Appendix

A.1 Simple Linear Algebra

For the vector $x \in \mathbf{R}^n$ we use $|x|$ to denote the Euclidean norm. For two vectors $x, y \in \mathbf{R}^n$ we denote their inner product by

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i \ .$$

For an $n \times n$-matrix A we use $|A|$ to denote the operator norm, i.e.

$$|A| = \sup\{|Ax| : \ x \in \mathbf{R}^n \ \text{ and } |x| \leq 1\} \ .$$

It is easy to see that other norms are equivalent.

By I we denote the unit $n \times n$-matrix with 1 on the diagonal and 0 elsewhere. We use the notation $\operatorname{adj} A$ to denote the adjoint (or adjugate) matrix of matrix A. It contains the $(n-1) \times (n-1)$-subdeterminants of the matrix A and it satisfies the formula

$$A \operatorname{adj} A = I \det A, \tag{A.1}$$

where $\det A$ denotes the determinant of A.

A.2 Covering Theorems

We use covering theorems to select subcollections that consists of balls B_j that are disjoint or that have bounded overlap.

S. Hencl and P. Koskela, *Lectures on Mappings of Finite Distortion*, Lecture Notes in Mathematics 2096, DOI 10.1007/978-3-319-03173-6,

Theorem A.1 (Vitali). *Let $\mathscr{B}$ be a collection of closed balls in $\mathbf{R}^n$ such that*

$$\sup\{\operatorname{diam} B : B \in \mathscr{B}\} < \infty.$$

Then there are $B_1, B_2, \ldots$ (possibly a finite sequence) from this collection such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and

$$\bigcup_{B \in \mathscr{B}} B \subset \bigcup_j 5B_j.$$

For a proof we refer the reader to [95]. Let us anyhow briefly explain the idea in a simple case. Suppose that the family $\mathscr{B}$ consists of balls $B(x, r_x)$, where $x \in A$ and A is bounded. Let $M = \sup_{x \in A} r_x$. Choose a ball $B_1 = B(x, r_x)$ so that $r_x > 3M/4$. Continue by considering points in $A \setminus 3B_1$, and repeating the first step (now letting $M_1 = \sup_{y \in A \setminus 3B_1} r_y$) and after that continue by induction.

In the Euclidean setting, a subcollection often can be chosen so that we only have uniformly bounded overlap for the cover.

Theorem A.2 (Besicovitch). *Let $\mathscr{B}$ be a collection of closed balls in $\mathbf{R}^n$ such that the set A consisting of the centers is bounded. Then there is a countable (possibly finite) subcollection $B_1, B_2, \ldots$ such that*

$$\chi_A(x) \leq \sum_j \chi_{B_j}(x) \leq C(n)$$

for all x.

In more general settings, say, in the Heisenberg group, Besicovitch fails. The reason it holds in the Euclidean setting, is basically the following geometric fact:

Suppose that we are given $B(x_1, r_1)$ and $B(x_2, r_2)$ so that $0 \in B(x_1, r_1) \cap B(x_2, r_2)$, $x_1 \notin B(x_2, r_2)$ and $x_2 \notin B(x_1, r_1)$. Then the angle between the vectors x_1 and x_2 is at least $60°$.

For a proof of the Besicovitch covering theorem, we again refer to [95].

A.3 L^p-Spaces

Recall that $L^p(\Omega)$, $1 \leq p < \infty$, consists of (equivalence classes) of measurable functions u with

$$\int_\Omega |u|^p < \infty.$$

We write

$$\|u\|_{L^p} = \|u\|_p := \left(\int_\Omega |u|^p \right)^{1/p}.$$

Furthermore, $L^\infty(\Omega)$ consists of those measurable functions on Ω that are essentially bounded. Then $\|u\|_{L^\infty} = \|u\|_\infty$ is the essential supremum of $|u|$ over Ω. If $1 < p < \infty$, we set $p' = p/(p-1)$, and we define $1' = \infty$. With this notation, we have the Minkowski

$$\|u + v\|_p \le \|u\|_p + \|v\|_p$$

and Hölder

$$\|uv\|_1 \le \|u\|_p \|v\|_{p'}$$

inequalities.

One often needs the following spherical coordinates. Given a Borel function $u \in L^1(B(0,1))$ we have that

$$\int_{B(0,1)} u = \int_{S^{n-1}(0,1)} \int_0^1 u(tw)t^{n-1}\, dtdw.$$

Here we use the notation $\int_{S^{n-1}(c,t)} f(x)\, dx$ to denote integration with respect to the surface measure, which is a constant multiple of the Hausdorff measure $\mathscr{H}^{n-1}$.

We say that a sequence $\{u_i\}_i$ converges to u in $L^p(\Omega)$ if all these functions belong to $L^p(\Omega)$ and if $\|u - u_i\|_p \to 0$ when $i \to \infty$. We then write $u_i \to u$ in $L^p(\Omega)$. If $u_i \to u$ in $L^p(\Omega)$, then there is a subsequence $\{u_{i_k}\}_k$ of $\{u_i\}_i$ which converges to u pointwise almost everywhere. For $1 \le p < \infty$, continuous functions are dense in $L^p(\Omega)$: given $u \in L^p(\Omega)$ one can find continuous u_i with $u_i \to u$ both in $L^p(\Omega)$ and almost everywhere. This can be easily seen by first approximating u by simple functions, then approximating the associated measurable sets by compact sets and finally approximating the characteristic functions of the compact sets by continuous functions.

The dual of $L^p(\Omega)$ is $L^{p/(p-1)}(\Omega)$ when $1 < p < \infty$. Then

$$\|u\|_p = \sup_{\|\varphi\|_{\frac{p}{p-1}}=1} \|u\varphi\|_1.$$

One of the inequalities easily follows by Hölder's inequality and the other by choosing φ to be a suitable constant multiple of $|u|^{p-1}$.

We also need the following weak compactness property: if $\{u_j\}_j$ is a bounded sequence in $L^p(\Omega)$, $1 < p < \infty$, then there is a subsequence $\{u_{j_k}\}_k$ and a function $u \in L^p(\Omega)$ so that

$$\lim_{k\to\infty} \int_\Omega u_{j_k}\varphi = \int_\Omega u\varphi$$

for each $\varphi \in L^{p/(p-1)}(\Omega)$. We then write

$$u_{j_k} \rightharpoonup u.$$

This notation should in principle include the exponent p, but the exponent in question is typically only indicated when its value is not obvious. This function u, called the weak limit, is unique and satisfies

$$\|u\|_p \leq \liminf_{k\to\infty} \|u_{j_k}\|_p.$$

The existence of the weak limit u follows from the fact that $L^p(\Omega)$, $1 < p < \infty$, is reflexive. Furthermore, the norm estimate on u is a consequence of a general result according to which a norm is lower semicontinuous with respect to the associated weak convergence. In general, weak convergence is defined by considering bounded linear mappings $T : X \to \mathbf{R}$; in the case of $L^p(\Omega)$, $1 < p < \infty$, they can be identified with elements of $L^{p/(p-1)}(\Omega)$. If $v_j = (v_1^j, \cdots, v_n^j) \in L^p(\Omega)$, then

$$v_j \rightharpoonup u$$

means that

$$v_i^j \rightharpoonup u_i$$

for each $1 \leq i \leq n$.

When we apply the above to a sequence $A_j(x)$ of $n \times n$-matrix functions, we conclude that the boundedness in $L^p(\Omega)$, $1 < p < \infty$ of the sequence $\{|A_j(x)|\}_j$ guarantees the existence of an $n \times n$-matrix function $A(x) \in L^p(\Omega)$ so that the rows (or columns) of a subsequence of $\{|A_j(x)|\}_j$ converge weakly to the corresponding rows (or columns) of $A(x)$. Notice that boundedness above is independent of the initial norm (like the operator or Hilbert-Schmidt one). Then $\|A\|_p \leq C_n \liminf_{k\to\infty} \|A_{j_k}\|_p$. In fact, one can show that

$$\|A\|_p \leq \liminf_{k\to\infty} \|A_{j_k}\|_p;$$

the L^p-norms generated by the operator or Hilbert-Schmidt norms are equivalent and so the associated concepts of weak convergence coincide.

We need the following sufficient condition for weak compactness in L^1.

Lemma A.3. *Let $\{g_j\}_{j\in\mathbf{N}}$ be a sequence of measurable functions on a domain $\Omega \subset \mathbf{R}^n$ of finite measure. Suppose that there is $H \in L^1(\Omega)$ such that for almost every $y \in \Omega$ and for every $j \in \mathbf{N}$ we have $|g_j(y)| \leq H(y)$. Then there is a subsequence $\{\tilde{g}_j\}_{j\in\mathbf{N}}$ of $\{g_j\}_{j\in\mathbf{N}}$ and $g \in L^1(\Omega)$ such that the subsequence $\{\tilde{g}_j\}_{j\in\mathbf{N}}$ converges weakly to g in $L^1(\Omega)$.*

Proof. We may assume that $H > 0$ everywhere on Ω. Define

$$h_j = \frac{g_j}{H}.$$

Since $0 \le h_j \le 1$, the sequence $\{h_j\}_{j\in\mathbf{N}}$ is bounded in $L^2(\Omega)$. Hence, after passing to a suitable subsequence, we may assume that $h_j \rightharpoonup h \in L^2(\Omega)$ in $L^2(\Omega)$. Thus

$$\int_\Omega h_j \eta \, dx \to \int_\Omega h\eta \, dx \tag{A.2}$$

for each $\eta \in L^2(\Omega)$. It is easy to show that $0 \le h \le 1$ a.e. and hence $h \in L^\infty$. Given $k \in \mathbf{N}$, set

$$H_k = \min\{H, k\}.$$

Let $\varphi \in L^\infty(\Omega)$. Then $H_k\varphi \in L^\infty(\Omega) \subset L^2(\Omega)$. By the triangle inequality

$$\begin{aligned}\Big|\int_\Omega (g_j\varphi - hH\varphi)\, dx\Big| &= \Big|\int_\Omega (h_j H\varphi - hH\varphi)\, dx\Big| \\ &\le \Big|\int_\Omega (h_j H_k\varphi - hH_k\varphi)\, dx\Big| \\ &\quad + \Big|\int_\Omega (h_j\varphi - h\varphi)(H - H_k)\, dx\Big|.\end{aligned}$$

With the help of (A.2) we now obtain

$$\begin{aligned}\limsup_{j\to\infty}\Big|\int_\Omega (g_j\varphi - hH\varphi)\, dx\Big| &\le \limsup_{j\to\infty}\Big|\int_\Omega (h_j\varphi - h\varphi)(H - H_k)\, dx\Big| \\ &\le C\int_\Omega |H - H_k|\, dx.\end{aligned}$$

Since $H \in L^1(\Omega)$, we conclude that

$$\lim_{j\to\infty}\int_\Omega g_j\varphi \, dx = \int_\Omega hH\varphi.$$

Since $hH \in L^1(\Omega)$, the claim follows. □

On several occasions we will need Jensen's inequality.

Theorem A.4 (Jensen's Inequality). *Let $G \subset \mathbf{R}^n$ be a measurable set with $0 < |G| < \infty$ and let $\Phi : [0,\infty) \to [0,\infty)$ be a convex function. Then for every nonnegative measurable function $h : G \to [0,\infty)$ we have*

$$\Phi\Big(\fint_G h(x)\,dx\Big) \le \fint_G \Phi(h(x))\,dx\,.$$

A.4 Maximal Operator

Let $u \in L^1_{\text{loc}}(\mathbf{R}^n)$. The non-centered maximal function of u is

$$\mathrm{M}u(x) = \sup_{x\in B(y,r)} \fint_{B(y,r)} |u|.$$

Recall that for a measurable set A with $0 < |A| < \infty$,

$$\fint_A v = v_A = \frac{1}{|A|}\int_A v\,.$$

Remark A.5. (1) According to the Lebesgue differentiation theorem

$$\mathrm{M}u(x) \ge |u(x)|$$

almost everywhere.

(2) There are many other maximal functions. For example the restricted, centered maximal function

$$\mathrm{M}^C_\delta u(x) = \sup_{0<r<\delta} \fint_{B(x,r)} |u|.$$

(3) We always have $\mathrm{M}^C_\infty u(x) \le \mathrm{M}u(x) \le 2^n \mathrm{M}^C_\infty u(x)$.

(4) Notice that $\{\mathrm{M}u > t\}$ is open for each $t \ge 0$ and, consequently, $\mathrm{M}u$ is measurable. Indeed, if $x \in \{\mathrm{M}u > t\}$, then it immediately follows from the definition that $B(y,r) \subset \{\mathrm{M}u > t\}$, for some $B(y,r)$ containing x.

Theorem A.6. *If $u \in L^1(\mathbf{R}^n)$ and $t > 0$, then*

$$|\{Mu > t\}| \le \frac{5^n}{t}\int_{\{Mu>t\}} |u| \le \frac{5^n}{t}\|u\|_1. \tag{A.3}$$

Proof. We may assume that $M := \int_{\{\mathrm{M}u>t\}} |u| < \infty$. For each $x \in \{\mathrm{M}u > t\}$ there is a ball B such that $x \in B$ and

$$\fint_B |u| > t$$

and hence

$$|B| < t^{-1} \int_B |u|\,.$$

If $y \in B$, then $\mathrm{M}u(y) > t$ and thus $B \subset \{\mathrm{M}u > t\}$. So

$$|B| < \frac{1}{t} \int_B |u| \leq \frac{1}{t} \int_{\{\mathrm{M}u>t\}\cap B} |u|.$$

By the Vitali covering theorem, Theorem A.1, we find pairwise disjoint balls $B_1, B_2, \ldots$ as above so that $\{\mathrm{M}u > t\} \subset \bigcup 5B_j$. Then

$$|\{\mathrm{M}u > t\}| \leq \sum |5B_j| = 5^n \sum |B_j| \leq \frac{5^n}{t} \sum \int_{B_j} |u| \leq \frac{5^n}{t} \int_{\{\mathrm{M}u>t\}} |u|.$$

$\square$

The following lemma implies (by sending $\lambda \to 0$) that the maximal operator is bounded from L^p to L^p for $p > 1$.

Lemma A.7. *Let $p > 1$, $\lambda > 0$ and $v \in L^p$. Then*

$$\int_{\{Mv>\lambda\}} (Mv)^p \leq C(n,p) \int_{\{|v|>\frac{\lambda}{2}\}} |v|^p.$$

Proof. Using the Fubini theorem and the estimate (A.3) we see that

$$\begin{aligned}
\int_{\{Mv>\lambda\}} (Mv)^p &= \int_{\{Mv>\lambda\}} \int_0^{Mv(x)} p t^{p-1}\, dt\, dx \\
&= p \int_\lambda^\infty t^{p-1} |\{Mv > t\}|\, dt + \lambda^p |\{Mv > \lambda\}| \\
&\leq C \int_\lambda^\infty t^{p-2} \int_{\{|v|>\frac{t}{2}\}} |v(x)|\, dx\, dt + C\lambda^{p-1} \int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|\, dx \\
&\leq C \int_\lambda^\infty \int_{\{|v|>\frac{t}{2}\}} |v(x)|^{p-1}\, dx\, dt + C \int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^p\, dx \\
&= C \int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^{p-1} \int_\lambda^{2|v(x)|} dt\, dx + C \int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^p\, dx \\
&\leq C \int_{\{|v|>\frac{\lambda}{2}\}} |v(x)|^p\, dx.
\end{aligned}$$

$\square$

Remark A.8. Suppose that $u \in L^p(\Omega)$, $p > 1$. Applying the previous lemma to the zero extension of u we conclude that $\int_\Omega (Mu)^p \leq C(p,n) \int_\Omega |u|^p$. Similarly, the inequality (A.3) can be restricted to Ω when $u \in L^1(\Omega)$.

The case $p = 1$ was not left out by accident from the previous lemma.

Example A.9. If $u(x) = \chi_{B(0,1)}(x)$, then $\mathrm{M}u$ behaves like $\frac{C}{|x|^n}$ close to ∞ and hence $\mathrm{M}u \notin L^1(\mathbf{R}^n)$. In fact, $Mu \notin L^1(\mathbf{R}^n)$ unless u is the zero function.

We continue with a powerful tool from harmonic analysis, the Calderón-Zygmund decomposition, and some consequences of this decomposition.

The dyadic decomposition of a cube Q_0 consists of open cubes $Q \subset Q_0$ with faces parallel to the faces of Q_0 and of edge length $l(Q) = 2^{-i}l(Q_0)$, where $i = 1, 2, \dots$ refer to the generation in the construction. The cubes in each generation cover Q_0 up to a set of measure zero and the closures of the cubes in a fixed generation cover Q_0; there are 2^{in} cubes of edge length $2^{-i}l(Q_0)$ in the ith generation and the cubes corresponding to the same generation are pairwise disjoint. For almost every $x \in Q_0$, there is a (unique) decreasing sequence $Q_0 \supset Q_1 \supset \dots$ of cubes in the dyadic decomposition so that $\{x\} = \bigcap Q_i$. In what follows, Q, Q_0, Q_x etc. are cubes.

Theorem A.10 (Calderón-Zygmund Decomposition). *Let $Q_0 \subset \mathbf{R}^n$, $u \in L^1(Q_0)$, and suppose that*

$$0 \le \fint_{Q_0} u \le t.$$

Then there is a subcollection $\{Q_j\}$ from the dyadic decomposition of Q_0 so that $Q_i \cap Q_j = \emptyset$ when $i \neq j$,

$$t < \fint_{Q_j} u \le 2^n t$$

for each j, and $u(x) \le t$ for almost every $x \in Q_0 \setminus \bigcup Q_j$.

Proof. For almost every $x \in Q_0$ there is a decreasing sequence $\{Q_j\}$ of dyadic cubes so that $\{x\} = \bigcap Q_j$. By the Lebesgue differentiation theorem

$$\lim_{j \to \infty} \fint_{Q_j} u = u(x)$$

for almost every such x. Let $u(x) > t$ and assume that the above holds for x with the sequence $\{Q_j\}$. Then there must be maximal $Q_x := Q_{j(x)}$ so that

$$\fint_{Q_x} u > t.$$

For this cube we have

$$t < \fint_{Q_x} u \le 2^n \fint_{Q_{j(x)-1}} u \le 2^n t.$$

We can pick such a cube Q_x for almost every x with $u(x) > t$. It is then easy to choose the desired subcollection from the cubes Q_x. □

The dyadic maximal function of a measurable function u (with respect to a cube Q_0) is defined by

$$M_{Q_0}u(x) = \sup_{x \in \overline{Q} \subset Q_0} \fint_Q |u|,$$

where the supremum is taken over all cubes Q that belong to the dyadic decomposition of Q_0 and whose closures contain x.

Remark A.11. As for the usual maximal function, we have the weak-type estimate

$$|\{x \in Q_0 : M_{Q_0}u(x) > t\}| \le \frac{2 \cdot 5^n}{t} \int_{\{x \in Q_0 : |u(x)| > \frac{t}{2}\}} |u|$$

for the dyadic maximal function. Moreover,

$$\int_{Q_0} (M_{Q_0}u)^p \le C(p,n) \int_{Q_0} |u|^p$$

for $p > 1$. The proof of the weak type estimate is actually easier than for the usual maximal operator because no covering theorem is needed.

The following simple consequence of the Calderón-Zygmund decomposition is essentially the converse of the weak type estimate for the dyadic maximal function.

Lemma A.12. *Let $u \in L^1(Q_0)$ and suppose $t \ge \fint_{Q_0} |u|$. Then*

$$\int_{\{x \in Q_0 : |u(x)| > t\}} |u| \le 2^n t |\{x \in Q_0 : M_{Q_0}u(x) > t\}|.$$

Proof. By the Calderón-Zygmund decomposition we find pairwise disjoint cubes $Q_1, Q_2, \ldots$ so that

$$t < \fint_{Q_j} |u| \le 2^n t$$

for all j, and $|u(x)| \le t$ almost everywhere in $Q_0 \setminus \bigcup Q_j$. Then

$$\begin{aligned}\int_{\{x \in Q_0 : |u(x)| > t\}} |u| &\le \sum \int_{Q_j} |u| \\ &\le \sum 2^n t |Q_j| \\ &\le 2^n t |\{x \in Q_0 : M_{Q_0}u(x) > t\}|,\end{aligned}$$

because

$$M_{Q_0}u(x) \geq \fint_{Q_j} |u| > t$$

for each $x \in Q_j$. $\square$

A.5 Sobolev Spaces

Definition A.13. Let $\Omega \subset \mathbf{R}^n$ be open and $u \in L^1_{\mathrm{loc}}(\Omega)$. A function $v \in L^1_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ is called a weak derivative of u if

$$\int_\Omega \varphi(x)v(x)dx = -\int_\Omega u(x)\nabla\varphi(x)dx$$

for every $\varphi \in C^\infty_C(\Omega)$. We refer to v by Du. For $1 \leq p \leq \infty$ we define the Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : Du \in L^p(\Omega, \mathbf{R}^n)\}$$

and we define the norm

$$\|u\|_{W^{1,p}(\Omega)} = \Big(\int_\Omega |u|^p + \int_\Omega |Du|^p\Big)^{\frac{1}{p}}.$$

Further $W^{1,p}(\Omega, \mathbf{R}^n)$ refers to mappings $f : \Omega \to \mathbf{R}^n$ whose each component function f_j, $j = 1, \cdots, n$, belongs to $W^{1,p}(\Omega)$. The definitions of $W^{1,p}_{\mathrm{loc}}(\Omega)$ and $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ should then be obvious.

A function $\hat{u}$ is called a representative of $u \in W^{1,p}(\Omega)$ if $u = \hat{u}$ almost everywhere.

Definition A.14. Let $\Omega \subset \mathbf{R}^n$ be open and let $1 \leq p \leq \infty$. We define the Sobolev space of functions with zero boundary value $W^{1,p}_0(\Omega)$ as the collection of all functions $u \in W^{1,p}(\Omega)$ for which there are $u_j \in C^\infty_C(\Omega)$ such that $u_j \to u$ in $W^{1,p}(\Omega)$.

Theorem A.15 (Definitions of Sobolev Spaces). *Let $u \in L^p(\Omega)$, $1 \leq p < \infty$, $\Omega \subset \mathbf{R}^n$. Then the following are equivalent:*

(1) **(ACL)** *The function u has a representative $\tilde{u}$ that is absolutely continuous on almost all line segments in Ω parallel to the coordinate axes and whose (classical) partial derivatives belong to $L^p(\Omega)$.*

(2) **(H)** *There is a sequence $\{\varphi_j\}_j \subset C^\infty(\Omega)$ so that $\varphi_j \to u$ in $L^p(\Omega)$ and $\{\nabla\varphi_j\}_j$ is Cauchy in $L^p(\Omega)$.*

(3) **(W)** *The function u belongs to* $W^{1,p}(\Omega)$.

Proof (sketch).

(2) ⇒ (1): Passing to a subsequence, we may assume that $\{\varphi_j(x)\}_j$ converges for almost every x. We define

$$\tilde{u}(x) = \lim_{j\to\infty} \varphi_j(x)$$

whenever the limit exists, and set, say, $\tilde{u}(x) = 0$ for the remaining $x \in \Omega$. Then $\tilde{u}(x) = u(x)$ almost everywhere in Ω. Since $\nabla\varphi_j$ is Cauchy in L^p it is easy to see that $\nabla\varphi_j$ converge to Du in L^p.
We fix $[a_1, b_1] \times \ldots \times [a_n, b_n] \subset \Omega$. By the Fubini theorem we obtain that for $\mathscr{L}_{n-1}$-a.e. $[x_2, \ldots, x_n] \in [a_2, b_2] \times \ldots \times [a_n, b_n]$ we have

$$\int_I |Du - \nabla\varphi_j|^p \underset{j\to\infty}{\longrightarrow} 0$$

where $I = [a_1, a_2] \times [x_2, \ldots, x_n]$ and moreover, we may assume that $\varphi_j(x) \to u(x)$ for $\mathscr{H}^1$-a.e. $x \in I$. Let us fix disjoint intervals $[c_i, d_i] \subset [a_1, a_2]$ such that $\varphi_j(c_i, x_2, \ldots, x_n) \to u(c_i, x_2, \ldots, x_n)$ and $\varphi_j(d_i, x_2, \ldots, x_n) \to u(d_i, x_2, \ldots, x_n)$. By the fundamental theorem of calculus applied to the functions φ_j we obtain

$$\left|\varphi_j(c_i, x_2, \ldots, x_n) - \varphi_j(d_i, x_2, \ldots, x_n)\right| \le \int_{I_i} |\nabla\varphi_j|$$

where $I_i = [c_i, d_i] \times [x_2, \ldots, x_n]$. We let $j \to \infty$ and then sum over i to obtain

$$\sum_i \left|u(c_i, x_2, \ldots, x_n) - u(d_i, x_2, \ldots, x_n)\right| \le \sum_i \int_{I_i} |Du| \,.$$

As $Du \in L^1$ this implies that u is absolutely continuous on I by the absolute continuity of the integral. More precisely, u has a representative which is absolutely continuous on I.

(1) ⇒ (3): Integration by parts is valid in one dimension for absolutely continuous functions. Hence we can integrate by parts over line segments and then use the Fubini theorem. The weak derivatives v_j are the classical partial derivatives of absolutely continuous functions.

(3) ⇒ (2): We use the (smooth) convolution approximation: Let

$$\psi_1(x) = \begin{cases} 0, & |x| \ge 1 \\ C\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1\,, \end{cases}$$

where C is chosen so that $\int_{\mathbf{R}^n} \psi_1\, dx = 1$. Define

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^n}\, \psi_1\left(\frac{x}{\varepsilon}\right). \tag{A.4}$$

If $v \in L^p_{\text{loc}}$, set

$$v^\varepsilon(x) = (\psi_\varepsilon * v)(x) = \int_\Omega \psi_\varepsilon(x-y)v(y)\, dy\,,$$

when $B(x,\varepsilon) \subset\subset \Omega$. If $v \in L^p(\mathbf{R}^n)$, then $v^\varepsilon \to v$ in $L^p(\mathbf{R}^n)$. Indeed, this is easy to see for a continuous function v and for a general function we find a sequence of continuous functions that converge to it in L^p. Also $v^\varepsilon(x) \to v(x)$ when x is a Lebesgue point of u.

Fix $x \in \Omega$ and $\varepsilon > 0$ small compared to $\operatorname{dist}(x, \partial\Omega)$. Now

$$\begin{aligned}\frac{u^\varepsilon(x+he_i) - u^\varepsilon(x)}{h} &= \frac{1}{\varepsilon^n}\int_\Omega \underbrace{\frac{1}{h}\left[\psi_1\left(\frac{x+he_i-y}{\varepsilon}\right) - \psi_1\left(\frac{x-y}{\varepsilon}\right)\right]}_{\displaystyle\underset{h\to 0}{\longrightarrow}\ \frac{1}{\varepsilon}\,\frac{\partial\psi_1}{\partial x_i}\left(\frac{x-y}{\varepsilon}\right) = \varepsilon^n\,\frac{\partial\psi_\varepsilon}{\partial x_i}(x-y)} u(y)\, dy \\ &\underset{h\to 0}{\longrightarrow} \int_\Omega \frac{\partial\psi_\varepsilon}{\partial x_i}(x-y)\, u(y)\, dy\end{aligned}$$

by the dominated convergence theorem:

$$\int_\Omega \left|\frac{1}{h}\Big[\psi_1\Big(\frac{x+he_i-y}{\varepsilon}\Big) - \psi_1\Big(\frac{x-y}{\varepsilon}\Big)\Big]u(y)\right|\, dy \le \frac{1}{\varepsilon}\int_\Omega \|\nabla\psi_1\|_{L^\infty}|u(y)|\, dy\,.$$

Thus there exists a derivative of u^ε and

$$\frac{\partial u^\varepsilon}{\partial x_i}(x) = \int_\Omega \frac{\partial\psi_\varepsilon}{\partial x_i}(x-y)\, u(y)\, dy$$

and because ψ_ε is smooth, we see that u^ε is C^1. By repeating this argument one can check that u_ε is actually C^∞. Moreover, when $u \in W^{1,p}$,

$$\begin{aligned}\frac{\partial u^\varepsilon}{\partial x_i}(x) &= \int \frac{\partial\psi_\varepsilon(x-y)}{\partial x_i} u(y)\, dy \\ &= -\int \frac{\partial\psi_\varepsilon(x-y)}{\partial y_i} u(y)\, dy \\ &= \int \psi_\varepsilon(x-y)\, v_i(y)\, dy\,.\end{aligned}$$

If $v_i \in L^p(\mathbf{R}^n)$, then this convolution sequence converges to v_i in $L^p(\mathbf{R}^n)$. When u is given on Ω, use a partition of unity to reduce the setting to that of $\mathbf{R}^n$. □

The weak derivative coincides with the usual derivative if both derivatives exist.

Corollary A.16. *Let $\Omega \subset \mathbf{R}^n$, $d \in \mathbf{N}$ and let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^d)$ be differentiable a.e. Then the classical derivative $\nabla f(x)$ equals to the weak derivative $Df(x)$ for a.e. $x \in \Omega$.*

Proof. From the proof of previous theorem **(1)** $\Rightarrow$ **(3)** we see that $Df(x)$ equals almost everywhere to the classical derivatives of the absolutely continuous representative. If $\nabla f(x)$ exists, then also classical partial derivatives exist at this point and they must equal a.e. to the derivatives of the absolutely continuous representative. □

It is well known that a Sobolev function satisfies the Poincaré inequality (see [29, Sect. 5.6.1]).

Theorem A.17. *Let $B \subset \mathbf{R}^n$ be a ball. If $u \in W^{1,1}(B)$, then*

$$\int_B |u(x) - u_B|\, dx \le C \operatorname{diam}(B) \int_B |Du(x)|\, dx.$$

Moreover, we also use the Sobolev-Poincaré inequality (see [29, Sect. 4.5.2]).

Theorem A.18. *Let $B \subset \mathbf{R}^n$ be a ball, $1 \le p < n$ and $p^* = \frac{np}{n-p}$. If $u \in W^{1,p}(B)$, then $u \in L^{p^*}(B)$ and*

$$\left(\fint_B |u(x) - u_B|^{p^*}\, dx\right)^{\frac{1}{p^*}} \le C \operatorname{diam}(B) \left(\fint_B |Du(x)|^p\, dx\right)^{\frac{1}{p}}.$$

The following theorem tells us that Sobolev functions are Hölder continuous for $p > n$ (see [122, proof of Theorem 2.4.4]).

Theorem A.19. *Let $u \in W^{1,p}(5B)$ and let $p > n$. Then*

$$|u(x) - u(y)| \le C(n,p)|x-y|^{1-n/p} \left(\int_{B(x,2|x-y|)} |Du|^p\right)^{1/p}$$

for all Lebesgue points $x, y \in B$ of u.

Actually, with some work one can relax the assumption to $u \in W^{1,p}(B)$ and replace $B(x, 2|x-y|)$ with $B(x, 2|x-y|) \cap B$.

Regarding the Poincaré inequality, each function that satisfies a Poincaré inequality is in fact a Sobolev function. The proof of the following theorem is from [32].

Theorem A.20. *Let $\Omega \subset \mathbf{R}^n$ be open and let $u, g \in L^1_{\mathrm{loc}}(\Omega)$. Assume that there is $C > 0$ such that for every ball $B \subset \Omega$ we have*

$$\int_B |u(x) - u_B|\, dx \le C \operatorname{diam}(B) \int_B |g(x)|\, dx.$$

Then $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$.

Proof. Let $A \subset\subset \Omega$ be a fixed domain. First we construct approximations of u. Let $k \in \mathbf{N}$ be such that $\{x : \operatorname{dist}(x, A) < \frac{4n}{k}\} \subset \Omega$ and denote the $1/k$-grid in $\mathbf{R}^n$ by

$$G_k = \left\{z \in (\tfrac{1}{k}\mathbf{Z} \times \ldots \times \tfrac{1}{k}\mathbf{Z}) : \operatorname{dist}(z, A) < \tfrac{n}{k}\right\}.$$

Pick a partition of unity $\{\phi_z\}_{z \in G_k}$ such that

$$\begin{aligned} &\text{each } \phi_z : \mathbf{R}^n \to \mathbf{R} \text{ is continuously differentiable;} \\ &\operatorname{spt} \phi_z \subset B(z, \tfrac{n}{k}) \text{ and } |\nabla \phi_z| \le Ck; \\ &\sum_{z \in G_k} \phi_z(y) = 1 \text{ for every } y \in A. \end{aligned} \tag{A.5}$$

Now we set

$$u_k(y) = \sum_{z \in G_k} \phi_z(y) u_{B(z, \frac{1}{k})} \text{ for every } y \in A. \tag{A.6}$$

The supports of ϕ_z have bounded overlap and hence this sum is locally finite and $u_k \in C^1(A)$. It is not difficult to show that $u_k \to u$ in $L^1(A)$. Indeed, this is simple for a continuous function u by uniform continuity, and the general case follows by approximation by continuous functions.

Next we need to estimate the derivative of u_k. For $y \in A$ we choose $z_0 \in G_k$ so that $y \in B(z_0, \frac{n}{k})$. Since $\sum \phi_z = 1$ and we have a locally finite sum we may write

$$Du_k(y) = D\Big(\sum_{z \in G_\varepsilon} \phi_z(y)\big(u_{B(z, \frac{1}{k})} - u_{B(z_0, \frac{2n}{k})}\big)\Big) = \sum_{z \in G_k} D\phi_z(y)\big(u_{B(z, \frac{1}{k})} - u_{B(z_0, \frac{2n}{k})}\big). \tag{A.7}$$

Since $y \in B(z_0, \frac{n}{k})$ it is easy to see that

$$\phi_z(y) \ne 0 \Rightarrow B(z, \frac{1}{k}) \subset B(z_0, \frac{2n}{k}).$$

Hence we may use (A.7) and (A.5) to estimate

$$
\begin{aligned}
|Du_k(y)| &\le \sum_{\{z\in G_k:\ \phi_z(y)\neq 0\}} Ck\left|u_{B(z,\frac{1}{k})} - u_{B(z_0,\frac{2n}{k})}\right| \\
&\le \sum_{\{z\in G_k:\ \phi_z(y)\neq 0\}} Ck\left|\fint_{B(z,\frac{1}{k})} (u(x) - u_{B(z_0,\frac{2n}{k})})\, dx\right| \\
&\le \sum_{\{z\in G_k:\ \phi_z(y)\neq 0\}} Ck\frac{1}{|B(z_0,\frac{2n}{k})|}\frac{|B(z_0,\frac{2n}{k})|}{|B(z,\frac{1}{k})|}\int_{B(z_0,\frac{2n}{k})} \left|u(x) - u_{B(z_0,\frac{2n}{k})}\right| dx\,.
\end{aligned}
$$

Only a bounded number of terms above are nonzero and hence we can use our assumption and $B(z_0, \frac{2n}{k}) \subset B(y, \frac{3n}{k})$ to obtain

$$
\begin{aligned}
|Du_k(y)| &\le Ck\fint_{B(z_0,\frac{2n}{k})} \left|u(x) - u_{B(z_0,\frac{2n}{k})}\right| dx \\
&\le C\fint_{B(z_0,\frac{2n}{k})} |g(x)|\, dx \le C\fint_{B(y,\frac{3n}{k})} |g(x)|\, dx\,.
\end{aligned}
\tag{A.8}
$$

Since $\fint_{B(y,\frac{3n}{k})} |g| \to |g(y)|$ in $L^1(A)$ as $k\to\infty$, there is a subsequence $k_j \to 0$ such that $\fint_{B(y,\frac{3n}{k_j})} |g|$ has a majorant $H \in L^1(A)$. From this, (A.8) and Lemma A.3, we obtain that there is a subsequence $k_i \to \infty$ and $g \in L^1(A, \mathbf{R}^n)$ such that $Du_{k_i} \to g$ weakly in L^1. Since $u_k \in C^1$ we have

$$
\int_A Du_{k_i}(y)\varphi(y)dy = -\int_A u_{k_i}(y)D\varphi(y)dy \tag{A.9}
$$

for every test function $\varphi \in C_c^\infty(A)$. Since $u_k \to u$ in L^1, we obtain, after passing to the limit, that

$$
\int_A g(y)\varphi(y)dy = -\int_A u(y)D\varphi(y)dy
$$

which means that g is a weak gradient of u in A and therefore $u \in W^{1,1}(A)$. □

There is also a version of the previous theorem for BV-functions (recall that BV-functions were defined in Definition 5.1).

Theorem A.21. *Let $\Omega \subset \mathbf{R}^n$ be open, $u \in L^1_{\text{loc}}(\Omega)$ and let μ be a Radon measure on Ω. Assume that there is $C > 0$ such that for every ball $B \subset \Omega$ we have*

$$
\int_B |u(x) - u_B|\, dx \le C\, \text{diam}(B)\mu(B).
$$

Then $u \in BV_{\text{loc}}(\Omega)$.

Proof. We proceed similarly to the proof of the previous theorem. Again we fix $A \subset\subset \Omega$ and we define u_k by the same formula (A.6). Again $u_k \in C^1$ and they converge in L^1 to u. Analogously to the estimate (A.8) we obtain using our assumption that

$$|Du_k(y)| \leq Ck \fint_{B(z_0,\frac{2n}{k})} \left|u(x) - u_{B(z_0,\frac{2n}{k})}\right| dx \leq C \frac{\mu(B(z_0,\frac{2n}{k}))}{|B(z_0,\frac{2n}{k}))|} .$$

By the integration of this inequality over A we obtain

$$\int_A |Du_k(y)|\, dy \leq C \sum_{z \in G_k} \mu(B(z,\tfrac{2n}{k})) \leq C\mu\big(\{x \in \Omega : \operatorname{dist}(x, A) < \tfrac{3n}{k}\}\big) .$$

It follows that Du_k form a bounded sequence in L^1. Recalling that $u_k \to u$ in L^1, we conclude [29, paragraph 5.2.3] that $u \in BV(A)$. In fact, there is a subsequence and vector ν of Radon measures such that Du_{k_i} converge to ν weak star in measures. As before we have (A.9) and by passing to a limit we have

$$\int_A \varphi(y) d\nu(y) = -\int_A u(y) D\varphi(y) dy$$

which means that ν is a weak gradient of u in A and therefore $u \in BV(A)$. □

We have seen in Theorem A.15 that $W^{1,1}$-functions can be characterized using the ACL-property. Similarly, it is possible to characterize BV-functions using the BVL-property, i.e. that the function in question has bounded variation on almost all lines parallel to coordinate axes (see [2, Sect. 3.11]).

More precisely, let $i \in \{1, 2, \dots, n\}$, $Q_0 = (0, 1)^n$ and by π_i denote the projection to the hyperplane perpendicular to i-th coordinate axis. For $y \in \pi_i(Q_0)$ we denote $g_{i,y}(t) = g(y + t\mathbf{e}_i)$.

Theorem A.22. *Let $g \in L^1(Q_0)$. Then $g \in BV(Q_0)$ if and only if for every $i \in \{1, \dots, n\}$ the function $g_{i,y}(t) \in BV\big((0, 1)\big)$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_i(Q_0)$ and moreover*

$$\int_{\pi_i(Q_0)} \left|Dg_{i,y}\right|\big((0, 1)\big)\, d\mathcal{H}^{n-1}(y) < \infty,$$

where $|Dg_{i,y}|((0, 1))$ denotes the total variation of our BV-function of a single variable. In this case we can estimate the total variation of Dg by

$$|Dg|(Q_0) \leq C \sum_{i=1}^{n} \int_{\pi_i(Q_0)} \left|Dg_{i,y}\right|\big((0, 1)\big)\, d\mathcal{H}^{n-1}(y).$$

A.6 Lipschitz Approximation of Sobolev Functions

We know by Theorem A.15 that, given a Sobolev function u, we can find C^1-smooth functions u_k such that $u_k \to u$ in $W^{1,1}$, but in some applications this approximation is not good enough. In this section we construct Lipschitz functions u_k that converge to u in $W^{1,1}$ and moreover $u = u_k$ on a big set. First we need a couple of lemmata.

Lemma A.23 (McShane Extension). *Let $A \subset \mathbf{R}^n$ and $f : A \to \mathbf{R}^m$ be L-Lipschitz, that is*

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in A$. Then there exists a $(\sqrt{m}L)$-Lipschitz $\tilde{f} : \mathbf{R}^n \to \mathbf{R}^m$ such that $\tilde{f}|_A = f$.

Proof. Let $m = 1$. Define

$$\tilde{f}(x) = \inf_{a \in A} \{f(a) + L|x - a|\}.$$

Then $\tilde{f}(x) = f(x)$ when $x \in A$: Since f is L-Lipschitz on A,

$$f(x) \leq f(a) + L|x - a| \quad \text{when } x, a \in A,$$

and so $f(x) \leq \tilde{f}(x)$. By the choice $a = x$ in the definition of $\tilde{f}(x)$ we obtain $\tilde{f}(x) \leq f(x)$.

Given $x, y \in \mathbf{R}^n$, we have that

$$\begin{aligned}\tilde{f}(x) &= \inf_{a \in A} \{f(a) + \underbrace{L|x - a|}_{\leq L(|y-a|+|y-x|)}\} \\ &\leq L|y - x| + \tilde{f}(y).\end{aligned}$$

Because this also holds with x replaced by y, we conclude that $\tilde{f}$ is L-Lipschitz.

Let us then consider the case $m \geq 2$. For given $f = (f_1, \ldots, f_m)$ define $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_m)$ as in the previous case. Now

$$|\tilde{f}(x) - \tilde{f}(y)|^2 = \sum_1^m |\tilde{f}_i(x) - \tilde{f}_i(y)|^2 \leq mL^2|x - y|^2,$$

and the claim follows. □

Remark A.24. By choosing a suitable extension different from the McShane extension, one could require above $\tilde{f}$ to be L-Lipschitz. This can be done using the so-called Kirszbaum extension.

In the following theorem we use the modified maximal function

$$M_{3r_0}u(x) = \sup_{x\in B(y,r)\subset B(x_0,3r_0)} \fint_{B(y,r)} |u| \, .$$

Lemma A.25. *Let $B = B(x_0, r_0)$ be a ball in $\mathbf{R}^n$ and let $u \in W^{1,1}(B(x_0, 3r_0))$. For $\lambda > 0$ we define*

$$F_\lambda = \{x \in B : M_{3r_0}|Du(x)| < \lambda\} \cap \{x \in B : \ x \text{ is a Lebesgue point of } u\}. \tag{A.10}$$

There is a constant $C > 0$ such that,

$$|u(x) - u(y)| \le C\lambda|x - y| \text{ for all } x, y \in F_\lambda.$$

Moreover, the measure of the remaining set satisfies

$$|B \setminus F_\lambda| = o(\tfrac{1}{\lambda}) \, .$$

Proof. Let $x, y \in F_\lambda$. Choose $B_j = B(x, 2^{-j}|x - y|)$ for $j \ge 0$ and $B_j = B(y, 2^{j+1}|x-y|)$ for $j < 0$. As x and y are Lebesgue points we obtain $u_{B_j} \to u(x)$ as $j \to \infty$ and $u_{B_j} \to u(y)$ as $j \to -\infty$ and hence

$$|u(x) - u(y)| \le \sum_{-\infty}^{\infty} |u_{B_{j+1}} - u_{B_j}| \, .$$

Moreover, for $j > 0$ (and thus $B_{j+1} \subset B_j$) we can estimate the difference

$$\begin{aligned} |u_{B_{j+1}} - u_{B_j}| &= \Big|\frac{1}{|B_{j+1}|}\int_{B_{j+1}} (u(x) - u_{B_j}) \, dx\Big| \\ &\le \frac{1}{|B_{j+1}|}\int_{B_{j+1}} |u(x) - u_{B_j}| \, dx \le \frac{C(n)}{|B_j|}\int_{B_j} |u(x) - u_{B_j}| \, dx \end{aligned} \tag{A.11}$$

and we have similar estimate for $j < 0$. For $|u_{B_0} - u_{B_1}|$ we add and subtract the term $u_{B_0\cap B_1}$ and easily obtain the bound

$$\frac{1}{|B_0|}\int_{B_0} |u(x) - u_{B_0}| \, dx + \frac{1}{|B_1|}\int_{B_1} |u(x) - u_{B_1}| \, dx.$$

Hence we can use the Poincaré inequality, Theorem A.17, to obtain

$$
\begin{aligned}
|u(x)-u(y)| &\le \sum_{-\infty}^{\infty} |u_{B_{j+1}} - u_{B_j}| \le \sum_{-\infty}^{\infty} C(n) \fint_{B_j} |u - u_{B_j}| \\
&\le C(n) \sum_{-\infty}^{\infty} r_j \fint_{B_j} |Du| \\
&\le C(n)|x-y|\big(\tilde{M}_{3r_0}|Du(x)| + \tilde{M}_{3r_0}|Du(y)|\big) \\
&\le 2C(n)|x-y|\lambda\ .
\end{aligned}
$$

Thus we have $C(n)\lambda$-Lipschitz continuity on the set F_λ. By Remark A.8 and Theorem A.6 we have

$$
|B \setminus F_\lambda| \le \frac{5^n 2}{\lambda} \underbrace{\int_{\{M_{3r_0}|Du(z)|>\lambda\}\cap 3B} |Du|}_{\underset{\lambda\to\infty}{\longrightarrow} 0} = o\left(\tfrac{1}{\lambda}\right)
$$

and the claim follows. □

Remark A.26. The above proof shows that u is $C(n)\lambda$-Lipschitz in F_λ, where $|B \setminus F_\lambda| = o\left(\frac{1}{\lambda}\right)$. Use the McShane extension theorem to extend the restriction of u to this set as a $C(n)\lambda$-Lipschitz function u_λ, defined in entire B. Then

$$
\int_B |Du - Du_\lambda| \le \int_{B\setminus F_\lambda} (|Du| + |Du_\lambda|) \le \int_{B\setminus F_\lambda} |Du| + C(n)\lambda o\left(\tfrac{1}{\lambda}\right) \underset{\lambda\to\infty}{\longrightarrow} 0
$$

because

$$
Du_\lambda(x) = Du(x) \tag{A.12}
$$

at almost every point $x \in F_\lambda$.
Reason: If $E \subset \Omega$ is measurable, $\partial_i v$ and $\partial_i w$ exist almost everywhere in E and $v = w$ on E, then $\partial_i v = \partial_i w$ almost everywhere in E. Simply notice that almost every point x of E is of linear density one in the x_i-direction.

One can do even better. Consider the set

$$
\mathrm{Bad}'_\lambda = \{x \in B :\ \tilde{M}_{3r_0} u(x) \ge \lambda\}.
$$

Then $|\mathrm{Bad}'_\lambda| = o\left(\frac{1}{\lambda}\right)$. So, when λ is large, the distance from any point in Bad'_λ to $B \setminus \mathrm{Bad}'_\lambda$ is at most one. Thus the McShane extension u_λ of u from $F_\lambda \setminus \mathrm{Bad}'_\lambda$ is $C(n)\lambda$-Lipschitz and bounded in absolute value by $2C(n)\lambda$ on B. It follows that

$$
\int_B |u - u_\lambda| + |Du - Du_\lambda| \underset{\lambda\to\infty}{\longrightarrow} 0.
$$

The final estimate of the preceding remark yields the following corollary:

Corollary A.27. *If $u \in W^{1,1}(3B)$, then there is a sequence $\{u_j\}_{j=1}^{\infty}$ of Lipschitz functions such that*

$$\{x \in B : u_{j+1}(x) \neq u(x)\} \subset \{x \in B : u_j(x) \neq u(x)\}, \quad |\{x \in B : u_j(x) \neq u(x)\}| \overset{j\to\infty}{\to} 0$$

and

$$\int_B |u - u_j| + |Du - Du_j| \overset{j\to\infty}{\to} 0\,.$$

A.7 Differentiability and Approximative Differentiability

The following differentiability result due to Menchoff was independently also proved by Gehring and Lehto.

Lemma A.28. *Suppose that $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a homeomorphism that belongs to $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^2)$. Then f is differentiable almost everywhere.*

Proof. It suffices to prove the claim for the two component functions of f. So, let $u : \mathbf{R}^2 \to \mathbf{R}$ be one of them. Then, by the Lebesgue differentiation theorem,

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |Du(x) - Du(x_0)|dx = 0 \tag{A.13}$$

for almost every x_0. Fix such a point x_0, let $r > 0$, and suppose $x \in B(x_0, r)$. Given an open rectangle R_x contained in $B(x_0, 2r)$ with $x \in R_x$, we define a function $v : \overline{R_x} \to \mathbf{R}$ by setting

$$v(y) = \big|u(y) - u(x_0) - \langle Du(x_0), x - x_0\rangle\big|.$$

Since f is a homeomorphism, the maximum of v on $\overline{R_x}$ occurs on ∂R_x, say at $y_x^R \in \partial R_x$. We conclude that

$$\begin{aligned}\big|u(x) - u(x_0) - \langle Du(x_0), x - x_0\rangle\big| &\leq v(y_x^R) \\ &\leq \big|u(y_x^R) - u(x_0) - \langle Du(x_0), y_x^R - x_0\rangle\big| \\ &\quad + |Du(x_0)||y_x^R - x|.\end{aligned}$$

Thus, differentiability of u at x_0 follows if, given $\varepsilon > 0$, we can find $r > 0$ so that, for each $x \in B(x_0, r)$ we can make the right-hand side above smaller than εr, via a suitable choice of R_x.

Towards this end, we first deal with the term $|u(y_x^R)-u(x_0)-\langle Du(x_0), y_x^R-x_0\rangle|$. We may assume that $Du(x_0) = 0$ by replacing u with the function w defined by setting $w(y) = u(y)-\langle Du(x_0), y-x_0\rangle$. By yet other replacements, we may assume that $x_0 = 0 = u(x_0)$. Then (A.13) guarantees that

$$\lim_{r\to 0} \frac{1}{r^2} \int_{-r}^{r} \int_{-r}^{r} |Du(s,t)|\, ds\, dt = 0.$$

Set $A_r = \{-r < s < r : \int_{-r}^{r} |Du(s,t)|\, dt \geq \varepsilon r\}$ and $A^r = \{-r < t < r : \int_{-r}^{r} |Du(s,t)|\, ds \geq \varepsilon r\}$. Then, for each $\varepsilon > 0$, we can find $r_\varepsilon > 0$ so that

$$|A_r| \leq \varepsilon r \text{ and } |A^r| \leq \varepsilon r \tag{A.14}$$

for all $0 < r < r_\varepsilon$. By Theorem A.15 we know that u is absolutely continuous on almost all lines parallel to coordinate axes and hence we may assume that u is absolutely continuous on $y_1 \times [-r, r]$ for every $y_1 \notin A_r$ and on $[-r, r] \times y_2$ for every $y_2 \notin A^r$. We conclude that $|u(y_1, y_2)| \leq 2\varepsilon r$ provided $y_1 \notin A_r$, $y_2 \notin A^r$ and $0 < r < r_\varepsilon$. Thus we have found plenty of rectangles R_x, on the boundary of whose, $|u(y_x^R) - u(x_0) - \langle Du(x_0), y_x^R - x_0\rangle| \leq 2\varepsilon r$.

By (A.14) we may moreover assume that the side-length of each rectangle R_x is at most εr. Finally,

$$|Du(x_0)||y_x^R - x| \leq \big(\operatorname{dist}(x, \partial R_x) + \operatorname{diam}(R_x)\big)|Du(x_0)|$$

and hence this term is handled by the estimate for the size of R_x. □

For the study of the regularity of the inverse we need the following elementary observation.

Lemma A.29. *Let $f : \Omega \to \mathbf{R}^n$ be a homeomorphism which is differentiable at $x \in \Omega$ with $J_f(x) > 0$. Then f^{-1} is differentiable at $f(x)$ and $Df^{-1}(f(x)) = (Df(x))^{-1}$.*

Proof. Let us denote $y = f(x)$. We know that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - Df(x)h}{\|h\|} = 0 \tag{A.15}$$

and we would like to show that

$$\lim_{t\to 0} \frac{f^{-1}(y+t) - f^{-1}(y) - (Df(x))^{-1}t}{\|t\|} = 0\,.$$

Given $t \in \mathbf{R}^n$ we set $h = f^{-1}(y+t) - f^{-1}(y)$ which implies that

$$f(x+h) - f(x) = f\big(x + f^{-1}(y+t) - x\big) - f(x) = t\,.$$

By (A.15) and $J_f(x) > 0$ we obtain for small enough $\|h\|$ that

$$\|t\| = \|f(x+h) - f(x)\| \approx \|Df(x)h\| \approx \|h\| \, .$$

Now (A.15) implies that

$$0 = \lim_{h\to 0} \frac{\big(f(x+h) - f(x) - Df(x)h\big)(Df(x))^{-1}}{\|h\|}$$

which implies that

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{(Df(x))^{-1}(f(x+h) - f(x)) - h}{\|t\|} \\ &= \lim_{t\to 0} \frac{(Df(x))^{-1}t - f^{-1}(y+t) - f^{-1}(y)}{\|t\|} \end{aligned}$$

and we get the desired conclusion. □

Definition A.30. Let $\Omega \subset \mathbf{R}^n$, $d \in \mathbf{N}$ and let $f : \Omega \to \mathbf{R}^d$ be a mapping. We say that f is approximatively differentiable at $x \in \Omega$ with approximative derivative $Df(x)$ if there is a set $A \subset \Omega$ of density one at x, i.e. $\lim_{r\to 0} \frac{|A\cap B(x,r)|}{|B(x,r)|} = 1$, such that

$$\lim_{y\to x, y\in A} \frac{f(y) - f(x) - Df(x)(y-x)}{\|y-x\|} = 0 \, .$$

It is well known that Sobolev functions are differentiable a.e.

Theorem A.31. *Let $\Omega \subset \mathbf{R}^n$, $d \in \mathbf{N}$ and let $f \in W^{1,1}(\Omega, \mathbf{R}^d)$. Then f is approximatively differentiable a.e. in Ω and its approximative derivative coincides with its weak derivative a.e.*

Proof. By Corollary A.27 we know that we can find a sequence of Lipschitz mappings $f_j : \Omega \to \mathbf{R}^d$ such that for

$$A_j := \{x \in \Omega : f_j(x) \neq f(x)\} \text{ we have } A_{j+1} \subset A_j \text{ and } |A_j| \to 0 \, .$$

The Lipschitz mappings f_j are differentiable a.e. (see Theorem 2.23) and their classical derivatives coincide with the weak derivatives Df_j (see Corollary A.16).

It is easy to see that $\Omega = \bigcup_j A_j \cup S$ where $|S| = 0$. Almost every point of A_j is a point of density of A_j and hence for almost every $x \in \Omega$ we can find j such that

$$\lim_{r\to 0} \frac{|A_j \cap B(x,r)|}{|B(x,r)|} = 1 \, .$$

Let us pick a point $x \in \Omega$ such that x is a point of density of A_j and f_j is differentiable at x. Since $f = f_j$ on A_j we obtain from the differentiability of f_j that

$$
\begin{aligned}
0 &= \lim_{y \to x} \frac{f_j(y) - f_j(x) - Df_j(x)(y-x)}{\|y-x\|} \\
&= \lim_{y \to x,\ y \in A_j} \frac{f(y) - f(x) - Df_j(x)(y-x)}{\|y-x\|}
\end{aligned}
$$

which implies the approximative derivative of f at x exists and equals to $Df_j(x)$.
□

We need an analogue of Lemma A.29 for approximatively differentiable mappings. For its proof we need the following observation.

Lemma A.32. *Let $\Omega \subset \mathbf{R}^n$ and let $g : \Omega \to \mathbf{R}^n$ be Lipschitz mapping that is differentiable at x. Suppose that $J_g(x) \neq 0$ and let A be a set of density 1 at $x \in \Omega$. Then the density of the set $g(A)$ at $g(x)$ is 1.*

Proof. Let us denote $y = g(x)$ and let $r > 0$ be small enough such that $B(y,r) \subset f(\Omega)$. Since g is differentiable at x, continuous and $J_g(x) \neq 0$ we obtain that for r small enough we have

$$
B(y,r) \subset g\big(B(x, C_1 r)\big) \quad \text{for} \quad C_1 = 2|(Dg(x))^{-1}|\,. \tag{A.16}
$$

We know that

$$
\lim_{\tilde{r} \to 0} \frac{|A \cap B(x,\tilde{r})|}{|B(x,\tilde{r})|} = 1
$$

and hence for given $\varepsilon > 0$ we obtain that for small enough r we have

$$
\big|B(x, C_1 r) \setminus A\big| < \varepsilon |B(x, C_1 r)| = \varepsilon C_1^n |B(x,r)|\,.
$$

Since g is Lipschitz with constant L, (A.16) gives

$$
\big|B(y,r) \backslash g(A)\big| \le \big|g(B(x, C_1 r)) \backslash g(A)\big| \le \big|g\big(B(x, C_1 r) \backslash A\big)\big| < L^n \varepsilon C_1^n |B(x,r)|\,.
$$

This shows that

$$
\lim_{r \to 0} \frac{|g(A) \cap B(y,r)|}{|B(y,r)|} = 1\,. \qquad \square
$$

Now we prove the desired analog of Lemma A.29, following [33, Lemma 2.1].

Lemma A.33. *Let $f : \Omega \to \Omega'$ be a homeomorphism such that $f \in W^{1,1}(\Omega, \Omega')$ and $f^{-1} \in W^{1,1}(\Omega', \Omega)$. Set*

$$E := \{y \in \Omega' : \ f^{-1} \text{ is approximatively differentiable at } y \text{ and } |J_{f^{-1}}(y)| > 0\} .$$

Then, there exists a Borel set $A \subset E$ with $|E \setminus A| = 0$ such that

$$f^{-1}(A) \subset F := \{x \in \Omega : \ f \text{ is approximatively differentiable at } x \text{ and } |J_f(x)| > 0\}$$

and we have

$$Df^{-1}(y) = \big(Df(f^{-1}(y))\big)^{-1} \text{ for every } y \in A .$$

Proof. From the proof of Theorem A.31 we know that there is a null set N_E such that for every $y \in E \setminus N_E$ we can find a Lipschitz mapping h and set $\tilde{E} \subset E$ such that $\tilde{E}$ has density one at y, $f^{-1} = h$ on $\tilde{E}$, h is differentiable at y and $Df^{-1}(y) = Dh(y)$. We can also require that $E \setminus N_E$ is a Borel set. Analogously, we can find a null set N_F so that for every $x \in F \setminus N_F$ we can find a Lipschitz mapping g and set $\tilde{F} \subset F$ such that $\tilde{F}$ has density one at x, $f = g$ on $\tilde{F}$, g is differentiable at x and $Df(x) = Dg(x)$.

By Corollary A.36 we obtain that there is a Borel set $J \subset \{J_f = 0\}$ such that $|f(J)| = 0$ and $|J| = |\{J_f = 0\}|$. This and the a.e. approximative differentiability of f (see Theorem A.31) shows that the set $N_F \cup (\Omega \setminus (F \cup J))$ has measure zero. We can find a Borel set N of measure zero such that $N_F \cup (\Omega \setminus (F \cup J)) \subset N$ and by the Area formula (A.19) applied to f^{-1} we know that

$$\int_{f(N)} |J_{f^{-1}}(y)|\, dy \le \int_{f^{-1}(f(N))} dx = |N| = 0 .$$

Since $J_{f^{-1}} > 0$ on E, this implies that $|f(N) \cap E| = 0$. For the set

$$A := E \setminus (f(N) \cup N_E \cup f(J))$$

we thus have $|E \setminus A| = 0$ and $f^{-1}(A) \subset F$.

Given $y \in A$ we denote $x = f^{-1}(y) \in F$ and we can find functions h, g and sets $\tilde{E}$ and $\tilde{F}$ as in the first paragraph. Since g is differentiable at x and h is differentiable at y we obtain

$$D(g \circ h)(y) = Dg(h(y))Dh(y) = Dg(x)Dh(y) . \tag{A.17}$$

Since $|J_g(x)| = |J_f(x)| > 0$ we obtain by Lemma A.32 that the set $g(\tilde{F}) \cap \tilde{E}$ has density one at y. For every $z \in g(\tilde{F}) \cap \tilde{E}$ we have $g(h(z)) = f(f^{-1}(z)) = z$ and hence $D(g \circ h)(y) = I$. Now (A.17) implies $Dh(y) = (Dg(x))^{-1}$. $\square$

A.8 Area and Coarea Formula

Let us recall the definition of the Lusin (N) condition.

Definition A.34. Let $\Omega \subset \mathbf{R}^n$ be open. We say that $f : \Omega \to \mathbf{R}^n$ satisfies the Lusin (N) condition if

$$\text{for each } E \subset \Omega \text{ such that } |E| = 0 \text{ we have } |f(E)| = 0\,.$$

Theorem A.35. *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and let η be a nonnegative Borel measurable function on $\mathbf{R}^n$. Then*

$$\int_\Omega \eta(f(x))\,|J_f(x)|\,dx \le \int_{\mathbf{R}^n} \eta(y)\,N(f,\Omega,y)\,dy\,, \tag{A.18}$$

where the multiplicity function $N(f,\Omega,y)$ of f is defined as the number of preimages of y under f in Ω. Moreover, there is an equality in (A.18) if we assume in addition that f satisfies the Lusin (N) condition.

Proof (Sketch of the Proof). It is known that each $f \in W^{1,1}$ is a.e. approximatively differentiable (see Theorem A.31), and by Corollary A.27 we can decompose $\Omega = S \cup \bigcup_{n=1}^\infty \Omega_i$ in a such a way that $|S| = 0$ and the restriction $f|_{\Omega_i}$ to each set Ω_i is Lipschitz (see the proof of Theorem A.31). It is well-known that the Area formula holds for Lipschitz mappings and hence on each Ω_i we get

$$\int_{\Omega_i} \eta(f(x))\,|J_f(x)|\,dx = \int_{\mathbf{R}^n} \eta(y)\,N(f,\Omega_i,y)\,dy\,.$$

By summing of these equalities we obtain the left-hand side of (A.18) since $|S| = 0$. The right-hand is bigger or equal since

$$\int_{\mathbf{R}^n} \eta(y)\,N(f,\Omega,y)\,dy = \int_{\mathbf{R}^n} \eta(y)\,N(f,S,y)\,dy + \sum_{i=1}^\infty \int_{\mathbf{R}^n} \eta(y)\,N(f,\Omega_i,y)\,dy$$

and the first term is nonnegative and potentially strictly positive if $|f(S)| > 0$. Moreover, if f satisfies the Lusin (N) condition then $|f(S)| = 0$ and hence the first term vanishes and the equality holds in (A.18). □

Corollary A.36. *(a) Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ be a homeomorphism, $\tilde\eta$ a nonnegative Borel measurable function on $\mathbf{R}^n$ and let $A \subset \Omega$ be a Borel measurable set. Then*

$$\int_A \tilde\eta(f(x))\,|J_f(x)|\,dx \le \int_{f(A)} \tilde\eta(y)\,dy\,. \tag{A.19}$$

(b) *Let* $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ *be a homeomorphism and let* η *be a nonnegative Borel measurable function on* $\mathbf{R}^n$. *Then there is a set* $\Omega' \subset \Omega$ *of full measure* $|\Omega'| = |\Omega|$ *such that*

$$\int_{\Omega'} \eta(f(x))\,|J_f(x)|\,dx = \int_{f(\Omega')} \eta(y)\,dy\,. \tag{A.20}$$

(c) *Let* $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ *be a homeomorphism, let* η *be a nonnegative Borel measurable function on* $\mathbf{R}^n$ *and let* A *denote the set where* f *is differentiable. Then*

$$\int_{A} \eta(f(x))\,|J_f(x)|\,dx = \int_{f(A)} \eta(y)\,dy\,.$$

(d) *Let* $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ *be a mapping whose multiplicity is essentially bounded by* N *and let* $A \subset \Omega$ *be a measurable set. Then*

$$\int_{A} |J_f(x)|\,dx \le N \int_{f(A)} dy = N|f(A)|\,.$$

Especially we get that the Jacobian of a mapping with essentially bounded multiplicity is locally integrable.

For the part (a) above, the multiplicity of a homeomorphism is bounded by one and we apply the previous theorem for $\tilde{\eta} = \chi_{f(A)}\eta$. Regarding (b), it is enough to set $\Omega' = \bigcup_{n=1}^{\infty} \Omega_i$, where the sets Ω_i are as in the proof of the previous theorem. The part (c) follows from the previous theorem and the fact that the Lusin condition (N) holds on the set A of differentiability. This can be easily shown from the definition of differentiability with the help of the Vitali covering theorem.

The Sard Theorem [90, Theorem 7.6] tells that the image of the set of critical points is of measure zero.

Theorem A.37 (Sard). *Let* $f : \mathbf{R}^n \to \mathbf{R}^n$ *be Lipschitz mapping. Then*

$$\mathscr{L}_n\big(\{f(x) : \; J_f(x) = 0\}\big) = 0\,.$$

The coarea formula is very useful tool in analysis and its form for Lipschitz functions can be found in Federer [31, 3.2.12].

Theorem A.38. *Let* $\Omega \subset \mathbf{R}^n$ *be an open set and let* $f : \Omega \to \mathbf{R}^m$ *be Lipschitz. Then for every measurable set* $E \subset \Omega$ *we have*

$$\int_{E} |J_m f(x)|\,dx = \int_{\mathbf{R}^m} \mathscr{H}^{n-m}(E \cap f^{-1}(y))\,dy$$

where $J_m f$ *denotes the square root of the sum of the squares of the determinants of the* m*-by-*m *minors of the differential of* f.

A.9 Estimates for q-Capacity

The following estimate on q-capacity is standard, see e.g. [58, Theorem 5.9].

Theorem A.39. *Let $n-1 \le q < n$ and $\varepsilon > 0$. Let $B_0 \subset \mathbf{R}^n$ be a ball of radius r, $n \ge 2$, and let $E, F \subset \frac{1}{2}B_0$. Suppose that $u \in W^{1,q}(B_0)$ is continuous and satisfies $u \le 0$ on E and $u \ge 1$ on F. Then*

$$r^\varepsilon \int_{B_0} |Du|^q \ge C \min\{\mathscr{H}_\infty^{n-q+\varepsilon}(E), \mathscr{H}_\infty^{n-q+\varepsilon}(F)\} .$$

Moreover, for $n = 2$ and $q = 1$ this estimate is valid also for $\varepsilon = 0$.

Proof. Without loss of generality we may assume that $u_{B_0} \ge \frac{1}{2}$; otherwise we switch the role of E and F. For every $x \in E$ we set $B_i(x) = B(x, 2^{-i+1}r)$, $i \in \mathbf{N}$. As u is continuous we have $u_{B_i(x)} \to u(x)$ and hence

$$\tfrac{1}{2} \le |u(x) - u_{B_0}| \le \sum_{i=1}^\infty |u_{B_i(x)} - u_{B_{i-1}(x)}| .$$

Since $B_i \subset B_{i-1}$, we may estimate the last term similarly to (A.11) and by the Poincaré inequality, Theorem A.17, and Hölder's inequality we obtain

$$\tfrac{1}{2} \le \sum_{i=0}^\infty \fint_{B_i(x)} |u(y) - u_{B_i(x)}|\, dy \le C \sum_{i=0}^\infty 2^{-i} r \Big(\fint_{B_i(x)} |Du(y)|^q\, dy\Big)^{\frac{1}{q}} . \quad \text{(A.21)}$$

Let $\delta > 0$. We claim that for every $x \in E$ we can find $i_x \in \mathbf{N}$ such that

$$\delta r^{-\varepsilon} (2^{-i_x} r)^{n-q+\varepsilon} \le \int_{B_{i_x}(x)} |Du(y)|^q\, dy , \quad \text{(A.22)}$$

provided δ is sufficiently small (in terms of q, n, ε). Otherwise (A.21) implies that

$$\tfrac{1}{2} \le C \sum_{i=0}^\infty 2^{-i} r \Big(\frac{1}{|B_i(x)|} \delta r^{-\varepsilon} (2^{-i} r)^{n-q+\varepsilon}\Big)^{\frac{1}{q}} \le C \delta^{\frac{1}{q}} \sum_{i=0}^\infty (2^{-i})^{\frac{\varepsilon}{q}} \le C \delta^{\frac{1}{q}}$$

which gives us a contradiction for small enough $\delta > 0$. Let $\delta > 0$ be fixed and so small that the above holds. For each $x \in E$ we choose a ball $B_{i_x}(x)$ such that (A.22) holds. By the Vitali covering theorem, Theorem A.1, we choose a subcollection of pairwise balls B_k with radii r_k such that $E \subset \bigcup_k 5B_k$. Using (A.22) for B_k we now obtain the desired estimate

$$\mathscr{H}_\infty^{n-q+\varepsilon}(E) \le \sum_k (5r_k)^{n-q+\varepsilon} \le C \sum_k \tfrac{1}{\delta} r^\varepsilon \int_{B_k} |Du(y)|^q\, dy \le C r^\varepsilon \int_{B_0} |Du(y)|^q\, dy . \quad \text{(A.23)}$$

The proof for $n = 2$ and $q = 1$ with $\varepsilon = 0$ is more demanding and we will not give it here. It follows from a stronger estimate for continuous functions in $W^{1,1}(\mathbf{R}^n)$ (see [1]):

$$\int_0^\infty \mathscr{H}^1_\infty(\{x \in \mathbf{R}^n : M|u(x)| > t\})\, dt \le C(n) \int_{\mathbf{R}^n} |Du(x)|\, dx\,.$$

To have a geometric idea, consider the simple situation

$$E = \{0\} \times [0, \operatorname{diam} E] \text{ and } F = \{t\} \times [0, \operatorname{diam} F] \text{ for some } t > 0\,.$$

Then we can use the fundamental theorem of calculus for each $y \in [0, \min(\operatorname{diam} E, \operatorname{diam} F)]$, $u(0, y) \le 0$ and $u(t, y) \ge 1$ to obtain

$$\int_0^t |Du(s, y)|\, ds \ge 1\,.$$

By Fubini we get the desired estimate

$$\int_{B_0} |Df| \ge \min(\operatorname{diam} E, \operatorname{diam} F)$$

in this simple situation. □

The following result is a consequence of the proof above.

Corollary A.40. *Let $n \ge 2$ and $n - 1 < q < n$. Let $\Omega \subset \mathbf{R}^n$ be an open set and let $F \subset \Omega$ be a continuum. Suppose that $u \in W^{1,q}(\Omega)$ is continuous, has compact support in Ω and satisfies $u \ge 1$ on F. Then*

$$C(q, n) \int_\Omega |Du|^q \ge \operatorname{diam}^{n-q}(F)\,.$$

Proof. We extend u outside Ω as zero. Without loss of generality assume that $u \le 1$. We can clearly fix a ball B_1 such that the support of u is contained in B_1 and we can find a ball $B_0 \subset 2B_1$ such that $F \subset B_0$ and $\operatorname{diam} B_0 \le 2 \operatorname{diam} F$.

In the case $u_{B_0} \le \frac{1}{2}$ we proceed similarly to the previous proof. For each $x \in F$ we can find balls $B_i(x)$ such that $u_{B_i(x)} \to u(x) \ge 1$ and we have (A.21). We can choose ε, so that $n - q + \varepsilon = 1$. Since F is a continuum we obtain $\operatorname{diam} F \le \mathscr{H}^1_\infty(F)$. As before we obtain (A.23) and hence

$$\operatorname{diam}^{n-q+\varepsilon} F \le C \operatorname{diam}^\varepsilon B_0 \int_{B_0} |Du(y)|^q\, dy$$

which gives us the desired estimate as $\operatorname{diam} B_0 \le 2 \operatorname{diam} F$.

It remains to consider the case $u_{B_0} > \frac{1}{2}$. Note that clearly $u_{2B_1} \leq \frac{1}{4}$ since u is supported in B_1. Now we can find balls $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_k$ such that $\tilde{B}_1 = B_0$, $\tilde{B}_k = 2B_1$,

$$\tilde{B}_i \subset \tilde{B}_{i+1},\ 2\,\mathrm{diam}\,\tilde{B}_i \leq \mathrm{diam}\,\tilde{B}_{i+1},\ \text{ and } |\tilde{B}_{i+1}| < C|\tilde{B}_i| \text{ for all } i \in \{1, \ldots, k-1\}.$$

Similarly to (A.21) we have

$$\tfrac{1}{4} \leq |u_{B_0} - u_{2B_1}| \leq \sum_{i=1}^{k} \fint_{\tilde{B}_i} |u(y) - u_{\tilde{B}_i}|\,dy \leq C \sum_{i=1}^{k} \mathrm{diam}(\tilde{B}_i)\Big(\fint_{\tilde{B}_i} |Du(y)|^q\,dy\Big)^{\frac{1}{q}}.$$

Since $2\,\mathrm{diam}\,\tilde{B}_i \leq \mathrm{diam}\,\tilde{B}_{i+1}$ and $q < n$, this implies that

$$C \leq \sum_{i=1}^{k} \mathrm{diam}^{1-\frac{n}{q}}(\tilde{B}_i)\Big(\int_{\tilde{B}_i} |Du(y)|^q\,dy\Big)^{\frac{1}{q}} \leq C\,\mathrm{diam}^{1-\frac{n}{q}}(\tilde{B}_1)\Big(\int_{\Omega} |Du(y)|^q\,dy\Big)^{\frac{1}{q}},$$

which gives us the desired estimate as $\mathrm{diam}\,\tilde{B}_1 = \mathrm{diam}\,B_0 \geq \mathrm{diam}\,F$. □

The following capacitary estimate can be found in [34].

Theorem A.41. *Suppose that Ω is a bounded open set, $E \subset \Omega$ is a continuum and that a continuous function $u \in W^{1,n}(\Omega)$ satisfies $u \geq 1$ on E and has compact support in Ω. Then*

$$\int_{\Omega} |Du(x)|^n\,dx \geq \frac{\omega_{n-1}}{\Big(\log\Big(\frac{C(n)\,\mathrm{diam}\,\Omega}{\mathrm{diam}\,E}\Big)\Big)^{n-1}}.$$

On the other hand, for $\Omega = B(0, R)$ and $E = \overline{B(0, r)}$ with $0 < r < R$, the above estimate is sharp as can be shown by taking $u(x) = \min\{1, \log \frac{R}{|x|} / \log(\frac{R}{r})\}$.

A.10 Solvability of $\Delta u = \varphi$

Solvability of the Poisson equation follows by convolution with the Green's function, see [28, Chap. 2.2.1 (b)].

Theorem A.42. *Let $\varphi \in C_C(\mathbf{R}^n)$. Then there is $u \in C^2(\mathbf{R}^n)$ such that $\Delta u(x) = \varphi(x)$.*

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