characteristic zero and I a differential ideal of R. Then the radical of I, i.e.,

$$\sqrt{I} := \{\, r \in R \mid r^e \in I \text{ for some } e \in \mathbb{N} \,\},$$

is a differential ideal of R. A differential ideal which equals its radical is said to be *radical* (also called a *perfect differential ideal*). For any subset G of R, the *radical*

differential ideal generated by G is defined to be the smallest radical differential ideal of R which contains G. For the importance of radical differential ideals, cf. Subsect. A.3.4.

The differential polynomial ring $K\{u_1,\ldots,u_m\}$ contains the polynomial algebra $K[u_1,\ldots,u_m]$. Kolchin's Irreducibility Theorem (cf. [Kol73, Sect. IV.17, Prop. 10]) states that the radical differential ideal of $K\{u_1,\ldots,u_m\}$ which is generated by a prime ideal of $K[u_1,\ldots,u_m]$ is a prime differential ideal. However, the radical differential ideal generated by an (algebraically) irreducible differential polynomial (involving proper derivatives) may not be prime, as the example of Clairaut's equation (cf. also [Inc56, pp. 39-40])

$$u - x\,u_x - f(u_x) = 0, \quad \text{e.g., with} \quad f(u_x) = -\frac{1}{4}u_x^2,$$

shows (cf. [Kol99, p. 575]).

A.3.2 Characteristic Sets

The basic notions of the theory of characteristic sets (cf., e.g., [Rit50], [Wu00]) are recalled in this section; cf. also [ALMM99], [Hub03a, Hub03b], [Wan01]. Differential algebra in general and the methods introduced by J. M. Thomas (cf. Sect. 2.2) in particular, make extensive use of the concepts which are discussed below.

Let K be a differential field of characteristic zero with n commuting derivations.

Definition A.3.1. Let $R := K\{u_1,\ldots,u_m\}$ be the differential polynomial ring with commuting derivations $\partial_1, \ldots, \partial_n$, let $\Delta := \{\partial_1,\ldots,\partial_n\}$, and denote by $\mathrm{Mon}(\Delta)$ the (commutative) monoid of monomials in $\partial_1, \ldots, \partial_n$. A *ranking* $>$ on R is a total ordering on

$$\mathrm{Mon}(\Delta)u := \{\, (u_k)_J \mid 1 \le k \le m,\, J \in (\mathbb{Z}_{\ge 0})^n \,\} = \{\, \partial^J u_k \mid 1 \le k \le m,\, J \in (\mathbb{Z}_{\ge 0})^n \,\}$$

which satisfies the following two conditions.

a) For all $1 \le k \le m$ and all $1 \le j \le n$ we have $\partial_j u_k > u_k$.
b) For all $1 \le k_1, k_2 \le m$ and all $J_1, J_2 \in (\mathbb{Z}_{\ge 0})^n$ we have

$$(u_{k_1})_{J_1} > (u_{k_2})_{J_2} \quad \Longrightarrow \quad \partial_j\,(u_{k_1})_{J_1} > \partial_j\,(u_{k_2})_{J_2} \qquad \text{for all } j = 1,\ldots,n.$$

A ranking $>$ on $K\{u_1,\ldots,u_m\}$ is said to be *orderly* if

$$|J_1| > |J_2| \quad \Longrightarrow \quad (u_{k_1})_{J_1} > (u_{k_2})_{J_2} \qquad \text{for all } 1 \le k_1, k_2 \le m,\, J_1, J_2 \in (\mathbb{Z}_{\ge 0})^n.$$

Remark A.3.2. Every ranking is a well-ordering, i.e., every non-empty subset of $\mathrm{Mon}(\Delta)u$ has a least element. Equivalently, every descending sequence of elements of $\mathrm{Mon}(\Delta)u$ terminates.

Example A.3.3. Let $R := K\{u\}$ be the differential polynomial ring with one differential indeterminate u and commuting derivations $\partial_1, \ldots, \partial_n$. A ranking $>$ on R which is analogous to the degree-reverse lexicographical ordering of monomials (cf. Ex. 2.1.27, p. 23) is defined for jet variables u_J, $u_{J'}$, $J = (j_1, \ldots, j_n)$, $J' = (j'_1, \ldots, j'_n) \in (\mathbb{Z}_{\geq 0})^n$, by

$$u_J > u_{J'} \quad :\Longleftrightarrow \quad \begin{cases} j_1 + \ldots + j_n > j'_1 + \ldots + j'_n \quad \text{or} \\ \quad (\ j_1 + \ldots + j_n = j'_1 + \ldots + j'_n \quad \text{and} \quad J \neq J' \quad \text{and} \\ \quad j_i < j'_i \quad \text{for} \quad i = \max\{\, 1 \leq k \leq n \mid j_k \neq j'_k \,\}\ ). \end{cases}$$

In this case the ranking of the first order jet variables is given by

$$\partial_1 u > \partial_2 u > \ldots > \partial_n u.$$

The ranking can be extended to more than one differential indeterminate in ways analogous to the term-over-position or position-over-term orderings (cf. Ex. 2.1.28, p. 23). In the former case the ranking is orderly, in the latter case it is not.

In what follows, we fix a ranking $>$ on $R = K\{u_1, \ldots, u_m\}$.

Remark A.3.4. With respect to the chosen ranking, for any non-constant differential polynomial $p \in R - K$ the *leader* of p is defined to be the greatest jet variable with respect to $>$ which occurs in p. It is denoted by $\mathrm{ld}(p)$. Any such differential polynomial p may be viewed as a univariate polynomial in $\mathrm{ld}(p)$ and its coefficients may be considered recursively in the same way (if not constant).

We recall the pseudo-reduction process of differential polynomials.

Definition A.3.5. Let $p \in R$ and $q \in R - K$. The differential polynomial p is said to be *partially reduced with respect to* q if no proper derivative of $\mathrm{ld}(q)$ occurs in p. It is said to be *reduced with respect to* q if it is constant or if it is partially reduced with respect to q and $\deg_x(p) < \deg_x(q)$ for $x := \mathrm{ld}(q)$. A subset S of R is said to be *auto-reduced* if $S \cap K = \emptyset$ and for every $p, q \in S$, $p \neq q$, the differential polynomial p is reduced with respect to q.

Remarks A.3.6. Let $p \in R$ and $q \in R - K$.

a) If p is not partially reduced with respect to q, then Euclidean pseudo-division transforms p into a differential polynomial p' which has this property. There exist a jet variable v which occurs in p and a monomial $\theta \in \mathrm{Mon}(\Delta)$, $\theta \neq 1$, such that $v = \theta\, \mathrm{ld}(q)$. By the defining property b) of a ranking (cf. Def. A.3.1), the leader of θq is v. The rules of differentiation imply that the degree of θq as a polynomial in v is one, and the coefficient of v in θq equals the partial derivative of q with respect to $\mathrm{ld}(q)$, which is called the *separant* of q, denoted by $\mathrm{sep}(q)$. Let d be the degree of v in p and c the coefficient of v^d in p. Then

$$\mathrm{sep}(q) \cdot p - c \cdot v^{d-1} \cdot \theta q$$

eliminates v^d from p. Iteration of this pseudo-division yields, after finitely many steps, a differential polynomial p' which is partially reduced with respect to q.

b) If p is partially reduced, but not reduced with respect to q, then pseudo-division can also be used to produce a differential polynomial p' which is reduced with respect to q. Suppose that $v := \mathrm{ld}(p) = \mathrm{ld}(q)$, and let $d := \deg_v(p)$, $d' := \deg_v(q)$. Then we have $d \geq d'$. The *initial* of p is defined to be the coefficient of v^d in p and denoted by $\mathrm{init}(p)$. Similarly, $\mathrm{init}(q)$ is the coefficient of $v^{d'}$ in q. Then

$$\mathrm{init}(q) \cdot p - \mathrm{init}(p) \cdot v^{d-d'} \cdot q$$

eliminates v^d from p. Again, iteration of this pseudo-division yields, after finitely many steps, a differential polynomial p' which is reduced with respect to q. The same method can be applied recursively to each coefficient of p, if p is not reduced with respect to q and $\mathrm{ld}(p) \neq \mathrm{ld}(q)$.

c) Let $S \subset R - K$ be finite. We apply differential and algebraic reductions (as in a) and b)) to pairs (p,q) of distinct elements of S and after each reduction, we replace p with p' in S if $p' \neq 0$ and remove p from S otherwise. Each reduction step transforms a polynomial p into another one called p' which either has smaller degree in $\mathrm{ld}(p)$ if $\mathrm{ld}(p') = \mathrm{ld}(p)$ or has a leader which is smaller than $\mathrm{ld}(p)$ with respect to the ranking $>$. Since degrees can decrease only finitely many times and since $>$ is a well-ordering, we obtain after finitely many steps either a subset of R which contains a non-zero constant or an auto-reduced subset of R.

d) Both the differential and the algebraic reduction perform a pseudo-division in the sense that p is multiplied by a possibly non-constant polynomial. If p originates from a set of differential polynomials representing a system of partial differential equations, then replacing p with the pseudo-remainder p' in this set may lead to an inequivalent system. If $\mathrm{init}(q)$ and $\mathrm{sep}(q)$ are non-zero when evaluated at any solution of the system, then the replacement of p with the differential polynomial p' computed by the above methods does not change the solution set. Assuming that this condition holds for every initial and separant which is used for pseudo-division, the process in c) computes for a given finite system of partial differential equations an equivalent one which either is recognized to be inconsistent (because of a non-zero constant left hand side) or is auto-reduced.

Remark A.3.7. Every auto-reduced subset of R is finite. This is due to the facts that the elements of an auto-reduced subset of R have pairwise distinct leaders and that every sequence of such leaders in which no element is a derivative of a previous one is finite.

Ritt introduced the following binary relation on the set of all auto-reduced subsets of R.

Definition A.3.8. Let p, $q \in R - K$. The differential polynomial p is said to have *higher rank* than q if either $\mathrm{ld}(p) > \mathrm{ld}(q)$, or if we have $\mathrm{ld}(p) = \mathrm{ld}(q) =: x$ and $\deg_x(p) > \deg_x(q)$. If $\mathrm{ld}(p) = \mathrm{ld}(q) =: x$ and $\deg_x(p) = \deg_x(q)$, then p and q are said to have *the same rank*.

Definition A.3.9. Let $A = \{p_1, \ldots, p_s\}$, $B = \{q_1, \ldots, q_t\} \subseteq R - K$ be auto-reduced sets. We assume that p_{i+1} has higher rank than p_i for all $i = 1, \ldots, s-1$ and that q_{i+1} has higher rank than q_i for all $i = 1, \ldots, t-1$. Then A is said to have *higher rank* than B if one of the following two conditions holds.

a) There exists $j \in \{1, \ldots, \min(s,t)\}$ such that p_i and q_i have the same rank for all $i = 1, \ldots, j-1$ and p_j has higher rank than q_j.
b) We have $s < t$, and p_i and q_i have the same rank for all $i = 1, \ldots, s$.

Remark A.3.10. In every non-empty set $\mathscr{A}$ of auto-reduced subsets of R there exists one which does not have higher rank than any other auto-reduced set in $\mathscr{A}$. Each such auto-reduced set is also said to be of *lowest rank* among those in $\mathscr{A}$. Assuming that each auto-reduced set is sorted as in the previous definition, the existence follows from Remark A.3.7 by considering first those sets in $\mathscr{A}$ whose first elements do not have higher rank than any other first element of sets in $\mathscr{A}$, considering then, if necessary, among these sets the ones whose second elements do not have higher rank than any other second element of these sets and so on.

Definition A.3.11. Let I be a differential ideal of R, $I \neq R$, and define $\mathscr{A}$ to be the set of auto-reduced subsets A of I which satisfy that the separant of each element of A is not an element of I. An auto-reduced subset A of I in $\mathscr{A}$ of lowest rank is called a *characteristic set* of I.

Remark A.3.12. Let I be a differential ideal of R, $I \neq R$. By Remark A.3.10, a characteristic set A of I exists. The definition of a characteristic set implies that the only element of I which is reduced with respect to every element of A is the zero polynomial.

We denote by $\langle A \rangle$ the differential ideal of R which is generated by A, and we define the product q of the initials and separants of all elements of A. Moreover, the *saturation* of $\langle A \rangle$ with respect to q is defined by

$$\langle A \rangle : q^{\infty} := \{\, p \in R \mid q^r \cdot p \in \langle A \rangle \text{ for some } r \in \mathbb{Z}_{\geq 0} \,\}.$$

Then we have

$$\langle A \rangle \subseteq I \subseteq \langle A \rangle : q^{\infty}.$$

If A is a characteristic set of $\langle A \rangle : q^{\infty}$, then not only is zero the unique element of $\langle A \rangle : q^{\infty}$ which is reduced with respect to every element of A, but membership to the ideal $\langle A \rangle : q^{\infty}$ can be decided by applying A in the pseudo-reduction process discussed in Remarks A.3.6.

Let P be a prime differential ideal of R and A a characteristic set of P (with respect to the chosen ranking on R). Then we have $P = \langle A \rangle : q^{\infty}$, where q is the product of the initials and separants of all elements of A. In particular, no initial and no separant of any element of A is an element of P.

It is not known how one could decide effectively whether a prime differential ideal is contained in another one or not (cf. also [Kol73, Sect. IV.9]).

A.3.3 Homogeneous Differential Polynomials

In this subsection we collect a few results about differential ideals which are generated by differential polynomials of a special kind, namely homogeneous differential polynomials, i.e., those for which each term has the same degree, and isobaric differential polynomials, i.e., those for which each term has the same total number of differentiations.

Let Ω be an open and connected subset of $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$, and let K be the differential field of meromorphic functions on Ω with commuting derivations $\partial_{z_1}, \ldots, \partial_{z_n}$ that are defined by partial differentiation with respect to the coordinates $z_1, \ldots, z_n$, respectively.

Let $R := K\{u_1, \ldots, u_m\}$ be the differential polynomial ring in the differential indeterminates $u_1, \ldots, u_m$ with commuting derivations which restrict to the derivations $\partial_{z_1}, \ldots, \partial_{z_n}$ on K and which we again denote by $\partial_{z_1}, \ldots, \partial_{z_n}$. We endow R with the standard grading, i.e., each jet variable $(u_k)_J$, $k = 1, \ldots, m$, $J \in (\mathbb{Z}_{\geq 0})^n$, is homogeneous of degree 1. Of course, every homogeneous component of $K\{u_1, \ldots, u_m\}$ of degree greater than zero is an infinite dimensional K-vector space.

For notational convenience we restrict our attention now to differential polynomial rings in one differential indeterminate u, i.e., $R = K\{u\}$; the results of this subsection can easily be generalized to several differential indeterminates.

For any set S of analytic functions on Ω we define the *vanishing ideal of S* by

$$\mathscr{I}_R(S) := \{\, p \in R \mid p(f) = 0 \text{ for all } f \in S \,\},$$

where $p(f)$ is obtained from p by substitution of f for u and of the partial derivatives of f for the corresponding jet variables in u. For any differential ideal I of $K\{u\}$ we denote by $\mathrm{Sol}_\Omega(I)$ the set of analytic functions on Ω that are solutions of the system of partial differential equations $\{\, p = 0 \mid p \in I \,\}$.

Definition A.3.13. A differential ideal of $K\{u\}$ is said to be *homogeneous*, if it is generated by homogeneous differential polynomials, i.e., differential polynomials that are homogeneous with respect to the standard grading of $K\{u\}$ (not necessarily of the same degree).

Lemma A.3.14. *Let S be a set of analytic functions on Ω having the property that for each f in S the function $c \cdot f$ also is in S for all c in some infinite subset of $\mathbb{C}$. Then the differential ideal $\mathscr{I}_R(S)$ of $R = K\{u\}$ is homogeneous. Conversely, if I is a homogeneous differential ideal of $K\{u\}$, then for every $f \in \mathrm{Sol}_\Omega(I)$ we have $c \cdot f \in \mathrm{Sol}_\Omega(I)$ for all $c \in \mathbb{C}$.*

Proof. For every $c \in \mathbb{C}$ and every analytic function f on Ω, the result of substituting $c \cdot f$ into any non-constant differential monomial in u differs from the one for f exactly by the factor c^d, where d is the total degree of the monomial. Moreover, for every $p \in K\{u\}$ we may consider $p(c \cdot u)$ as a polynomial in c with coefficients

in $K\{u\}$. Then, for any fixed analytic function f on Ω, the equality $p(c \cdot f) = 0$ for infinitely many $c \in \mathbb{C}$ implies that $p(c \cdot f)$ is the zero polynomial. Therefore, the assumption on S implies that the homogeneous components of a differential polynomial which annihilates S are annihilating polynomials as well. The statement claiming the converse also follows easily. □

Example A.3.15. Let $S = \{\, c \cdot \exp(z) \mid c \in \mathbb{C} \,\}$. Then $\mathscr{I}_R(S)$ is generated by $u_z - u$ and hence is homogeneous.

Remark A.3.16. The notion of homogeneity depends on the chosen coordinate system. Expressed in terms of jet coordinates, homogeneity is not invariant under coordinate changes of the dependent variables. If we introduce in the previous example a coordinate v which is related to u by $u = v + 1$, then $u_z - u$ is transformed into the non-homogeneous differential polynomial $v_z - v - 1$, which annihilates exactly the set of analytic functions $\{\, c \cdot \exp(z) - 1 \mid c \in \mathbb{C} \,\}$.

Definition A.3.17. We denote by $L = \mathbb{C}((z_1, \ldots, z_n))$ the field of formal Laurent series in z_1, ..., z_n. Let $L\{u\}$ be the differential polynomial ring with commuting derivations $\partial_{z_1}, \ldots, \partial_{z_n}$ that act on L by partial differentiation.

a) A differential polynomial $p \in L\{u\}$ is said to be[1] *isobaric of weight d* if $p = 0$ or if it can be written as $p = \sum_{i=1}^{r} a_i\, c_i\, m_i$ for some $r \in \mathbb{N}$, where a_i are non-zero complex numbers, c_i are Laurent monomials in z_1, ..., z_n, and m_i are non-constant differential monomials in $L\{u\}$, such that $\mathrm{ord}(m_i) - \deg(c_i)$ equals d for all $i = 1, \ldots, r$. (In this case, each coefficient of p in L is a homogeneous Laurent polynomial.)
b) Let K be the field of meromorphic functions on Ω as above and let $w \in \Omega$. A differential polynomial $p \in K\{u\}$ is said to be *isobaric of weight d around w* if the coefficient-wise expansion around w of its representation as sum of terms is isobaric of weight d as differential polynomial in $\mathbb{C}((y_1, \ldots, y_n))\{u\}$, where $y_i := z_i - w_i$, $i = 1, \ldots, n$.

A differential ideal of $L\{u\}$ (or $K\{u\}$) is said to be *isobaric (around $w \in \Omega$)* if it is generated by differential polynomials which are isobaric (around w). (The generators need not be isobaric of the same weight.)

Example A.3.18. Every differential monomial m of $K\{u\}$ is an isobaric differential polynomial of weight $\mathrm{ord}(m)$ (around any point). The differential polynomials $z_2\, u_{z_1,z_2} + u_{z_1}$ and $u_{z_1} + (\frac{1}{z_1} + \frac{1}{z_2})\, u$ in $\mathbb{C}((z_1, z_2))\{u\}$ are isobaric of weight 1.

Remark A.3.19. Under a change of coordinates of Ω an isobaric differential polynomial is not necessarily transformed to an isobaric one. For instance, let Ω be a simply connected open subset of $\mathbb{C} - \{0\}$ with coordinate z. Under the coordinate transformation defined by $\exp(\widetilde{z}) = z$, the differential polynomial u_z, which is isobaric of weight 1 around any point in Ω, is mapped to $\exp(-\widetilde{z})\, u_{\widetilde{z}}$, which is not isobaric. However, as another example, if $\Omega = \mathbb{C}$ is chosen with coordinate z, then the

[1] The notion of isobaric differential polynomial differs from that defined in [Kol73, Sect. I.7] (cf. also [GMO91]) inasmuch as degrees of formal Laurent series coefficients are taken into account.

translation $\widetilde{z}+1=z$ transforms the differential polynomial $z^d\,\partial_z^k u$, $d\in\mathbb{Z}$, $k\in\mathbb{Z}_{\geq 0}$, which is isobaric of weight $k-d$ around 0, into $(\widetilde{z}+1)^d\,\partial_{\widetilde{z}}^k u$, which is isobaric of the same weight around -1.

We define

$$X(\Omega) := \{\, c\in\mathbb{C} \mid c\cdot z\in\Omega \text{ for all } z\in\Omega \,\}.$$

For any analytic function f on Ω and any $c\in X(\Omega)$ we denote by $f(c\cdot -)$ the analytic function on Ω defined by

$$f(c\cdot -)(z_1,\ldots,z_n) := f(c\cdot z_1,\ldots,c\cdot z_n), \qquad (z_1,\ldots,z_n)\in\Omega.$$

Lemma A.3.20. *Assume that $X(\Omega)$ is an infinite set and $0\in\Omega$. Let S be a set of analytic functions on Ω having the property that for each f in S the function $f(c\cdot -)$ also is in S for all c in some infinite subset of $X(\Omega)$. Then the differential ideal $\mathscr{I}_R(S)$ of $R=K\{u\}$ is isobaric around* 0*. Conversely, if I is a differential ideal of $K\{u\}$ that is isobaric around* 0*, then for every $f\in\mathrm{Sol}_\Omega(I)$ we have $f(c\cdot -)\in\mathrm{Sol}_\Omega(I)$ for all $c\in X(\Omega)$.*

Proof. The chain rule implies that for every jet variable $u_J\in K\{u\}$, $J\in(\mathbb{Z}_{\geq 0})^n$, and every analytic function f on Ω we have $u_J(f(c\cdot -)) = c^{|J|}\cdot u_J(f)(c\cdot -)$ for all $c\in X(\Omega)$. We may now argue in the same way as in the proof of Lemma A.3.14. A differential polynomial $p\in K\{u\}$ is isobaric of weight d around 0 if and only if for every analytic function f on Ω we have $p(f(c\cdot -)) = c^d\cdot p(f)(c\cdot -)$ for infinitely many $c\in X(\Omega)$. □

Examples A.3.21. Let x, y be coordinates of $\Omega=\mathbb{C}^2$, K the field of meromorphic functions on Ω, and $R=K\{u\}$ as above. Then we have $X(\Omega)=\mathbb{C}$.

a) Let $S=\{\, f_1(x^2)+f_2(xy) \mid f_1,f_2 \text{ analytic}\,\}$. Then the vanishing ideal $\mathscr{I}_R(S)$ is isobaric (around 0) because $\mathscr{I}_R(S)$ is generated by the differential polynomial

$$x\,u_{x,y} - y\,u_{y,y} - u_y,$$

which is isobaric (around 0) of weight 1. Note that $\mathscr{I}_R(S)$ is also generated by

$$u_{x,y} - \frac{y}{x}\,u_{y,y} - \frac{1}{x}\,u_y,$$

which is an isobaric differential polynomial of weight 2. (We refer to Sect. 3.2 for more details on methods which determine the vanishing ideal of such a set S.)

b) Let $S=\{\, f_1(x+2y)+f_2(x+y)\cdot e^x \mid f_1,f_2 \text{ analytic}\,\}$. Then the vanishing ideal $\mathscr{I}_R(S)$ is not isobaric (around 0) because it is generated by the differential polynomial

$$2\,u_{x,x} - 3\,u_{x,y} + u_{y,y} - 2\,u_x + u_y,$$

and the zero polynomial is therefore the only isobaric element of $\mathscr{I}_R(S)$.

A.3.4 Basis Theorem and Nullstellensatz

We recall two fundamental theorems of differential algebra, which are analogs of Hilbert's Basis Theorem and Hilbert's Nullstellensatz in commutative algebra. For more details, we refer to [Rit34, §§ 77–85], [Rau34], [Rit50, Sects. I.12–14, II.7–11, IX.27], [Sei56], [Kol73, Sect. III.4], [Kap76, § 27], [Kol99, pp. 572–583].

Let K be a differential field of characteristic zero and $R := K\{u_1, \ldots, u_m\}$ the differential polynomial ring in the differential indeterminates $u_1, \ldots, u_m$ with commuting derivations $\partial_1, \ldots, \partial_n$.

There are differential ideals of R which are not finitely generated, as the example of the differential ideal generated by the infinitely many differential monomials

$$(\partial u)(\partial^2 u), \quad (\partial^2 u)(\partial^3 u), \quad \ldots, \quad (\partial^k u)(\partial^{k+1} u), \quad \ldots$$

shows, where $n = 1, m = 1, u = u_1, \partial = \partial_1$; cf. [Rit34, p. 12]. However, the following theorem shows that radical differential ideals admit a representation in terms of a finite generating set.

Theorem A.3.22 (Basis Theorem of Ritt-Raudenbush). *Every radical differential ideal of R is finitely generated.*

For the sake of completeness, we include a proof following [Kap76, § 27].

Proof. The theorem is proved by induction on m. The statement is trivial for $m = 0$. We assume now that $m > 0$ and that the statement is true for smaller values of m, and we consider $K\{u_1, \ldots, u_{m-1}\}$ as a differential subring of $R = K\{u_1, \ldots, u_m\}$.

Let us assume that there exists a radical differential ideal of R which is not finitely generated and derive a contradiction. The non-empty set of such differential ideals is partially ordered by set inclusion and every totally ordered subset has an upper bound given by the union of its elements. Therefore, by Zorn's Lemma, there exists a maximal element I of that set.

We show that I is prime. Assuming the contrary, there exist p, $q \in R$ such that $pq \in I$, but $p \notin I$ and $q \notin I$. Then the radical differential ideals I_1 and I_2 of R which are generated by I and p and by I and q, respectively, are finitely generated, say, by $G_1 = \{p, p_1, \ldots, p_s\}$ and $G_2 = \{q, q_1, \ldots, q_t\}$, respectively, where we may assume that $p_1, \ldots, p_s, q_1, \ldots, q_t \in I$. Then $I_1 \cdot I_2$ is contained in the radical differential ideal which is generated by $\{g_1 \cdot g_2 \mid g_1 \in G_1, g_2 \in G_2\}$. On the other hand, the latter ideal contains I because $I \subseteq I_1$ and $I \subseteq I_2$ imply that it contains the square of every element of I and it is a radical ideal. Since G_1 and G_2 are finite, we conclude that I is finitely generated as a radical differential ideal, which is a contradiction.

Let J be the radical differential ideal of $R = K\{u_1, \ldots, u_m\}$ which is generated by $I \cap K\{u_1, \ldots, u_{m-1}\}$. Since the radical differential ideal $I \cap K\{u_1, \ldots, u_{m-1}\}$ of $K\{u_1, \ldots, u_{m-1}\}$ is finitely generated by the induction hypothesis, J is also finitely generated. By the assumption on I, the ideal J is a proper subset of I.

We choose an arbitrary ranking $>$ on R (cf. Def. A.3.1). Among the differential polynomials in $I - J$ with least leader v with respect to $>$ let r be one of least degree

d in v. Then we have $\mathrm{init}(r) \notin J$ and $\mathrm{sep}(r) \notin J$ because otherwise $r - \mathrm{init}(r)\, v^d$ or $r - \frac{1}{d}\,\mathrm{sep}(r)\, v$ would be either zero or in $I - J$, both possibilities being contradictions to the choice of r. Since $\mathrm{init}(r)$ and $\mathrm{sep}(r)$ have leader smaller than v with respect to $>$ or degree in v less than d and since both differential polynomials are not in J, we also have $\mathrm{init}(r) \notin I$ and $\mathrm{sep}(r) \notin I$. This implies $\mathrm{init}(r) \cdot \mathrm{sep}(r) \notin I$ because I is a prime ideal. By the choice of I, the radical differential ideal H of R which is generated by I and $\mathrm{init}(r) \cdot \mathrm{sep}(r)$ is finitely generated, say, by the differential polynomials $\mathrm{init}(r) \cdot \mathrm{sep}(r)$, $h_1, \ldots, h_k$, where we may assume that $h_1, \ldots, h_k \in I$.

We claim that $\mathrm{init}(r) \cdot \mathrm{sep}(r) \cdot I$ is contained in the radical differential ideal L of R which is generated by J and r. In fact, for every element $a \in I$, the pseudo-reduction process described in Remarks A.3.6 yields an element $b \in I$ of the form $b = \mathrm{init}(r)^i \cdot \mathrm{sep}(r)^j \cdot a - c \cdot r$ for some $i, j \in \mathbb{Z}_{\geq 0}$ and $c \in R$ such that b is reduced with respect to r. Then we have $b \in J$ and $\mathrm{init}(r)^{\max(i,j)} \cdot \mathrm{sep}(r)^{\max(i,j)} \cdot a^{\max(i,j)} \in L$ and therefore $\mathrm{init}(r) \cdot \mathrm{sep}(r) \cdot a \in L$, which proves the claim.

Finally, we obtain a contradiction to the choice of I by showing that I is equal to the radical differential ideal of R which is generated by L and $h_1, \ldots, h_k$. The latter ideal is finitely generated as a radical differential ideal because J and L are so. Clearly it is contained in I. Conversely, the square of every element of I is contained in $H \cdot I$ by the definition of H, hence in the radical differential ideal of R which is generated by $\mathrm{init}(r) \cdot \mathrm{sep}(r) \cdot I$, $h_1 \cdot I, \ldots, h_k \cdot I$, and therefore in the one generated by L and $h_1, \ldots, h_k$. This proves the reverse inclusion. $\square$

The following theorem can be found, e.g., in [Kap76, § 29], [Rit50, Sect. II.3], [Rau34, Thms. 5 and 6].

Theorem A.3.23. *Every radical differential ideal of R is an intersection of finitely many prime differential ideals. A minimal representation as such an intersection (i.e., one in which none of the prime differential ideals is contained in another one) is uniquely determined up to reordering of the components.*

Proof. Let us assume that there exists a radical differential ideal I of R which has no representation as intersection of finitely many prime differential ideals and derive a contradiction. By the Basis Theorem of Ritt-Raudenbush (Theorem A.3.22), every ascending chain of radical differential ideals of R terminates. Hence, we may assume that I is chosen to be maximal among those radical differential ideals of R having the above property.

Since I is not prime, there exist $p, q \in R$ such that $pq \in I$, but $p \notin I$ and $q \notin I$. Then the radical differential ideals I_1 and I_2 of R which are generated by I and p and by I and q, respectively, are intersections of finitely many prime differential ideals. Moreover, $I_1 \cdot I_2$ is contained in the radical differential ideal which is generated by I and pq, which is equal to I. Therefore, I contains the square of every element of $I_1 \cap I_2$. Since I is radical, this implies $I_1 \cap I_2 \subseteq I$, and then we conclude from $I \subseteq I_1$ and $I \subseteq I_2$ that we have $I = I_1 \cap I_2$. Hence, I has a representation as intersection of finitely many prime differential ideals, which is a contradiction.

Let $P_1 \cap \ldots \cap P_r = Q_1 \cap \ldots \cap Q_s$ be two minimal representations of a radical differential ideal of R as intersection of prime differential ideals. Then we have

$P_1 \cap \ldots \cap P_r \subseteq Q_1$. Since Q_1 is prime, there exists $i \in \{1,\ldots,r\}$ such that $P_i \subseteq Q_1$. On the other hand, we have $Q_1 \cap \ldots \cap Q_s \subseteq P_i$ and, therefore, $Q_j \subseteq P_i$ for some $j \in \{1,\ldots,s\}$. By the minimality of the representation $Q_1 \cap \ldots \cap Q_s$, we have $j = 1$. An iteration of this argument shows that we have $\{P_1,\ldots,P_r\} = \{Q_1,\ldots,Q_s\}$. □

Let Ω be a connected open subset of $\mathbb{C}^n$ and denote by K the differential field of meromorphic functions on Ω. Analytic solutions of systems of differential equations with coefficients in K may have domains of definition which are properly contained in Ω. For more details, we refer to [Rit50, Sects. II.7–11, IX.27], [Rau34, Thm. 9].

Theorem A.3.24 (Nullstellensatz for Analytic Functions). *Let $p_1, \ldots, p_s \in R$ and I the differential ideal of R they generate. Moreover, let $q \in R$ be a differential polynomial which vanishes for all analytic solutions of I. Then some power of q is an element of I.*

Proof (Sketch). The assertion is equivalent to the following statement. If $q \in R$ is not an element of the radical differential ideal $\sqrt{I}$ which is generated by $p_1, \ldots, p_s$, then there exists an analytic solution of $p_1 = 0, \ldots, p_s = 0$ which is not a solution of $q = 0$.

Let $\sqrt{I} = P_1 \cap \ldots \cap P_r$ be a representation of $\sqrt{I}$ as intersection of prime differential ideals (cf. Thm. A.3.23). Since we assume that $q \notin \sqrt{I}$, there exists $j \in \{1,\ldots,r\}$ such that $q \notin P_j$. Let A be a characteristic set of P_j (with respect to a chosen Riquier ranking on R) defining a passive differential system (i.e., one which incorporates all integrability conditions, cf. also the introduction to Sect. 2.1). It is enough to show the existence of an analytic solution of this differential system for which q does not vanish.

Note that no initial and no separant of any element of A is contained in P_j. We consider A as a system of algebraic equations (in finitely many variables) for analytic functions, neglecting the differential relationships among the jet variables. By a version of Hilbert's Nullstellensatz for analytic functions (cf. [Rit50, Sects. IV.13–14]), a solution of this system exists for which neither q nor any initial or separant of elements of A vanishes. A point $w \in \Omega$ may be chosen such that all coefficients of q and all coefficients of all equations defined by A are analytic around w and such that w is not a zero of the evaluation of q at the above solution or those of the initials and separants of elements of A. The equations given by A are solved for their leaders and now considered as differential equations with boundary values at $z = w$ determined by evaluating the above (algebraic) solution at w. If a differential indeterminate occurs in an equation, but none of its derivatives is a leader of an equation, then it is replaced with an arbitrary analytic function taking the value at w which is prescribed by the above (algebraic) solution. A solution of this boundary value problem exists by Riquier's Existence Theorem. By construction, it is not a solution of q. □

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List of Algorithms

D. Robertz, *Formal Algorithmic Elimination for PDEs*,
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List of Examples

Some families of analytic functions dealt with in the text are referenced below.

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