Appendix A
Beyond Matrices

A good part of what we have done in the preceding chapters in the case of matrices can be done in a more general setting. We will consider three types of generalizations, which can be combined:

(1) Extension in space;
(2) Extension in time;
(3) Nonlinear analogues.

In (1) and (2) "space" and "time" refer to the probabilistic interpretation of potential theory. In Chap. 2 we considered Markov chains (discrete and continuous time) with finite state space. If extensions (1) and (2) are considered together, we meet the general Markov processes and the general potential theory: many old and new books are consecrated to these theories and we will not try to give an outline of these; let us mention nevertheless [9], a nice (old, but still incomparable) place where to gain a precise idea of all aspects of the subject. Here we will have a look only at generalizations which do not go too far away in spirit from what was done for matrices.

There are three natural ways to address the extension in space. The first is to consider a denumerable (infinite) set I of indexes and so look at infinite matrices; in that case the difficulty lies in the fact that counting measure is no more finite. The second is to replace I by a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where the matrices become continuous operators on the corresponding L^2 space. We will look at these two first extensions when generalizing below the concepts of ultrametric and filtered matrices in Chaps. 3 and 5, where we also make a modest extension in time, in considering filtrations indexed by $\mathbb{R}_+$. The third is to replace I by a compact metrisable space F, the Z- and M-matrices becoming linear or nonlinear (third extension), continuous or not (see below), operators defined on a subspace $\mathscr{D}$ of the space $\mathscr{C}$ of continuous functions. We begin with this generalization (which may implicitly contain a time extension) without straying too far from the Chap. 2: it's amazing to see how the

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principal features of Z- and M-matrices can be easily extended, usefully to a wider context.

Extension of Z-Matrices, M-Matrices and Inverses

Instead of the set I of indices for matrices we take a compact metrizable space F; beware that "x" will denote here a generic point of F whereas "i" denotes a generic point of I. Let $\mathscr{C}$ be the space of (real) continuous functions on F equipped with the uniform norm and consider a map A from a subset $\mathscr{D}$ of $\mathscr{C}$ into $\mathbb{R}^F$ verifying

$$\forall u, v \in \mathscr{D}\ \forall x \in F\ [u \leq v \text{ and } u(x) = v(x)] \Longrightarrow [(Au)(x) \geq (Av)(x)] \tag{A.1}$$

which, if $\mathscr{D}$ is a vector subspace and A is linear, is equivalent to

$$\forall v \in \mathscr{D}\ \forall x \in F\ [0 \leq v \text{ and } v(x) = 0] \Longrightarrow [(Av)(x) \leq 0]. \tag{A.2}$$

Actually (A.2) is a very known "principle" of potential theory where it bears different names, but what matters here is if F is finite, identified to $\{1, \ldots, n\}$, $\mathscr{D}$ is $\mathbb{R}^n$ and A is linear, identified to a matrix, then it can be recognized in (A.2) and so in (A.1) a "functional" way to say A is a Z-matrix, no more, no less. As we will see below, (A.1) is the right statement to acquiring comparison theorems, even in the linear case.

They are numerous examples of operators verifying (A.1). The most sophisticated are probably the so called "fully nonlinear degenerate second order elliptic operators" [16] (in despite of the name, these include all of the usual linear elliptic and parabolic operators) and will be called **differential elliptic operators** subsequently. In that case, F is the closure of a relatively compact open subset E of some $\mathbb{R}^n$, $\mathscr{D}$ is the restriction to F of the $\mathscr{C}^2$-functions on $\mathbb{R}^n$ and A is defined on E by

$$(Au)(x) = \Phi[x, u(x), \nabla u(x), Hu(x)] \tag{A.3}$$

with $u \in \mathscr{D}$, $x \in E$, ∇u the gradient of u, Hu the hessian matrix of u, Φ any function with appropriate arguments (the last one is a symmetric matrix), and nonincreasing in its last argument w.r.t. the usual order on semidefinite matrices—here is the ellipticity property. One completes the definition of A on F by taking $Au(x) = u(x)$ for $x \in \partial E$, that is, in other words, by the Dirichlet's condition at the boundary. To check that A verifies (A.1) is just a short exercise using Taylor's formula.

We propose to say that operators verifying (A.1) are elliptic operators (they are called "derivers" in [19]).

Moreover, the most simple examples are what we call the **elementary elliptic operators**. In that case, F being an arbitrary metrizable compact space, $\mathscr{D}$ is $\mathscr{C}$ and by definition A can be written $A = \varphi(I - P)$ where φ is (the multiplication by) a

nonnegative function, I is the identity and P is a nondecreasing map from $\mathscr{C}$ into $\mathbb{R}^F$; sure, we met often very particular cases in the Chap. 2. It is quite simple to confirm that such an A verifies (A.1) and so is elliptic in our terminology.

A final example before we look at a generalization of an M-matrix. This time, as for matrices, F is a finite set identified with $I = \{1, \ldots, n\}$. A continuous map $\Psi = (\Psi_i)_{i \in I}$ from $\mathbb{R}^n$ into $\mathbb{R}^n$ is called **quasimonotone** (see [57]) if each Ψ_i is nondecreasing in each of its arguments except the ith. So if Ψ is linear, $-\Psi$ can be identified with a Z-matrix, and in the general case $-\Psi$ is a nonlinear elliptic operator (often elementary, but not always). The quasimonotone maps are important ingredients in the study of some dynamical systems called competitive or cooperative systems [55]. An easy and useful result, even in the case of a Z-matrix, is the following: let $-A$ be quasimonotone and v, f two points of $\mathbb{R}^n$; if the system of inequalities $u \geq v$, $Au \geq f$ has at least a solution then it has a smallest one $\tilde{u}$, which verifies $[\tilde{u}_i = v_i$ or $(A\tilde{u})_i = f_i]$ for each $i \in I$. A good part of linear or non linear potential theory is devoted to hard extensions of this kind of result.

Coming back to the general situation (F metrizable compact, A defined on a subset $\mathscr{D}$ of $\mathscr{C}$) we suppose here $\mathscr{D}$ is a vector subspace and now introduce a condition implying that our elliptic operator A is injective and its inverse on $A(\mathscr{D})$ is nondecreasing, namely we suppose the existence of $\chi \in \mathscr{D}$

$$\chi \text{ is positive and } \forall u \in \mathscr{D}\ \forall x \in F\ t \mapsto (Au + t\chi)(x) \text{ is increasing on } \mathbb{R} \tag{A.4}$$

which is, if A is linear, equivalent to

$$\chi > 0 \text{ and } A\chi > 0. \tag{A.5}$$

With a nod to mathematical economics we will say that the elliptic operator A is **productive** if it verifies (A.4) and the function χ can be called a **witness of productivity**. One recognizes in (A.5) one of the usual necessary and sufficient condition for a Z-matrix A to be an M-matrix. At the other end of complexity, where A is a differential elliptic operator as in (A.3), one often sees the hypothesis that the function Φ is increasing in its second argument: that implies at once that the positive constant functions are witnesses to productivity. This brings us to a fundamental theorem; here we consider an arbitrary partition $(E, \partial E)$ of F in a subset E and its complementary ∂E, but the notation suggests the possibility of applying it to Dirichlet's problem. The following result is known as the **comparison Theorem**.

Theorem A.1 (Theorem of Comparison) *Let A be a productive elliptic operator, $(E, \partial E)$ a partition of F and u, v two elements of $\mathscr{D}$. Then*

$$[u \leq v \text{ on } \partial E \text{ and } Au \leq Av \text{ on } E] \Longrightarrow u \leq v \text{everywhere}.$$

Proof Suppose there is $x \in E$ such that $u(x) > v(x)$, let χ be a witness to productivity and $\tau = \inf\{t \geq 0 \colon u \leq v + t\chi\}$, which is positive and finite. Setting

$w = v + \tau\chi$, we have $w \geq u$, and since we are looking at continuous functions on a compact space, there is $\xi \in F$ such that $w(\xi) = u(\xi)$. From our hypotheses we have $\xi \in E$, from the ellipticity of A we have $Aw(\xi) \leq Au(\xi)$ and from the productivity of A we have $Av(\xi) < Aw(\xi)$, and so finally a contradiction with our hypothesis $Av(\xi) \geq Au(\xi)$.

Corollary A.2 *Any productive elliptic operator A is injective and its inverse A^{-1} is increasing on its domain of definition $A(\mathscr{D})$.*

Proof Take ∂E empty in the theorem: you get $Au \leq Av \Rightarrow u \leq v$, and so the injectivity of A and the nondecreasingness of A^{-1}, even its increasingness by again using (A.1).

So if F is finite, identified to $\{1, \ldots, n\}$ and A is linear identified to a matrix, then A is an M-matrix since here the injectivity implies the surjectivity. If F is just a singleton and $\mathscr{C}$ identified to $\mathbb{R}$, then a (nonlinear) elliptic operator A defined on $\mathscr{C}$ is just a (real) function, and a productive one is just an increasing function: the corollary can be viewed as a generalization of a part of the usual theorem on the inversion of an increasing function; one can delve further in this direction: under mild conditions, it can be quite easily proved that A^{-1} is continuous for the uniform norm on the interior of $A(\mathscr{D})$. The major difficulty is to identify $A(\mathscr{D})$ (remember, the Dirichlet's problem is an instance of our problem of inversion); below we will briefly look at the case when A is continuous from $\mathscr{C}$ into $\mathscr{C}$ (which is certainly not the case for a differential elliptic operator which is not completely degenerated).

Let us now state what is called the **Domination Principle**.

Corollary A.3 (Domination Principle) *Let A be a productive elliptic operator and u ,v two elements of $\mathscr{D}$. Then*

$$u \leq v \text{ on } \{Au > Av\} \Longrightarrow u \leq v \text{ everywhere}.$$

Proof Take $\partial E = \{Au > Av\}$ in the theorem, and so E=$\{Au \leq Av\}$ (do not be afraid by the tautology "$Au \leq Av$ on $\{Au \leq Av\}$"!).

We retrieve, in a general setting, the domination principle of Chap. 2, except here it is explicitly written with A, not with A^{-1}. If one supposes that A verifies a "submarkovianity" property, namely $t \mapsto (Au + t)(x)$ is nondecreasing in $t \in \mathbb{R}$ for any $u \in \mathscr{D}$, $x \in F$, one can easily deduce from the corollary a general statement of the complete maximum principle seen in Chap. 2.

Suppose now that A is a continuous productive elliptic operator from $\mathscr{C}$ into $\mathscr{C}$. Adapting modern tools used in the study of Dirichlet's problem [16] (essentially here a modern variant of the classical Perron's method in potential theory), it is possible but not trivial to prove that $A(\mathscr{C})$ verifies an intermediary property, namely if f, g belong to $A(\mathscr{C})$, then any $h \in \mathscr{C}$ verifying $f \leq h \leq g$ belongs also to $A(\mathscr{C})$. So if A is linear, A is surjective since $tA\chi$, $t \in \mathbb{R}$, χ witness, can be as great or as little as you wish. But, if A is linear, there is a more elementary way to look at A^{-1}: A is linear and continuous, so lipschitzian and so A is a elementary elliptic operator

and can be written $A = k(I - P)$ where k is a constant, I the identity and P is a nonnegative continuous linear operator. Knowing that $A\chi$, χ a witness, belongs of course to $A(\mathscr{C})$, it is a good exercise on series to prove that the geometric series $\sum_{n\geq 0} P^n$ is normally convergent and that its sum is equal to kA^{-1}, a series we saw for matrices in Chap. 2 and others.

Extension of Ultrametricity to Infinite Matrices

Here I is an infinite denumerable set. As in Definition 3.2, we will say an infinite matrix U indexed by I is **ultrametric** if U is symmetric, the diagonal of U is pointwise dominant and the ultrametric inequality

$$U_{ik} \geq \inf(U_{ij}, U_{jk})$$

is verified for any i, j, k in I. Is such an ultrametric matrix a potential matrix in some way as in Theorem 3.5.

Before providing some elements in response, we need to introduce substochastic matrices and potential matrices into our infinite context.

An infinite matrix P will be said **substochastic** if we have $0 \leq P_{ij} \leq 1$ and $\sum_{j\in I} P_{ij} \leq 1$ for any i; the **potential**-matrix V generated by such a P, when it exists—that is when the series is convergent, is

$$V = \sum_{n\geq 0} P^n$$

(cf Sect. 2.3). There are some important differences with the finite case: suppose V exists; it is a right inverse of $\mathbb{I} - P$, but generally $\mathbb{I} - P$ has many different right inverses; nevertheless it can be proved V is the smallest nonnegative right inverse of $\mathbb{I} - P$. Let us have a closer look. An (infinite) vector v is said **invariant** (or **harmonic**) w.r.t. the substochastic matrix P if $Pv = v$; in the finite case, it is impossible to have a non-null invariant vector and a potential matrix, but it is not the case for infinite matrices—it is even possible to have $\mathbb{1}$ as an **invariant** (P is stochastic) and $\sum_{n\geq 0} P^n < \infty$—and if H is an infinite matrix whose columns are invariant vectors and V the potential of P, then $V + H$ is another right inverse of $\mathbb{I} - P$.

Let us return to the study of an ultrametric matrix U, looking only at the particularly important case where U is the canonical ultrametric matrix associated with a (infinite denumerable) rooted tree T with I as set of nodes and r as root. We suppose T locally finite, that is any node i has a finite number s_i of successors, and, in order to avoid referring to the potential of a continuous time Markov chain, we suppose $\sup_i s_i < +\infty$. We will extend in our context all the vocabulary and notations introduced for a finite rooted tree in the Chap. 3. So our ultrametric matrix

$U = (U_{ij})$ is given by

$$U_{ij} = |i| + 1,$$

where $|i|$ is the length of the geodesic from the root r to the node i. Further the canonical M-matrix M associated to T is given by:

$$\begin{cases} M_{ij} = 0 \text{ if } i \neq j \text{ and } (i, j) \notin T; \\ M_{ij} = M_{j\,i} = -1 \text{ if } j \text{ is the predecessor of } i; \\ M_{ii} = 1 + s_i \text{ where } s_i \text{ is the number of successors of } i. \end{cases}$$

It is evident that M is diagonally dominant (the sum of each line is 0 except for the line r where the sum is 1) and easy to check as a left inverse of U; moreover M can be written $M = k(\mathbb{I} - P)$ where $k = \sup_i(1 + s_i)$ and P is a substochastic matrix, stochastic at any node except for r which therefore is attached to an added trap ∂. Consequently we have $k(\mathbb{I} - P)U = \mathbb{I}$ and so kU is candidate to be the potential of P, but it often misses the point. For example, if T is the infinite linear tree ($s_i = 1$ for any i), then $2U$ is the potential V of P, but if T is the infinite dyadic tree ($s_i = 2$ for any i), then $3U$ is not the potential V of T: $3U - V$ is a positive harmonic matrix for P. This situation is enlightened by the probabilistic interpretation. Let $\mathbb{P}$ be the probability attached to the Markov chain (X_n) on T starting from r with P as transition matrix, and $\tau = \inf\{n : X_n = \partial\}$ the lifetime of the chain. Then either we have $\mathbb{P}\{\tau < \infty\} = 1$, and kU is the potential of P, or $\mathbb{P}\{\tau = \infty\} > 0$, and kU is not the potential. In this last case a detailed study of the situation requires the addition of T as an "exit" boundary (a kind of Cantor set); as in [24].

Extension of Ultrametricity to L^2-Operators

One can see the finite index set I of our finite matrices as a probability space equipped tacitly with the field $\mathscr{P}$ of all the parts of I and more or less tacitly with the normalized counting measure. Then any (symmetric) matrix is identified with a (symmetric) L^2-operator. More generally, we replace I by any probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and define and study ultrametric operators. In this general context, ultrametric distances and partitions are not good tools, but filtrations (nondecreasing families of conditional expectations) are excellent ones; as said in Definition 5.1 a conditional expectation is a symmetric projection in L^2 which is nonnegative and leaves $\mathbb{1}$ invariant. We know from Chaps. 3 and 5 that in the finite case looking to ultrametric distances or increasing sets of partitions or filtrations are equivalent; in particular following Proposition 5.17 a symmetric matrix is ultrametric if and only if it is filtered. So, in the general case we can say that an operator U is **ultrametric** if it is filtered, that is there exists a finite (for the moment) nondecreasing sequences

$(\mathbb{E}_k : 0 \leq k \leq n)$ of conditional expectations (shortly, a filtration) and a sequence $(a_k : 0 \leq k \leq n)$ of nonnegative bounded random variables such that $\mathbb{E}_k a_k = a_k$ for any k and

$$U = \sum_{k=0}^{n} a_k \mathbb{E}_k$$

where in the formula a_k should be interpreted as the multiplication operator by a_k, which commutes with $\mathbb{E}_k$. Then, if U is such a filtered or ultrametric operator, the operator $U + t\mathbb{I}$ is, for any $t > 0$ (and sometimes for $t = 0$), invertible and its inverse can be written $k(\mathbb{I} - P)$, k, P functions of t, where k is a positive constant and P is a substochastic operator, that is positive and verifies $P\mathbb{1} \leq \mathbb{1}$. As such, we observe the same result as in Chap. 3. In fact, the notion of a filtered operator and its study were first introduced in this probabilistic context [21], and the comments in Chap. 5 on filtered matrices are a restriction of more general statements.

More generally one can study weakly filtered operators. Such an operator U can be expressed

$$U = \sum_{k=0}^{n} a_k \mathbb{E}_k z_k$$

where a_k, z_k are nonnegative bounded random variables such that $\mathbb{E}_{k+1} a_k = a_k$ and $\mathbb{E}_{k+1} z_k = z_k$ for $k < n$. The set forth in Chap. 5 is still valid, except for the elements revolving around the consideration of "non-normalized" conditional expectations (whose entries are just 0 and 1), which do not exist outside the matrix case.

Coming back to an ultrametric operator $U = \sum_{k=0}^{n} a_k \mathbb{E}_k$, let us set $A_i = \sum_{k=0}^{i-1} a_k$ and $A_{-1} = 0$ so that we can write $U = \sum_{k=0}^{n} \mathbb{E}_k (A_k - A_{k-1})$ where (A_k) is an adapted $(\mathbb{E}_k A_k = A_k)$ nondecreasing $(A_{k-1} \leq A_k)$ process. This suggests an extension of the notion of an ultrametric (or filtered) operator in the case of a continuous time filtration $(\mathbb{E}_t : 0 \leq t)$, right continuous, by considering

$$U := \int_0^\infty \mathbb{E}_t \, dA_t$$

where (A_t) is an adapted nondecreasing process and is bounded (A_∞ is a bounded random variable)—or we encounter difficulties as in the case of infinite matrices. A classical approach to the continuous by the discrete (see [21]) permits us to prove that again $U + t\mathbb{I}$, for any $t > 0$, is invertible, with inverse of the form $k(\mathbb{I} - P)$ like above. It is also possible to consider a weakly filtered operator U in this context, but the integral defining U in this case is more involved, so we won't comment further on this subject here.

Appendix B
Basic Matrix Block Formulae

In this appendix we include some basic facts about matrices than we need in our exposition.

Take $n \geq 2$, $1 \leq p < n$ and $q = n - p$. We define $J = \{1, \cdots, p\}$ and $K = \{p+1, \cdots, n\}$. Consider the following partition of an $n \times n$ matrix U

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = U_{JJ}$, $D = U_{KK}$. We assume that U and D are nonsingular. Then, if we apply the Gauss algorithm by blocks we obtain

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix}$$

and therefore

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ C & D \end{pmatrix}.$$

In particular, one has the well known formula for the determinant of U given by $|U| = |D|\,|A - BD^{-1}C|$. This shows that $A - BD^{-1}C$, which is called the **Schur complement of** D, is nonsingular.

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On the other hand the inverse of U can be computed as

$$U^{-1} = \begin{pmatrix} \mathbb{I} & 0 \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \mathbb{I} & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix} \begin{pmatrix} [A - BD^{-1}C]^{-1} & -[A - BD^{-1}C]^{-1}BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix}$$

$$= \begin{pmatrix} [A - BD^{-1}C]^{-1} & -[A - BD^{-1}C]^{-1}BD^{-1} \\ -D^{-1}C[A - BD^{-1}C]^{-1} & D^{-1} + D^{-1}C[A - BD^{-1}C]^{-1}BD^{-1} \end{pmatrix}.$$

Thus, the block of U^{-1} associated with J is

$$(U^{-1})_{JJ} = [A - BD^{-1}C]^{-1}.$$

Given a permutation σ of $\{1, \cdots, n\}$, we construct the matrix Π given by

$$\Pi_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases}$$

Then the matrix ΠU is obtained from U by permuting the rows of U according to σ, that is the i-th row of ΠU is the $\sigma(i)$-th row of U. Similarly $U\Pi'$ is the matrix obtained from U by permuting the columns of U according to σ. Finally, the matrix $\Pi U \Pi'$ is similar to U and is obtained from it by permuting the rows and columns according to σ.

Appendix C
Symbolic Inversion of a Diagonally Dominant M-Matrix

Following [26], which is too rarely cited in the bibliography of books on matrices or graphs, we present without proofs (see [8, 26] and [27]) a wonderful method for inverse symbolic diagonally dominant symmetric M-matrices. The examples provided here are for a 3×3 matrix but the results are true for any order.

We start with the following matrix

$$A = \begin{pmatrix} x+a+b & -a & -b \\ -a & y+a+c & -c \\ -b & -c & z+b+c \end{pmatrix}$$

where x, y, z, a, b, c are symbolic variables intended to be replaced by nonnegative reals. In fact, any 3×3 symbolic matrix can be written like this after obvious change in variables, but it does not seems very interesting. The associated graph to A is the complete K_4 graph; it is allowed to annulate (and drop) a variable—if you set $c = 0$, the associated graph is the graph of a Wheatstone bridge—, but it is not allowed to identify some variables (for example $x = y$ or $a = b$) if the intent is to use the Wang algebra (but of course it is allowed in the ordinary algebra).

Wang algebra will be seen here as a commutative algebra $\mathfrak{A}$ such that $x + x = 0$ and $x.x = 0$ for any $x \in \mathfrak{A}$ (after Duffin it is the Grassmann algebra on $\mathbb{Z}/2\mathbb{Z}$, and he uses that in the proofs). Such a property implies spectacular simplifications in algebraic calculations: for example one has $(a+b).(b+c).(c+a) = 0$ for any $a, b, c \in \mathfrak{A}$.

We are going to calculate the determinant and the principal cofactors of A with the help of the Wang algebra generated by x, y, z, a, b, c, and deduce from them the remaining cofactors with the help of the ordinary algebra.

Determinant The determinant of A is merely the product of the diagonal elements, but in the Wang algebra generated by x, y, z, a, b, c. Indeed if one expands $D = (x+a+b)(y+a+c)(z+b+c)$ in this algebra one finds first (recall that $c^2 = 0$)

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$$D = (x + a + b)(yz + by + cy + az + ab + ac + cz + bc)$$

and finally one finds D equal to

$$\begin{aligned} &xyz + bxy + cxy + axz + abx + acx + cxz + bcx+ \\ &+ayz + aby + acy + acz + byz + bcy + abz + bcz \end{aligned}$$

since the terms $a^2z, a^2b, a^2c, b^2y, ab^2, abc+abc, b^2c$ vanish. This long polynomial of six variables, called DD in the sequel, is what is found if one computes as usual the determinant of A after simplification.

The monomials are linked only by "+" signs, and so DD was called an **unisignant** by Sylvester about 1850, a few years before Kirchhoff discovered this determinant in his study of electrical networks. Both of them, independently, gave a wonderful graphical interpretation of DD: let us consider the graph associated to A and let us label the six edges by x, y, z, a, b, c respectively (which can be interpreted as probabilities, or resistances, or conductances, etc.). Then each triplet in DD can be seen as the name of a tree generating the graph, and DD supplies the list of the generating trees, without omission or repetition. This kind of result, famous in graph theory and electrical networks, is often called the **matrix-tree theorem** and generally ascribed to Kirchhoff alone, with matrices like A being called **Kirchhoff's matrix** when they are not called **laplacian** of the graph. The concept of Wang algebra is more recent, being initiated by Wang and other Chinese electricians between 1930 and 1940 (see [13]) and, despite Duffin's popularization, seems to be alive only among electricians.

Principal Cofactors The computations, via Wang algebra, are analogous but easier since we have only two terms in the diagonals. For example the cofactor C_{11} à la Wang is equal to $(y + a + c)(z + b + c)$; we expanded it already when computing DD and got

$$C_{11} = (y + a + c)(z + b + c) = yz + by + cy + az + ab + ac + cz + bc.$$

In the same way we have

$$C_{22} = (x + a + b)(z + b + c) = xz + bx + cx + az + ab + ac + bz + bc$$

and

$$C_{33} = (x + a + b)(y + a + c) = xy + ax + cx + ay + ac + by + ab + bc.$$

Remaining Cofactors Here we will not use any more the Wang algebra; nevertheless we have a nice result due to Bott and Duffin. For example, in order to compute C_{13}, one looks, in ordinary algebra, at the expanded forms of C_{11} and C_{33} and simply keeps of them their "intersection", that is the common monomials, $by+ab+ac+bc$.

So we have

$$C_{13} = C_{31} = C_{11} \cap C_{33} = by + ab + ac + bc.$$

In the same way we have

$$C_{12} = C_{21} = C_{11} \cap C_{22} = az + ab + ac + bc$$

and

$$C_{23} = C_{32} = C_{22} \cap C_{33} = cx + ab + ac + bc.$$

We leave to the reader the pleasure of writing the inverse of A from the determinant and the cofactors. Note that all of them are unisignant.

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Index of Notations

A

$A^{(\alpha)}$: Hadamard power of A 166
$\hat{a}^{\mathbb{E}}$: Envelop of a 136

C

CBF : Constant block form 63
CMP : Complete maximum principle 8

D

D_x : Diagonal matrix, $x \in \mathbb{R}^n$ 121
$d_{\mathscr{G}}(i, j)$: Geodesic distance 67

E

$\mathbb{E}$: Conditional expectation 120
$e(i)$: The i-th element of the canonical basis 5

F

$\mathbb{F}$: Incidence matrix 129
F, **G** : Filtration 126
$\mathfrak{F}$: Finest partition 58
$f(A)$: Hadamard function 166
f_{ij}^U : Probability that the chain ever visits j starting from i 40

G

γ : Path 66
GUM : Generalized ultra. matrix 64
Geod(i, j) : The geodesic in a tree 67
$\mathscr{G}$: Graph 66
$\mathscr{G}^W$: Incidence graph of W^{-1} 86

H

$H(T)$: Height of T 67

I

$\langle\, ,\, \rangle_\mu$: Inner product in $\mathbb{R}^n$ 121
$\mathbb{I}$: Identity matrix 5
$\mathbb{1}_B$: Indicator of B 5
$\mathbb{1}$: Vector of ones 5
Im(A) : Image of A 123
i^- : Immediate predecessor 67

L

$\bar{\lambda}$: Total mass of λ 7
LUM : Linear ultrametric matrix 188

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λ^U : Right equilibrium potential 7
$\mathscr{L}(T)$: The set of leaves of T 67

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$\mathbb{M}_n$: Mean value matrix 120
$\min\{U\}$: minimum value of U 60
μ^U : Left equilibrium potential 7

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NBF : Nested block form 63
$\mathfrak{N}$: Coarsest partition 58

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$\odot$: Hadamard product 121

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$bi\mathscr{P}$: Bi-potential 7

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