Appendix A

Classical Valuation Theory

In the following, let K be a field with a non-Archimedean absolute value denoted by $|\cdot|: K \longrightarrow \mathbb{R}_{\geq 0}$; cf. 2.1/1. We will always assume that such an absolute value is *non-trivial*, i.e. that its values in $\mathbb{R}_{\geq 0}$ are not restricted to 0 and 1. Furthermore, let V be a K-vector space. A *vector space norm* on V (cf. 2.3/4) is a map $\|\cdot\|: V \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the following conditions for elements $x, y \in V$ and $\alpha \in K$:

(i) $\|x\| = 0 \Longleftrightarrow x = 0$,
(ii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$.
(iii) $\|\alpha x\| = |\alpha| \, \|x\|$,

When no confusion is possible, we will usually make no notational difference between the absolute value $|\cdot|$ on K and the vector space norm $\|\cdot\|$ on V, thus always writing $|x|$ instead of $\|x\|$ for elements $x \in V$. To give an example of a K-vector space norm, let V be a finite dimensional K-vector space and fix a basis $v_1, \ldots, v_d$ on it. Then we define the corresponding *maximum norm* $|\cdot|_{\max}$ on V as follows. Given an element $x \in V$, write it as a linear combination $x = \sum_{i=1}^{d} \alpha_i v_i$ with coefficients $\alpha_i \in K$ and set

$$|x|_{\max} = \max_{i=1\ldots d} |\alpha_i|.$$

One easily checks that $|\cdot|_{\max}$ defines a vector space norm on V. Furthermore, if K is complete under its absolute value, V is complete under such a maximum norm.

As usual, any vector space norm on a K-vector space V defines a topology on V. Two such norms $|\cdot|_1$ and $|\cdot|_2$ are called *equivalent* if they induce the same topology on V. The latter amounts to the fact that there exist constants $c, c' > 0$ such that $|x|_1 \leq c|x|_2 \leq c'|x|_1$ for all $x \in V$; use the fact that the absolute value on K is non-trivial. It is clear that any two maximum norms, attached to certain K-bases on

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a finite dimensional K-vector space V, are equivalent. If K is complete, a stronger assertion is possible.

Theorem 1. *Let V be a finite dimensional K-vector space and assume that K is complete. Then all K-vector space norms on V are equivalent. In particular, V is complete under such a norm.*

Proof. Choose a K-basis $v_1, \ldots, v_d$ of V and consider the attached maximum norm $|\cdot|_{\max}$ on V. Let $|\cdot|$ be a second K-vector space norm on V. Then there is a constant $c > 0$ such that $|x| \leq c|x|_{\max}$ for all $x \in V$. In fact, if $x = \sum_{i=1}^{d} \alpha_i v_i$, we have

$$|x| \leq \max_{i=1\ldots d} |\alpha_i||v_i| \leq \max_{i=1\ldots d} |\alpha_i| \max_{i=1\ldots d} |v_i| = c|x|_{\max}$$

for $c = \max_{i=1\ldots d} |v_i|$. Thus, it remains to show that there is a constant $c' > 0$ satisfying $|x|_{\max} \leq c'|x|$ for all $x \in V$.

We want to do this by induction on the dimension d of V. For $d = 0$ the assertion is trivial. Thus, let $d > 0$ and assume that a constant c' as desired does not exist. Then we can construct a sequence $x_n \in V$ such that

$$|x_n|_{\max} = 1 \text{ for all } n \qquad \text{and} \qquad \lim_{n\to\infty} |x_n| = 0.$$

Write $x_n = \sum_{i=1}^{d} \alpha_{ni} v_i$ with coefficients $\alpha_{ni} \in K$ and consider the elements α_{nd} for $i = d$ fixed as a sequence in K. If it is a zero sequence, look at the sequence $x'_n = x_n - \alpha_{nd} v_d$ in $V' = \sum_{i=1}^{d-1} K v_i$. Then, due to the non-Archimedean triangle inequality, $|x'_n|_{\max} = 1$ for almost all indices n and $\lim_{n\to\infty} |x'_n| = 0$. However, this is impossible by the induction hypothesis, since $|\cdot|_{\max}$ and $|\cdot|$ must be equivalent on the subspace $V' \subset V$, which is of dimension $d - 1$. Therefore α_{nd} cannot be a zero sequence. Replacing the x_n by a suitable subsequence, we may assume that there is some $\varepsilon > 0$ satisfying $|\alpha_{nd}| \geq \varepsilon$ for all n. Then

$$y_n = \alpha_{nd}^{-1} x_n = v_d + \sum_{i=1}^{d-1} \alpha_{nd}^{-1} \alpha_{ni} v_i$$

is still a zero sequence in V. Hence, we see that

$$v_d = -\lim_{n\to\infty} \sum_{i=1}^{d-1} \alpha_{nd}^{-1} \alpha_{ni} v_i .$$

In other words, v_d belongs to the closure of V' in V. However, by induction hypothesis, V' is complete and, hence, closed in V. As $v_d \notin V'$, we get a contradiction. $\square$

Corollary 2. *Let $|\cdot|_1$ and $|\cdot|_2$ be two absolute values on an algebraic field extension L/K restricting to the given absolute value $|\cdot|$ on K. Assume that K is complete with respect to $|\cdot|$. Then $|\cdot|_1$ and $|\cdot|_2$ coincide on L.*

Proof. Since L is a union of finite subextensions of L/K, we may assume that the extension L/K is finite. Then, viewing L as a normed K-vector space under $|\cdot|_1$ and $|\cdot|_2$, these norms are equivalent by Theorem 1. Thus, there are constants $c, c' > 0$ such that $|\alpha|_1 \le c|\alpha|_2 \le c'|\alpha|_1$ for all $\alpha \in L$. Replacing α by α^n for any integer $n > 0$ and using the multiplicativity of $|\cdot|_1$ and $|\cdot|_2$, we get

$$|\alpha|_1 \le c^{\frac{1}{n}}|\alpha|_2 \le c'^{\frac{1}{n}}|\alpha|_1$$

and therefore, by taking limits, $|\alpha|_1 = |\alpha|_2$ for all $\alpha \in L$. □

We have just seen that for any algebraic field extension L/K, there is at most one way to extend the given absolute value $|\cdot|$ from K to L, provided K is complete with respect to $|\cdot|$. We want to show now that such an extension will always exist.

Theorem 3. *Let L/K be an algebraic extension of fields where K is complete with respect to a given absolute value $|\cdot|$. Then there is a unique way to extend $|\cdot|$ to an absolute value $|\cdot|'$ of L. In fact,*

$$|\alpha|' = |N_{K(\alpha)/K}(\alpha)|^{\frac{1}{d}}$$

for elements $\alpha \in L$ where $N_{K(\alpha)/K}$ denotes the norm of $K(\alpha)$ over K and where d is the degree of α over K.

If L is finite over K, we see from Theorem 1 that L is complete with respect to the absolute value $|\cdot|'$.

Proof. As $N_{K(\alpha)/K}(\alpha) = \alpha$ for elements $\alpha \in K$, it is clear that $|\cdot|'$ extends $|\cdot|$. To show that $|\cdot|'$ defines a non-Archimedean absolute value on L, let us verify the conditions of 2.1/1. Clearly, $N_{K(\alpha)/K}(\alpha) = 0$ if and only if $\alpha = 0$ and therefore $|\alpha|' = 0$ if and only if $\alpha = 0$. Furthermore, if $\alpha \in L$ is contained in a finite subextension L' of L/K, say of degree n, then we conclude from the definition of norms that $|\alpha|' = |N_{L'/K}(\alpha)|^{\frac{1}{n}}$. Since the norm $N_{L'/K}$ is multiplicative, we see that $|\cdot|'$ is multiplicative as well.

Thus, it remains to show $|\alpha + \beta|' \le \max\{|\alpha|', |\beta|'\}$ for $\alpha, \beta \in L$. This estimate does not follow right away from properties of the norm, some more work is necessary. First note that for $|\alpha|' \le |\beta|'$ and $\beta \ne 0$, we can divide by β and thereby are reduced to showing $|1 + \alpha|' \le 1$ for $\alpha \in L$ satisfying $|\alpha|' \le 1$. Let $R = \{\alpha \in K ; |\alpha| \le 1\}$ be the valuation ring of K. With the aid of Hensel's Lemma, see Lemma 4 below, we will show in Lemma 5 that an element $\alpha \in L$ is integral over R if and only if $N_{K(\alpha)/K}(\alpha) \in R$, i.e. if and only if $|\alpha|' \le 1$. But then the non-Archimedean triangle inequality is easily derived. If $|\alpha|' \le 1$ for some $\alpha \in L$, then α is integral over R. Hence, the same is true for $1 + \alpha$ and we get $|1 + \alpha|' \le 1$. □

In order to state Hensel's Lemma, let $R = \{\alpha \in K\,;\,|\alpha| \le 1\}$ be the valuation ring of K and let $k = R/\{\alpha \in R\,;\,|\alpha| < 1\}$ be the attached residue field. The canonical projection $R \longrightarrow k$, which will be denoted by $\alpha \longmapsto \tilde{\alpha}$, induces for a variable (or a system of variables) X a projection

$$R[X] \longrightarrow k[X], \qquad f = \sum c_i X^i \longmapsto \tilde{f} = \sum \tilde{c}_i X^i,$$

on the level of polynomial rings.

Hensel's Lemma 4. *Let $f \in R[X]$ be a polynomial in one variable X such that there exists a factorization $\tilde{f} = \tilde{p} \cdot \tilde{q}$ with coprime factors $\tilde{p}, \tilde{q} \in k[X]$, i.e. where $\tilde{p}$ and $\tilde{q}$ are non-zero and their greatest common divisor in $k[X]$ is 1. Then $\tilde{p}, \tilde{q}$ can be lifted to polynomials $p, q \in R[X]$ satisfying*

$$f = p \cdot q, \qquad \deg q = \deg \tilde{q}.$$

Before giving the proof, let us derive the statement on integral dependence that was used in the proof of Theorem 3.

Lemma 5. *As in Theorem 3, let L/K be an algebraic extension of fields and let R be the valuation ring of K. Then, for elements $\alpha \in L$, the following are equivalent:*

(i) *α is integral over R.*
(ii) *$N_{K(\alpha)/K}(\alpha) \in R$.*

Proof. To begin with, assume condition (i), namely that α is integral over R. Then there is a monic polynomial $h \in R[X]$ satisfying $h(\alpha) = 0$. Let $f \in K[X]$ be the minimal polynomial of α over K. As f must divide h in $K[X]$, there is a decomposition of type $h = fg$ in $K[X]$. We claim that both, f and g belong to $R[X]$. To justify this, consider the Gauß norm on $K[X]$, which is given by

$$\left\| \sum_{i=0}^{n} a_i X^i \right\| = \max_{i=0\ldots n} |a_i|.$$

As in Sect. 2.2, one shows that the Gauß norm is multiplicative and this implies $1 = \|h\| = \|f\| \cdot \|g\|$. Since f is a monic polynomial, we have $\|f\| \ge 1$ and there is a constant $c \in K$ such that $|c| = \|f\|^{-1}$. Setting $f' = cf$ and $g' = c^{-1}g$, we get $h = f'g'$ with $\|f'\| = \|g'\| = 1$. In particular, $h = f'g'$ is a decomposition in $R[X]$, which can be transported into $k[X]$, thus implying the decomposition $\tilde{h} = \tilde{f}'\tilde{g}'$. As

$$\deg \tilde{f}' + \deg \tilde{g}' = \deg \tilde{h} = \deg h = \deg f + \deg g,$$

$$\deg \tilde{f}' \le \deg f, \qquad \deg \tilde{g}' \le \deg g,$$

we have necessarily $\deg \tilde{f}' = \deg f$ and $\deg \tilde{g}' = \deg g$. However, since f is monic, $\|f\| > 1$ would imply $\deg \tilde{f}' < \deg f$. Therefore we must have $|c| = 1$ and, hence, $f, g \in R[X]$. Thus, if $f = \sum_{i=0}^{n} c_i X^i \in R[X]$ is the minimal polynomial of α over K, we get $N_{K(\alpha)/K}(\alpha) = (-1)^n c_0 \in R$, which implies condition (ii).

Conversely, assume $N_{K(\alpha)/K}(\alpha) \in R$ as in condition (ii). As before, consider the minimal polynomial $f = \sum_{i=0}^{n} c_i X^i \in K[X]$ of α over K. We want to show that $f(\alpha) = 0$ is, in fact, an integral equation of α over R. Proceeding indirectly, assume that $f \notin R[X]$. Then we have $\|f\| > 1$ and we can choose a constant $c \in K$ such that $|c| = \|f\|^{-1} < 1$. Writing $f' = cf$, we get $\|f'\| = 1$. Since $c_0 = (-1)^n N_{K(\alpha)/K}(\alpha) \in R$ and $c_n = 1$, it follows $0 < \deg \tilde{f}' < \deg f$. Now look at the decomposition $\tilde{f}' = \tilde{p}\tilde{q}$ with $\tilde{p} = 1$ and $\tilde{q} = \tilde{f}'$. Due to Hensel's Lemma 4, we can lift $\tilde{p}$ and $\tilde{q}$ to polynomials $p, q \in R[X]$ such that $cf = f' = pq$ and $\deg q = \deg \tilde{q}$. Since $\deg \tilde{q}$ is strictly between 0 and $\deg f$, we see that $cf = pq$ is a non-trivial decomposition which, however, contradicts the fact that f is irreducible. Therefore we must have $f \in R[X]$, thus, implying condition (i). □

It remains to do the *proof of Hensel's Lemma*. Starting out from the decomposition $\tilde{f} = \tilde{p}\tilde{q}$, we choose a lifting $q_0 \in R[X]$ of $\tilde{q}$ satisfying $\deg q_0 = \deg \tilde{q}$. Then the highest coefficient of q_0 is a unit in R and, by Euclid's division, there is an equation $f = p_0 q_0 + r_1$ with suitable polynomials $p_0, r_1 \in R[X]$ where $\deg r_1 < \deg q_0$. From this we get $\tilde{f} = \tilde{p}_0 \tilde{q} + \tilde{r}_1$. Since we have

$$\deg \tilde{r}_1 \leq \deg r_1 < \deg q_0 = \deg \tilde{q}$$

and $\tilde{q}$ divides $\tilde{f}$, Euclid's division in $k[X]$ implies $\tilde{r}_1 = 0$. In particular, $\|r_1\| < 1$ and p_0 is a lifting of $\tilde{p}$. Let $m = \deg p_0$ and $n = \deg q_0$. It is now our strategy, to construct polynomials $a, b \in R[X]$ with

$$\|a\|, \|b\| \leq \|r_1\|, \qquad \deg a < m, \qquad \deg b < n,$$

such that

$$f = p_0 q_0 + r_1 = (p_0 + a)(q_0 + b),$$

or, equivalently

$$bp_0 + aq_0 + ab = r_1. \tag{$*$}$$

Then the decomposition $f = (p_0 + a)(q_0 + b)$ will be a lifting of $\tilde{f} = \tilde{p}\tilde{q}$, as required.

To do this, we neglect the quadratic term ab in the Eq. ($*$) for a moment. Let $K[X]_i$ for $i \in \mathbb{N}$ be the R-submodule of $K[X]$ consisting of all polynomials in $K[X]$ of degree $\leq i$. For the valuation ring R and its residue field k the notations $R[X]_i$ and $k[X]_i$ are used in a similar way. Then consider the R-linear map

$$\varphi: R[X]_{m-1} \oplus R[X]_{n-1} \longrightarrow R[X]_{m+n-1}, \qquad (a,b) \longmapsto bp_0 + aq_0,$$

as well as its versions $\varphi \otimes K$ over K and $\varphi \otimes k$ over k. We claim that all of these are isomorphisms. In fact, start with $\varphi \otimes k$. This map is injective, since $\tilde{b}\tilde{p} + \tilde{a}\tilde{q} = 0$ implies that $\tilde{q}$ divides $\tilde{b}$, due to the fact that $\tilde{p}$ and $\tilde{q}$ are coprime. However, since $\deg \tilde{b} < m = \deg \tilde{q}$, we get $\tilde{a} = \tilde{b} = 0$. But then, by reasons of dimensions, $\varphi \otimes k$ is surjective and, hence, bijective. From this we can conclude that φ and $\varphi \otimes K$ are isometric in the sense that

$$\bigl\|(\varphi \otimes K)(b,a)\bigr\| = \max\bigl\{\|a\|, \|b\|\bigr\}, \qquad a \in K[X]_{m-1}, \qquad b \in K[X]_{n-1}.$$

In particular, $\varphi \otimes K$ is injective, and the same dimension argument, as used before, shows that $\varphi \otimes K$ is bijective. Furthermore, relying on the fact that $\varphi \otimes K$ is isometric, we finally see that φ is bijective. Now, to lift the decomposition $\tilde{f} = \tilde{p}\tilde{q}$ as stated, let $\varepsilon = \|r\|$. We claim:

There are sequences $p_i \in R[X]_{m-1}$, $q_i \in R[X]_{n-1}$, and $r_{i+1} \in R[X]_{m+n-1}$, starting with the initial elements p_0, q_0, r_1 constructed above, such that

$$f = \Bigl(\sum_{i=1}^{j} p_i\Bigr)\Bigl(\sum_{i=1}^{j} q_i\Bigr) + r_{j+1}, \qquad j = 0, 1, \ldots,$$

where

$$\|p_j\|, \|q_j\| \le \varepsilon^j, \qquad \|r_{j+1}\| \le \varepsilon^{j+1}.$$

Then, as the field K is complete, $p = \sum_{i=1}^{\infty} p_i$ and $q = \sum_{i=1}^{\infty} q_i$ make sense as polynomials in $R[X]$ of degree m, respectively n, and by a limit argument, we get the desired decomposition $f = pq$.

To justify the claim, we proceed by induction on j. So assume that the polynomials p_i, q_i and r_{i+1} have already been constructed, up to some index $j \ge 0$. Then, writing $p' = \sum_{i=1}^{j} p_i$ and $q' = \sum_{i=1}^{j} q_i$ and applying the above properties of the R-linear map φ, now with p', q' in place of p_0, q_0, we can solve the equation

$$r_{j+1} = q_{j+1}p' + p_{j+1}q'$$

for some elements $p_{j+1} \in R[X]_{m-1}$ and $q_{j+1} \in R[X]_{n-1}$ satisfying

$$\|p_{j+1}\|, \|q_{j+1}\| \le \varepsilon^{j+1}.$$

But then we have

$$f = (p' + p_{j+1})(q' + q_{j+1}) + r_{j+2}$$

with $r_{j+2} = -p_{j+1}q_{j+1} \in R[X]_{m+n-1}$ where $\|r_{j+2}\| \le \varepsilon^{2(j+1)} \le \varepsilon^{j+2}$. Thus, our claim is justified, and Hensel's Lemma is proved. $\square$

The problem of extending a non-Archimedean absolute value $|\cdot|$ from a field K to an algebraic extension L has been settled in Theorem 3 for the case where K is complete. If K is not complete with respect to $|\cdot|$, we can pass to its completion $\hat{K}$, which can be constructed as follows. Consider the ring $K^{\mathbb{N}}$ of all infinite sequences in K, addition and multiplication being defined componentwise. The Cauchy sequences define a subring $C(K)$ of $K^{\mathbb{N}}$ and the zero sequences an ideal $Z(K) \subset C(K)$. It is easy to see that the quotient $\hat{K} = C(K)/Z(K)$ is a field and that the canonical map $K \longrightarrow \hat{K}$ sending an element α to the residue class of the constant sequence $\alpha, \alpha, \ldots$ is a homomorphism of fields. In particular, we can view K as a subfield of $\hat{K}$. We can even define an absolute value $|\cdot|'$ on $\hat{K}$ extending the one given on K. Indeed, given any $\alpha \in \hat{K}$, choose a representing Cauchy sequence (α_i) in K. Then the sequence $(|\alpha_i|)$ is a zero sequence or, due to the non-Archimedean triangle inequality, it becomes constant at a certain index i_0. Therefore the limit $c = \lim_{i\to\infty} |\alpha_i|$ exists and is well-defined, and we can set $|\alpha|' = c$. One can show that $\hat{K}$ is complete with respect to $|\cdot|'$ and that it contains K as a dense subfield.

Now if L/K is an algebraic field extension, we can consider the completion $\hat{K}$ of K and its algebraic closure $\hat{K}^{\mathrm{alg}}$. Extending the absolute value of K to $\hat{K}$, as just described, and prolonging it to $\hat{K}^{\mathrm{alg}}$ with the help of Theorem 3, we get a canonical non-Archimedean absolute value on $\hat{K}^{\mathrm{alg}}$, which may be denoted by $|\cdot|$ again. Then we can choose a K-morphism $\tau\colon L \longrightarrow \hat{K}^{\mathrm{alg}}$ and pull back the absolute value from $\hat{K}^{\mathrm{alg}}$ to L via τ. Thereby we obtain an absolute value on L extending the one given on K. However, the latter will not be unique in general, which corresponds to the fact that the K-morphism $\tau\colon L \longrightarrow \hat{K}^{\mathrm{alg}}$ may not be unique.

Taking the algebraic closure of a complete field, we may loose completeness. In particular, the field $\hat{K}^{\mathrm{alg}}$ may not be complete again. However, if we start with an algebraically closed field, its completion will remain algebraically closed. This way it is possible to construct extension fields that are algebraically closed and complete at the same time.

Krasner's Lemma 6. *Let K an algebraically closed field with a non-Archimedean absolute value $|\cdot|$. Then its completion $\hat{K}$ is algebraically closed.*

Proof. Consider an algebraic closure L of $\hat{K}$ and extend the absolute value of $\hat{K}$ to L, using the assertion of Theorem 3. Let $f = \sum_{i=0}^{n} c_i X^i$ be a monic polynomial of degree > 0 in $\hat{K}[X]$. Then f admits a zero $\alpha \in L$, and it is enough to show that α can be approximated by elements in K. To verify this, choose $\varepsilon > 0$ and approximate the coefficients c_i by elements $d_i \in K$ in such a way that the polynomial $g = \sum_{i=0}^{n} d_i X^i \in K[X]$ satisfies $|g(\alpha)| \leq \varepsilon^n$. Assuming $d_n = 1$, write $g = \prod_{i=1}^{n}(X - \beta_i)$ with zeros $\beta_i \in K$. Then $|g(\alpha)| = \prod_{i=1}^{n} |\alpha - \beta_i| \leq \varepsilon^n$ implies that there is an index i such that $|\alpha - \beta_i| \leq \varepsilon$. Consequently, α can be approximated by elements in K. □

The argument used in the proof is referred to as the principle of *continuity of roots*.

Appendix B

Completed Tensor Products

In the following we want to show that the category of affinoid K-algebras admits amalgamated sums where, as usual, K is a field endowed with a non-trivial complete non-Archimedean absolute value. Such amalgamated sums are constructed as completions of ordinary tensor products.

To handle completed tensor products, we need a slightly more general setting. Let R be a ring with a ring norm $|\cdot|$ on it, see 2.3/1, and M a *normed R-module*. Thereby we mean an R-module M together with a map $M \longrightarrow \mathbb{R}_{\geq 0}$, denoted by $|\cdot|$ again, such that for all $x, y \in M$ and $a \in R$ we have

(i) $|x| = 0 \Longleftrightarrow x = 0$,
(ii) $|x + y| \leq \max\{|x|, |y|\}$,
(iii) $|ax| \leq |a| \cdot |x|$.

The map $|\cdot|: M \longrightarrow \mathbb{R}_{\geq 0}$ is called a *semi-norm* on M if only conditions (ii) and (iii) are satisfied and (i) possibly not. Furthermore, an R-linear map $\varphi: M \longrightarrow N$ between normed R-modules is called *bounded* if there exists a real constant $\gamma > 0$ such that $|\varphi(x)| \leq \gamma |x|$ for all $x \in M$. In this case γ is referred to as a *bound* for φ.

Looking at topologies that are generated by module norms, we see immediately that bounded morphisms of normed R-modules are continuous. The converse is not always true. However, if there exists a subfield $K \subset R$ such that the norm on R restricts to a non-trivial absolute value on K, then every continuous morphism of normed R-modules is bounded. To justify this, assume that R contains a field K with the stated properties. Then, by restriction of scalars, any R-module M can be viewed as a K-vector space and, in fact, as a normed K-vector space in the sense of 2.3/4. Clearly we have $|ax| \leq |a| \cdot |x|$ for $a \in K$ and $x \in M$, but also

$$|a| \cdot |x| \leq |a| \cdot |a^{-1}ax| \leq |a| \cdot |a^{-1}| \cdot |ax| = |a| \cdot |a|^{-1} \cdot |ax| = |ax|$$

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for $a \neq 0$, which shows $|ax| = |a| \cdot |x|$ for all $a \in K$ and $x \in M$. Then, if $\varphi: M \longrightarrow N$ is a continuous morphism of normed R-modules, there exists a constant $\delta > 0$ such that $|\varphi(x)| \leq 1$ for all $x \in M$ satisfying $|x| \leq \delta$. Fixing an element $t \in K$ such that $0 < |t| < 1$, we choose an integer $n \in \mathbb{Z}$ such that $|t|^{n-1} \leq \delta$. Now, considering an arbitrary element $x \in M$, there exists an integer $r \in \mathbb{Z}$ satisfying $|t|^n \leq |t|^r |x| \leq |t|^{n-1}$. Then $|t^r x| \leq \delta$ and, hence, $|\varphi(t^r x)| \leq 1$, as well as $1 \leq |t|^{r-n}|x|$, and we get

$$\left|\varphi(x)\right| = \left|t\right|^{-r} \cdot \left|\varphi(t^r x)\right| \leq \left|t\right|^{-r} \leq \left|t\right|^{-r} \cdot \left|t\right|^{r-n} \cdot \left|x\right| = \left|t\right|^{-n} \cdot \left|x\right|,$$

which shows that $|t|^{-n}$ is a bound for φ. Thus, we have shown:

Lemma 1. (i) *Any bounded morphism of normed R-modules is continuous.*

(ii) *Conversely, assume that R contains a field K such that the norm on R restricts to a non-trivial absolute value on K. Then every continuous morphism of R-modules is bounded.*

Note that the assumption in (ii) is satisfied if R is a non-zero affinoid K-algebra, for K a field with a non-trivial complete non-Archimedean absolute value. Thus, in this case a morphism of normed R-modules is continuous if and only if it is bounded.

Now let us turn to tensor products and their related bilinear maps. Let M, N, E be normed modules over a normed ring R. An R-bilinear map $\Phi: M \times N \longrightarrow E$ is called *bounded* if there exists a real constant $\gamma > 0$ such that $|\Phi(x, y)| \leq \gamma \cdot |x| \cdot |y|$ for all $x \in M$ and $y \in N$. Again, γ is called a *bound* for Φ. An R-linear or R-bilinear map that is bounded by 1 is called *contractive*.

Proposition 2. *Let M, N be normed modules over a normed ring R. There exists a contractive R-bilinear map $\tau: M \times N \longrightarrow T$ into a complete normed R-module T such that the following universal property holds*:

Given any R-bilinear map $\Phi: M \times N \longrightarrow E$, bounded by some $\gamma > 0$, into a complete normed R-module E, there exists a unique R-linear map $\varphi: T \longrightarrow E$, bounded by γ as well, such that the diagram

$$\begin{tikzcd}
M \times N \arrow[r, "\tau"] \arrow[d, "\Phi"'] & T \arrow[dl, dashed, "\varphi"] \\
E &
\end{tikzcd}$$

is commutative.

Proof. To construct the map τ, we view the ordinary tensor product $M \otimes_R N$ as a semi-normed R-module using the semi-norm $|\cdot|: M \otimes_R N \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$|z| = \inf\Big(\max_{i=1,\dots,r} |x_i| \cdot |y_i|\Big), \qquad z \in M \otimes_R N,$$

where the infimum runs over all possible representations

$$z = \sum_{i=1}^{r} x_i \otimes y_i, \qquad x_i \in M, \qquad y_i \in N.$$

That we really get a semi-norm on $M \otimes_R N$ is easily verified. Thus, we can define $T = M \hat{\otimes}_R N$ as the separated completion of $M \otimes_R N$. It is an R-module again and, in fact, a complete normed R-module, since the semi-norm on $M \otimes_R N$ gives rise to an R-module norm on $M \hat{\otimes}_R N$. For elements $x \in M$ and $y \in N$, we write $x \hat{\otimes} y$ for the element in $M \hat{\otimes}_R N$ that is induced by the tensor $x \otimes y \in M \otimes_R N$. Then it is clear that the map

$$\tau\colon M \times N \longrightarrow M \hat{\otimes}_R N, \qquad (x, y) \longmapsto x \hat{\otimes} y,$$

is R-bilinear and contractive. The R-module $M \hat{\otimes}_R N$, together with its R-module norm, is called the *completed tensor product* of M and N over R.

Now let us show that the R-bilinear map τ satisfies the universal property of the assertion. So let $\Phi\colon M \times N \longrightarrow E$ be a bounded R-bilinear map into a complete normed R-module E and let $\gamma > 0$ be a bound for Φ. Using the universal property of ordinary tensor products in terms of the canonical R-bilinear map $\tau'\colon M \times N \longrightarrow M \otimes_R N$ sending a pair (x, y) to the tensor $x \otimes y$, there is a unique R-linear map $\varphi'\colon M \otimes_R N \longrightarrow E$ making the following diagram commutative:

$$\begin{tikzcd} M \times N \arrow[r, "\tau'"] \arrow[d, "\Phi"'] & M \otimes_R N \arrow[dl, dashed, "\varphi'"] \\ E & \end{tikzcd}$$

Then consider some element $z = \sum_{i=1}^{r} x_i \otimes y_i \in M \otimes_R N$ where $x_i \in M$ and $y_i \in N$. Since $\varphi'(z) = \sum_{i=1}^{r} \Phi(x_i, y_i)$, we get

$$\big|\varphi'(z)\big| \le \max_{i=1,\dots,r} \big|\Phi(x_i, y_i)\big| \le \gamma \max_{i=1,\dots,r} \big|x_i\big| \cdot \big|y_i\big|.$$

Taking the infimum over all representations of z as a sum of tensors $\sum_{i=1}^{r} x_i \otimes y_i$ yields $|\varphi'(z)| \le \gamma |z|$, and we see that φ' is bounded by γ.

Since E is complete, φ' gives rise to an R-linear map $\varphi\colon M \hat{\otimes}_R N \longrightarrow E$ that is bounded by γ as well. Furthermore, we can enlarge the above diagram to obtain the following commutative diagram:

$$\begin{tikzcd}
\tau : M \times N \arrow[r, "\tau'"] \arrow[d, "\Phi"'] & M \otimes_R N \arrow[r, "\mathrm{can}"] \arrow[dl, "\varphi'"'] & M \mathbin{\widehat{\otimes}}_R N \arrow[dll, "\varphi"] \\
E & &
\end{tikzcd}$$

It remains to show that φ is uniquely determined by the relation $\Phi = \varphi \circ \tau$. However, this is clear since φ is unique on the image $\tau(M \times N)$, which generates a dense R-submodule in $M \mathbin{\widehat{\otimes}}_R N$. □

In the situation of Proposition 2, the normed R-module T together with the contractive R-bilinear map $\tau : M \times N \longrightarrow T$ is uniquely determined up to isometric isomorphism and will be denoted by $M \mathbin{\widehat{\otimes}}_R N$. It is called the *completed tensor product* of M and N over R. For the attached contractive R-bilinear map $\tau : M \times N \longrightarrow M \mathbin{\widehat{\otimes}}_R N$ we will use the notation $(x, y) \longmapsto x \mathbin{\widehat{\otimes}} y$. In other words, we set $x \mathbin{\widehat{\otimes}} y = \tau(x, y)$ for $(x, y) \in M \times N$. Note that, independent of the construction in the proof of Proposition 2, there is a canonical R-linear map $M \otimes_R N \longrightarrow M \mathbin{\widehat{\otimes}}_R N$, namely the one given by $x \otimes y \longmapsto x \mathbin{\widehat{\otimes}} y$. It has a dense image in $M \mathbin{\widehat{\otimes}}_R N$, since the closure of this image, just as $M \mathbin{\widehat{\otimes}}_R N$, satisfies the universal property of completed tensor products.

As in the case of ordinary tensor products, the universal property defining completed tensor products can be used to derive various standard facts. To list some of them, look at normed R-modules M, N, P. Then there are canonical isometric isomorphisms

$$R \mathbin{\widehat{\otimes}}_R M \simeq M,$$

$$M \mathbin{\widehat{\otimes}}_R N \simeq N \mathbin{\widehat{\otimes}}_R M,$$

$$\big(M \mathbin{\widehat{\otimes}}_R N\big) \mathbin{\widehat{\otimes}}_R P \simeq M \mathbin{\widehat{\otimes}}_R \big(N \mathbin{\widehat{\otimes}}_R P\big),$$

$$\big(M \oplus N\big) \mathbin{\widehat{\otimes}}_R P \simeq \big(M \mathbin{\widehat{\otimes}}_R P\big) \oplus \big(N \mathbin{\widehat{\otimes}}_R P\big),$$

where the norm on a direct sum like $M \oplus N$ is given by $|x \oplus y| = \max\big(|x|, |y|\big)$.

Furthermore, the completed tensor product of two bounded morphisms of normed R-modules can be constructed. Indeed, let $\varphi_i : M_i \longrightarrow N_i$ for $i = 1, 2$ be morphisms of normed R-modules that are bounded by constants $\gamma_i > 0$. Then the R-bilinear map

$$M_1 \times M_2 \longrightarrow N_1 \mathbin{\widehat{\otimes}}_R N_2, \qquad (x_1, x_2) \longmapsto \varphi_1(x_1) \mathbin{\widehat{\otimes}} \varphi_2(x_2),$$

is bounded by $\gamma_1\gamma_2$ and, thus, gives rise to an R-linear map

$$\varphi_1 \mathbin{\widehat{\otimes}} \varphi_2 : M_1 \mathbin{\widehat{\otimes}}_R M_2 \longrightarrow N_1 \mathbin{\widehat{\otimes}}_R N_2, \qquad x_1 \mathbin{\widehat{\otimes}} x_2 \longmapsto \varphi_1(x_1) \mathbin{\widehat{\otimes}} \varphi_2(x_2),$$

that is bounded by $\gamma_1\gamma_2$ as well. The map $\varphi_1 \mathbin{\hat{\otimes}} \varphi_2$ is referred to as the *completed tensor product* of φ_1 and φ_2.

Also note that the associativity isomorphism above admits the following generalization:

Proposition 3. *Let $S \longrightarrow R$ be a contractive homomorphism between normed rings and let M be a normed S-module, as well as N and P normed R-modules. Then there is a canonical isometric isomorphism of normed S-modules*

$$(M \mathbin{\hat{\otimes}}_S N) \mathbin{\hat{\otimes}}_R P \simeq M \mathbin{\hat{\otimes}}_S (N \mathbin{\hat{\otimes}}_R P)$$

where $M \mathbin{\hat{\otimes}}_S N$ is a normed R-module via the R-module structure of N.

The *proof* is straightforward, see [BGR], 2.1.7/7.

Next let us discuss completed tensor products on the level of *normed algebras*. To do this, fix a normed ring R and consider two normed R-algebras A_1, A_2; by the latter we mean normed rings A_i that are equipped with a contractive ring homomorphism $R \longrightarrow A_i$. In particular, we may view the A_i as normed R-modules, which implies that the completed tensor product $A_1 \mathbin{\hat{\otimes}}_R A_2$ exists as a complete normed R-module. We want to show that $A_1 \mathbin{\hat{\otimes}}_R A_2$ is, in fact, a normed R-algebra, based on the R-algebra structure of the ordinary tensor product $A_1 \otimes_R A_2$. Using the semi-norm on $A_1 \otimes_R A_2$ as defined in the proof of Proposition 2, we see that the canonical ring homomorphism $R \longrightarrow A_1 \otimes_R A_2$ is contractive. Furthermore, for two elements

$$z = \sum_{i=1}^{m} x_i \otimes y_i, \qquad z' = \sum_{j=1}^{n} x'_j \otimes y'_j \qquad \in A_1 \otimes_R A_2,$$

we get

$$\begin{aligned} |z \cdot z'| = \Big|\sum_{i=1}^{m}\sum_{j=1}^{n} x_i x'_j \otimes y_i y'_j\Big| &\le \max_{i,j} |x_i x'_j| \cdot |y_i y'_j| \\ &\le \max_{i=1,\ldots,m} |x_i| \cdot |y_i| \cdot \max_{j=1,\ldots,n} |x'_j| \cdot |y'_j|, \end{aligned}$$

which yields

$$|zz'| \le |z| \cdot |z'|.$$

when taking the infimum over all representations of z and z' as sums of tensors. Thus, passing from $A_1 \otimes_R A_2$ to its completion, it follows that, indeed, the completed tensor product $A_1 \mathbin{\hat{\otimes}}_R A_2$ is a normed R-algebra where the multiplication is characterized by

$$(x \mathbin{\widehat{\otimes}} y) \cdot (x' \mathbin{\widehat{\otimes}} y') = xx' \mathbin{\widehat{\otimes}} yy'$$

and the structural morphism $R \longrightarrow A_1 \mathbin{\widehat{\otimes}}_R A_2$ by $a \longmapsto a \mathbin{\widehat{\otimes}} 1 = 1 \mathbin{\widehat{\otimes}} a$.

We want to characterize $A_1 \mathbin{\widehat{\otimes}}_R A_2$ in terms of a universal property for normed R-algebras.

Proposition 4. *Let R be a normed ring and A_1, A_2 normed R-algebras. Then the contractive R-algebra homomorphisms*

$$\sigma_1 \colon A_1 \longrightarrow A_1 \mathbin{\widehat{\otimes}}_R A_2, \qquad a_1 \longmapsto a_1 \mathbin{\widehat{\otimes}} 1,$$
$$\sigma_2 \colon A_2 \longrightarrow A_1 \mathbin{\widehat{\otimes}}_R A_2, \qquad a_2 \longmapsto 1 \mathbin{\widehat{\otimes}} a_2,$$

admit the following universal property of amalgamated sums:

Let $\varphi_1 \colon A_1 \longrightarrow D$ and $\varphi_2 \colon A_2 \longrightarrow D$ be two homomorphisms of normed R-algebras that are bounded by constants $\gamma_1 > 0$ and $\gamma_2 > 0$ and assume that D is complete. Then there is a unique R-algebra homomorphism $\varphi \colon A_1 \mathbin{\widehat{\otimes}}_R A_2 \longrightarrow D$, bounded by $\gamma_1 \gamma_2$, such that the diagram

$$\begin{tikzcd}
A_1 \arrow[d, "\sigma_1"'] \arrow[dr, "\varphi_1"] & \\
A_1 \mathbin{\widehat{\otimes}}_R A_2 \arrow[r, dashed, "\varphi"] & D \\
A_2 \arrow[u, "\sigma_2"] \arrow[ur, "\varphi_2"] &
\end{tikzcd}$$

is commutative.

Proof. Consider homomorphisms of normed R-algebras $\varphi_1 \colon A_1 \longrightarrow D$ as well as $\varphi_2 \colon A_2 \longrightarrow D$ where D is complete and assume that φ_1 and φ_2 are bounded by constants $\gamma_1 > 0$ and $\gamma_2 > 0$. Then

$$A_1 \times A_2 \longrightarrow D, \qquad (a_1, a_2) \longmapsto \varphi_1(a_1) \cdot \varphi_2(a_2),$$

is an R-bilinear map that is bounded by $\gamma_1 \gamma_2$. Thus, by the universal property of completed tensor products in Proposition 2, it gives rise to an R-linear map

$$\varphi \colon A_1 \mathbin{\widehat{\otimes}}_R A_2 \longrightarrow D, \qquad a_1 \mathbin{\widehat{\otimes}} a_2 \longmapsto \varphi_1(a_1) \cdot \varphi_2(a_2),$$

that is bounded by $\gamma_1 \gamma_2$. Furthermore, φ satisfies

$$\begin{aligned}\varphi\big((a_1 \mathbin{\hat{\otimes}} a_2)\cdot(a_1' \mathbin{\hat{\otimes}} a_2')\big) &= \varphi(a_1a_1' \mathbin{\hat{\otimes}} a_2a_2') = \varphi_1(a_1a_1')\cdot\varphi_2(a_2a_2')\\ &= \varphi_1(a_1)\cdot\varphi_2(a_2)\cdot\varphi_1(a_1')\cdot\varphi_2(a_2')\\ &= \varphi(a_1 \mathbin{\hat{\otimes}} a_2)\cdot\varphi(a_1' \mathbin{\hat{\otimes}} a_2')\end{aligned}$$

for $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$. This shows that φ is multiplicative on the image of $A_1 \otimes_R A_2$ in $A_1 \mathbin{\hat{\otimes}}_R A_2$ and, hence, by continuity, on $A_1 \mathbin{\hat{\otimes}}_R A_2$ itself. Since

$$\varphi(a_1 \mathbin{\hat{\otimes}} a_2) = \varphi\big((a_1 \mathbin{\hat{\otimes}} 1)\cdot(1 \mathbin{\hat{\otimes}} a_2)\big) = \varphi_1(a_1)\cdot\varphi_2(a_2)$$

for $a_1 \in A_1$ and $a_2 \in A_2$, it is clear by a continuity argument as before that φ is unique on $A_1 \mathbin{\hat{\otimes}}_R A_2$. □

If $\psi_i\colon A_i \longrightarrow B_i$, $i = 1, 2$, are bounded morphisms of normed R-algebras, their completed tensor product

$$\psi_1 \mathbin{\hat{\otimes}} \psi_2\colon A_1 \mathbin{\hat{\otimes}}_R A_2 \longrightarrow B_1 \mathbin{\hat{\otimes}}_R B_2, \qquad a_1 \mathbin{\hat{\otimes}} a_2 \longmapsto \psi_1(a_1) \mathbin{\hat{\otimes}} \psi_2(a_2),$$

is defined as a bounded R-linear map, but can also be obtained within the context of normed R-algebras using the universal property of Proposition 4; both versions coincide.

Next we want to study the behavior of restricted power series under completed tensor products. To do this, let A be a complete normed ring and $\zeta = (\zeta_1, \ldots, \zeta_n)$ a set of variables. Then, as usual, the A-algebra of *restricted power series* in ζ with coefficients in A is given by

$$A\langle\zeta\rangle = \Big\{\sum_{\nu\in\mathbb{N}^n} a_\nu\zeta^n \in A[\![\zeta]\!]\ ;\ a_\nu \in A,\ \lim_{\nu\in\mathbb{N}^n} a_\nu = 0\Big\}.$$

It is a complete normed A-algebra under the *Gauß norm*

$$\Big|\sum_{\nu\in\mathbb{N}^n} a_\nu\zeta^n\Big| = \max_{\nu\in\mathbb{N}^n}|a_\nu|.$$

Proposition 5. *Let R be a complete normed ring, A a complete normed R-algebra, and $\zeta = (\zeta_1, \ldots, \zeta_n)$ a set of variables. Then, using the Gauß norm on $R\langle\zeta\rangle$ and $A\langle\zeta\rangle$, there is a canonical isometric isomorphism of normed R-algebras*

$$A \mathbin{\hat{\otimes}}_R R\langle\zeta_1, \ldots, \zeta_n\rangle \xrightarrow{\ \sim\ } A\langle\zeta_1, \ldots, \zeta_n\rangle.$$

Proof. We want to show that the canonical maps

$$\sigma_1\colon A \longrightarrow A\langle\zeta\rangle, \qquad \sigma_2\colon R\langle\zeta\rangle \longrightarrow A\langle\zeta\rangle,$$

which are contractive, satisfy the universal property mentioned in Proposition 4. To do this, consider two morphisms of R-algebras

$$\varphi_1\colon A \longrightarrow D, \qquad \varphi_2\colon R\langle\zeta\rangle \longrightarrow D$$

into a complete normed R-algebra D such that φ_1 and φ_2 are bounded by constants $\gamma_1, \gamma_2 > 0$. Then there is a well-defined R-algebra homomorphism

$$\varphi\colon A\langle\zeta\rangle \longrightarrow D, \qquad \sum_{\nu\in\mathbb{N}^n} a_\nu\zeta^\nu \longmapsto \sum_{\nu\in\mathbb{N}^n} \varphi_1(a_\nu)\cdot\varphi_2(\zeta^\nu).$$

Indeed, if the a_ν form a zero sequence in A, their images form a zero sequence in D since $|\varphi_1(a_\nu)| \le \gamma_1|a_\nu|$. Furthermore, we have $|\varphi_2(\zeta^\nu)| \le \gamma_2$ for all ν so that the infinite sums of type $\sum_\nu \varphi_1(a_\nu)\cdot\varphi_2(\zeta^\nu)$ are converging. Hence, φ is well-defined, and it is bounded by $\gamma_1\gamma_2$, as shown by the estimate

$$\Big|\sum_{\nu\in\mathbb{N}^n} \varphi_1(a_\nu)\cdot\varphi_2(\zeta^\nu)\Big| \le \gamma_1\gamma_2\cdot\max_\nu\big|a_\nu\big| = \gamma_1\gamma_2\cdot\Big|\sum_{\nu\in\mathbb{N}^n} a_\nu\zeta^\nu\Big|.$$

By continuity, φ is even a homomorphism of R-algebras and, in fact, the unique bounded homomorphism making the diagram

$$\begin{tikzcd}
A \arrow[d, "\sigma_1"'] \arrow[dr, "\varphi_1"] & \\
A\langle\zeta\rangle \arrow[r, dashed, "\varphi"] & D \\
R\langle\zeta\rangle \arrow[u, "\sigma_2"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

commutative. Thus, we are done. □

For the remainder of this section, we want to look at affinoid K-algebras where, as usual, K is a field with a complete non-Archimedean absolute value that is non-trivial. Any such algebra A may be viewed as a complete normed K-algebra by choosing a residue norm on it. Furthermore, we know from 3.1/20 that any two residue norms $|\cdot|$ and $|\cdot|'$ on A are equivalent in the sense that they induce the same topology on A. In particular, the identity map $(A, |\cdot|) \longrightarrow (A, |\cdot|')$ and its inverse are bounded due to Lemma 1.

Now let $\tau_1\colon R \longrightarrow A_1$ and $\tau_2\colon R \longrightarrow A_2$ be two homomorphisms of affinoid K-algebras. In order to construct the completed tensor product $A_1 \hat{\otimes}_R A_2$, we need to specify appropriate norms on R, A_1, and A_2 in such a way that τ_1 and τ_2 are

contractive. We do this in terms of residue norms. In fact, choosing epimorphisms $\alpha: T_m \longrightarrow R$ and $\alpha_i: T_{n_i} \longrightarrow A_i$, $i = 1, 2$, we can use 3.1/19 in conjunction with 3.1/7 and 3.1/9 to construct commutative diagrams

$$\begin{array}{ccc} T_m & \hookrightarrow & T_{m+n_1} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_1'} \\ R & \xrightarrow{\tau_1} & A_1 \end{array} \qquad \begin{array}{ccc} T_m & \hookrightarrow & T_{m+n_2} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2'} \\ R & \xrightarrow{\tau_2} & A_2 \end{array}$$

where α_1' and α_2' are extensions of α_1 and α_2 and, hence, are surjective. Considering the residue norms associated to α, α_1', and α_2' on R and the A_i, it is clear that the maps τ_1 and τ_2 are contractive and, hence, that the completed tensor product $A_1 \hat{\otimes}_R A_2$ can be constructed. If we consider a second set of residue norms on R, A_1, and A_2 such that τ_1 and τ_2 are contractive, then the resulting semi-norms on $A_1 \otimes_R A_2$ that are used to construct the completed tensor product, are seen to be equivalent. As a result, the attached completions can canonically be identified and it follows that, indeed, the completed tensor product $A_1 \hat{\otimes}_R A_2$ is well-defined, up to a set of equivalent ring norms on it, just as is the case for affinoid K-algebras and their possible residue norms on them. We will keep this in mind and talk about "the" completed tensor product of A_1 and A_2 over R. However, when it comes to particular norms on $A_1 \hat{\otimes}_R A_2$, we have to be more specific.

Our main objective for the remainder of this section is to show:

Theorem 6. *Let $\tau_1: R \longrightarrow A_1$ and $\tau_2: R \longrightarrow A_2$ be homomorphisms of affinoid K-algebras. Then the completed tensor product $A_1 \hat{\otimes}_R A_2$ is an affinoid K-algebra as well. In other words, the category of affinoid K-algebras admits amalgamated sums.*

To prepare the proof of the theorem, we start with some consequences of Proposition 5.

Proposition 7. *Let $\xi_1, \ldots, \xi_m$ and $\zeta_1, \ldots, \zeta_n$ be sets of variables, and K' an extension field of K with a complete absolute value extending the one given on K. Then there are canonical isometric isomorphisms*

$$K\langle \xi_1, \ldots, \xi_m \rangle \hat{\otimes}_K K\langle \zeta_1, \ldots, \zeta_n \rangle \xrightarrow{\sim} K\langle \xi_1, \ldots, \xi_m, \zeta_1, \ldots, \zeta_n \rangle,$$
$$K' \hat{\otimes}_K K\langle \zeta_1, \ldots, \zeta_n \rangle \xrightarrow{\sim} K'\langle \zeta_1, \ldots, \zeta_n \rangle,$$

with respect to the Gauß norm on the occurring Tate algebras.

Proposition 8. *Let A_1 and A_2 be affinoid K-algebras. Then $A_1 \hat{\otimes}_K A_2$ is an affinoid K-algebra as well. Similarly, if K' is an extension field of K with a complete absolute value extending the one given on K, then $K' \hat{\otimes}_K A_i$ is an affinoid K'-algebra.*

More specifically, choose epimorphisms of K-algebras $\alpha_i : T_{n_i} \longrightarrow A_i$ for $i = 1, 2$, and consider the attached residue norms on A_1 and A_2. Then the canonical morphism of K-algebras

$$\alpha : T_{n_1+n_2} = T_{n_1} \hat{\otimes}_K T_{n_2} \longrightarrow A_1 \hat{\otimes}_K A_2$$

is surjective and its kernel is generated by $\ker \alpha_1$ *and* $\ker \alpha_2$, *thus giving rise to an isomorphism of K-algebras*

$$T_{n_1+n_2}\big/\big(\ker \alpha_1, \ker \alpha_2\big) \xrightarrow{\sim} A_1 \hat{\otimes}_K A_2.$$

The latter is an isometric isomorphism if we consider on $T_{n_1+n_2}/(\ker \alpha_1, \ker \alpha_2)$ *its canonical residue norm. Likewise, the homomorphisms of K-algebras*

$$\alpha'_i : K'\langle \zeta_1, \ldots, \zeta_{n_i} \rangle = K' \hat{\otimes}_K T_{n_i} \longrightarrow K' \hat{\otimes}_K A_i, \qquad i = 1, 2,$$

are surjective, and their kernels are generated by $\ker \alpha_i$, *thus giving rise to isometric isomorphisms*

$$\big(K' \hat{\otimes}_K T_{n_i}\big)\big/\big(\ker \alpha_i\big) \xrightarrow{\sim} K' \hat{\otimes}_K A_i, \qquad i = 1, 2.$$

Proof. We show that $T_{n_1+n_2}/(\ker \alpha_1, \ker \alpha_2)$ and, likewise, $K'\langle \zeta_1, \ldots, \zeta_{n_i} \rangle/(\ker \alpha_i)$ satisfy the universal property of completed tensor products. To do this, consider a commutative diagram of type

$$\begin{tikzcd}
T_{n_1} \arrow[r, "\alpha_1"] \arrow[d] & A_1 \arrow[d, "\sigma_1"] \arrow[dr, "\varphi_1"] & \\
T_{n_1+n_2} \arrow[r, "\tilde{\alpha}"] & T_{n_1+n_2}/(\ker \alpha_1, \ker \alpha_2) \arrow[r, dashed, "\varphi"] & D \\
T_{n_2} \arrow[r, "\alpha_2"] \arrow[u] & A_2 \arrow[u, "\sigma_2"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

where σ_i is induced by the inclusion $T_{n_i} \hookrightarrow T_{n_1+n_2}$, $i = 1, 2$, and where $\tilde{\alpha}$ is the canonical projection. Concerning the right part of the diagram, D is a complete normed K-algebra and the $\varphi_i : A_i \longrightarrow D$, $i = 1, 2$, are homomorphisms that are bounded by constants $\gamma_1, \gamma_2 > 0$. Using Proposition 7 and interpreting $T_{n_1+n_2}$ as the completed tensor product $T_{n_1} \hat{\otimes}_K T_{n_2}$, there exists a canonical homomorphism of K-algebras $T_{n_1+n_2} \longrightarrow D$ that is bounded by $\gamma_1\gamma_2$ and that, apparently, will factor through the quotient $T_{n_1+n_2}/(\ker \alpha_1, \ker \alpha_2)$ via a unique homomorphism of K-algebras

$$\varphi\colon T_{n_1+n_2}\big/\big(\ker\alpha_1, \ker\alpha_2\big) \longrightarrow D$$

making the above diagram commutative. Let us equip now the affinoid K-algebra $T_{n_1+n_2}/(\ker\alpha_1, \ker\alpha_2)$ with its residue norm via $\tilde{\alpha}$. Then, by the definition of residue norms, we see that the maps σ_1 and σ_2 are contractive since the canonical inclusions of T_{n_i} into $T_{n_1+n_2}$ preserve Gauß norms. Furthermore, by the definition of residue norms again, φ is bounded by $\gamma_1\gamma_2$ since the same is true for the composition $\varphi\circ\tilde{\alpha}$; one may also use the fact that for every $\overline{f} \in T_{n_1+n_2}/(\ker\alpha_1, \ker\alpha_2)$ there is an inverse image $f \in T_{n_1+n_2}$ satisfying $|f| = |\overline{f}|$, cf. 3.1/5. Altogether we conclude that $T_{n_1+n_2}/(\ker\alpha_1, \ker\alpha_2)$ along with the contractions σ_1, σ_2 satisfy the universal property of a completed tensor product $A_1 \hat{\otimes}_K A_2$. Thus, we are done with the first part of the assertion. The completed tensor products of type $K' \hat{\otimes}_K A_i$ are dealt with similarly. □

Proposition 9. *Let $\sigma\colon S \longrightarrow R$ as well as $\tau_1\colon R \longrightarrow A_1$ and $\tau_2\colon R \longrightarrow A_2$ be homomorphisms of affinoid K-algebras. Then there is a canonical homomorphism of normed K-algebras $A_1 \hat{\otimes}_S A_2 \longrightarrow A_1 \hat{\otimes}_R A_2$, and the latter is an epimorphism.*

More specifically, consider residue norms on R, S, A_1, and A_2, and assume that σ and the τ_i are contractive. Then the norm on $A_1 \hat{\otimes}_R A_2$ coincides with the residue norm derived from the norm on $A_1 \hat{\otimes}_S A_2$.

Proof. We proceed similarly as in the proof of Proposition 8 and consider a commutative diagram of type

$$\begin{tikzcd}
A_1 \arrow[r, equal] \arrow[d, "\sigma_1'"'] & A_1 \arrow[d, "\sigma_1"'] \arrow[dr, "\varphi_1"] & \\
A_1 \hat{\otimes}_S A_2 \arrow[r, "\alpha"] & A_1 \hat{\otimes}_R A_2 \arrow[r, dashed, "\varphi"] & D \\
A_2 \arrow[r, equal] \arrow[u, "\sigma_2'"] & A_2 \arrow[u, "\sigma_2"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

where D is a complete normed R-algebra and the $\varphi_i\colon A_i \longrightarrow D$, $i = 1, 2$, are homomorphisms of R-algebras that are bounded by constants $\gamma_1, \gamma_2 > 0$. Furthermore, φ is the unique homomorphism of R-algebras, bounded by $\gamma_1\gamma_2$, that is derived from the universal property of $A_1 \hat{\otimes}_R A_2$. It follows that $\varphi\circ\alpha$ is the unique homomorphism of S-algebras derived from the universal property of $A_1 \hat{\otimes}_S A_2$; it is bounded by $\gamma_1\gamma_2$ as well. Now consider the factorization

$$\alpha\colon A_1 \hat{\otimes}_S A_2 \longrightarrow \big(A_1 \hat{\otimes}_S A_2\big)\big/\ker\alpha \hookrightarrow A_1 \hat{\otimes}_R A_2$$

where $\ker\alpha$ is a closed ideal in $A_1 \hat{\otimes}_S A_2$ since α is contractive and, hence, continuous. Thus, proceeding in the manner of 3.1/5 (i) and (ii), we can equip

the quotient $(A_1 \hat{\otimes}_S A_2)/\ker\alpha$ with the residue norm derived from the norm on $A_1 \hat{\otimes}_S A_2$. Clearly, the homomorphisms σ_1 and σ_2 factor through contractive homomorphisms of R-algebras $\tilde{\sigma}_i: A_i \longrightarrow (A_1 \hat{\otimes}_S A_2)/\ker\alpha$, $i = 1, 2$, and it is easily seen that $(A_1 \hat{\otimes}_S A_2)/\ker\alpha$ along with $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ satisfy the universal property of the completed tensor product $A_1 \hat{\otimes}_R A_2$. Thus, we are done. □

Now the *Proof of Theorem* 6 can be carried out without problems. We assume that $\tau_1: R \longrightarrow A_1$ and $\tau_2: R \longrightarrow A_2$ are contractive homomorphisms of affinoid K-algebras, the latter being equipped with suitable residue norms. Then the completed tensor product $A_1 \hat{\otimes}_K A_2$ is an affinoid K-algebra by Proposition 8 and so is the completed tensor product $A_1 \hat{\otimes}_R A_2$, since it is a quotient of $A_1 \hat{\otimes}_K A_2$ by Proposition 9.

Finally, we want to mention the following generalization of the first part of Proposition 8:

Proposition 10. *Let $\tau_1: R \longrightarrow A_1$ and $\tau_2: R \longrightarrow A_2$ be homomorphisms of affinoid K-algebras, and consider ideals $\mathfrak{a}_1 \subset A_1$ as well as $\mathfrak{a}_2 \subset A_2$. Furthermore, fix residue norms on R, A_1, and A_2 such that τ_1 and τ_2 are contractive, and provide the quotients $A_1/\mathfrak{a}_1$ and $A_2/\mathfrak{a}_2$ with the residue norms derived from the given residue norms on A_1 and A_2 via the canonical projections $\alpha_i: A_i \longrightarrow A_i/\mathfrak{a}_i$. Then*

$$\alpha_1 \hat{\otimes} \alpha_2 \colon A_1 \hat{\otimes}_R A_2 \longrightarrow (A_1/\mathfrak{a}_1) \hat{\otimes}_R (A_2/\mathfrak{a}_2)$$

is surjective and its kernel is generated by the images of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in $A_1 \hat{\otimes}_R A_2$. This way $\alpha_1 \hat{\otimes} \alpha_2$ gives rise to an isomorphism of R-algebras

$$\big(A_1 \hat{\otimes}_R A_2\big)/\big(\mathfrak{a}_1, \mathfrak{a}_2\big) \xrightarrow{\sim} \big(A_1/\mathfrak{a}_1\big) \hat{\otimes}_R \big(A_2/\mathfrak{a}_2\big),$$

which is isometric if we consider on $(A_1 \hat{\otimes}_R A_2)/(\mathfrak{a}_1, \mathfrak{a}_2)$ the residue norm derived from the completed tensor product norm on $A_1 \hat{\otimes}_R A_2$.

Proof. Use the same arguments as in the proof of Proposition 8. □

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Addresses:

Professor J.-M. Morel, CMLA,
École Normale Supérieure de Cachan,
61 Avenue du Président Wilson, 94235 Cachan Cedex, France
E-mail: morel@cmla.ens-cachan.fr

Professor B. Teissier, Institut Mathématique de Jussieu,
UMR 7586 du CNRS, Équipe "Géométrie et Dynamique",
175 rue du Chevaleret
75013 Paris, France
E-mail: teissier@math.jussieu.fr

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