Appendix A
Some Useful Variations of Gronwall's Lemma

In (numerical) analysis of differential equations Gronwall's Lemma plays an important role. The original version is due to T.H. Grönwall [31], but there exist a huge number of variations. In this chapter we collect some useful versions and provide those proofs which are not easily found elsewhere in the literature.

The first result is a standard integral version of Gronwall's Lemma, which together with a proof can be found in, for example, [56, Chap. 1,Th. 8.1].

Lemma A.1. *Let $T > 0$ and $c \geq 0$. Let $\varphi, v \colon [0,T] \to \mathbb{R}$ be continuous and nonnegative functions. If*

$$\varphi(t) \leq c + \int_0^t v(\sigma)\varphi(\sigma)\,\mathrm{d}\sigma \quad \text{for all } t \in [0,T],$$

then

$$\varphi(t) \leq c \exp\Big(\int_0^t v(\sigma)\,\mathrm{d}\sigma \Big) \quad \text{for all } t \in [0,T].$$

The next variation of Gronwall's Lemma is a generalization with weak singularities. A reference for this version is, for example, [36, Lemma 7.1.1], while the presented proof follows [23].

Lemma A.2. *Let $T > 0$ and $C_1, C_2 \geq 0$ and let $\varphi \colon [0,T] \to \mathbb{R}$ be a nonnegative and continuous function. Let $\beta > 0$. If we have*

$$\varphi(t) \leq C_1 + C_2 \int_0^t (t-\sigma)^{-1+\beta}\varphi(\sigma)\,\mathrm{d}\sigma \quad \text{for all } t \in (0,T], \tag{A.1}$$

then there exists a constant $C = C(C_2, T, \beta)$ such that

$$\varphi(t) \leq C C_1, \quad \text{for all } t \in (0,T].$$

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Proof. For the proof we recall the following identity

$$\int_0^t (t-\sigma)^{-1+\alpha}\sigma^{-1+\beta}\,\mathrm{d}\sigma = B(\alpha,\beta)t^{-1+\alpha+\beta}, \tag{A.2}$$

where $B(x, y)$ denotes the beta function.

Choose the smallest $n = n(\beta) \in \mathbb{N}$ such that $-1 + n\beta \geq 0$ and iterate the inequality (A.1) $n-1$ times, then by applying (A.2) we obtain

$$\begin{aligned}\varphi(t) &\leq D_1C_1 + C_2\int_0^t (t-\sigma)^{-1+n\beta}\varphi(\sigma)\,\mathrm{d}\sigma\\ &\leq D_1C_1 + D_2T^{-1+n\beta}\int_0^t \varphi(\sigma)\,\mathrm{d}\sigma,\end{aligned}$$

with constants $D_1 = D_1(C_2, T, \beta)$ and $D_2 = D_2(C_2, \beta)$. Now Lemma A.1 yields the desired results. □

While the last two lemmas yield estimates for continuous functions, there also exist discrete analogues. The next lemma is a slightly generalized version of [30, 2.2. (9)] (see J.M. Holte[1] for a more recent presentation).

Lemma A.3. *Let $c \geq 0$ and $(\varphi_j)_{j\geq 1}$ and $(v_j)_{j\geq 1}$ be nonnegative sequences. If*

$$\varphi_j \leq c + \sum_{i=1}^{j-1} v_i\varphi_i \quad \text{for } j \geq 1,$$

then

$$\varphi_j \leq c\prod_{i=1}^{j-1}(1+v_i) \leq c\exp\Big(\sum_{i=1}^{j-1} v_i\Big) \quad \text{for } j \geq 1.$$

We also have a discrete version of Lemma A.2. Here we follow the proof of [23, Lemma 7.1].

Lemma A.4. *For $T > 0$ and $k > 0$ consider $t_j = jk$ with $j = 1, \ldots, N_k$ such that $N_k k \leq T < (N_k+1)k$. Let $C_1, C_2 \geq 0$ and let $(\varphi_j)_{j=1,\ldots,N_k}$ be a nonnegative sequence.*

If for $\beta \in (0, 1]$ we have

$$\varphi_j \leq C_1 + C_2k\sum_{i=1}^{j-1} t_{j-i}^{-1+\beta}\varphi_i \quad \text{for all } j = 1, \ldots, N_k, \tag{A.3}$$

[1] http://homepages.gac.edu/~holte/publications/gronwallTALK.pdf

then there exists a constant $C = C(C_2, T, \beta)$ *such that*

$$\varphi_j \le CC_1 \quad \textit{for all } j = 1, \dots, N_k.$$

In particular, the constant C *does not depend on* k.

Proof. By using (A.2) we get for $\alpha, \beta \in (0, 1]$

$$\begin{aligned} k\sum_{i=0}^{j-1} t_{j-i}^{-1+\alpha} t_{i+1}^{-1+\beta} &\le \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} (t_j - \sigma)^{-1+\alpha} \sigma^{-1+\beta} \, \mathrm{d}\sigma \\ &\le \int_0^{t_j} (t_j - \sigma)^{-1+\alpha} \sigma^{-1+\beta} \, \mathrm{d}\sigma = B(\alpha, \beta) t_j^{-1+\alpha+\beta}. \end{aligned} \tag{A.4}$$

As in the proof of Lemma A.2 we choose the smallest $n = n(\beta) \in \mathbb{N}$ such that $-1 + n\beta \ge 0$ and iterate the inequality (A.3) $n - 1$ times. Then by applying (A.4) we obtain

$$\begin{aligned} \varphi_j &\le D_1 C_1 + D_2 k \sum_{i=1}^{j-n} t_{j-i}^{-1+n\beta} \varphi_i \\ &\le D_1 C_1 + D_2 T^{-1+n\beta} k \sum_{i=1}^{j-1} \varphi_i, \end{aligned}$$

with constants $D_1 = D_1(C_2, T, \beta)$ and $D_2 = D_2(C_2, \beta)$. Now the desired result follows by an application of Lemma A.3. □

Appendix B
Results on Semigroups and Their Infinitesimal Generators

The first section of this chapter provides a short review of the theory of strongly continuous semigroups on Banach spaces. The content is primarily based on [60, 69]. The second section deals with semigroups on Hilbert spaces, whose infinitesimal generators are self-adjoint and have compact inverses. In this case we define fractional powers $(-A)^r$, $r \in (0,\infty)$, of the generator and introduce a characterization of the dual space of the domain $\mathrm{dom}((-A)^r)$.

B.1 Strongly Continuous Semigroups of Bounded Operators

In this section we consider a Banach space B.

Definition B.1. A family $(E(t))_{t\in[0,\infty)}$ of bounded linear operators from B into B is called a *strongly continuous semigroup* (or a C_0-*semigroup*) if:

(i) $E(0) = \mathrm{Id}_B$,
(ii) $E(t+s) = E(t)E(s)$ for all $t, s \geq 0$,
(iii) $\lim_{t\searrow 0} E(t)x = x$ for all $x \in B$.

Strongly continuous semigroups enjoy the following properties.

Lemma B.2. *Let $(E(t))_{t\in[0,\infty)}$ be a C_0-semigroup on a Banach space B. There exist constants $c \geq 0$ and $M \geq 1$ such that*

$$\|E(t)\| \leq M\mathrm{e}^{ct}$$

for all $t \in [0,\infty)$.

For the proof we refer to [60, Chap. 1.2, Th. 2.2]. If $c = 0$ the semigroup is *uniformly bounded*. In addition, if we also have $M = 1$ then we call $(E(t))_{t\in[0,\infty)}$ a *semigroup of contractions*.

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Lemma B.3. *Let $(E(t))_{t\in[0,\infty)}$ be a C_0-semigroup on a Banach space B. Then the mapping*

$$[0,\infty)\times B\to B,\quad (t,x)\mapsto E(t)x$$

is continuous. In particular, for every $x\in B$ the mapping $t\mapsto E(t)x$ is uniformly continuous on compact subintervals of $[0,\infty)$.

A proof of this lemma is found in [69, Lem. VII.4.3] or [60, Chap. 1.2, Cor. 2.3].

The next definition assigns a linear operator to a semigroup.

Definition B.4. Consider a C_0-semigroup $(E(t))_{t\in[0,\infty)}$ on a Banach space B. The linear operator A defined by

$$Ax=\lim_{h\searrow 0}\frac{E(h)x-x}{h}$$

with domain

$$\mathrm{dom}(A)=\left\{x\in B:\lim_{h\searrow 0}\frac{E(h)x-x}{h}\text{ exists in }B\right\}$$

is called the *infinitesimal generator* of the semigroup $(E(t))_{t\in[0,\infty)}$.

Lemma B.5. *Let $(E(t))_{t\in[0,\infty)}$ be a C_0-semigroup on a Banach space B with infinitesimal generator A. Then we have the following properties:*

(i) For all $x\in B$ it holds that

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}E(\sigma)x\,\mathrm{d}\sigma=E(t)x,\quad \textit{for all } t\in[0,\infty).$$

(ii) For all $x\in B$ we have $\int_0^t E(\sigma)x\,\mathrm{d}\sigma\in\mathrm{dom}(A)$ and

$$E(t)x-x=A\Big(\int_0^t E(\sigma)x\,\mathrm{d}\sigma\Big),\quad \textit{for all } t\in[0,\infty).$$

(iii) For all $x\in\mathrm{dom}(A)$, $t\in[0,\infty)$ we have $E(t)x\in\mathrm{dom}(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)x=AE(t)x=E(t)Ax.$$

(iv) For all $x\in\mathrm{dom}(A)$ and $s,t\in[0,\infty)$, $s<t$, it holds that

$$E(t)x-E(s)x=\int_s^t AE(\sigma)x\,\mathrm{d}\sigma=\int_s^t E(\sigma)Ax\,\mathrm{d}\sigma.$$

This lemma coincides with [60, Chap. 1.2, Th. 2.4]. As the next result from [60, Chap. 1.2, Th. 2.6] shows a semigroup is uniquely determined by its infinitesimal generator.

Lemma B.6. *Two C_0-semigroups with the same infinitesimal generator A coincide.*

The following theorem gives a characterization of the infinitesimal generator of a C_0-semigroup of contractions.

Theorem B.7 (Hille–Yosida). *A linear, possibly unbounded operator $A\colon \mathrm{dom}(A) \subset B \to B$ is the infinitesimal generator of a C_0-semigroup of contractions $(E(t))_{t\in[0,\infty)}$ if and only if:*

(i) A is closed and $\mathrm{dom}(A)$ *is dense in B.*
(ii) The resolvent set $\rho(A)$ of A contains the positive real line and

$$\|R(\lambda, A)\| = \|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda}, \quad \textit{for all } \lambda > 0.$$

A proof is given in [60, Chap. 1.3, Th. 3.1] and [69, Th. VII.4.11]. A version of this theorem, which gives a corresponding characterization of the infinitesimal generator of general C_0-semigroups, is found in [69, Th. VII.4.13].

B.2 Fractional Powers of A and the Spaces $\dot{H}^s$

This section deals with semigroups on a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We consider a densely defined, linear, self-adjoint and positive definite operator $A\colon \mathrm{dom}(A) \subset H \to H$, which is not necessarily bounded but with compact inverse. Under these conditions Theorem B.7 yields that $-A$ is the infinitesimal generator of a C_0-semigroup of contractions $(E(t))_{t\in[0,\infty)}$ on H.

By applying the spectral theorem for linear compact and self-adjoint operators [69, Th. VI.3.2] to A^{-1} we obtain the existence of an increasing sequence of real numbers $(\lambda_n)_{n\ge1}$ and an orthonormal basis of eigenvectors $(e_n)_{n\ge1}$ in H such that $Ae_n = \lambda_n e_n$, $n \in \mathbb{N}$, and

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n (\to \infty).$$

We have the following characterization of the domain of A

$$\mathrm{dom}(A) = \Big\{x \in H : \sum_{n=1}^{\infty} \lambda_n^2 (x, e_n)^2 < \infty\Big\}.$$

In fact, from [60, Chap. 2.5, Th. 5.2] it follows that $(E(t))_{t\in[0,\infty)}$ is an analytic semigroup.

The above conditions on A are more restrictive as in [60, Chap. 2.6] but they allow us to define fractional powers of A in a much simpler way (see also [57, Ex. 6.1.2, Ex. 6.1.7]). For any $r \geq 0$ let the operator $A^{\frac{r}{2}} \colon \mathrm{dom}(A^{\frac{r}{2}}) \subset H \to H$ be given by

$$A^{\frac{r}{2}} x = \sum_{n=1}^{\infty} \lambda_n^{\frac{r}{2}} (x, e_n) e_n \tag{B.1}$$

for all

$$x \in \mathrm{dom}(A^{\frac{r}{2}}) = \Big\{ x \in H : \|x\|_r^2 := \sum_{n=1}^{\infty} \lambda_n^r (x, e_n)^2 < \infty \Big\}.$$

By setting $\dot{H}^r := \mathrm{dom}(A^{\frac{r}{2}})$ and $(\cdot,\cdot)_r := (A^{\frac{r}{2}}\cdot, A^{\frac{r}{2}}\cdot)$ we obtain a separable Hilbert space $(\dot{H}^r, (\cdot,\cdot)_r, \|\cdot\|_r)$ for every $r > 0$.

The next result gives a characterization of the dual space $(\dot{H}^r)'$ with $r > 0$. For this we consider the set

$$\dot{H}^{-r} := \Big\{ x = \sum_{n=1}^{\infty} x_n e_n \ : \ x_n \in \mathbb{R},\ n = 1, 2, \dots,$$
$$\text{such that } \|x\|_{-r}^2 := \sum_{n=1}^{\infty} \lambda_n^{-r} x_n^2 < \infty \Big\}$$

and, analogously to (B.1), we define the fractional power of A for negative exponents by

$$A^{-\frac{r}{2}} x = \sum_{n=1}^{\infty} \lambda_n^{-\frac{r}{2}} x_n e_n$$

for all $x = \sum_{n=1}^{\infty} x_n e_n \in \dot{H}^{-r}$. It always holds that $H \subset \dot{H}^{-r}$ but in general we have $H \neq \dot{H}^r$ for every $r > 0$. It follows that $\dot{H}^{-r}$ is the largest set such that $A^{-\frac{r}{2}}$ maps into H. In this sense $\dot{H}^{-r} = \mathrm{dom}(A^{-\frac{r}{2}})$.

As above we endow $\dot{H}^{-r}$ with the inner product $(\cdot,\cdot)_{-r} := (A^{-\frac{r}{2}}\cdot, A^{-\frac{r}{2}}\cdot)$ and the norm $\|\cdot\|_{-r} = \|A^{-\frac{r}{2}} \cdot\|$.

Theorem B.8. *For $r > 0$ the dual space $(\dot{H}^r)'$ is isometrically isomorphic to $\dot{H}^{-r}$. In particular, $\dot{H}^{-r}$ is a separable Hilbert space.*

Proof. We follow the lines of the proof of [69, Th. II.2.3] together with some suitable generalizations.

Let us define a linear operator $T \colon \dot{H}^{-r} \to (\dot{H}^r)'$ by

$$(Tx)(y) = \sum_{n=1}^{\infty} x_n (y, e_n)$$

for $x = \sum_{n=1}^{\infty} x_n e_n \in \dot{H}^{-r}$ and $y = \sum_{n=1}^{\infty}(y, e_n)e_n \in \dot{H}^r$. Clearly, $T\colon \dot{H}^{-r} \to (\dot{H}^r)'$ and $Tx\colon \dot{H}^r \to \mathbb{R}$ are both linear mappings for every $x \in \dot{H}^{-r}$. Further,

$$\begin{aligned}|(Tx)(y)| &= \Big|\sum_{n=1}^{\infty} x_n (y, e_n)\Big| = \Big|\sum_{n=1}^{\infty} \lambda_n^{-\frac{r}{2}} x_n \lambda_n^{\frac{r}{2}} (y, e_n)\Big| \\ &\leq \Big(\sum_{n=1}^{\infty} \lambda_n^{-r} x_n^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \lambda_n^{r} (y, e_n)^2\Big)^{\frac{1}{2}} = \|x\|_{-r} \|y\|_r .\end{aligned}$$

Consequently,

$$\sup_{\substack{y\in\dot{H}^r \\ \|y\|_r=1}} |(Tx)(y)| \leq \|x\|_{-r},$$

which shows that T indeed maps into $(\dot{H}^r)'$.

Next, we prove that T is one-to-one. For this we consider $x \in \dot{H}^{-r}$ such that $Tx = 0 \in (\dot{H}^r)'$. Then we obtain

$$0 = (Tx)(e_n) = x_n \quad \text{for every } n = 1, 2, \ldots .$$

Hence, $x = 0 \in \dot{H}^{-r}$.

It remains to show that T is onto and isometric. For this we consider an arbitrary element $z \in (\dot{H}^r)'$. We set $x_n := z(e_n)$, $n = 1, 2, \ldots$, and prove that

$$x := \sum_{n=1}^{\infty} x_n e_n \in \dot{H}^{-r}, \quad \|x\|_{-r} \leq \sup_{\substack{y\in\dot{H}^r \\ \|y\|_r=1}} |z(y)| \tag{B.2}$$

and $Tx = z$. Let us define

$$y_n := \lambda_n^{-r} x_n \quad \text{for } n = 1, 2, \ldots .$$

Then, it holds for every $N \in \mathbb{N}$

$$\begin{aligned}0 &\leq \sum_{n=1}^{N} \lambda_n^{-r} x_n^2 = \sum_{n=1}^{N} \lambda_n^{r} y_n^2 \\ &= \sum_{n=1}^{N} y_n x_n = \sum_{n=1}^{N} y_n z(e_n) = z\Big(\sum_{n=1}^{N} y_n e_n\Big) \\ &\leq \sup_{\substack{y\in\dot{H}^r \\ \|y\|_r=1}} |z(y)| \Big\|\sum_{n=1}^{N} y_n e_n\Big\|_r = \sup_{\substack{y\in\dot{H}^r \\ \|y\|_r=1}} |z(y)| \Big(\sum_{n=1}^{N} \lambda_n^{r} y_n^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\Big(\sum_{n=1}^{N} \lambda_n^{-r} x_n^2\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{N} \lambda_n^{r} y_n^2\Big)^{\frac{1}{2}} \leq \sup_{\substack{y \in \dot{H}^r \\ \|y\|_r = 1}} |z(y)|$$

for every $N \in \mathbb{N}$. By taking the limit $N \to \infty$ we obtain (B.2).

In addition, it holds

$$(Tx)(e_n) = x_n = z(e_n) \quad \text{for all } n = 1, 2, \dots$$

and both linear mappings Tx and z coincide for every $N \in \mathbb{N}$ on $\operatorname{span}\{e_n \,:\, n = 1, \dots, N\}$. Since both mappings are continuous in $\dot{H}^r$ they also coincide on the closure of $\operatorname{span}\{e_n \,:\, n = 1, 2, \dots\}$ with respect to the norm $\|\cdot\|_r$. But this closure is equal to $\dot{H}^r$, which completes the proof. □

Next, it is worth noting that the spectral structure of A carries over to the semigroup $(E(t))_{t \in [0,T]}$. In fact, by setting

$$\tilde{E}(t)x := \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t}(x, e_n)e_n \tag{B.3}$$

for all $x \in H$ and $t \in [0, \infty)$, we obtain a C_0-semigroup on H. Consider $x \in H$ such that

$$y := \lim_{h \searrow 0} \frac{\tilde{E}(h)x - x}{h} \quad \text{exists in } H. \tag{B.4}$$

Then, we get by Parseval's identity

$$0 = \lim_{h \searrow 0} \Big\| \frac{\tilde{E}(h)x - x}{h} - y \Big\|^2 = \lim_{h \searrow 0} \sum_{n=1}^{\infty} \Big(\frac{\mathrm{e}^{-\lambda_n h} - 1}{h}(x, e_n) - (y, e_n) \Big)^2,$$

which yields

$$(y, e_n) = \lim_{h \searrow 0} \frac{\mathrm{e}^{-\lambda_n h} - 1}{h}(x, e_n) = -\lambda_n (x, e_n).$$

Since $y \in H$ it follows that $x \in \operatorname{dom}(A) = \dot{H}^2$ and $y = -Ax$.

That the limes in (B.4) exists for all $x \in \operatorname{dom}(A)$ follows in a similar way by applying Lebesgue's dominated convergence theorem. Therefore, the infinitesimal generator of $\tilde{E}$ coincides with $-A$ and from Lemma B.6 we obtain that (B.3) is indeed a spectral representation of E.

The next lemma gives some very useful norm estimates of $(E(t))_{t\in[0,\infty)}$. Since we make use of them very frequently, we present a proof, but only under the above conditions on A. The estimates *(i)*, *(ii)* and *(iv)* are also valid for analytic semigroups in general. For this we refer to [60, Chap. 2.6, Th. 6.13] and [57, Th. 6.1.8].

Lemma B.9. *Under the above conditions on the infinitesimal generator $-A$ of the semigroup $(E(t))_{t\in[0,\infty)}$ the following properties hold true:*

(i) For any $\mu \geq 0$ it holds that

$$A^{\mu} E(t)x = E(t)A^{\mu} x \quad \textit{for all } x \in \dot{H}^{2\mu}$$

and there exists a constant $C = C(\mu)$ such that

$$\|A^{\mu} E(t)\| \leq C t^{-\mu} \quad \textit{for } t > 0.$$

(ii) For any $0 \leq \nu \leq 1$ there exists a constant $C = C(\nu)$ such that

$$\|A^{-\nu}(E(t) - \mathrm{Id}_H)\| \leq C t^{\nu} \textit{ for } t \geq 0.$$

(iii) For any $0 \leq \rho \leq 1$ there exists a constant $C = C(\rho)$ such that

$$\int_{\tau_1}^{\tau_2} \|A^{\frac{\rho}{2}} E(\tau_2 - \sigma)x\|^2 \,\mathrm{d}\sigma \leq C(\tau_2 - \tau_1)^{1-\rho} \,\|x\|^2 \textit{ for all } x \in H,\ 0 \leq \tau_1 < \tau_2.$$

(iv) For any $0 \leq \rho \leq 1$ there exists a constant $C = C(\rho)$ such that

$$\left\| A^{\rho} \int_{\tau_1}^{\tau_2} E(\tau_2 - \sigma)x \,\mathrm{d}\sigma \right\| \leq C(\tau_2 - \tau_1)^{1-\rho} \|x\| \textit{ for all } x \in H,\ 0 \leq \tau_1 < \tau_2.$$

Proof. The first part of *(i)* follows directly from the spectral representations (B.1) and (B.3).

In order to prove the second part of *(i)* we make use of the fact that the function $x \mapsto x^{\mu}\mathrm{e}^{-x}$ is bounded for $x \in [0,\infty)$. Consequently, for a constant $C = C(\mu) > 0$ we have

$$\|A^{\mu} E(t)\| = \sup_{n\geq 1} |\lambda_n^{\mu} \mathrm{e}^{-t\lambda_n}| \leq C t^{-\mu}.$$

For *(ii)* we apply the fact that the function $x \mapsto x^{-\nu}(1 - \mathrm{e}^{-x})$ is bounded for $x \in [0,\infty)$ and $\nu \in [0,1]$. Hence, for a constant $C > 0$ which only depends on $\nu \in [0,1]$

$$\|A^{-\nu}(E(t) - \mathrm{Id}_H)\| = \sup_{n\geq 1} \left| \frac{1 - \mathrm{e}^{-t\lambda_n}}{\lambda_n^{\nu}} \right| \leq C t^{\nu}.$$

For the proof of *(iii)* we use the expansion of $x \in H$ in terms of the eigenbasis $(e_n)_{n\geq 1}$ of the operator A. By Parseval's identity we get

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \left\|A^{\frac{\rho}{2}} E(\tau_2-\sigma)x\right\|^2 \mathrm{d}\sigma &= \int_{\tau_1}^{\tau_2} \left\|\sum_{n=1}^{\infty} A^{\frac{\rho}{2}} E(\tau_2-\sigma)(x,e_n)e_n\right\|^2 \mathrm{d}\sigma \\ &= \sum_{n=1}^{\infty} \int_{\tau_1}^{\tau_2} (x,e_n)^2 \lambda_n^{\rho} e^{-2\lambda_n(\tau_2-\sigma)}\,\mathrm{d}\sigma \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(x,e_n)^2\lambda_n^{\rho-1}\big(1-e^{-2\lambda_n(\tau_2-\tau_1)}\big).\end{aligned}$$

Again, by the boundedness of the function $x \mapsto x^{\rho-1}(1-\mathrm{e}^{-x})$ for $x \in [0,\infty)$ and $\rho \in [0,1]$ there exists a constant $C = C(\rho) > 0$ such that

$$\int_{\tau_1}^{\tau_2} \left\|A^{\frac{\rho}{2}} E(\tau_2-\sigma)x\right\|^2 \mathrm{d}\sigma \leq C(\rho)(\tau_2-\tau_1)^{1-\rho}\sum_{n=1}^{\infty}(x,e_n)^2,$$

which completes the proof of *(iii)*.

The following proof of *(iv)* also works for analytic semigroups in general. By Lemma B.5*(ii)* we first notice that

$$\begin{aligned}\left\|A^{\rho}\int_{\tau_1}^{\tau_2} E(\tau_2-\sigma)x\ \mathrm{d}\sigma\right\| &= \left\|A^{\rho-1}A\int_{0}^{\tau_2-\tau_1} E(\sigma)x\ \mathrm{d}\sigma\right\| \\ &= \left\|A^{\rho-1}\big(E(\tau_2-\tau_1)-I\big)x\right\|\end{aligned}$$

Then, *(iv)* follows from *(ii)*. □

The following result is concerned with the continuity of the semigroup in the border case $\rho = 1$ of Lemma B.9*(iii)* and *(iv)*.

Lemma B.10. *Let* $0 \leq \tau_1 < \tau_2$. *Then we have:*

(i)

$$\lim_{\tau_2-\tau_1\to 0}\int_{\tau_1}^{\tau_2} \left\|A^{\frac{1}{2}} E(\tau_2-\sigma)x\right\|^2 \mathrm{d}\sigma = 0 \quad \textit{for all } x \in H,$$

(ii)

$$\lim_{\tau_2-\tau_1\to 0}\left\|A\int_{\tau_1}^{\tau_2} E(\tau_2-\sigma)x\,\mathrm{d}\sigma\right\| = 0 \quad \textit{for all } x \in H.$$

Proof. As in the proof of Lemma B.9 we use the orthogonal expansion of $x \in H$ with respect to the eigenbasis $(e_n)_{n\geq 1}$ of the operator A. Thus, for *(i)* we get, as in the proof of Lemma B.9*(iii)*,

$$\int_{\tau_1}^{\tau_2} \left\| A^{\frac{1}{2}} E(\tau_2 - \sigma)x \right\|^2 \mathrm{d}\sigma = \frac{1}{2} \sum_{n=1}^{\infty} (x, e_n)^2 \left(1 - \mathrm{e}^{-2\lambda_n(\tau_2 - \tau_1)}\right).$$

We apply Lebesgue's dominated convergence theorem. Note that the sum is dominated by $\frac{1}{2}\|x\|^2$ for all $\tau_2 - \tau_1 \geq 0$. Moreover, for every $n \geq 1$ we have

$$\lim_{\tau_2 - \tau_1 \to 0} \left(1 - \mathrm{e}^{-2\lambda_n(\tau_2 - \tau_1)}\right)(x, e_n)^2 = 0.$$

Hence, Lebesgue's theorem gives us *(i)*.

The case *(ii)* is actually true for all strongly continuous semigroups, since by Lemma B.5*(ii)* and Lemma B.3 it holds

$$\lim_{\tau_2 - \tau_1 \to 0} \left\| A \int_{\tau_1}^{\tau_2} E(\tau_2 - \sigma)x \,\mathrm{d}\sigma \right\| = \lim_{\tau_2 - \tau_1 \to 0} \left\| E(\tau_2 - \tau_1)x - x \right\| = 0.$$

The proof is complete. □

We close this section with an extension of the linear operators $(E(t))_{t\in[0,\infty)}$ to the spaces $\dot{H}^r$ with $r < 0$. In the same way as in (B.3) we define

$$E^r(t) := \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} x_n e_n \tag{B.5}$$

for all $t \in [0, \infty)$ and $x = \sum_{n=1}^{\infty} x_n e_n \in \dot{H}^r$ with $r < 0$. As above, the family of linear operators $(E^r(t))_{t\in[0,\infty)}$ is a strongly continuous semigroup on $\dot{H}^r$ and for all $x \in H$ it holds that $E^r(t)x = E(t)x$. More precisely, it holds that $E(t)$ and $E^r(t)$ are similar, that is

$$E^r(t)x = A^{\frac{r}{2}} E(t) A^{-\frac{r}{2}},$$

for all $t > 0$, which implies that $E(t)$ and $E^r(t)$ have the same spectrum [24, Chap. II, Cor. 5.3]. Further, as in [24, Chap. II, Cor. 5.5] one can show that the infinitesimal generator of $(E^r(t))_{t\in[0,\infty)}$ is the unique continuous extension of A to an isometry from $\dot{H}^{r+2}$ to $\dot{H}^r$.

In most occasions we drop the index r in the notation of $E^r(t)$ and also write $E(t)$ for the extended semigroup.

Appendix C
A Generalized Version of Lebesgue's Theorem

Lebesgue's dominated convergence theorem is an important and well-known tool in measure theory and probability. However, the standard formulation of the theorem turns out to be too restrictive for some proofs in this book and we rely on the following generalized version which is due to H.W. Alt [1] and [2, 1.23].

Theorem C.1. *Let $(S, \mathscr{B}, \mu)$ denote a measure space and Y a Banach space with norm $|\cdot|$. Consider Borel measurable mappings $f, f_n \colon S \to Y$, $n = 1, 2, \dots$, and mappings $g, g_n \in L^1(S;\mathbb{R})$, $n = 1, 2, \dots$, such that $g_n \to g$ as $n \to \infty$ with respect to the norm in $L^1(S;\mathbb{R})$. If it holds that*

$$\begin{aligned} |f_n| &\le |g_n| && \mu\text{-almost everywhere for all } n \in \mathbb{N} \text{ and} \\ f_n &\to f && \mu\text{-almost everywhere for } n \to \infty, \end{aligned}$$

then $f, f_n \in L^1(S;Y)$ and $f_n \to f$ in $L^1(S;Y)$ as $n \to \infty$.

In particular, it follows that

$$\lim_{n\to\infty} \int_S f_n \, \mathrm{d}\mu = \int_S \lim_{n\to\infty} f_n \, \mathrm{d}\mu.$$

For the proof we refer to [1] and [2, 1.23].

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