

# Appendix A

## Adding Places

In this paragraph we show that augmenting the set of places can only increase the finiteness length of an  $S$ -arithmetic subgroup of an almost simple group. Since the proof of the Rank Conjecture in [BKW13], the finiteness length of any such group is known, so one can verify the statement by just looking at the number there. Still it is interesting to observe that this fact is clear a priori for relatively elementary reasons. The proof works as in the special case considered in [Abr96].

**Theorem A.1.** *Let  $k$  be a global function field,  $\mathbf{G}$  a  $k$ -isotropic, connected, almost simple  $k$ -group, and  $S$  a non-empty, finite set of places of  $k$ . If  $\mathbf{G}(\mathcal{O}_S)$  is of type  $F_n$  and  $S' \supseteq S$  is a larger finite set of places then  $\mathbf{G}(\mathcal{O}_{S'})$  is also of type  $F_n$ .*

*Proof.* Proceeding by induction it suffices to prove the case where only one place is added to  $S$ , i.e.,  $S' = S \cup \{s\}$  for some place  $s$ . Also note that as far as finiteness properties are concerned, we may (and do) assume that  $\mathbf{G}$  is simply connected.

Let  $X_s$  be the Bruhat–Tits building that belongs to  $\mathbf{G}(k_s)$ . The group  $\mathbf{G}(\mathcal{O}_{S'}) \subseteq \mathbf{G}(k_s)$  acts continuously on  $X_s$ . We claim that this action is cocompact and that cell stabilizers are abstractly commensurable to  $\mathbf{G}(\mathcal{O}_S)$ . With these two statements the result follows from Theorem 1.22.

Note that the stabilizer of a cell is commensurable to the stabilizers of its faces and cofaces since the building is locally finite. Also all cells of same type are conjugate by the action of  $\mathbf{G}(k_s)$ . Hence it remains to see that some cell-stabilizer is commensurable to  $\mathbf{G}(\mathcal{O}_S)$ . To see this note that  $\mathbf{G}(\mathcal{O}_s)$  is a maximal compact subgroup of  $\mathbf{G}(k_s)$ . The Bruhat–Tits Fixed Point Theorem [BT72b, Lemme 3.2.3] (see also [BH99, Corollary II.2.8]) implies that it has a fixed point and by maximality the fixed point is a vertex and  $\mathbf{G}(\mathcal{O}_s)$  is its full stabilizer. Now  $\mathbf{G}(\mathcal{O}_S) = \mathbf{G}(\mathcal{O}_{S'}) \cap \mathbf{G}(\mathcal{O}_s)$  so  $\mathbf{G}(\mathcal{O}_S)$  is the stabilizer in  $\mathbf{G}(\mathcal{O}_{S'})$  of that vertex.

For cocompactness we use that  $\mathbf{G}(\mathcal{O}_{S'})$  is dense in  $\mathbf{G}(k_s)$ , see Lemma A.2 below. Let  $x$  be an interior point of some chamber of  $X_s$ . The orbit  $\mathbf{G}(k_s).x$  is a discrete space which, by strong transitivity, contains one point from every chamber of  $X_s$ . The orbit map  $\mathbf{G}(k_s) \rightarrow \mathbf{G}(k_s).x$  is continuous by continuity of the action, so the

image of the dense subgroup  $\mathbf{G}(\mathcal{O}_{S'})$  is dense in the discrete space  $\mathbf{G}(k_s).x$ . Hence  $\mathbf{G}(\mathcal{O}_{S'})$  acts transitively on chambers and, in particular, cocompactly.  $\square$

It remains to provide the density statement used in the proof. It is known and a consequence of the Strong Approximation Theorem:

**Lemma A.2.** *Let  $k$  be a global field and let  $\mathbf{G}$  be a  $k$ -isotropic, connected, simply connected, absolutely almost simple  $k$ -group. Let  $S$  be a non-empty finite set of places and let  $s \notin S$ . Then  $\mathbf{G}(\mathcal{O}_{S \cup \{s\}})$  is dense in  $\mathbf{G}(k_s)$ .*

*Proof.* For a place  $s$  of  $k$  let  $k_s$  denote the local field at  $s$  and  $\mathcal{O}_s$  the ring of integers in  $k_s$ . For a finite set  $S$  of places of  $k$  let  $\mathbb{A}_S = \prod_{s \in S} k_s \times \prod_{s \notin S} \mathcal{O}_s$  denote the ring of  $S$ -adeles. Recall that the ring of adeles is  $\mathbb{A} = \lim_S \mathbb{A}_S$  (see [Wei82]).

We know that  $\mathbf{G}_S := \prod_{s \in S} \mathbf{G}(k_s)$  is non-compact by Margulis [Mar91, Proposition 2.3.6].

Recall that  $k_s$  embeds into  $\mathbb{A}$  at  $s$ , and that  $k$  discretely embeds into  $\mathbb{A}$  diagonally. With these identifications  $\mathbf{G}(k) \cdot \mathbf{G}_S$  is dense in  $\mathbf{G}(\mathbb{A})$  by Prasad [Pra77, Theorem A], that is, if  $U$  is an open subset of  $\mathbf{G}(\mathbb{A})$  then  $\mathbf{G}(k) \cap U \mathbf{G}_S \neq \emptyset$ .

If  $V$  is an open subset of  $\mathbf{G}(k_s)$  then

$$U = V \times \prod_{s' \in S} \mathbf{G}(k_{s'}) \times \prod_{s' \notin S \cup \{s\}} \mathbf{G}(\mathcal{O}_{s'})$$

is open in  $\mathbf{G}(\mathbb{A})$ . Hence there is a  $g \in \mathbf{G}(k)$  with  $g \in V$  and  $g \in \mathbf{G}(\mathcal{O}_{S \cup \{s\}})$  (where we now consider  $\mathbf{G}(k)$  and  $\mathbf{G}(\mathcal{O}_{S \cup \{s\}})$  as subgroups of  $\mathbf{G}(k_s)$ ). Thus  $V \cap \mathbf{G}(\mathcal{O}_{S \cup \{s\}}) \neq \emptyset$  as desired.  $\square$

Theorem A.1 is the natural generalization to higher finiteness properties of Behr's Proposition 2 in [Beh98], the proof of which is not given but attributed to Kneser [Kne64].

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# Index of Symbols

We use various symbols that are specific to the present notes. Each of these symbols is listed below together with a reference to the page where it is introduced.

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**Addresses:**

Professor J.-M. Morel, CMLA,  
École Normale Supérieure de Cachan,  
61 Avenue du Président Wilson, 94235 Cachan Cedex, France  
E-mail: [morel@cmla.ens-cachan.fr](mailto:morel@cmla.ens-cachan.fr)

Professor B. Teissier, Institut Mathématique de Jussieu,  
UMR 7586 du CNRS, Équipe “Géométrie et Dynamique”,  
175 rue du Chevaleret  
75013 Paris, France  
E-mail: [teissier@math.jussieu.fr](mailto:teissier@math.jussieu.fr)

*For the “Mathematical Biosciences Subseries” of LNM:*

Professor P. K. Maini, Center for Mathematical Biology,  
Mathematical Institute, 24-29 St Giles,  
Oxford OX1 3LP, UK  
E-mail: [maini@maths.ox.ac.uk](mailto:maini@maths.ox.ac.uk)

Springer, Mathematics Editorial, Tiergartenstr. 17,  
69121 Heidelberg, Germany,  
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