Appendix A
General Concepts

In the first part of the appendix, we give an overview of some of the basic mathematical tools used in this book.

A.1 Linear and Multilinear Algebra

In this section, we present some of the (multi-)linear algebra used in this book which may not be as well known as other linear algebraic concepts.

Singular Values of a Linear Operator

Let $(E, \langle\cdot,\cdot\rangle_E)$ and $(F, \langle\cdot,\cdot\rangle_F)$ be d-dimensional Euclidean spaces and $L : E \to F$ a linear operator. Then the *adjoint* of L is the unique linear operator $L^* : F \to E$ such that $\langle Lx, y\rangle_F = \langle x, L^*y\rangle_E$ for all $x \in E$ and $y \in F$. The *singular values* of L are the nonnegative square roots of the eigenvalues of the positive semi-definite self-adjoint operator $L^*L : E \to E$. We denote them by $\sigma_1(L) \geq \cdots \geq \sigma_d(L) \geq 0$. The number of the positive singular values equals the rank of L. Using the singular values we can define the *absolute determinant* of L by

$$|\det L| := |\det L^*L|^{1/2} = \sigma_1(L) \cdot \ldots \cdot \sigma_d(L). \tag{A.1}$$

If L is an isomorphism, the singular values of L are all positive and $\sigma_i(L)^{-1}$, $i = 1,\ldots,d$, are the singular values of L^{-1}. The geometric meaning of the singular values becomes clear in the following proposition which can be found in Boichenko et al. [9, Chap. I, Propositions 1.2.2 and 7.2.1] or Temam [108, Sect. V.1.3].

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Proposition A.1. *Let $L : E \to F$ be a linear operator between d-dimensional Euclidean spaces. If B is a closed (or open) ball in E of radius r, then LB is a closed (or open) ellipsoid in* $\operatorname{im} L \subset F$ *with semi-axes of lengths $r\sigma_i(L)$, $\sigma_i(L)>0$.*

Furthermore, we define the *singular value function of order* k $(0 \leq k \leq d)$ of a linear operator $L : E \to F$ by

$$\alpha_k(L) := \begin{cases} \sigma_1(L)\sigma_2(L)\cdots\sigma_k(L) & \text{for } k > 0, \\ 1 & \text{for } k = 0. \end{cases}$$

Note that singular value functions can also be defined for non-integer values of k (cf. Boichenko et al. [9]). If $T : E \to F$ and $S : F \to G$ are linear operators between d-dimensional Euclidean spaces, then *Horn's inequality*

$$\alpha_k(ST) \leq \alpha_k(S)\alpha_k(T) \quad \text{for } k = 0, 1, \ldots, d$$

holds. In the case that $k = d$ the singular value functions coincide with the absolute determinants and therefore equality holds (cf. [9, Chap. I, Proposition 7.4.3]).

Tensors on a Vector Space

Let V be a d-dimensional vector space over $\mathbb{R}$. As usual, V^* denotes the dual space of V, the space of *covectors* or real-valued linear functionals on V. A *covariant k-tensor* on V is a multilinear map

$$F : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

Similarly, a *contravariant l-tensor* is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \to \mathbb{R}.$$

A *tensor of type* (k, l) or (k, l)*-tensor* is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \times \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

By convention, a tensor of type $(0, 0)$ is just a real number.

The space of all covariant k-tensors is denoted by $T^k(V)$, the space of contravariant l-tensors by $T_l(V)$, and the space of (k, l)-tensors by $T^k_l(V)$. There are some natural identifications: $T^k_0(V) = T^k(V)$, $T^0_l(V) = T_l(V)$, $T^1(V) = V^*$, and $T_1(V) = V^{**} = V$.

The *tensor product* $F \otimes G$ of two tensors $F \in T^k_l(V)$ and $G \in T^p_q(V)$ is the $(k+p, l+q)$-tensor defined by

$$\begin{aligned}&F \otimes G(\omega^1, \dots, \omega^{l+q}, X_1, \dots, X_{k+p})\\ &\quad := F(\omega^1, \dots, \omega^l, X_1, \dots, X_k)G(\omega^{l+1}, \dots, \omega^{l+q}, X_{k+1}, \dots, X_{k+p}).\end{aligned}$$

If $(E_1, \dots, E_d)$ is a basis of V, we denote by $(\varphi^1, \dots, \varphi^d)$ the corresponding dual basis of V^*, defined by $\varphi^i(E_j) = \delta^i_j$. A basis of $T^k_l(V)$ is given by the set of tensors of the form

$$E_{j_1} \otimes \dots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \dots \otimes \varphi^{i_k},$$

where the indices i_p, j_q range from 1 to d. Hence, the dimension of $T^k_l(V)$ is d^{k+l}. Every tensor of type (k, l) can be written in terms of this basis as

$$F = F^{j_1 \dots j_l}_{i_1 \dots i_k} E_{j_1} \otimes \dots \otimes E_{j_l} \otimes \varphi^{i_1} \otimes \dots \otimes \varphi^{i_k}.$$

Here we use the *Einstein summation convention*, that is, if in any term the same index appears twice, once as a lower and once as an upper index, that term is assumed to be summed over the possible values of this index (which is usually from 1 to the dimension of the space).

An important class of tensors are the *alternating tensors*, those which change sign whenever two arguments are interchanged. We denote by $\bigwedge^k V^*$ the space of all covariant alternating k-tensors on V, also called *(exterior) k-forms*. The space $\bigwedge^k V^*$ is called the *k-th exterior power of V*. By $\bigwedge^k V$ we denote the space of all contravariant alternating k-tensors, also called *k-multivectors*.[1] By convention, $\bigwedge^0 V = \bigwedge^0 V^* = \mathbb{R}$.

There is a natural bilinear associative product on forms called the *wedge product*, defined on one-forms by setting

$$\omega^1 \wedge \dots \wedge \omega^k(X_1, \dots, X_k) := \det\left(\omega^i(X_j)\right),$$

and extending by linearity. If $(\omega_1, \dots, \omega_d)$ is a basis of V^*, an associated basis of $\bigwedge^k V^*$ is given by the tensors of the form

$$\omega_{i_1} \wedge \dots \wedge \omega_{i_k},$$

where $1 \le i_1 < i_1 < \dots < i_k \le d$. Consequently, the dimension of $\bigwedge^k V^*$ is $\binom{d}{k} = d!/(k!(d-k)!)$.

[1] A word of caution: Some authors write $\bigwedge^k V$ for the alternating covariant tensors and $\bigwedge^k V^*$ for the alternating contravariant tensors. A discussion of the reason why can be found in Lee [75, Chap. 12].

The wedge product can also be defined on k-multivectors by setting

$$\xi^1 \wedge \ldots \wedge \xi^k(\omega_1, \ldots, \omega_k) := \det\left(\omega_i(\xi^j)\right)$$

for one-multivectors (which are the elements of $V^{**} = V$), and extending by linearity.

Taking the direct sum of the spaces $\bigwedge^k V$ (or alternatively $\bigwedge^k V^*$) for $0 \leq k \leq d$, one obtains another vector space

$$\bigwedge V := \bigoplus_{k=0}^{d} \bigwedge^k V,$$

of dimension 2^d, which is called the *exterior algebra* of V. If $L : V \to W$ is a linear operator between real vector spaces of dimension d_1 and d_2, respectively, then the *k-th exterior power* of L is the linear operator defined by

$$L^{\wedge k} : \bigwedge^k V \to \bigwedge^k W, \quad \xi^1 \wedge \ldots \wedge \xi^k \mapsto L\xi^1 \wedge \ldots \wedge L\xi^k$$

for all $\xi^1, \ldots, \xi^k \in V$ and extending by linearity. This definition naturally gives an induced operator $L^{\wedge} : \bigwedge V \to \bigwedge W$, called the *exterior power* of L.

Now assume that V is endowed with an inner product $\langle \cdot, \cdot \rangle$. Then an associated inner product on $\bigwedge^k V$ is defined by

$$\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle_{\bigwedge^k V} := \det\left(\langle v_i, w_j \rangle\right),$$

and extending by linearity in each argument. The associated norm is denoted by $|\cdot|_{\bigwedge^k V}$. If V and W are Euclidean spaces of the same dimension and $L : V \to W$ is a linear operator, the operator norm of $L^{\wedge k} : \bigwedge^k V \to \bigwedge^k W$ with respect to the norms $|\cdot|_{\bigwedge^k V}$ and $|\cdot|_{\bigwedge^k W}$ is the product of the k greatest singular values of L, $\|L^{\wedge k}\| = \sigma_1(L) \cdots \sigma_k(L)$ (cf. Arnold [4, Proposition 3.2.7]). The operator norm of the exterior power $L^{\wedge} : \bigwedge V \to \bigwedge W$ then is $\|L^{\wedge}\| = \max_{0 \leq k \leq d}(\sigma_1(L) \cdots \sigma_k(L))$.

An operator $L \in \mathscr{L}(V, V)$ induces an operator $L_k \in \mathscr{L}(\bigwedge^k V, \bigwedge^k V)$ by

$$\begin{aligned} L_k(v_1 \wedge \ldots \wedge v_k) := {} & Lv_1 \wedge \ldots \wedge v_k + v_1 \wedge Lv_2 \wedge \ldots \wedge v_k + \ldots \\ & + v_1 \wedge \ldots \wedge Lv_k, \end{aligned}$$

called the *k-th derivation operator of L*. The eigenvalues of this operator are the sums $\lambda_{i_1} + \cdots + \lambda_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq d$ and $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of L. Moreover, we have the relation $(\mathrm{e}^{tL})^{\wedge k} = \mathrm{e}^{tL_k}$ (cf. Arnold [4, Lemma 3.2.6]).

The following lemma can be used to prove generalizations of the Liouville formula for ordinary differential equations. It can be found in Temam [108, Chap. V, Lemma 1.2].

Lemma A.1. *For all $k \in \{1, \ldots, d\}$ and all $v_1, \ldots, v_k \in V$ it holds that*

$$\langle L_k(v_1 \wedge \ldots \wedge v_k), v_1 \wedge \ldots \wedge v_k \rangle_{\bigwedge^k V} = |v_1 \wedge \ldots \wedge v_k|^2_{\bigwedge^k V} \operatorname{tr}(L \circ Q),$$

where $Q = Q(v_1, \ldots, v_k)$ denotes the orthogonal projection in V onto the linear subspace spanned by $v_1, \ldots, v_k$.

A.2 Differentiable Manifolds

The natural state space of a control system given by ordinary differential equations is a differentiable manifold. Usually, this is a submanifold of some Euclidean space $\mathbb{R}^n$. But for the analysis of bilinear systems, for instance, also systems on more abstractly defined manifolds like projective spaces play an important role. In this section, we provide the necessary background on differentiable manifolds which is needed for the treatment of smooth systems in this book. In particular, for the understanding of Chaps. 4–6, the reader should be familiar with the material presented here. Good references are, for instance, the books Gallot et al. [48], Lee [75] or Bullo and Lewis [15]. However, we note that in the last reference (which is a control theory book) no proofs for the differential-geometric results can be found, but the exposition is very clear, many examples are given, and, in contrast to almost all books on differential geometry, the theory is exposed under minimal differentiability assumptions.

Definition of a Manifold

Let M be a second-countable Hausdorff space.[2] A family $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ is called a $\mathscr{C}^k$*-atlas* on M for some $k \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$ if the following axioms are satisfied:

(i) $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M;
(ii) For each $\alpha \in A$, $\phi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism onto an open subset V_α of $\mathbb{R}^d$ for some $d \in \mathbb{N}$;

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[2]We recall that a topological space X is called *Hausdorff* if any two distinct points $x, y \in X$ have disjoint open neighborhoods. The space X is called *second-countable* if its topology has a countable basis, that is, there is a countable family of open sets such that every open set can be written as the union of sets in this family.

(iii) For all $\alpha, \beta \in A$, the transition function

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is of class $\mathscr{C}^k$ (if $U_\alpha \cap U_\beta = \emptyset$, this is trivially satisfied).

If $k = 0$, the transition functions are only assumed to be continuous, which is already satisfied by Axiom (ii). In the case $k = \omega$, the transition functions are assumed to be real-analytic. Every $\mathscr{C}^k$-atlas $\mathscr{A}$ is contained in a unique maximal $\mathscr{C}^k$-atlas $\mathscr{A}_{\max}$, that is, a $\mathscr{C}^k$-atlas with the property that no further charts can be added without violating Axiom (iii). The pair $(M, \mathscr{A}_{\max})$ is then called a $\mathscr{C}^k$-*manifold* and if $k \geq 1$, $\mathscr{A}_{\max}$ is called a *differentiable structure* on M. In the case that $k = 0$ we also speak of a *topological manifold*, in the case $k \geq 1$ of a *differentiable manifold of class* $\mathscr{C}^k$, in the case $k = \infty$ of a *smooth manifold*, and in the case $k = \omega$ of a *real-analytic manifold*. In the rest of this section we restrict ourselves to the case $k \geq 1$. Usually, when we speak of a $\mathscr{C}^k$-manifold, we do not explicitly mention the atlas, that is, we only write M instead of $(M, \mathscr{A})$ or $(M, \mathscr{A}_{\max})$.

The elements (ϕ_α, U_α) of a $\mathscr{C}^k$-atlas $\mathscr{A}$ are called *charts* and the inverse maps $\phi_\alpha^{-1} : V_\alpha \to U_\alpha$ *local coordinate systems* of M. A chart (ϕ_α, U_α) is said to be a *chart around* $p \in M$ if $p \in U_\alpha$. If the natural number d (the dimension of the Euclidean space where ϕ_α takes its values) is the same for all charts, we call this number the *dimension* of M and write $d = \dim M$. If M is connected, this is automatically satisfied. Throughout this book we assume that all manifolds have a well-defined dimension.

The definition of a $\mathscr{C}^k$-manifold implies several topological properties of the underlying topological space M. In particular, M is locally compact, locally path-connected, and metrizable. If M is connected, it is automatically path-connected. In general, the connected components of M coincide with the path-connected components.

When speaking of a d-dimensional real vector space V as a differentiable (real-analytic) manifold, we mean V endowed with the maximal $\mathscr{C}^\omega$-atlas which contains a chart (ϕ, V), where $\phi : V \to \mathbb{R}^d$ is a linear isomorphism.

Every open subset N of a d-dimensional $\mathscr{C}^k$-manifold $(M, \mathscr{A})$ is itself a d-dimensional $\mathscr{C}^k$-manifold with atlas $\{(\phi|_{U \cap N}, U \cap N) : (\phi, U) \in \mathscr{A}\}$. Given two $\mathscr{C}^k$-manifolds $(M, \mathscr{A})$ and $(N, \mathscr{B})$ of dimensions k and l, respectively, their Cartesian product $M \times N$ (endowed with the product topology) becomes a $(k + l)$-dimensional $\mathscr{C}^k$-manifold with the maximal $\mathscr{C}^k$-atlas which contains the *product atlas*

$$\{(\phi \times \psi, U \times V) : (\phi, U) \in \mathscr{A},\ (\psi, V) \in \mathscr{B}\}.$$

A manifold of this type is called a *product manifold*. Inductively, the product of any finite number of $\mathscr{C}^k$-manifolds can be defined.

Tangent Spaces and Derivatives

In order to develop a differential calculus on $\mathscr{C}^k$-manifolds ($k \geq 1$), the notions of tangent vectors and tangent spaces have to be introduced. In this and the following sections, when we speak of a $\mathscr{C}^{r-1}$-manifold or $\mathscr{C}^{r-1}$-map, we use the convention that $r - 1 = \infty$ if $r = \infty$ and $r - 1 = \omega$ if $r = \omega$.

Let M be a d-dimensional $\mathscr{C}^k$-manifold. On the set of all triples (p, ϕ, ξ), where $p \in M$, (ϕ, U) is a chart around p, and $\xi \in \mathbb{R}^d$, we introduce an equivalence relation by

$$(p, \phi, \xi) \sim (p, \psi, \eta) \quad :\Leftrightarrow \quad \eta = \mathrm{D}(\psi \circ \phi^{-1})(\phi(p))\xi.$$

The equivalence class $[p, \phi, \xi]$ of a triple (p, ϕ, ξ) is called a *tangent vector at* p. The *tangent space at* p, denoted by T_pM, is defined as the set of all tangent vectors at p, and is endowed with the structure of a real vector space, given by

- $[p, \phi, \xi] + [p, \phi, \eta] := [p, \phi, \xi + \eta]$ for all $\xi, \eta \in \mathbb{R}^d$;
- $\lambda[p, \phi, \xi] := [p, \phi, \lambda\xi]$ for all $\lambda \in \mathbb{R}$, $\xi \in \mathbb{R}^d$.

It can easily be shown that these operations are well-defined and give T_pM the structure of a vector space isomorphic to $\mathbb{R}^d$. The zero vector $[p, \phi, 0] \in T_pM$ is denoted by 0_p.

A map $f : M \to N$ between $\mathscr{C}^k$-manifolds M and N is said to be *differentiable at* $p \in M$ if there are charts (ϕ, U) of M around p and (ψ, V) of N around $f(p)$ such that $f(U) \subset V$ and the *local representation*

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$$

of f is differentiable at $\phi(p)$ in the usual sense. If $\psi \circ f \circ \phi^{-1}$ is of class $\mathscr{C}^r$, $r \in \{1, \dots, k\}$, in a neighborhood of $\phi(p)$, then f is said to be $\mathscr{C}^r$*-differentiable at* p. It follows from Axiom (iii) in the definition of $\mathscr{C}^k$-manifolds that this definition is independent of the chosen charts. If f is $\mathscr{C}^r$-differentiable at every $p \in M$, then f is called a $\mathscr{C}^r$*-map*. If f is additionally invertible and also $f^{-1} : N \to M$ is a $\mathscr{C}^r$-map, then f is called a $\mathscr{C}^r$*-diffeomorphism*. It is easy to see that every $\mathscr{C}^r$-map is continuous and hence every $\mathscr{C}^r$-diffeomorphism is a homeomorphism. For the set of all $\mathscr{C}^r$-maps $f : M \to N$ we use the notation $\mathscr{C}^r(M, N)$.

Given a $\mathscr{C}^1$-map $f : M \to N$ between $\mathscr{C}^k$-manifolds M and N, the *derivative of* f *at* $p \in M$ is the linear map $\mathrm{d}_p f : T_pM \to T_{f(p)}N$, defined by

$$\mathrm{d}_p f[p, \phi, \xi] := [f(p), \psi, \mathrm{D}(\psi \circ f \circ \phi^{-1})(\phi(p))\xi],$$

where (ϕ, U) and (ψ, V) are charts of M and N around p and $f(p)$, respectively. One easily shows that this definition is independent of the choice of the charts.

A $\mathscr{C}^r$*-curve* is a continuous map $c : I \to M$ defined on an interval $I \subset \mathbb{R}$ with values in a $\mathscr{C}^k$-manifold M such that the restriction of c to the interior of I is a

$\mathscr{C}^r$-map. Given a $\mathscr{C}^1$-curve $c : I \to M$ and $t \in \operatorname{int} I$, the *tangent vector to c at t* is an element of $T_{c(t)}M$, given by

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = \dot{c}(t) := \left[c(t), \phi, \frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ c)(t)\right],$$

where (ϕ, U) is any chart around $c(t)$. Every tangent vector can be obtained as the tangent vector to a $\mathscr{C}^1$-curve and hence

$$T_pM = \left\{\dot{c}(0) \mid c : (-\varepsilon, \varepsilon) \to M \text{ is a } \mathscr{C}^1\text{-curve with } c(0) = p\right\}.$$

The derivative satisfies the following properties:

- The *chain rule* holds:

$$\mathrm{d}_p(f \circ g) = \mathrm{d}_{g(p)}f \circ \mathrm{d}_pg$$

  for all $\mathscr{C}^1$-maps $g : M \to N$ and $f : N \to P$.
- If $f : M \to N$ is a $\mathscr{C}^1$-map and $c : (-\varepsilon, \varepsilon) \to M$ is a $\mathscr{C}^1$-curve with $c(0) = p$, then

$$\mathrm{d}_pf(\dot{c}(0)) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)(0).$$

- If $f : M \to N$ is a $\mathscr{C}^1$-diffeomorphism, then $\mathrm{d}_pf : T_pM \to T_{f(p)}N$ is an isomorphism for all $p \in M$.
- The *inverse function theorem* holds: If $f : M \to N$ is a $\mathscr{C}^r$-map, $r \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, and d_pf is invertible for some $p \in M$, then there are open neighborhoods U of p and V of $f(p)$ such that $V = f(U)$ and the restriction $f|_U : U \to V$ is a $\mathscr{C}^r$-diffeomorphism with

$$\mathrm{d}_{f(p)}f^{-1} = (\mathrm{d}_pf)^{-1}.$$

To every chart (ϕ, U) around a point $p \in M$ we can associate an isomorphism $T_pM \to \mathbb{R}^d$ by $\alpha_{p,\phi} : [p, \phi, \xi] \mapsto \xi$. The preimages of the standard basis vectors $e_1, \dots, e_d \in \mathbb{R}^d$ under $\alpha_{p,\phi}$ form a basis of T_pM. They are denoted by

$$\partial_i\phi(p) := \alpha_{p,\phi}^{-1}e_i = [p, \phi, e_i], \quad i = 1, \dots, d.$$

The reason for this notation stems from the fact that every tangent vector $[p, \phi, \xi]$ can be identified canonically with a directional derivative acting on differentiable functions on M via

$$[p, \phi, \xi]f := \mathrm{D}(f \circ \phi^{-1})(\phi(p))\xi \quad \text{for all } f \in \mathscr{C}^1(M, \mathbb{R}). \tag{A.2}$$

Hence, $\partial_i\phi(p)$ corresponds to the i-th partial derivative. To make formulas better readable, we introduce another (more common) notation for the expression $\partial_i\phi(p)f$. Namely, we write

$$\frac{\partial f}{\partial \phi^i}(p) := \partial_i\phi(p)f = \frac{\partial(f\circ\phi^{-1})}{\partial x_i}(\phi(p)).$$

If V is a d-dimensional real vector space with its standard $\mathscr{C}^\omega$-atlas, the tangent space T_pV at any point $p \in V$ can be identified canonically with V itself via

$$T_pV \ni [p,\phi,\xi] \mapsto \phi^{-1}\xi \in V$$

for every chart (ϕ, V) such that $\phi : V \to \mathbb{R}^d$ is a linear isomorphism.

The *tangent bundle* TM of the d-dimensional $\mathscr{C}^k$-manifold M is defined as the disjoint union of all tangent spaces T_pM, $p \in M$. It can be endowed with an atlas in a canonical way such that it becomes a $2d$-dimensional $\mathscr{C}^{k-1}$-manifold. The charts of this atlas are defined as follows: If (ϕ, U) is a chart of M, every tangent vector $v \in T_pM$ with $p \in U$ can be written uniquely as $v = v^i\,\partial_i\phi(p)$. Then a chart of TM is given by (ψ, TU) with

$$\psi(v) = (\phi(p), v^1, \dots, v^d) \quad \text{for } v \in T_pM,\ p \in U.$$

For every $p \in M$ we denote by T_p^*M the dual space of T_pM, that is, $T_p^*M := (T_pM)^*$. The disjoint union T^*M of all these dual spaces is called the *cotangent bundle* of M, and it can also be endowed with a canonical $\mathscr{C}^{k-1}$-atlas.

If (ϕ, U) is a chart of M around p, then a basis $\mathrm{d}\phi^1(p), \dots, \mathrm{d}\phi^d(p)$ of T_p^*M is given by

$$\mathrm{d}\phi^i(p)[p,\phi,\xi] := \xi_i \quad \text{for } \xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d.$$

This basis is the dual basis of $\partial_1\phi(p), \dots, \partial_d\phi(p)$, that is, $\mathrm{d}\phi^i(p)\partial_j\phi(p) = \delta^i_j$.

Given a product manifold $M_1 \times M_2$ such that $k = \dim M_1$ and $l = \dim M_2$, the tangent space $T_{(p,q)}(M_1 \times M_2)$ at some point $(p,q) \in M_1 \times M_2$ can be identified canonically with $T_pM_1 \times T_qM_2$ by

$$[(p,q), \phi\times\psi, \xi] \mapsto ([p,\phi,(\xi_1,\dots,\xi_k)], [q,\psi,(\xi_{k+1},\dots,\xi_{k+l})]),$$

where $\xi_1,\dots,\xi_{k+l}$ are the coordinates of ξ in the standard basis of $\mathbb{R}^{k+l}$. Using this identification, the derivative of a $\mathscr{C}^1$-map $f : M_1 \times M_2 \to P$ can be computed as

$$\mathrm{d}_{(p,q)}f(v,w) = \mathrm{d}_pf(\cdot,q)v + \mathrm{d}_qf(p,\cdot)w \quad \text{for all } (v,w) \in T_pM_1 \times T_qM_2$$

with the partial maps $f(\cdot,q) : M_1 \to P$ and $f(p,\cdot) : M_2 \to P$. For the partial derivatives we also use the common notation $\partial f/\partial x_1$, $\partial f/\partial x_2$.

Tensor Fields

All the objects defined on $\mathscr{C}^k$-manifolds which are interesting for us, in particular vector fields and Riemannian metrics, can be regarded as special tensor fields. Tensor fields are the natural extensions of tensors on a vector space (see Sect. A.1) to the tangent bundle of a manifold.

Given a $\mathscr{C}^k$-manifold M with $k \geq 2$, the *bundle of* (k,l)*-tensors* $T_l^k M$ is defined as the disjoint union of the spaces $T_l^k(T_pM)$, $p \in M$. Analogously, the *bundle of* k*-forms* $\bigwedge^k M$ is the disjoint union of the spaces $\bigwedge^k T_pM$, $p \in M$. There are the usual identifications $T_1^0 M = TM$ and $T_0^1 M = \bigwedge^1 M = T^*M$. Each of these spaces can be endowed with a canonical $\mathscr{C}^{k-1}$-atlas (and also with the structure of a differentiable vector bundle).

A *tensor field* on M of type (k,l) is a map $t : M \to T_l^k M$, $p \mapsto t_p$, such that $t_p \in T_l^k(T_pM)$ for all $p \in M$. Given a chart (ϕ, U) of M, we can express each t_p in terms of the bases of T_pM and T_p^*M introduced above, that is

$$t_p = t_{i_1 \dots i_k}^{j_1 \dots j_l}(p) \partial_{j_1}\phi(p) \otimes \cdots \otimes \partial_{j_l}\phi(p) \otimes \mathrm{d}\phi^{i_1}(p) \otimes \cdots \otimes \mathrm{d}\phi^{i_k}(p).$$

If the so-defined coordinate functions $t_{i_1 \dots i_k}^{j_1 \dots j_l} : U \to \mathbb{R}$ are of class $\mathscr{C}^r$ ($r \in \{0, \dots, k-1\}$) for every chart (ϕ, U), we say that the tensor field t is of class $\mathscr{C}^r$. A *differential* k*-form* is a tensor field $\omega : M \to \bigwedge^k M$ of class $\mathscr{C}^r$, $r \geq 1$.

Obviously, the tensor product and the wedge product for tensors on vector spaces extend to tensor fields by performing these operations pointwise. For the corresponding operations on tensor fields we use the same notation as we do for tensors. For instance, the wedge product of two differential forms ω and μ is denoted by $\omega \wedge \mu$ and defined by $(\omega \wedge \mu)_p := \omega_p \wedge \mu_p$ for all $p \in M$.

A function $f : M \to \mathbb{R}$ can be regarded as a tensor field of type $(0,0)$ (since tensors of type $(0,0)$ are by convention just real numbers). We introduce the notation $\mathscr{C}^r(M)$ for the space of all $\mathscr{C}^r$-functions from M to $\mathbb{R}$. For the tensor product $f \otimes t$ of a function $f \in \mathscr{C}^r(M)$ and an arbitrary $\mathscr{C}^r$-tensor field t we simply write ft.

Vector Fields

To define ordinary differential equations on $\mathscr{C}^k$-manifolds, we need the notion of a vector field. Given a $\mathscr{C}^k$-manifold M with $k \geq 2$, a tensor field X of type $(0,1)$ and class $\mathscr{C}^r$, $r \in \{0, \dots, k-1\}$, is called a $\mathscr{C}^r$*-vector field* on M. Such X assigns to each $p \in M$ a tangent vector $X_p \in T_pM$ (using the natural identification $T_1 M = TM$). For the real vector space of all $\mathscr{C}^r$-vector fields on M we introduce the notation $\mathscr{X}^r(M)$.

Each vector field $X \in \mathscr{X}^r(M)$ defines an ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x).$$

Assuming that $r \geq 1$, for each $x \in M$ there exists a unique maximal solution $\lambda : I \to M$ with initial condition $\lambda(0) = x$ whose domain I is an open interval containing 0. That is, the curve λ is of class $\mathscr{C}^1$ and satisfies $\dot{\lambda}(t) = X(\lambda(t))$ for all $t \in I$. The solutions to all initial conditions $(0, x)$, $x \in M$, can be condensed into one map $(t, x) \mapsto X_t(x)$, that is, $X_0(x) = x$ and $(\partial/\partial t)X_t(x) \equiv X(X_t(x))$. For fixed $t \in \mathbb{R}$, the map $X_t : x \mapsto X_t(x)$ is a local $\mathscr{C}^r$-diffeomorphism of M in the sense that the domain $\operatorname{dom} X_t$ of X_t is an open set in M and $X_t : \operatorname{dom} X_t \to X_t(\operatorname{dom} X_t)$ is a $\mathscr{C}^r$-diffeomorphism. The domain $\operatorname{dom} X_t$ is the set of all elements of M whose maximal solutions extend up to time t, that is, their interval of definition (α, ω) contains t. The vector field X is called *complete* if $\operatorname{dom} X_t = M$ for all $t \in \mathbb{R}$. Equivalently, X is complete if all maximal solutions are defined on $\mathbb{R}$. If the vector field X has the property that the image of every maximal solution of the associated differential equation is relatively compact, X is automatically complete. In particular, this is the case if M is compact.

The map $(t, x) \mapsto X_t(x)$ is called the *flow* of the vector field. Restricted to the domains, the flow satisfies the homomorphism property: $X_{t+s} = X_t \circ X_s$, that is, if $X_s(x)$ and $X_t(X_s(x))$ are defined, then $X_{t+s}(x)$ is defined and the equality $X_{t+s}(x) = X_t(X_s(x))$ holds. It is clear that $\operatorname{dom} X_{t+s} = X_t(\operatorname{dom} X_s) \cap \operatorname{dom} X_s$. In particular, the elements of the flow commute with each other: $X_t \circ X_s = X_s \circ X_t$ and $X_{-t} = (X_t)^{-1}$.

Given two vector fields $X, Y \in \mathscr{X}^r(M)$, $r \geq 1$, another vector field of class $\mathscr{C}^{r-1}$, called the *Lie bracket of* X *and* Y and denoted by $[X, Y]$, is defined via

$$[X, Y](p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(\mathrm{d}_{X_t(p)} X_{-t}\big)\, Y(X_t(p)).$$

The Lie bracket satisfies the following properties:

- Bilinearity over $\mathbb{R}$: If X, Y, W and Z are vector fields and $a, b \in \mathbb{R}$, then

$$[aX + Y, bZ + W] = ab[X, Z] + a[X, W] + b[Y, Z] + [Y, W].$$

- Anti-symmetry: $[X, Y] = -[Y, X]$.
- Jacobi-identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

In particular, for a smooth manifold M the vector space $\mathscr{X}^\infty(M)$ becomes a Lie algebra when endowed with the Lie bracket of vector fields.

Using the interpretation of tangent vectors as derivations (A.2), a vector field $X \in \mathscr{X}^r(M)$ can be applied to a function $f \in \mathscr{C}^r(M)$, $r \geq 1$:

$$(Xf)(p) := X_p f \quad \text{for all } p \in M.$$

The resulting function $Xf : M \to \mathbb{R}$ is of class $\mathscr{C}^{r-1}$. If $X, Y \in \mathscr{X}^r(M)$ and $f \in \mathscr{C}^r(M)$, then $[X, Y]f = X(Yf) - Y(Xf)$.

Riemannian Metrics

Every $\mathscr{C}^k$-manifold is metrizable, but there is no canonical way to measure distances. However, there exists a large class of nice "smooth" metrics. These are defined as follows. Let M be a connected $\mathscr{C}^k$-manifold with $k \geq 2$. Given a symmetric $\mathscr{C}^r$-tensor field g of type $(2, 0)$ on M, $r \in \{0, \ldots, k-1\}$, every tangent space T_pM becomes endowed with a bilinear symmetric form[3]

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

We assume that g_p is positive definite for every $p \in M$, that is, g_p is an inner product on T_pM. The induced norms $|\cdot|_p$ on T_pM, $p \in M$, allow to measure the lengths of tangent vectors and therefore also the lengths of differentiable curves on M. Precisely, let $c : [a, b] \to M$ be a piecewise $\mathscr{C}^1$-curve, that is, there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that each of the restrictions $c|_{[t_i, t_{i+1}]}$, $i = 0, \ldots, n-1$, is a $\mathscr{C}^1$-curve. Then the *length* of c is defined as

$$\mathscr{L}(c) := \int_a^b |\dot{c}(t)|_{c(t)}\, \mathrm{d}t.$$

From the assumption that M is connected it follows that each two points $p, q \in M$ can be joined by a piecewise $\mathscr{C}^1$-curve. In this case, a metric which is compatible with the topology of M is given by

$$\varrho(p, q) := \inf \left\{ \mathscr{L}(c) \mid c : [a, b] \to M \text{ piecewise } \mathscr{C}^1 \text{ with } c(a) = p,\ c(b) = q \right\}. \tag{A.3}$$

The tensor field g is called a *Riemannian metric* on M and the metric ϱ the *Riemannian distance* associated with g. If M is a $\mathscr{C}^k$-manifold and g a $\mathscr{C}^{k-1}$-Riemannian metric on M, then (M, g) is called a *Riemannian manifold of class* $\mathscr{C}^k$ or a *Riemannian* $\mathscr{C}^k$*-manifold*. Using partitions of unity, one can construct a Riemannian metric of class $\mathscr{C}^{k-1}$ on every $\mathscr{C}^k$-manifold, $2 \leq k \leq \infty$.

With respect to a chart (ϕ, U) of M, the Riemannian metric g has a local expression

$$g_p = g_{ij}(p)\mathrm{d}\phi^i(p) \otimes \mathrm{d}\phi^j(p),$$

[3]The tensor field g is called symmetric if it is symmetric at every point, that is, $g_p(v, w) = g_p(w, v)$ for all $v, w \in T_pM$ and $p \in M$.

where the real numbers $g_{ij}(p)$, $1 \le i, j \le d$, define a positive definite symmetric matrix $(g_{ij}(p))$. The entries of the inverse of this matrix are denoted by $g^{ij}(p)$, that is, $g_{ik}g^{kj} = \delta_i^j$.

For two points $p, q \in M$ the infimum in (A.3) need not be attained, that is, a curve of minimal length joining p and q not necessarily exists. However, locally (in sufficiently small neighborhoods of points) such shortest curves do exist. A curve which locally realizes the shortest distance between two points in its image is called a *geodesic*. However, it is not convenient to define geodesics via the property of realizing shortest distances, but rather by the property of being "straight lines" in M, that is, being as straight as possible. In $\mathbb{R}^d$, a straight line, given as a curve $t \mapsto a + tv$, is characterized by the property that its second derivative vanishes. To adapt this criterion to Riemannian manifolds, the notion of a *connection* needs to be introduced.

Let (M, g) be a d-dimensional Riemannian manifold of class $\mathscr{C}^k$ with $k \ge 3$. To each chart (ϕ, U) of M one can associate d^3 $\mathscr{C}^{k-2}$-functions by

$$\Gamma_{ij}^k := \frac{g_{kl}}{2}\left(\frac{\partial g_{il}}{\partial \phi^j} + \frac{\partial g_{jl}}{\partial \phi^i} - \frac{\partial g_{ij}}{\partial \phi^l}\right), \quad \Gamma_{ij}^k : U \to \mathbb{R}.$$

These functions are called the *Christoffel symbols* of (M, g) with respect to the chart (ϕ, U). They have the property that $\Gamma_{ij}^k = \Gamma_{ji}^k$, that is, they are symmetric in the lower two indices.

Using the Christoffel symbols, one can define the *Levi–Civita connection* associated with (M, g), which is an operator assigning to a pair (X, Y) of $\mathscr{C}^r$-vector fields with $r \in \{1, \ldots, k-2\}$ a $\mathscr{C}^{r-1}$-vector field $\nabla_X Y$. Locally, we can write any vector fields X and Y as $X = X^i \partial_i \phi$ and $Y = Y^j \partial_j \phi$. Then $\nabla_X Y$ is defined by

$$(\nabla_X Y)(p) := X^i(p)\left(Y^j(p)\Gamma_{ij}^k(p)\partial_k \phi(p) + \frac{\partial Y^j}{\partial \phi^i}(p)\partial_j \phi(p)\right).$$

It can be checked easily that this definition is independent of the chosen charts. The Levi–Civita connection satisfies the following identities for all $X, X_1, X_2, Y, Y_1, Y_2 \in \mathscr{X}^r(M)$ and $f \in \mathscr{C}^r(M)$:

- $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$;
- $\nabla_{fX} Y = f \nabla_X Y$;
- $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$;
- $\nabla_X (fY) = f \nabla_X Y + (Xf)Y$;
- $[X, Y] = \nabla_X Y - \nabla_Y X$;
- $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

We can interpret $\nabla_X Y$ as the vector field obtained by computing (pointwise) the directional derivative of Y in direction X. In fact, $(\nabla_X Y)_p$ depends only on $X(p)$ and the values of Y in an arbitrarily small neighborhood of p. Hence, to every

$\mathscr{C}^r$-vector field we can assign its *covariant derivative* at $p \in M$ by $\nabla X(p)v := (\nabla_v X)(p)$, $\nabla X(p) : T_pM \to T_pM$. The *symmetrized covariant derivative* of X at p is defined by

$$S\nabla X(p) := \frac{1}{2}\left(\nabla X(p) + \nabla X(p)^*\right),$$

where $\nabla X(p)^*$ denotes the adjoint operator (with respect to the Riemannian metric). With respect to a chart (ϕ, V) and the associated basis $\partial_1\phi(p), \ldots, \partial_d\phi(p)$, we have a matrix expression for $S\nabla X(p)$, which is given by

$$s_\nu^\mu = \frac{1}{2}\left(\frac{\partial X^\mu}{\partial \phi^\nu} + \frac{\partial X^\kappa}{\partial \phi^\theta} g^{\mu\theta} g_{\kappa\nu} + X^i g^{\mu l} \frac{\partial g_{\nu l}}{\partial \phi^i}\right). \tag{A.4}$$

In order to define geodesics as the "straight lines" in M, we need a way to compute the second derivative of a curve. To this end, a further concept derived from the Levi–Civita connection has to be introduced. Given a $\mathscr{C}^r$-curve $c : I \to M$, a *vector field along* c is a map $X : I \to TM$ such that $X(t) \in T_{c(t)}M$ for all $t \in I$. The vector field X is of class $\mathscr{C}^r$ if the coordinate functions $X^1, \ldots, X^d : I \to \mathbb{R}$, defined by writing $X(t) = X^i(t)\partial_i\phi(c(t))$ with respect to a chart (ϕ, U), are of class $\mathscr{C}^r$ for every chart around some point $c(t_0)$, $t_0 \in I$. The real vector space of all $\mathscr{C}^r$-vector fields along a $\mathscr{C}^r$-curve c is denoted by $\mathscr{X}_c^r$. Now we can differentiate vector fields along c by the local formula

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) := (X^i)'(t)\partial_i\phi(c(t)) + X^i(t)\left(\nabla_{\dot{c}(\cdot)}\partial_i\phi\right)(c(t)).$$

This defines an operator $\mathrm{D}/\mathrm{d}t$, called the *covariant derivative along* c, which assigns to a $\mathscr{C}^r$-vector field along c a $\mathscr{C}^{r-1}$-vector field, and has the following properties:

- For all $X_1, X_2 \in \mathscr{X}_c^r$,

$$\frac{\mathrm{D}(X_1 + X_2)}{\mathrm{d}t} = \frac{\mathrm{D}X_1}{\mathrm{d}t} + \frac{\mathrm{D}X_2}{\mathrm{d}t};$$

- For each $X \in \mathscr{X}_c^r$ and $f \in \mathscr{C}^r(M)$,

$$\frac{\mathrm{D}(fX)}{\mathrm{d}t} = f'X + f\frac{\mathrm{D}X}{\mathrm{d}t};$$

- For every $Y \in \mathscr{X}^r(M)$,

$$\frac{\mathrm{D}(Y \circ c)}{\mathrm{d}t} = \left(\nabla_{\dot{c}(\cdot)}Y\right) \circ c;$$

- For all $X, Y \in \mathscr{X}_c^r$,

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{c(t)}(X(t), Y(t)) = g_{c(t)}\left(\frac{\mathrm{D}X}{\mathrm{d}t}(t), Y(t)\right) + g_{c(t)}\left(X(t), \frac{\mathrm{D}Y}{\mathrm{d}t}(t)\right). \quad \text{(A.5)}$$

Finally, we can define a *geodesic* as a $\mathscr{C}^2$-curve $c : I \to M$ such that $(\mathrm{D}\dot{c}/\mathrm{d}t) \equiv 0$, that is, the covariant derivative of the tangent vector field $t \mapsto \dot{c}(t)$ along c vanishes. In local coordinates, this reads

$$\ddot{c}^k(t) + \dot{c}^i(t)\dot{c}^j(t)\Gamma_{ij}^k(c(t)) = 0 \quad \text{for } k = 1, \ldots, d,$$

where $(c^1(t), \ldots, c^d(t)) = \phi(c(t))$. This is a second-order ordinary differential equation and the Picard–Lindelöf theorem guarantees existence and uniqueness of solutions. Therefore, for every $p \in M$ and $v \in T_pM$ there exists a unique maximal geodesic $c_v : I \to M$ with $c(0) = p$ and $\dot{c}(0) = v$. Geodesics have the following desired properties: Every $\mathscr{C}^1$-curve $c : [a,b] \to M$ with $\mathscr{L}(c) \leq \mathscr{L}(\tilde{c})$ for all $\mathscr{C}^1$-curves $\tilde{c} : [a,b] \to M$ with $\tilde{c}(a) = c(a)$ and $\tilde{c}(b) = c(b)$, is a geodesic. On the other hand, for every $p \in M$ there exists $\varepsilon > 0$ such that for all $\delta \in [0, \varepsilon)$ and for every $v \in T_pM$ with $|v|_p = 1$ the geodesic $c_v : [0, \delta] \to M$ is the shortest curve between its endpoints. Furthermore, it can be seen easily that every geodesic is parametrized proportionally to its arclength, that is, $|\dot{c}(t)|_{c(t)}$ is constant.

The subset of T_pM, where $c_v(1)$ is defined, contains an open neighborhood U_p of 0_p, such that the map

$$\exp_p : U_p \to M, \quad \exp_p(v) := c_v(1),$$

is a $\mathscr{C}^{k-2}$-diffeomorphism onto its image. In particular, it holds that

$$\mathrm{d}_{0_p} \exp_p = \mathrm{id}_{T_pM} .$$

The map $\exp_p$ is called the *Riemannian exponential map* at $p \in M$.

By the *theorem of Hopf–Rinow* (cf., for instance, Gallot et al. [48, Theorem 2.103]), the following assertions are equivalent for a Riemannian manifold:

(a) All maximal geodesics are defined on $\mathbb{R}$;
(b) There exists a point $p_0 \in M$ such that all geodesics starting at p_0 are defined on $\mathbb{R}$;
(c) Every bounded and closed subset of M is compact;
(d) Endowed with the Riemannian distance, M is a complete metric space.

On a Riemannian manifold (M, g) one can define the absolute determinant $|\det \mathrm{d}_p f|$ for a $\mathscr{C}^r$-map $f : M \to M$ by using the definition (A.1) via the singular values. Then $|\det \mathrm{d}_{(\cdot)} f| : M \to \mathbb{R}_+$ is a $\mathscr{C}^{r-1}$-function. Moreover, one can define the divergence of a vector field $X \in \mathscr{X}^r(M)$ by

$$\operatorname{div} X(p) := \operatorname{tr}\big(\nabla X(p) : T_pM \to T_pM\big) .$$

Then $\operatorname{div} X : M \to \mathbb{R}$ is a $\mathscr{C}^{r-1}$-function.

On every Riemannian $\mathscr{C}^k$-manifold (M, g), $k \geq 2$, one can define a canonical Borel measure $\mathrm{vol} = \mathrm{vol}_g$, called the *Riemannian volume*, as follows. For a Borel set $A \subset M$ which is contained in the domain of a chart (ϕ, U), we set

$$\mathrm{vol}(A) := \int_{\phi(A)} \sqrt{\det[g_{ij}(\phi^{-1}(x))]}\mathrm{d}x,$$

where the integral is the usual Lebesgue integral on $\mathbb{R}^d$, $d = \dim M$. Using the transformation theorem for Lebesgue integration, one shows that this definition is independent of the chosen chart. Then vol is extended naturally to all Borel subsets of M. Using these definitions, one finds that the transformation rule

$$\int_{f(A)} \varphi \, \mathrm{dvol} = \int_A \varphi \circ f |\det \mathrm{d}f| \, \mathrm{dvol}$$

holds for every $\mathscr{C}^1$-diffeomorphism $f : M \to M$ and every integrable function $\varphi : M \to \mathbb{R}$.

A.3 Carathéodory Differential Equations

Continuous-time control systems are usually given by ordinary differential equations of the form $\dot{x}(t) = F(x(t), \omega(t))$ with measurable control functions ω. The standard results about ordinary differential equations such as the Picard–Lindelöf theorem about existence and uniqueness of solutions assume that the right-hand side of the equation is continuous in t. Differential equations whose right-hand side is only measurable are called *Carathéodory differential equations* and most of the theory for equations with continuous right-hand side is also valid for those (with minor modifications which are mostly obvious). In this section, we present the results about Carathéodory equations needed for the treatment of control systems in this book.

The Carathéodory Flow Box Theorem

Recall that an *absolutely continuous curve* is a map $\gamma : I \to \mathbb{R}^d$, defined on some interval $I \subset \mathbb{R}$, with the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite system $\{[\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]\}$ of disjoint subintervals of I the implication

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta \quad \Rightarrow \quad \sum_{i=1}^{n} |\gamma(\beta_i) - \gamma(\alpha_i)| < \varepsilon$$

holds (for some norm $|\cdot|$ on $\mathbb{R}^d$). Equivalently, we can require that every coordinate function $\gamma_i : I \to \mathbb{R}$, $i = 1, \ldots, d$, has this property. Any absolutely continuous curve is continuous and differentiable Lebesgue almost everywhere. A function $\gamma : I \to \mathbb{R}$ is absolutely continuous if and only if its derivative exists almost everywhere and defines a Lebesgue integrable function $\dot{\gamma} : I \to \mathbb{R}$ such that $\gamma(t) \equiv \gamma(t_0) + \int_{t_0}^{t} \dot{\gamma}(s)\mathrm{d}s$ for some $t_0 \in I$. A curve $\gamma : I \to \mathbb{R}^d$ is called *locally absolutely continuous* if the restriction of γ to every compact subinterval $J \subset I$ is absolutely continuous. A locally absolutely continuous curve on a differentiable manifold is defined as follows.

Definition A.1. Let M be a $\mathscr{C}^k$-manifold, $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$. A map $\gamma : I \to M$, defined on some interval $I \subset \mathbb{R}$, is called a *locally absolutely continuous curve* if $\varphi \circ \gamma : I \to \mathbb{R}$ is locally absolutely continuous for every $\varphi \in \mathscr{C}^k(M)$.

To describe the properties of the right-hand sides of Carathéodory differential equations, we need to introduce the following notions.

Definition A.2. Let M be a $\mathscr{C}^k$-manifold, $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, and $I \subset \mathbb{R}$ a nonempty interval. A *Carathéodory function* on M is a map $\varphi : I \times M \to \mathbb{R}$ with the property that $\varphi_t : x \mapsto \varphi(t, x)$ is continuous for each $t \in I$ and $\varphi_x : t \mapsto \varphi(t, x)$ is Lebesgue measurable for each $x \in M$. A Carathéodory function is *locally integrally bounded* if, for each compact subset $K \subset M$, there exists a positive locally integrable function $\psi_K : I \to \mathbb{R}$ such that $|\varphi(t, x)| \leq \psi_K(t)$ for all $t \in I$ and $x \in K$. A Carathéodory function $\varphi : I \times M \to \mathbb{R}$ is of *class* $\mathscr{C}^k$ if $\varphi_t = \varphi(t, \cdot) : M \to \mathbb{R}$ is of class $\mathscr{C}^k$ for each $t \in I$. If $r \in \mathbb{N}$, then a Carathéodory function φ is *locally integrally of class* $\mathscr{C}^r$ if it is of class $\mathscr{C}^r$, and $X_1 \cdots X_r(\varphi_t)$[4] is locally integrally bounded for all $t \in I$ and $X_1, \ldots, X_r \in \mathscr{X}^r(M)$. If φ is locally integrally of class $\mathscr{C}^r$ for every $r \in \mathbb{N}$, then it is *locally integrally of class* $\mathscr{C}^\infty$.

For $k = \omega$ locally integrally class $\mathscr{C}^k$-functions are defined in a different way. We refer to Bullo and Lewis [15, Sect. A.2.1] for further details.

Definition A.3. Let M be a $\mathscr{C}^{k+1}$-manifold, $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, and $I \subset \mathbb{R}$ a nonempty interval. A map $f : I \times M \to TM$ with the property that $f(t, x) \in T_x M$ for all $(t, x) \in I \times M$, is called a *Carathéodory vector field* on M if for every continuous one-form $\alpha : M \to T^*M$ the function $\alpha \cdot f : I \times M \to \mathbb{R}$, $(t, x) \mapsto \alpha(x) f(t, x)$, is a Carathéodory function. A Carathéodory vector field f is *locally integrally of class* $\mathscr{C}^k$ if $\alpha \cdot f$ is locally integrally of class $\mathscr{C}^k$ for every $\mathscr{C}^k$-one-form α. If $f : I \times M \to TM$ is a Carathéodory vector field on M, the equation

$$\dot{x}(t) = f(t, x(t)) \tag{A.6}$$

[4] Here we mean the application of the vector fields $X_1, \ldots, X_r$ as differential operators acting on functions $M \to \mathbb{R}$.

is called a *Carathéodory differential equation* or a *differential equation of Carathéodory type*. A *solution* of (A.6) is a locally absolutely continuous curve $\gamma : J \to M$, defined on some open subinterval $J \subset I$, such that $\dot{\gamma}(t) = f(t, \gamma(t))$ for Lebesgue almost all $t \in J$.

The following result about existence and uniqueness of solutions for Carathéodory differential equations can be found in Bullo and Lewis [15, Theorem A.11] as the *time-dependent flow box theorem*.

Theorem A.1 (Flow Box Theorem). *Let $f : I \times M \to TM$ be a locally integrally class $\mathscr{C}^k$-vector field, $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, and let $(t_0, x_0) \in I \times M$. Then there exists a triple (U, T, Φ) (called a* flow box *of f at (t_0, x_0)) with the following properties:*

(i) U is an open subset of M containing x_0;
(ii) $T > 0$ or $T = \infty$;
(iii) $\Phi : (t_0 - T, t_0 + T) \times U \to M$ is a map having the following properties:

(a) the map $t \mapsto \Phi(t, x)$ is locally absolutely continuous for each $x \in U$;
(b) the map $x \mapsto \Phi(t, x)$ is of class $\mathscr{C}^k$ for each $t \in (t_0 - T, t_0 + T)$;
(c) $t \mapsto \Phi(t, x)$ is the unique solution of $\dot{x}(t) = f(t, x(t))$ with $\Phi(t_0, x) = x$;

(iv) for all $t \in (t_0 - T, t_0 + T)$, $\Phi_t : U \to M$ is a $\mathscr{C}^k$-diffeomorphism onto its image, where $\Phi_t(x) = \Phi(t, x)$.

Furthermore, if $(\tilde{U}, \tilde{T}, \tilde{\Phi})$ is another such triple, then Φ and $\tilde{\Phi}$ agree when restricted to $((t_0 - T, t_0 + T) \cap (t_0 - \tilde{T}, t_0 + \tilde{T})) \times (U \cap \tilde{U})$.

For linear equations of Carathéodory type the usual *variation-of-constants formula* holds (see Aulbach and Wanner [5, Theorem 2.10]).

Proposition A.2. *Let $I \subset \mathbb{R}$ be a nonempty interval and $A : I \to \mathbb{R}^{d \times d}$, $b : I \to \mathbb{R}^d$, locally integrable mappings. Then the equation*

$$\dot{x}(t) = A(t)x(t) + b(t) \tag{A.7}$$

is a Carathéodory differential equation. The solution $\Phi(t; t_0, x_0)$ of the corresponding initial value problem (A.7), $x(t_0) = x_0$, exists and is unique with

$$\Phi(t; t_0, x_0) = \Psi(t, t_0)x_0 + \int_{t_0}^{t} \Psi(t, s)b(s)\mathrm{d}s$$

for all $(t, t_0, x_0) \in I \times I \times \mathbb{R}^d$, where $t \mapsto \Psi(t, t_0) \in \mathrm{GL}(\mathbb{R}^d)$ is the unique solution of the initial value problem

$$\dot{X}(t) = A(t)X(t), \quad X(t_0) = I \in \mathbb{R}^{d \times d}.$$

The Variational Equation and Applications

For a Carathéodory differential equation on a Riemannian manifold (M, g) the variational equation can be written in a covariant way. See the following proposition whose proof is standard and will be omitted.

Proposition A.3. *Let (M, g) be a Riemannian $\mathscr{C}^k$-manifold, $k \geq 2$. Consider a locally integrally class $\mathscr{C}^{k-1}$-vector field $f : I \times M \to TM$ and the corresponding differential equation*

$$\dot{x}(t) = f(t, x(t))$$

with flow box (U, T, Φ) at $(t_0, x_0) \in I \times M$. Then for any $v \in T_{x_0}M$ the curve

$$c(t) := \mathrm{d}_{x_0}\Phi_t(v), \quad c : (t_0 - T, t_0 + T) \to TM,$$

is locally absolutely continuous and satisfies the Riemannian variational equation

$$\frac{\mathrm{D}z}{\mathrm{d}t}(t) = \nabla f_t(\Phi_t(x_0))z(t) \tag{A.8}$$

almost everywhere, where $\mathrm{D}/\mathrm{d}t$ denotes the covariant derivative along the solution $\Phi(\cdot, x_0)$.[5]

The preceding proposition has two important applications, the *Wazewski inequality* and the *(generalized) Liouville formula*. The Wazewski inequality gives an estimate for the operator norm of the derivative $\mathrm{d}_x\Phi_t$ (given a flow box (U, T, Φ) of a Carathéodory vector field).

Proposition A.4 (Wazewski Inequality). *Let (M, g) be a Riemannian $\mathscr{C}^k$-manifold, $k \geq 3$. Consider a locally integrally class $\mathscr{C}^{k-1}$-vector field $f : I \times M \to TM$ and the corresponding differential equation*

$$\dot{x}(t) = f(t, x(t))$$

with flow box (U, T, Φ) at $(t_0, x_0) \in I \times M$. Then it holds that

$$\|\mathrm{d}_{x_0}\Phi_t\| \leq \exp\left(\int_{t_0}^{t} \lambda_{\max}(S\nabla f_s(\Phi_s(x_0)))\mathrm{d}s\right)$$

for all $t \in [t_0, t_0 + T)$, where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue and $S\nabla(\cdot)$ the symmetrized covariant derivative.

[5] Although we have only defined the covariant derivative of a $\mathscr{C}^r$-vector field along a $\mathscr{C}^r$-curve, this notion also makes sense if both the curve and the vector field are only locally absolutely continuous.

Proof. Let $x_t :\equiv \Phi_t(x_0)$ and $\lambda(t) :\equiv \lambda_{\max}(S\nabla f_t(x_t))$. Let $z : J \to TM$, $t_0 \in J \subset I$, be a locally absolutely continuous vector field along x_t which solves the variational equation (A.8). Then for almost all $t \in J$ we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|z(t)|^2 &= \frac{\mathrm{d}}{\mathrm{d}t} g_{x_t}(z(t),z(t)) \overset{(A.5)}{=} g_{x_t}\left(\frac{\mathrm{D}z}{\mathrm{d}t}(t), z(t)\right) + g_{x_t}\left(z(t), \frac{\mathrm{D}z}{\mathrm{d}t}(t)\right) \\
&= g_{x_t}(\nabla f_t(x_t)z(t), z(t)) + g_{x_t}(z(t), \nabla f_t(x_t)z(t)) \\
&= g_{x_t}(\nabla f_t(x_t)z(t), z(t)) + g_{x_t}\left(\nabla f_t(x_t)^* z(t), z(t)\right) \\
&= 2g_{x_t}\left(\frac{1}{2}\left[\nabla f_t(x_t) + \nabla f_t(x_t)^*\right] z(t), z(t)\right) \\
&\leq 2\lambda(t)|z(t)|^2.
\end{aligned}$$

Now we assume that $z(t) \neq 0$ for all $t \in J \cap [t_0, \infty)$. For almost all t, this implies

$$\begin{aligned}
\frac{\frac{\mathrm{d}}{\mathrm{d}t}|z(t)|^2}{|z(t)|^2} \leq 2\lambda(t) \Rightarrow \int_{t_0}^{t} \frac{\frac{\mathrm{d}}{\mathrm{d}s}|z(s)|^2}{|z(s)|^2}\mathrm{d}s \leq 2\int_{t_0}^{t}\lambda(s)\mathrm{d}s& \\
\Rightarrow \log\left(|z(t)|^2\right) - \log\left(|z(t_0)|^2\right) \leq 2\int_{t_0}^{t}\lambda(s)\mathrm{d}s& \\
\Rightarrow \log|z(t)| - \log|z(t_0)| \leq \int_{t_0}^{t}\lambda(s)\mathrm{d}s& \\
\Rightarrow |z(t)| \leq |z(t_0)|\exp\left(\int_{t_0}^{t}\lambda(s)\mathrm{d}s\right).&
\end{aligned}$$

It is easy to see that the integral over λ exists. Since for each nonzero $v \in T_{x_0}M$ the map $z(t) = \mathrm{d}_{x_0}\Phi_t(v)$ is a solution of (A.8) with $z(t) \neq 0$ for all $t \in (t_0 - T, t_0 + T)$, we obtain

$$\begin{aligned}
\|\mathrm{d}_{x_0}\Phi_t\| &= \max_{|v|=1}\|\mathrm{d}_{x_0}\Phi_t(v)\| \\
&\leq \max_{|v|=1}\|\underbrace{\mathrm{d}_{x_0}\Phi_{t_0}}_{=\mathrm{id}}(v)\|\exp\left(\int_{t_0}^{t}\lambda(s)\mathrm{d}s\right) = \exp\left(\int_{t_0}^{t}\lambda(s)\mathrm{d}s\right),
\end{aligned}$$

which finishes the proof. □

The classical Liouville formula expresses the absolute determinant $|\det \mathrm{d}_x\Phi_t|$ in terms of the integral over the divergence of f_t along the solution $\Phi_t(x)$. There exist several generalizations of this formula. We use the following one which involves an invariant subbundle of the tangent bundle.

Proposition A.5 (Generalized Liouville Formula). *Let (M, g) be a d-dimensional Riemannian $\mathscr{C}^k$-manifold, $k \geq 2$. Consider a locally integrally class $\mathscr{C}^{k-1}$-vector*

field $f : I \times M \to TM$ and the corresponding differential equation

$$\dot{x}(t) = f(t, x(t))$$

with flow box (U, T, Φ) at $(t_0, x_0) \in I \times M$. Let $E \to M$ be a subbundle of $I \times TM \to M$, $(t, v) \mapsto \pi_{TM}(v)$ (with the base point projection $\pi_{TM} : TM \to M$), of rank $n \in \{1, \ldots, d\}$, which is invariant under $\mathrm{d}\Phi$ in the sense that

$$\mathrm{d}_x \Phi_t(E_{t_0,x}) = E_{t,\Phi_t(x)}$$

holds for all $x \in U$ and $t \in (t_0 - T, t_0 + T)$. Then

$$\left|\det \mathrm{d}_{x_0} \Phi_t|_{E_{t_0,x_0}}\right| = \exp\left(\int_{t_0}^{t} \mathrm{tr}\left[\nabla f_s(\Phi_s(x)) \circ Q(s, \Phi_s(x))\right] \mathrm{d}s\right),$$

where $Q(t, x) : T_x M \to E_{t,x}$ denotes the orthogonal projection.

Proof. For every $t \in (t_0 - T, t_0 + T)$ we write

$$L(t) := \mathrm{d}_{x_0} \Phi_t|_{E_{t_0,x_0}} : E_{t_0,x_0} \to E_{t,\Phi_t(x_0)}.$$

Let $(v_1, \ldots, v_n)$ be an orthonormal basis of E_{t_0,x_0}. Then

$$\begin{aligned} |\det L(t)|^2 &= \det(L(t)^* L(t)) = \det\left(\langle L(t)^* L(t) v_i, v_j\rangle\right)_{i,j=1}^n \\ &= \det\left(\langle L(t) v_i, L(t) v_j\rangle\right)_{i,j=1}^n . \end{aligned}$$

Using that $v_i(t) := L(t) v_i$ solves the Riemannian variational equation for each $i \in \{1, \ldots, n\}$, we obtain for almost all $t \in (t_0 - T, t_0 + T)$ that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |\det L(t)|^2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \langle v_1(t) \wedge \ldots \wedge v_n(t), v_1(t) \wedge \ldots \wedge v_n(t)\rangle_{\bigwedge^n T_{\Phi_t(x)} M} \\ &= \left\langle \frac{\mathrm{D}v_1}{\mathrm{d}t}(t) \wedge \ldots \wedge v_n(t), v_1(t) \wedge \ldots \wedge v_n(t)\right\rangle_{\bigwedge^n T_{\Phi_t(x)} M} \\ &\quad + \ldots + \\ &\quad \left\langle v_1(t) \wedge \ldots \wedge \frac{\mathrm{D}v_n}{\mathrm{d}t}(t), v_1(t) \wedge \ldots \wedge v_n(t)\right\rangle_{\bigwedge^n T_{\Phi_t(x)} M} \\ &= \langle \nabla f_t(\Phi_t(x)) v_1(t) \wedge \ldots \wedge v_n(t), v_1(t) \wedge \ldots \wedge v_n(t)\rangle_{\bigwedge^n T_{\Phi_t(x)} M} \\ &\quad + \ldots + \\ &\quad \langle v_1(t) \wedge \ldots \wedge \nabla f_t(\Phi_t(x)) v_n(t), v_1(t) \wedge \ldots \wedge v_n(t)\rangle_{\bigwedge^n T_{\Phi_t(x)} M} . \end{aligned}$$

From Lemma A.1 and invariance of E it thus follows that $|\det L(t)|$ satisfies the scalar linear Carathéodory differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,|\det L(t)| &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\,|\det L(t)|^2}{2\,|\det L(t)|}\\ &= \operatorname{tr}\left[\nabla f_t(\Phi_t(x)) \circ Q(t, \Phi_t(x))\right] |\det L(t)|\,.\end{aligned}$$

Hence, the variation-of-constants formula gives

$$|\det L(t)| = \exp\left(\int_{t_0}^{t} \operatorname{tr}\left[\nabla f_s(\Phi_s(x)) \circ Q(s, \Phi_s(x))\right] \mathrm{d}s\right),$$

since $|\det L(t_0)| = |\det \mathrm{id}_{E_{t_0,x_0}}| = 1$. □

Cut-Off Functions

Every $\mathscr{C}^k$-manifold, $k \in \mathbb{N} \cup \{\infty\}$, admits partitions of unity of class $\mathscr{C}^k$.[6] As for instance shown in Lee [75, Proposition 2.26], one can construct cut-off functions from such partitions which yields the following proposition.

Proposition A.6. *Let M be a $\mathscr{C}^k$-manifold, $k \in \mathbb{N} \cup \{\infty\}$. For any closed set $A \subset M$ and any open set U containing A there exists a cut-off function $\theta : M \to \mathbb{R}$ of class $\mathscr{C}^k$, that is, $\theta(x) \in [0,1]$ for all $x \in M$, $\theta(x) \equiv 1$ on A, and $\theta(x) \equiv 0$ on U^c.*

Given an arbitrary $\mathscr{C}^k$-vector field f on a manifold M and a class $\mathscr{C}^k$ cut-off function $\theta : M \to [0,1]$ with compact support, one obtains a complete $\mathscr{C}^k$-vector field θf whose integral curves coincide with those of f on the set where $\theta(x) \equiv 1$.

A.4 Metric Spaces

In this short section we prove two simple lemmas about metric spaces. To this end, we first introduce some notation: Let (X, ϱ) be a metric space and $K \subset X$ a subset. Then for every $\varepsilon > 0$ the ε-neighborhood of K is defined by

$$N_\varepsilon(K) := \{x \in X \mid \exists y \in K :\ \varrho(x,y) < \varepsilon\}\,.$$

[6] A *partition of unity* is a family of nonnegative functions $f_\alpha : M \to \mathbb{R}$ such that for every $x \in M$ only finitely many of the values $f_\alpha(x)$ are different from zero and $\sum_\alpha f_\alpha(x) = 1$.

That is, $N_\varepsilon(K)$ is the union of the open balls $B(x, \varepsilon)$, $x \in K$, and thus an open neighborhood of K. For a point $x \in X$ and a nonempty set $A \subset X$ the distance from x to A is defined by

$$\operatorname{dist}(x, A) := \inf_{a \in A} \varrho(x, a).$$

Lemma A.2. *Let (X, ϱ) be a metric space and $A \subset X$ nonempty. Then the function*

$$x \mapsto \operatorname{dist}(x, A), \quad X \to \mathbb{R}_+,$$

is continuous.

Proof. For all $x, y \in X$ and $a \in A$ we have

$$\operatorname{dist}(x, A) \leq \varrho(x, a) \leq \varrho(x, y) + \varrho(a, y).$$

Hence, $\operatorname{dist}(x, A) - \varrho(x, y) \leq \varrho(a, y)$, which implies

$$\operatorname{dist}(x, A) - \varrho(x, y) \leq \inf_{a \in A} \varrho(y, a) = \operatorname{dist}(y, A).$$

Therefore, $\operatorname{dist}(x, A) - \operatorname{dist}(y, A) \leq \varrho(x, y)$. By changing the roles of x and y we obtain

$$|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq \varrho(x, y),$$

which proves the assertion. □

Recall that a topological space X is called *locally compact* if every neighborhood of a point $x \in X$ contains a compact neighborhood of x.

Lemma A.3. *Let (X, ϱ) be a locally compact metric space. Then for every compact set $K \subset X$ there exists some $\varepsilon > 0$ such that* $\operatorname{cl} N_\varepsilon(K)$ *is compact.*

Proof. Since X is locally compact, every $x \in K$ has an open neighborhood K_x with compact closure. Since K is compact, there are $x_1, \ldots, x_n \in K$ with $K \subset \bigcup_{i=1}^n K_{x_i}$. Let $W := \bigcup_{i=1}^n \operatorname{cl} K_{x_i}$. Then, as a finite union of compact sets, W is a compact neighborhood of K. Assume to the contrary that for each $\varepsilon > 0$ there is some $x \in X$ with $\operatorname{dist}(x, K) < \varepsilon$ and $x \notin W$. Then there are sequences $(y_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ with $y_n \in X \setminus W$, $z_n \in K$, and $\varrho(y_n, z_n) < 1/n$ for all $n \in \mathbb{N}$. By compactness of K we may assume that $z_n \to z \in K$ for $n \to \infty$. Consequently, also $y_n \to z$. Let $i \in \{1, \ldots, n\}$ such that $z \in K_{x_i}$. Then, for sufficiently large n we obtain $y_n \in K_{x_i} \subset W$ in contradiction to $y_n \in X \setminus W$. Hence, there exists some $\varepsilon > 0$ with $N_\varepsilon(K) \subset W$ which implies that $\operatorname{cl} N_\varepsilon(K) \subset W$ is compact. □

Appendix B
Dynamical Systems

In this part of the appendix, we recall basic concepts from the theory of dynamical systems. By a (classical) dynamical system we understand a mapping $\Phi : \mathbb{T} \times X \to X$, where $\mathbb{T} \in \{\mathbb{Z}_+, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$, which satisfies the axioms $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $x \in X$ and $t, s \in \mathbb{T}$. In other words, a dynamical system is a group or semigroup action of $\mathbb{T}$ on a set X. The set $\mathbb{T}$ is also called the time set of the dynamical system. In the case $\mathbb{T} = \mathbb{R}$ we also speak of a flow, or in the case $\mathbb{T} = \mathbb{R}_+$ of a semiflow. Alternatively, we speak of a continuous-time dynamical system if $\mathbb{T} \in \{\mathbb{R}_+, \mathbb{R}\}$ and of a discrete-time dynamical system if $\mathbb{T} \in \{\mathbb{Z}_+, \mathbb{Z}\}$. Often, we additionally assume that X is a topological or metric space and Φ is continuous. For fixed $t \in \mathbb{T}$, the map $\Phi_t : X \to X$, $x \mapsto \Phi(t, x)$, is called the time-t-map of the dynamical system. If $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$, this map is invertible with inverse Φ_{-t}. The orbit through a point $x \in X$ is the set $\mathcal{O}(x) = \{\Phi(t, x) : t \in \mathbb{T}\}$.

B.1 Chain Recurrence and Chain Transitivity

In this section, we collect some definitions and elementary results about continuous-time dynamical systems on compact metric spaces. Throughout we assume that $\Phi : \mathbb{R} \times X \to X$ is a continuous flow on a compact metric space (X, ϱ). All of the following definitions and results (together with proofs) can be found in Colonius and Kliemann [25, Appendix B]. Further references are Conley [29] and Katok and Hasselblatt [61].

Definition B.1. The *ω-limit set* of a subset $Y \subset X$ is defined as

$$\omega(Y) := \bigcap_{t>0} \mathrm{cl} \bigcup_{s \geq t} \Phi(s, Y).$$

The *α-limit set* of Y is

C. Kawan, *Invariance Entropy for Deterministic Control Systems*, Lecture Notes in Mathematics 2089, DOI 10.1007/978-3-319-01288-9,

$$\alpha(Y) := \bigcap_{t>0} \operatorname{cl} \bigcup_{s \le -t} \Phi(s, Y).$$

If Y consists of only one element y, we write $\omega(y) := \omega(\{y\})$ and $\alpha(y) := \alpha(\{y\})$.

Definition B.2. A compact set $K \subset X$ is called *invariant* if $\Phi_t(K) \subset K$ for all $t \in \mathbb{R}$. It is called *isolated invariant* if it is invariant and there is a neighborhood N of K such that $\Phi(\mathbb{R}, x) \subset N$ implies $x \in K$.

Definition B.3. A *Morse decomposition* of Φ is a finite collection $\{\mathcal{M}_i\}_{i=1}^n$ of nonempty, pairwise disjoint, and isolated compact invariant sets such that:

(i) For all $x \in X$ one has $\alpha(x), \omega(x) \subset \bigcup_{i=1}^n \mathcal{M}_i$.
(ii) If there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_l}$ and $x_1, \ldots, x_l \in X \setminus \bigcup_{i=1}^n \mathcal{M}_i$ with $\alpha(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \ldots, l$, then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The elements of a Morse decomposition are called *Morse sets*. A Morse decomposition is *finer* than another one if the elements of the first one are contained in those of the second one.

Definition B.4. For $x, y \in X$ and $\varepsilon, \tau > 0$, an (ε, τ)-*chain* from x to y is given by a natural number $n \in \mathbb{N}$ together with points

$$x_0 = x, x_1, \ldots, x_n = y \quad \text{and times} \quad \tau_0, \ldots, \tau_{n-1} \ge \tau,$$

such that $\varrho(\Phi(\tau_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, 1, \ldots, n-1$.

Definition B.5. A subset $Y \subset X$ is called *chain transitive* if for all $x, y \in Y$ and $\varepsilon, \tau > 0$ there exists an (ε, τ)-chain from x to y. A point $x \in X$ is *chain recurrent* if for all $\varepsilon, \tau > 0$ there exists an (ε, τ)-chain from x to x. The *chain recurrent set* $\mathcal{R}$ of Φ is the set of all chain recurrent points.

Proposition B.1. *The following assertions hold:*

(i) The set $\mathcal{R}$ is closed and invariant. The flow Φ restricted to a maximal (with respect to set inclusion) chain transitive subset of the chain recurrent set $\mathcal{R}$ is chain transitive. In particular, the flow restricted to $\mathcal{R}$ is chain recurrent.
(ii) A closed subset Y of X is chain transitive if it is chain recurrent and connected. Conversely, if Φ is chain transitive on X, then X is connected.
(iii) The connected components of the chain recurrent set $\mathcal{R}$ coincide with the maximal chain transitive subsets of $\mathcal{R}$.

Proposition B.2. *There exists a finest Morse decomposition $\{\mathcal{M}_1, \ldots, \mathcal{M}_l\}$ if and only if the chain recurrent set $\mathcal{R}$ has only finitely many connected components. In this case, the Morse sets coincide with the chain recurrent components of $\mathcal{R}$ and the flow restricted to each Morse set is chain transitive and chain recurrent.*

B.2 Vector Bundles and Linear Flows

In this section, we collect some definitions and results about real finite-dimensional vector bundles and linear flows. We start with the definition of a vector bundle following Lee [75, Chap. 5].

Definition B.6. Let B be a topological space. A *(real) vector bundle of rank k over* B is a topological space E together with a continuous surjective map $\pi : E \to B$ satisfying:

(i) For each $b \in B$ the set $E_b := \pi^{-1}(b) \subset E$ (called the *fiber* of E over b) is endowed with the structure of a k-dimensional real vector space;
(ii) For each $b \in B$ there exist a neighborhood U of b in B and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a *local trivialization* of E over U) such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Here π_1 is the projection onto the first factor. Furthermore, for each $c \in U$ the restriction of Φ to E_c is a linear isomorphism from E_c to $\{c\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

The space E is called the *total space* of the vector bundle, B is called the *base space*, and π the *projection*. Often we simply write $\pi : E \to B$, $E \to B$, or E for a vector bundle. The subset of E consisting of all the zero vectors of the fibers E_b, $b \in B$, is called the *zero section* of the vector bundle. A vector bundle $\pi : E \to B$ is called *trivial* if there exists a local trivialization over all of B (called a *global trivialization* of E). In this case, E itself is homeomorphic to the product $B \times \mathbb{R}^k$. A *subbundle* of a vector bundle $\pi : E \to B$ is a vector bundle $\pi' : E' \to B$ such that $E' \subset E$ is a closed subset of E which intersects each fiber E_b, $b \in B$, in a linear subspace, and such that $\pi' = \pi|_{E'}$ (E' is endowed with the subspace topology).

Definition B.7. Let $E \to B$ be a vector bundle and $E^1, E^2 \subset E$ subbundles with $E_b^1 \cap E_b^2 = \{0\}$ for each $b \in B$. The *Whitney sum* of E^1 and E^2 is the vector bundle $E' = E^1 \oplus E^2 \subset E$ with fibers

$$E_b' = E_b^1 \oplus E_b^2 = \left\{e_1 + e_2 \ : \ e_i \in E_b^i\right\}.$$

Then $E' \to B$ is a subbundle of $E \to B$.

Lemma B.1. *Consider a trivial vector bundle $\pi : B \times X \to B$, $\pi(b,x) = b$, where (B, ϱ) is a compact metric space and $(X, \langle\cdot,\cdot\rangle)$ a d-dimensional Euclidean space. Suppose that there is a decomposition*

$$B \times X = \mathscr{V} \oplus \mathscr{W}$$

into a Whitney sum of subbundles $\mathscr{V}$ *and* $\mathscr{W}$. *For each* $b \in B$ *let* P_b *denote the projection onto* $\mathscr{V}_b$ *along* $\mathscr{W}_b$. *Then the mapping*

$$b \mapsto P_b, \quad B \to \mathscr{L}(X, X),$$

is continuous.

Proof. Let $\pi_{\mathscr{V}} : \mathscr{V} \to B$ denote the projection of $\mathscr{V}$ (that is, $\pi_{\mathscr{V}} = \pi|_{\mathscr{V}}$), let k be the rank of $\mathscr{V}$, and fix $b_0 \in B$. Then, by definition, there exists an open neighborhood $U \subset B$ of b_0 and a homeomorphism $\varphi : \pi_{\mathscr{V}}^{-1}(U) \to U \times \mathbb{R}^k$ of the form

$$\varphi(b, x) = (b, \hat{\varphi}(b, x)).$$

Hence, for every $(b, y) \in U \times \mathbb{R}^k$, there exists a unique $x \in \mathscr{V}_b$ with $\hat{\varphi}(b, x) = y$. In particular, the map $\hat{\varphi}_b : \mathscr{V}_b \to \mathbb{R}^k$, $x \mapsto \hat{\varphi}(b, x)$, is a linear isomorphism, and it holds that

$$\mathscr{V}_b = \hat{\varphi}_b^{-1}(\mathbb{R}^k) \quad \text{for every } b \in U.$$

Now let $\{e_1(b_0), \dots, e_k(b_0)\}$ be a fixed basis of $\mathscr{V}_{b_0}$ and define $e_1(b), \dots, e_k(b) \in \mathscr{V}_b$ by

$$e_j(b) := \hat{\varphi}_b^{-1}(\hat{\varphi}_{b_0}(e_j(b_0)))$$

for all $b \in U$. It follows that $\{e_1(b), \dots, e_k(b)\}$ is a basis of $\mathscr{V}_b$ for all $b \in U$. Analogously, we find such a basis $\{e_{k+1}(b), \dots, e_d(b)\}$ for $\mathscr{W}_b$, depending continuously on b, and we can assume that both bases are defined on the same neighborhood U of b_0. Then for each $(b, x) \in U \times X$ there are unique $\alpha_1(b, x), \dots, \alpha_d(b, x) \in \mathbb{R}$ such that

$$x = \underbrace{\sum_{i=1}^{k} \alpha_i(b, x) e_i(b)}_{= P_b x} + \sum_{i=k+1}^{d} \alpha_i(b, x) e_i(b).$$

Let $a_{ij}(b) := \langle e_i(b), e_j(b) \rangle$ for all $b \in U$ and $i, j = 1, \dots, d$. Then $A(b) := (a_{ij}(b))_{1 \le i, j \le d}$ is a symmetric positive definite matrix and for $j = 1, \dots, d$, $x \in X$, it holds that

$$x_j(b) := \langle x, e_j(b) \rangle = \sum_{i=1}^{d} a_{ij}(b) \alpha_i(b, x).$$

Hence, the vectors $\hat{x} := (x_1(b), \dots, x_d(b))$ and $\alpha(b,x) := (\alpha_1(b,x), \dots, \alpha_d(b,x))$ satisfy $\hat{x} = A(b)\alpha(b,x)$ which implies $\alpha(b,x) = A(b)^{-1}\hat{x}$. Therefore, in particular $\alpha_1(b,x), \dots, \alpha_k(b,x)$ depend continuously on (b,x) and thus also $P_b x$. Continuity of P_b is then shown as follows. We have

$$\begin{aligned}\|P_b - P_{b_0}\| &= \max_{|x|=1} |P_b x - P_{b_0} x| \\ &= \max_{|x|=1} \underbrace{\left| \sum_{i=1}^{k} \alpha_i(b,x)e_i(b) - \sum_{i=1}^{k} \alpha_i(b_0,x)e_i(b_0) \right|}_{=: f(b,x)}.\end{aligned}$$

Since f is uniformly continuous on the compact set $W \times \mathrm{S}(X)$, where $W \subset U$ is a compact neighborhood of b_0 and $\mathrm{S}(X) = \{x \in X : |x| = 1\}$, for every $\varepsilon > 0$ we find $\delta > 0$ such that $\varrho(b, b_0) < \delta$ implies $|f(b,x) - f(b_0,x)| < \varepsilon$ for all $x \in \mathrm{S}(X)$. This implies continuity of $b \mapsto P_b$ at b_0. □

Definition B.8. A *(discrete- or continuous-time) linear flow* on a vector bundle $\pi : E \to B$ is a continuous flow $\Phi : \mathbb{T} \times E \to E$, $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$, such that for each $t \in \mathbb{T}$ the time-t-map $\Phi_t : E \to E$ maps fibers into fibers, that is, $\pi(\Phi(t,e_1)) = \pi(\Phi(t,e_2))$ if $\pi(e_1) = \pi(e_2)$, and the restrictions $\Phi_t|_{E_b} : E_b \to E_{\pi(\Phi_t(e))}$ are linear maps. Every linear flow induces a flow Θ on the base space B by $\Theta(t,b) := \pi(\Phi(t,e))$ for $b \in B$ and $e \in E_b$. Analogously, a *linear semiflow* on a vector bundle is defined by replacing $\mathbb{T}$ with $\mathbb{T}_+$ in the above definition.

If the base space B of the vector bundle $\pi : E \to B$ is trivial, that is, B consists of only one point, the space E is a finite-dimensional real vector space and each continuous-time linear semiflow on E has the form $(t,x) \mapsto \mathrm{e}^{At}x$ for some $A \in \mathscr{L}(E,E)$. This is proved in the following proposition. The arguments of the proof are borrowed from the theory of strongly continuous semigroups on Banach spaces (see, for instance, Pazy [89]).

Proposition B.3. *Let X be a finite-dimensional real vector space and $T : \mathbb{R}_+ \times X \to X$, $(t,x) \mapsto T(t)x$, a linear semiflow on X. Then the mapping $t \mapsto T(t)$, $\mathbb{R}_+ \to \mathscr{L}(X,X)$, is continuous and there exists a unique linear operator $A \in \mathscr{L}(X,X)$ such that $T(t) = \mathrm{e}^{At}$ for all $t \geq 0$, which for all $x \in X$ is given by*

$$Ax = \lim_{t \searrow 0} \frac{T(t)x - x}{t}. \tag{B.1}$$

Proof. Let X be endowed with some norm $|\cdot|$. Then continuity of $t \mapsto T(t)$ follows from uniform continuity of $(t,x) \mapsto T(t)x$ on compact sets of the form $[a,b] \times \mathrm{S}(X)$, where $[a,b] \subset \mathbb{R}_+$ and $\mathrm{S}(X) := \{x \in X : |x| = 1\}$. Now let $D(A) \subset X$ be the set of all $x \in X$ such that the limit in (B.1) exists and define $A : D(A) \to X$ according to (B.1). In the following, we show that $D(A) = X$, that is, that the definition of A is correct. It is easy to see that $D(A)$ is a linear subspace of X and

therefore a closed set. Hence, it suffices to prove that $D(A)$ is dense in X. From continuity of $T(\cdot)$ we can conclude that for every $x \in X$ it holds that

$$\frac{1}{t}\int_0^t T(s)x\mathrm{d}s \to x \quad \text{for } t \searrow 0. \tag{B.2}$$

For every $x \in X$ we have

$$\frac{T(s)-I}{s}\int_0^t T(r)x\mathrm{d}r = \frac{1}{s}\int_0^t T(s+r)x\mathrm{d}r - \frac{1}{s}\int_0^t T(r)x\mathrm{d}r.$$

Substituting $\rho = s + r$ in the second integral gives

$$\begin{aligned}
\frac{T(s)-I}{s}\int_0^t T(r)x\mathrm{d}r &= \frac{1}{s}\int_s^{t+s} T(\rho)x\mathrm{d}\rho - \frac{1}{s}\int_0^t T(r)x\mathrm{d}r \\
&= \frac{1}{s}\left(\int_t^{t+s} T(\rho)x\mathrm{d}\rho + \int_s^t T(\rho)x\mathrm{d}\rho \right.\\
&\qquad \left. - \int_s^t T(r)x\mathrm{d}r - \int_0^s T(r)x\mathrm{d}r\right) \\
&= \frac{1}{s}\left(\int_0^s (T(t+r)-T(r))x\mathrm{d}r\right) \\
&= \frac{1}{s}\int_0^s T(r)(T(t)-I)x\mathrm{d}r.
\end{aligned}$$

From (B.2) it follows that the right-hand side tends to $(T(t)-I)x$ as $s \searrow 0$. Hence,

$$\int_0^t T(r)x\mathrm{d}r \in D(A) \quad \text{and} \quad A\int_0^t T(r)x\mathrm{d}r = (T(t)-I)x.$$

Consequently, (B.2) implies that for any $x \in X$ there exists a sequence (x_n) in $D(A)$ such that $x_n \to x$, which proves that $D(A) = \mathrm{cl}\, D(A) = X$. It is clear that A is a linear operator. Now for $s > 0$ consider the equalities

$$\frac{T(t+s)x - T(t)x}{s} = T(t)\frac{(T(s)-I)x}{s} = \frac{T(s)-I}{s}T(t)x.$$

The limit for $s \searrow 0$ of the second term exists and is equal to $T(t)Ax$. Hence, also the other limits exist and the right derivative of $t \mapsto T(t)x$ equals $AT(t)x$. For $t > 0$ and $s > 0$ sufficiently small we have

$$\frac{T(t-s)x - T(t)x}{-s} = T(t-s)\frac{(T(s)-I)x}{s}.$$

Therefore, also the left derivative exists and equals $T(t)Ax$. We have thus proven that $(\mathrm{d}/\mathrm{d}t)T(t)x = AT(t)x$ for all $x \in X$ and $t > 0$, which implies $T(t) = \mathrm{e}^{At}$. Uniqueness of A is obvious. □

The following lemma gives an estimate for the growth of linear flows on Euclidean space.

Lemma B.2. *Let $A \in \mathbb{R}^{d\times d}$ and denote by $\alpha(A)$ the maximum of the real parts of all eigenvalues of A. Then it holds that*

$$\forall \delta > 0\ \exists c \geq 1\ \forall t \geq 0 : \ \left\| \mathrm{e}^{At} \right\| \leq c\mathrm{e}^{(\alpha(A)+\delta)t},$$

where $\|\cdot\|$ denotes the operator norm induced by an arbitrary vector norm on $\mathbb{R}^d$.

Proof. Given $\delta > 0$, define $B_\delta := A - (\alpha(A) + \delta)I$. Then all eigenvalues of B_δ have negative real parts, and hence, by Robinson [93, Chap. IV, Theorem 5.1], there exist constants $a > 0$ and $c \geq 1$ such that

$$\left\| \mathrm{e}^{B_\delta t} \right\| \leq c\mathrm{e}^{-at} \quad \text{for all } t \geq 0.$$

Since $\mathrm{e}^{B_\delta t} = \mathrm{e}^{-(\alpha(A)+\delta)t}\mathrm{e}^{At}$, this implies

$$\left\| \mathrm{e}^{At} \right\| \leq c\mathrm{e}^{-at}\mathrm{e}^{(\alpha(A)+\delta)t} \leq c\mathrm{e}^{(\alpha(A)+\delta)t},$$

which proves the assertion. □

Finally, we cite Selgrade's theorem (see, for instance, Colonius and Kliemann [25, Theorem 5.2.5]).

Theorem B.1 (Selgrade). *Consider a continuous-time linear flow Φ on a vector bundle $\pi : \mathscr{V} \to B$ of rank d with connected and compact metric base space B. Suppose that the induced flow on B is chain transitive. Then there exists a unique finest Morse decomposition $\{\mathscr{M}_1, \ldots, \mathscr{M}_r\}$ of the induced flow $\mathbb{P}\Phi$ on the projective bundle[1] $\mathbb{P}\mathscr{V}$ with $1 \leq r \leq d$. Every chain recurrent component $\mathscr{M}_i$ defines an invariant subbundle of $\mathscr{V}$ via*

$$\mathscr{V}^i = \mathbb{P}^{-1}(\mathscr{M}_i) = \{v \in \mathscr{V} \ : \ v \notin Z \Rightarrow \mathbb{P}v \in \mathscr{M}_i\},$$

where Z denotes the zero section of $\mathscr{V}$, and the following decomposition into a Whitney sum holds:

$$\mathscr{V} = \mathscr{V}^1 \oplus \cdots \oplus \mathscr{V}^r.$$

[1]The *projective bundle* $\mathbb{P}\mathscr{V} \to B$ of a vector bundle $\pi : \mathscr{V} \to B$ is the fiber bundle whose fibers are the projective spaces of the fibers $\pi^{-1}(b)$, $b \in B$, defined as the quotient space $\mathbb{P}\mathscr{V} := (\mathscr{V}\backslash Z)/\sim$ under the equivalence relation $\sim$ whose equivalence classes are the lines through the origins of the fibers $\pi^{-1}(b)$.

B.3 Dimension Theory and Topological Entropy

The Subadditivity Lemma

The following lemma is a well-known result in analysis frequently used in connection with exponential growth rates, in particular with entropy.[2] For the sake of completeness, we give a proof.

Lemma B.3 (Subadditivity Lemma). *Let* $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$ *and let* $f : \mathbb{T}_+ \to \mathbb{R}$ *be a subadditive function, that is,*

$$f(t+s) \leq f(t) + f(s) \quad \text{for all } t, s \in \mathbb{T}_+.$$

Suppose further that f *is bounded from above on an interval of the form* $\mathbb{T} \cap [0, t_0]$ *with* $t_0 > 0$. *Then the limit* $\lim_{t\to\infty} f(t)/t$ *exists and equals* $\inf_{t>0} f(t)/t$.

Proof. From boundedness of f on $\mathbb{T} \cap [0, t_0]$ and subadditivity it follows that f is bounded from above on any bounded interval. Let $\gamma := \inf_{t>0} f(t)/t$. Fix a positive number $T \in \mathbb{T}$ and write each $t \in \mathbb{T}$, $t > 0$, as $t = k(t)T + r(t)$ with $k(t) \in \mathbb{Z}_+$ and $r(t) \in \mathbb{T} \cap [0, T)$. Then $k(t)/t \to 1/T$ for $t \to \infty$. By subadditivity, for any $t, T > 0$ it holds that

$$\gamma \leq \frac{f(t)}{t} \leq \frac{1}{t}\left[k(t)f(T) + f(r(t))\right].$$

Hence, for every $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon, T)$ such that for all $t > T_0$

$$\gamma \leq \frac{f(t)}{t} \leq \frac{f(T)}{T} + \varepsilon,$$

where we used boundedness of f on $\mathbb{T} \cap [0, T]$. Since ε and T are arbitrary, the result follows. □

Remark B.1. The lemma also applies to subadditive functions $f : \mathbb{T} \cap (0, \infty) \to \mathbb{R}$, since one can extend such a function to $\mathbb{T}_+$ by setting $f(0) := 0$ without destroying subadditivity.

Hausdorff and Capacitive Dimension

There exist several notions of dimension for topological and metric spaces, generalizing the dimension concept in vector spaces. In the following, we introduce the notions of *Hausdorff* and *capacitive dimension* for metric spaces both used in several entropy estimates in this book.

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[2]This result is also known as *Fekete's Lemma* due to Michael Fekete.

Let (X, ϱ) be a metric space, $Z \subset X$, and $d \geq 0$, $\varepsilon > 0$. Define

$$\mu_H(Z, d, \varepsilon) = \mu_H(Z, d, \varepsilon; \varrho) := \inf \left\{ \sum_{j \geq 1} r_j^d \ : \ r_j \leq \varepsilon, \ Z \subset \bigcup_{j \geq 1} B(x_j, r_j) \right\},$$

where the infimum is taken over all countable covers of Z by metric balls $B(x_j, r_j)$ of radii $r_j \leq \varepsilon$ and centers $x_j \in X$.[3] It is easy to see that the function $\mu_H(\cdot, d, \varepsilon)$ is an outer measure on X. For fixed Z and d, the function $\mu_H(Z, d, \cdot)$ does not decrease with decreasing ε and hence the limit

$$\mu_H(Z, d) = \mu_H(Z, d; \varrho) := \lim_{\varepsilon \searrow 0} \mu_H(Z, d, \varepsilon) = \sup_{\varepsilon > 0} \mu_H(Z, d, \varepsilon)$$

exists (it may be ∞). The number $\mu_H(Z, d)$ is called the *d-dimensional outer Hausdorff measure of Z*. The function $\mu_H(\cdot, d)$ is a metric outer measure on X, that is, the restriction of $\mu_H(\cdot, d)$ to the Borel-σ-algebra of X is a measure. For every $Z \subset X$ there exists a critical value $d_{\text{crit}}(Z)$ such that

$$\mu_H(Z, d) = \begin{cases} 0 & \text{for } d > d_{\text{crit}}(Z), \\ \infty & \text{for } d < d_{\text{crit}}(Z). \end{cases}$$

This unique value is called the *Hausdorff dimension of Z* and is denoted by $\dim_H(Z)$.[4] For a totally bounded set $Z \subset X$ (that is, for every $\varepsilon > 0$ finitely many ε-balls are sufficient to cover Z), $d \geq 0$ and $\varepsilon > 0$ we also introduce the quantity

$$\mu_C(Z, d, \varepsilon) = \mu_C(Z, d, \varepsilon; \varrho) := \varepsilon^d n(\varepsilon, Z),$$

where $n(\varepsilon, Z)$ is the minimal number of ε-balls necessary to cover Z:

$$n(\varepsilon, Z) := \min \left\{ \#\mathscr{C} \ : \ \mathscr{C} = \{B(x_j, \varepsilon)\}_j, \ \ Z \subset \bigcup_j B(x_j, \varepsilon) \right\}.$$

It is easy to see that $\mu_H(Z, d, \varepsilon) \leq \mu_C(Z, d, \varepsilon)$. We define the *$d$-dimensional upper capacitive measure of Z* by

$$\mu_C(Z, d) = \mu_C(Z, d; \varrho) := \limsup_{\varepsilon \searrow 0} \mu_C(Z, d, \varepsilon).$$

The properties of $\mu_C(Z, d)$ are similar to those of $\mu_H(Z, d)$. In particular, $\mu_C(\cdot, d, \varepsilon)$ and $\mu_C(\cdot, d)$ are outer measures if X is totally bounded.

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[3]Taking balls centered at elements of Z makes no essential difference, that is, it does not change the value of the Hausdorff dimension of Z.

[4]Equivalently, one can introduce the Hausdorff measures and the Hausdorff dimension by replacing the covers of Z by metric balls with radii $\leq \varepsilon$ with covers consisting of arbitrary sets with diameters $\leq \varepsilon$.

The *upper capacitive dimension of* Z is defined by

$$\overline{\dim}_C(Z) := \limsup_{\varepsilon \searrow 0} \frac{\log n(\varepsilon, Z)}{\log(1/\varepsilon)}.$$

Analogously, one defines the *lower capacitive dimension* by replacing the limit superior by a limit inferior. In the literature, one finds several other names for this notion of dimension such as *(upper and lower) box dimension* or *fractal dimension*. Alternatively, one can introduce the upper capacitive dimension in the same way as the Hausdorff dimension as a critical value for the upper capacitive measure. The following proposition shows that the upper capacitive dimension of a totally bounded set Z does not depend on the space it is embedded in.

Proposition B.4. *Let* (X, ϱ) *be a metric space and* $Z \subset X$ *a totally bounded set. Let* $\overline{\dim}_C(Z; X)$ *denote the upper capacitive dimension of* Z *as a subspace of* (X, ϱ)*, and* $\overline{\dim}_C(Z; Z)$ *the upper capacitive dimension of* Z *as a subspace of* (Z, ϱ)*. Then* $\overline{\dim}_C(Z; X) = \overline{\dim}_C(Z; Z)$.

Proof. By $n(\varepsilon, Z; X)$ ($n(\varepsilon, Z; Z)$) we denote the minimal cardinality of a cover of Z with ε-balls in X (in Z). For given $\varepsilon > 0$, let $\mathscr{B} = \{B(x_1, \varepsilon), \ldots, B(x_n, \varepsilon)\}$, $x_i \in X$, be a minimal cover of Z with ε-balls in X (in particular, $n = n(\varepsilon, Z; X)$). Then for every $i \in \{1, \ldots, n\}$ there exists some $z_i \in B(x_i, \varepsilon) \cap Z$, since otherwise $\mathscr{B}$ would not be minimal. Let $\tilde{\mathscr{B}} := \{B(z_1, 2\varepsilon), \ldots, B(z_n, 2\varepsilon)\}$. Now take an arbitrary point $z \in Z$. Then there exists $i \in \{1, \ldots, n\}$ with $\varrho(z, x_i) < \varepsilon$. It follows that

$$\varrho(z, z_i) \leq \varrho(z, x_i) + \varrho(x_i, z_i) < \varepsilon + \varepsilon = 2\varepsilon.$$

Hence, $\tilde{\mathscr{B}}$ is a cover of Z consisting of n balls in Z of radius 2ε. This implies

$$n(2\varepsilon, Z; X) \leq n(2\varepsilon, Z; Z) \leq n(\varepsilon, Z; X).$$

Therefore, for all $\varepsilon \in (0, 1)$ it holds that

$$\frac{\log n(2\varepsilon, Z; X)}{\log(1/\varepsilon)} \leq \frac{\log n(2\varepsilon, Z; Z)}{\log(1/\varepsilon)} \leq \frac{\log n(\varepsilon, Z; X)}{\log(1/\varepsilon)}.$$

Using that $\log(1/\varepsilon) = \log(2) + \log(1/(2\varepsilon))$, we obtain

$$\limsup_{\varepsilon \searrow 0} \frac{\log n(2\varepsilon, Z; X)}{\log(2) + \log(1/(2\varepsilon))} \leq \limsup_{\varepsilon \searrow 0} \frac{\log n(2\varepsilon, Z; Z)}{\log(2) + \log(1/(2\varepsilon))} \leq \overline{\dim}_C(Z; X).$$

This implies $\overline{\dim}_C(Z; X) \leq \overline{\dim}_C(Z; Z) \leq \overline{\dim}_C(Z; X)$. □

Some more properties of the Hausdorff and upper capacitive dimensions are summarized in the following proposition. For proofs we refer to Boichenko et al. [9, Chap. III].

Proposition B.5. *Let (X, ϱ) be a metric space. Then the following assertions hold:*

(i) $0 \leq \dim_H(Z) \leq \overline{\dim}_C(Z)$ *for any totally bounded set* $Z \subset X$.
(ii) $\dim_H(\emptyset) = 0$ *and* $\overline{\dim}_C(\emptyset) = 0$.
(iii) $\dim_H(Z_1) \leq \dim_H(Z_2)$ *if* $Z_1 \subset Z_2 \subset X$.
(iv) $\dim_H(\bigcup_{j \geq 1} Z_j) = \sup_{j \geq 1} \dim_H(Z_j)$ *for any sequence* $Z_j \subset X$.
(v) $\overline{\dim}_C(Z_1) \leq \overline{\dim}_C(Z_2)$ *if* $Z_1 \subset Z_2 \subset X$ *are totally bounded sets.*
(vi) $\overline{\dim}_C(\bigcup_{j \geq 1} Z_j) \geq \sup_{j \geq 1} \overline{\dim}_C(Z_j)$ *for a sequence* $Z_j \subset X$ *of totally bounded sets whose union is totally bounded. For finite unions equality holds.*
(vii) *If* $Z \subset X$ *is a totally bounded set, then* $\overline{\dim}_C(Z) = \overline{\dim}_C(\mathrm{cl}\, Z)$.
(viii) *If* X *is a* d*-dimensional Riemannian manifold, then* $\dim_H(X) = d$*. If, additionally,* X *is compact, then* $\overline{\dim}_C(X) = d$.

Topological Entropy

The concept of topological entropy for discrete-time dynamical systems on compact topological spaces was first introduced by Adler et al. [1] as a topological analog to the measure-theoretic entropy of Kolmogorov [69] and Sinai [99]. Topological entropy can be regarded as a measure of the global exponential complexity of the orbit structure, and it has proved to be an important topological invariant. Later, equivalent definitions were given by Dinaburg [37] and Bowen [10] for maps on metric spaces. In Chap. 3, we use Bowen's definition of topological entropy and his result on the entropy of a linear dynamical system. In the following, we give the necessary background for understanding the concepts involved.

Let $f : X \to X$ be a uniformly continuous map on a metric space (X, ϱ). The iterates of f are defined inductively by $f^0 := \mathrm{id}_X$ and $f^{n+1} := f \circ f^n$ for all $n \in \mathbb{Z}_+$.[5] It is easy to see that for each integer $n \geq 1$ the following function defines a metric on X which induces the same topology as ϱ:

$$\varrho_{n,f}(x, y) := \max_{0 \leq i \leq n} \varrho(f^i(x), f^i(y)).$$

Usually, a metric of this form is called a *Bowen-metric* or a *Bowen–Dinaburg-metric*.[6] The metric balls with respect to $\varrho_{n,f}$ are also called *Bowen-balls of order* n. A set $E \subset X$ is called (n, ε, f)*-separated* if the distance of any two distinct points $x, y \in E$ measured by the metric $\varrho_{n,f}$ is at least ε. A set $F \subset X$ (n, ε, f)*-spans* another set $K \subset X$ if for every $x \in K$ there exists $y \in F$ such that $\varrho_{n,f}(x, y) < \varepsilon$.

[5] This defines a discrete-time dynamical system by $\Phi : \mathbb{Z}_+ \times X \to X$, $\Phi(n, x) := f^n(x)$.

[6] Usually, the maximum in the definition of $\varrho_{n,f}$ is only taken over $i \in \{0, \ldots, n-1\}$. However, this makes no essential difference, and we use the slightly different definition only for formal reasons.

Equivalently, K is covered by the ε-balls in the metric $\varrho_{n,f}$ centered at the elements of F.[7]

It is clear that an (n, ε, f)-separated subset of a compact set $K \subset X$ is finite and that there is an upper bound for its cardinality, since otherwise one could place infinitely many disjoint Bowen-balls of radius $\varepsilon/2$ and order n in K contradicting compactness. The maximal cardinality of an (n, ε, f)-separated subset of K is denoted by $r_{\text{sep}}(n, \varepsilon, K, f)$. For the minimal cardinality of a set which (n, ε, f)-spans K we write $r_{\text{span}}(n, \varepsilon, K, f)$. A maximal (n, ε, f)-separated subset E of K automatically (n, ε, f)-spans K. Otherwise there would exist a point $x \in K$ which has distance at least ε to every element of E, and $E \cup \{x\}$ would also be (n, ε, f)-separated. On the other hand, given an (n, ε, f)-separated subset E of K and a set F which $(n, \varepsilon/2, f)$-spans K, one finds that two distinct elements of E cannot be contained in the same Bowen-ball of radius $\varepsilon/2$ and order n around an element of F. This defines an injective map from E to F which shows that $r_{\text{sep}}(n, \varepsilon, K, f) \leq r_{\text{span}}(n, \varepsilon/2, K, f)$. Altogether,

$$r_{\text{span}}(n, \varepsilon, K, f) \leq r_{\text{sep}}(n, \varepsilon, K, f) \leq r_{\text{span}}\left(n, \frac{\varepsilon}{2}, K, f\right) < \infty.$$

Moreover, these quantities are non-decreasing with decreasing ε. Therefore, the following definitions make sense:

$$h_{\text{sep},\varrho}(\varepsilon, K, f) := \limsup_{n\to\infty} \frac{1}{n} \log r_{\text{sep}}(n, \varepsilon, K, f),$$
$$h_{\text{span},\varrho}(\varepsilon, K, f) := \limsup_{n\to\infty} \frac{1}{n} \log r_{\text{span}}(n, \varepsilon, K, f),$$
$$h_{\text{top},\varrho}(K, f) := \lim_{\varepsilon\searrow 0} h_{\text{sep},\varrho}(\varepsilon, K, f) = \lim_{\varepsilon\searrow 0} h_{\text{span},\varrho}(\varepsilon, K, f).$$

One defines the *topological entropy* of f as

$$h_{\text{top},\varrho}(f) := \sup_{K\subset X} h_{\text{top},\varrho}(K, f),$$

where the supremum is taken over all nonempty compact subsets of X. In general, this quantity depends on the metric ϱ. If two metrics ϱ_1 and ϱ_2 are uniformly equivalent, that is, if the identity maps $\text{id} : (X, \varrho_1) \to (X, \varrho_2)$ and $\text{id} : (X, \varrho_2) \to (X, \varrho_1)$ are uniformly continuous, then the corresponding entropies coincide. In particular, this is the case if X is compact. Then the topological entropy can also be defined in a purely topological way using open covers of the space X as done

[7] In the definitions of separated and spanning sets, Bowen requires that $\varrho_{n,f}(x, y) > \varepsilon$ and $\varrho_{n,f}(x, y) \leq \varepsilon$, respectively. For our purposes however it is more convenient to relax the strict inequality and vice versa. For the value of topological entropy this makes no difference.

in [1]. One elementary property of topological entropy which we use in Chap. 3 is the following power rule (see also Bowen [10, Proposition 4]).

Lemma B.4. *Let $f : X \to X$ be a uniformly continuous map on a metric space (X, ϱ) and $K \subset X$ a compact set. Then for each integer $m \geq 1$ it holds that*

$$h_{\mathrm{top},\varrho}(K, f^m) = m h_{\mathrm{top},\varrho}(K, f).$$

Proof. It is clear that $r_{\mathrm{span}}(n, \varepsilon, K, f^m) \leq r_{\mathrm{span}}(mn, \varepsilon, K, f)$ which implies

$$\begin{aligned} h_{\mathrm{span},\varrho}(\varepsilon, K, f^m) &= \limsup_{n\to\infty} \frac{1}{n} \log r_{\mathrm{span}}(n, \varepsilon, K, f^m) \\ &\leq m \limsup_{n\to\infty} \frac{1}{mn} \log r_{\mathrm{span}}(mn, \varepsilon, K, f) \\ &\leq m \limsup_{n\to\infty} \frac{1}{n} \log r_{\mathrm{span}}(n, \varepsilon, K, f) = m h_{\mathrm{span},\varrho}(\varepsilon, K, f). \end{aligned}$$

This gives $h_{\mathrm{top},\varrho}(K, f^m) \leq m h_{\mathrm{top},\varrho}(K, f)$. For the converse inequality, fix $m \geq 1$ and $\varepsilon > 0$. Choose $\delta = \delta(\varepsilon)$ such that $\varrho(x, y) < \delta$ implies $\varrho(f^j(x), f^j(y)) < \varepsilon$ for $0 \leq j \leq m$, which is possible by uniform continuity of f. Then an (n, δ, K, f^m)-spanning set is automatically (mn, ε, K, f)-spanning, which implies $r_{\mathrm{span}}(mn, \varepsilon, K, f) \leq r_{\mathrm{span}}(n, \delta, K, f^m)$. For each $k \geq 1$ let $n_k \geq 1$ be such that $m(n_k - 1) < k \leq mn_k$. Then we obtain

$$\begin{aligned} h_{\mathrm{span},\varrho}(\varepsilon, K, f) &= \limsup_{k\to\infty} \frac{1}{k} \log r_{\mathrm{span}}(k, \varepsilon, K, f) \\ &\leq \limsup_{k\to\infty} \frac{1}{k} \log r_{\mathrm{span}}(mn_k, \varepsilon, K, f) \\ &\leq \limsup_{k\to\infty} \frac{n_k}{k} \frac{1}{n_k} r_{\mathrm{span}}(n_k, \delta, K, f^m). \end{aligned}$$

Since $n_k/k \to 1/m$ for $k \to \infty$, it follows that

$$h_{\mathrm{span},\varrho}(\varepsilon, K, f) \leq \frac{1}{m} h_{\mathrm{span},\varrho}(\delta, K, f^m),$$

which implies the desired inequality. □

The following result can be found in Bowen [10, Theorem 15]. An elementary proof can also be found in Matveev and Savkin [79, Theorem 2.4.2].

Proposition B.6. *If $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map, then*

$$h_{\mathrm{top},\varrho}(T) = \sum_{\lambda\in\sigma(T)} \max\{0, n_\lambda \log|\lambda|\},$$

where ϱ denotes a metric on $\mathbb{R}^d$ induced by a norm and n_λ is the algebraic multiplicity of the eigenvalue λ.

In the same manner as for maps, topological entropy can be defined for continuous-time dynamical systems on metric spaces: Let $\Phi : \mathbb{R}_+ \times X \to X$ be a semiflow which is uniformly continuous in the sense of [10, Sect. 5], that is, for all $t_0 > 0$ it holds that

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall t \in [0, t_0] : \forall x, y \in X :$$
$$\varrho(x, y) < \delta \quad \Rightarrow \quad \varrho(\Phi_t(x), \Phi_t(y)) < \varepsilon. \tag{B.3}$$

As we did for maps, we define the Bowen-metrics

$$\varrho_{\tau,\Phi}(x, y) := \max_{t \in [0,\tau]} \varrho(\Phi_t(x), \Phi_t(y)), \quad \tau > 0.$$

For any real number $\tau > 0$ a set $E \subset X$ is called $(\tau, \varepsilon, \Phi)$-separated if $\varrho_{\tau,\Phi}(x, y) \geq \varepsilon$ for any two distinct points $x, y \in E$. A set $F \subset X$ $(\tau, \varepsilon, \Phi)$-spans another set K if for each $x \in K$ there is $y \in F$ with $\varrho_{\tau,\Phi}(x, y) < \varepsilon$. Then $r_{\mathrm{sep}}(\tau, \varepsilon, K, \Phi)$ and $r_{\mathrm{span}}(\tau, \varepsilon, K, \Phi)$ are the maximal and minimal cardinalities of (n, ε, Φ)-separated and (n, ε, Φ)-spanning sets, respectively. The topological entropy is again defined by

$$\begin{aligned} h_{\mathrm{top},\varrho}(K, \Phi) &:= \lim_{\varepsilon \searrow 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{sep}}(\tau, \varepsilon, K, \Phi) \\ &= \lim_{\varepsilon \searrow 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{span}}(\tau, \varepsilon, K, \Phi), \\ h_{\mathrm{top},\varrho}(\Phi) &:= \sup_{K \subset X \text{compact}} h_{\mathrm{top},\varrho}(K, \Phi). \end{aligned}$$

The following proposition shows that the topological entropy of a semiflow coincides with the entropy of its time-one-map. In particular, together with Proposition B.6, this shows that the entropy of a linear flow $(t, x) \mapsto \mathrm{e}^{At}x$ on a Euclidean space is given by the sum of the positive real parts of the eigenvalues of A (counting multiplicities):

$$h_{\mathrm{top},\varrho}(\{\mathrm{e}^{At}\}) = \sum_{\lambda \in \sigma(A)} \max\{0, n_\lambda \operatorname{Re}(\lambda)\}.$$

As in Proposition B.6, ϱ denotes a metric induced by a norm.

Proposition B.7. *The topological entropy of a uniformly continuous semiflow Φ on a metric space (X, ϱ) equals the topological entropy of its time-one-map: $h_{\mathrm{top},\varrho}(\Phi) = h_{\mathrm{top},\varrho}(\Phi_1)$.*

Proof. Fix a compact set $K \subset X$ and real numbers $\tau, \varepsilon > 0$. Let $F \subset X$ be a set which $(\tau, \varepsilon, \Phi)$-spans K and define $n(\tau) \in \mathbb{Z}_+$ to be the greatest integer such that $n(\tau) \leq \tau$. Then for every $x \in K$ there is some $y \in F$ with $\max_{t \in [0,\tau]} \varrho(\Phi_t(x), \Phi_t(y)) < \varepsilon$. Since $\Phi_j = (\Phi_1)^j$ for all $j \in \mathbb{Z}_+$, this implies

$$\varrho_{n(\tau),\Phi_1}(x, y) = \max_{0 \leq j \leq n(\tau)} \varrho((\Phi_1)^j(x), (\Phi_1)^j(y)) \leq \max_{t \in [0,\tau]} \varrho(\Phi_t(x), \Phi_t(y)) < \varepsilon.$$

Thus, F $(n(\tau), \varepsilon, \Phi_1)$-spans the set K, which implies $r_{\text{span}}(n(\tau), \varepsilon, K, \Phi_1) \leq r_{\text{span}}(\tau, \varepsilon, K, \Phi)$. It follows that

$$\begin{aligned} h_{\text{span},\varrho}(\varepsilon, K, \Phi_1) &= \limsup_{n \to \infty} \frac{1}{n} \log r_{\text{span}}(n, \varepsilon, K, \Phi_1) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log r_{\text{span}}(n, \varepsilon, K, \Phi) \\ &\leq \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\text{span}}(\tau, \varepsilon, K, \Phi) \\ &= h_{\text{span},\varrho}(\varepsilon, K, \Phi). \end{aligned}$$

Consequently, $h_{\text{top},\varrho}(\Phi_1) \leq h_{\text{top},\varrho}(\Phi)$.

In order to show the converse inequality, let $\tau, \varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ according to (B.3) with $t_0 = 1$. Let $n(\tau) \in \mathbb{Z}_+$ be the smallest integer such that $\tau \leq n(\tau)$ and let $F \subset X$ be a set which $(n(\tau), \delta, \Phi_1)$-spans K. Then for every $x \in K$ there is some $y \in F$ such that $\varrho_{n(\tau),\Phi_1}(x, y) < \delta$. For every $t \in [0, \tau]$ there are unique $j \in \{0, 1, \ldots, n(\tau)\}$ and $s \in [0, 1)$ such that $t = j + s$, which implies

$$\begin{aligned} \varrho(\Phi_t(x), \Phi_t(y)) &= \varrho(\Phi_s(\Phi_j(x)), \Phi_s(\Phi_j(y))) \\ &= \varrho(\Phi_s((\Phi_1)^j(x)), \Phi_s((\Phi_1)^j(y))) < \varepsilon. \end{aligned}$$

Consequently, F also $(\tau, \varepsilon, \Phi)$-spans the set K. Finally, we obtain

$$\begin{aligned} h_{\text{span},\varrho}(\varepsilon, K, \Phi) &= \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\text{span}}(\tau, \varepsilon, K, \Phi) \\ &\leq \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\text{span}}(n(\tau), \delta, K, \Phi_1) \\ &\leq \limsup_{n \to \infty} \frac{1}{n-1} \log r_{\text{span}}(n, \delta, K, \Phi_1) \\ &= \limsup_{n \to \infty} \frac{1}{n} \log r_{\text{span}}(n, \delta, K, \Phi_1) = h_{\text{span},\varrho}(\delta, K, \Phi_1). \end{aligned}$$

Thus, $h_{\text{top},\varrho}(K, \Phi) \leq h_{\text{top},\varrho}(K, \Phi_1)$ and $h_{\text{top},\varrho}(\Phi) \leq h_{\text{top},\varrho}(\Phi_1)$. $\square$

Finally, we prove a simple estimate for the topological entropy of a Lipschitz map. The proof is taken from Katok and Hasselblatt [61, Theorem 3.2.9].

Proposition B.8. *Let $f : X \to X$ be a map on a metric space (X, ϱ), which satisfies a global Lipschitz condition with Lipschitz constant $L(f)$. Assume further that $K \subset X$ is a compact set of finite upper capacitive dimension. Then*

$$h_{\mathrm{top},\varrho}(K, f) \leq \max\{0, \log L(f)\} \cdot \overline{\dim}_C(K) < \infty.$$

Proof. Let $L := \max\{1, L(f)\}$, $n \geq 1$ and $\varepsilon > 0$. Pick $x, y \in X$ with $\varrho(x, y) < L^{-n}\varepsilon$. Then for any $0 \leq i \leq n$ we have

$$\varrho(f^i(x), f^i(y)) \leq L^i \varrho(x, y) < L^{i-n}\varepsilon \leq \varepsilon.$$

Hence,

$$\varrho_{n,f}(x, y) = \max_{0 \leq i \leq n} \varrho(f^i(x), f^i(y)) < \varepsilon.$$

If $F \subset X$ is a minimal set which (n, ε, f)-spans K, then K is covered by the Bowen-balls of radius ε and order n, centered at the elements of F. Each of these balls contains an $(L^{-n}\varepsilon)$-ball (with respect to ϱ), as we have proved. We thus obtain

$$r_{\mathrm{span}}(n, \varepsilon, K, f) \leq n(L^{-n}\varepsilon, K).$$

For $L^{-n}\varepsilon < 1$ we have $|\log(L^{-n}\varepsilon)| = |-n \log L + \log \varepsilon| = n \log L - \log \varepsilon$, and therefore

$$n = \frac{|\log(L^{-n}\varepsilon)| + \log \varepsilon}{\log L} = \frac{|\log(L^{-n}\varepsilon)|}{\log L}\left(1 + \frac{\log \varepsilon}{|\log(L^{-n}\varepsilon)|}\right).$$

We may assume that $L > 1$ and hence

$$\lim_{n\to\infty}\left(1 + \frac{\log \varepsilon}{|\log(L^{-n}\varepsilon)|}\right) = 1.$$

This implies

$$\begin{aligned} h_{\mathrm{span},\varrho}(\varepsilon, K, f) &= \limsup_{n\to\infty} \frac{\log r_{\mathrm{span}}(n, \varepsilon, K, f)}{n} \leq \limsup_{n\to\infty} \frac{\log n(L^{-n}\varepsilon, K)}{n} \\ &= \log L \cdot \limsup_{n\to\infty} \frac{\log n(L^{-n}\varepsilon, K)}{|\log(L^{-n}\varepsilon)|} \leq \log L \cdot \overline{\dim}_C(K). \end{aligned}$$

It follows that $h_{\mathrm{top},\varrho}(K, f) \leq \log L \cdot \overline{\dim}_C(K)$, as claimed. □

B.4 Additive and Subadditive Cocycles

Let $\Phi : \mathbb{T} \times X \to X$ be a dynamical system on a set X with time set $\mathbb{T} \in \{\mathbb{Z}_+, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$. By an *additive cocycle over* Φ we understand a function

$$a : \mathbb{T} \times X \to \mathbb{R}$$

which satisfies the equality

$$a(t+s, x) = a(t, x) + a(s, \Phi(t, x)) \quad \text{for all } t, s \in \mathbb{T} \text{ and } x \in X.$$

In general, we do not impose any continuity assumptions on Φ and a. However, in a topological context, we have the following result proved in [66, Corollary 2] via investigation of the *uniform growth spectrum* introduced by Grüne [53].

Theorem B.2. *Let $\Phi : \mathbb{T} \times X \to X$ be a continuous dynamical system on a Hausdorff space X and $a : \mathbb{T} \times X \to \mathbb{R}$ a continuous additive cocycle over Φ. Then, given a compact Φ-invariant set $M \subset X$, that is, $\Phi_t(M) \subset M$ for all $t \in \mathbb{T}$, we have*

$$\begin{aligned} \inf_{x \in M} \limsup_{t \to \infty} \frac{1}{t} a(t, x) &= \inf_{x \in M} \liminf_{t \to \infty} \frac{1}{t} a(t, x) \\ &= \lim_{t \to \infty} \inf_{x \in M} \frac{1}{t} a(t, x) = \sup_{t > 0} \inf_{x \in M} \frac{1}{t} a(t, x) \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} \sup_{x \in M} \limsup_{t \to \infty} \frac{1}{t} a(t, x) &= \sup_{x \in M} \liminf_{t \to \infty} \frac{1}{t} a(t, x) \\ &= \lim_{t \to \infty} \sup_{x \in M} \frac{1}{t} a(t, x) = \inf_{t > 0} \sup_{x \in M} \frac{1}{t} a(t, x). \end{aligned} \tag{B.5}$$

Furthermore, there are $x_*, x^* \in M$ such that

$$\inf_{x \in M} \limsup_{t \to \infty} \frac{1}{t} a(t, x) = \lim_{t \to \infty} \frac{1}{t} a(t, x_*),$$

$$\sup_{x \in M} \limsup_{t \to \infty} \frac{1}{t} a(t, x) = \lim_{t \to \infty} \frac{1}{t} a(t, x^*).$$

A *subadditive cocycle* over the dynamical system Φ is a function

$$a : \mathbb{T} \times X \to \mathbb{R}$$

which satisfies the inequality

$$a(t+s,x) \leq a(t,x) + a(s,\Phi(t,x)) \quad \text{for all } t,s \in \mathbb{T} \text{ and } x \in X. \tag{B.6}$$

In the case where X is a compact metric space and both Φ and a are continuous, Schreiber [97, Theorem 1] shows that

$$\sup_{x\in X} \limsup_{t\to\infty} \frac{1}{t} a(t,x) = \lim_{t\to\infty} \sup_{x\in X} \frac{1}{t} a(t,x) = \inf_{t>0} \sup_{x\in X} \frac{1}{t} a(t,x),$$

using methods from ergodic theory, in particular Kingman's subadditive ergodic theorem. For a superadditive cocycle a (where the inequality in (B.6) is reversed), one has the analogous result with suprema replaced by infima and vice versa, and limsup replaced by liminf.

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