APPENDIX

The appendix contains elements of vector calculus, as well as notions on the field theory and on the theory of distributions. These results are presented without demonstration or with a concise one, representing a review of known results or complements of such results.

1. Elements of vector calculus

In the following, we deal with vector analysis and with exterior differential calculus; the notions of vector calculus can be found in several chapters of the work and are linked – especially – to the systems of forces. For a better understanding of the principal properties of the vectors and taking into account that we apply techniques of vector calculus to the study of mechanical systems in an Euclidean three-dimensional space E_3, we consider the vectors in the vector three-dimensional space L_3, introduced in Chap. 1, Subsec. 1.1.2, using oriented segments of line as geometric representations of them; their tensor properties have been emphasized in Chap. 3, Subsecs 1.2.2 and 1.2.3. However, some results which will be given hold in a *n*-dimensional vector space L_n too.

1.1 Vector analysis

A free, bound or sliding vector is a *function of the independent variable* $t \in [t_0, t_1]$ if the parameters which determine it are functions of this variable. In general, we suppose that we have to do with free vectors; however, the results obtained are valid also for the other types of vectors, excepting special cases. Let be the vector

$$\mathbf{V} = \mathbf{V}(t), \quad V_i = V_i(t), \quad i = 1,2,3, \tag{A.1.1}$$

with respect to an orthonormed frame of reference; thus, various operations which will be defined in connection with the vector $\mathbf{V}$ correspond to operations effected on its components in a system of orthogonal Cartesian co-ordinates or – eventually – in another system of co-ordinates. In the following, we deal with functions or vector mappings $t \to \mathbf{V}(t)$, in the mentioned case, in which a single variable is involved, as well as in the case in which they depend on several variables. Without many details, the results known for scalar functions may be adapted for vector ones.

1.1.1 Limits. Continuity

We say that the vector $\mathbf{V}(t)$ tends to a *limit* $\mathbf{V}^0$ for $t \to t^0$, $t^0 \in T \equiv [t_0, t_1]$, and we have

$$\lim_{t \to t^0} \mathbf{V}(t) = \mathbf{V}^0 , \tag{A.1.2}$$

if we may write

$$\lim_{t \to t^0} V_i(t) = V_i^0 , \quad i = 1,2,3 , \tag{A.1.2'}$$

for its components; analogously, we may define *the limits at the right* and *at the left*.

We say that the vector $\mathbf{V}$ is a continuous function (of class C^0) if its components are continuous functions. Obviously, the domain of definition of the vector function is specified by the domain (eventually, domains) of definition of its components; in general, we assume that all its components have the same domain of definition. Similar properties may be obtained, in the same way, in the case of vectors depending on several independent variables.

1.1.2 Differentiation of vectors

We say that *the vector function* $t \to \mathbf{V}(t)$ is *differentiable* in $t \in T$ if the limit

$$\lim_{h \to 0} \frac{\mathbf{V}(t+h) - \mathbf{V}(t)}{h} = \mathbf{V}'(t) = \dot{\mathbf{V}}(t) \tag{A.1.3}$$

exists; the notation by a "point" for the derivative is used in the case in which *the independent variable* t *is the time*, as it will be assumed in what follows. *The differential* of the vector $\mathbf{V}(t)$ is

$$\mathrm{d}\mathbf{V}(t) = \dot{\mathbf{V}}(t)\mathrm{d}t , \tag{A.1.4}$$

so that its *derivative* may be written in the form

$$\dot{\mathbf{V}}(t) = \frac{\mathrm{d}\mathbf{V}(t)}{\mathrm{d}t} \tag{A.1.3'}$$

too. If the vector is given in the form $\mathbf{V}(t) = V_j(t)\mathbf{i}_j$, we obtain

$$\dot{\mathbf{V}}(t) = \dot{V}_j(t)\mathbf{i}_j , \tag{A.1.5}$$

which may be a definition relation of the derivative. *The derivatives of higher order* $\ddot{\mathbf{V}}(t)$, $\dddot{\mathbf{V}}(t)$,…, $\mathbf{V}^{(n)}(t)$ can be analogously defined. We say that the vector $\mathbf{V}(t)$ is of class $C^n(T)$ if its components in a system of co-ordinate axes (in particular, in a system of orthogonal Cartesian co-ordinates) are of class $C^n(T)$ (the derivative of *n*th order exists and is differentiable; n finite or infinite).

The modulus of the derivative $\dot{\mathbf{V}}$ is given by

$$|\dot{\mathbf{V}}| = \sqrt{\dot{V}_i \dot{V}_i}\,, \tag{A.1.6}$$

while the modulus of the differential $\mathrm{d}\mathbf{V}$ reads

$$|\mathrm{d}\mathbf{V}| = \sqrt{\mathrm{d}V_i \mathrm{d}V_i}\,. \tag{A.1.6'}$$

Following formulae of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}_1 + \mathbf{V}_2) = \dot{\mathbf{V}}_1 + \dot{\mathbf{V}}_2\,, \tag{A.1.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda\mathbf{V}) = \dot{\lambda}\mathbf{V} + \lambda\dot{\mathbf{V}}\,, \quad \lambda = \lambda(t) \text{ scalar,} \tag{A.1.7'}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}_1 \cdot \mathbf{V}_2) = \dot{\mathbf{V}}_1 \cdot \mathbf{V}_2 + \mathbf{V}_1 \cdot \dot{\mathbf{V}}_2\,, \tag{A.1.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}_1 \times \mathbf{V}_2) = \dot{\mathbf{V}}_1 \times \mathbf{V}_2 + \mathbf{V}_1 \times \dot{\mathbf{V}}_2\,, \tag{A.1.8'}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) = (\dot{\mathbf{V}}_1, \mathbf{V}_2, \mathbf{V}_3) + (\mathbf{V}_1, \dot{\mathbf{V}}_2, \mathbf{V}_3) + (\mathbf{V}_1, \mathbf{V}_2, \dot{\mathbf{V}}_3)\,, \tag{A.1.8''}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}[u(t)] = \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{V}'_u \dot{u}\,, \quad u(t) \text{ scalar,} \tag{A.1.9}$$

are easily obtained. From the relation $\mathbf{V}^2(t) = V^2(t)$ one has $\mathbf{V} \cdot \mathrm{d}\mathbf{V} = V\mathrm{d}V$, so that we may write

$$\mathrm{d}|\mathbf{V}| \leq |\mathrm{d}\mathbf{V}|\,. \tag{A.1.10}$$

The equality takes place in the case of a vector of constant direction ($\mathbf{V} = V(t)\mathbf{u}$, $\mathbf{u} = \overrightarrow{\mathrm{const}}$). In the case of a vector $\mathbf{V}(t)$ of constant modulus ($V^2 = \mathrm{const}$) we have $\mathbf{V} \cdot \dot{\mathbf{V}} = 0$; hence, *the derivative of a vector of constant modulus is a vector normal to that one*. As well,

$$\mathbf{V} = \overrightarrow{\mathrm{const}} \Leftrightarrow \mathrm{d}\mathbf{V} = \mathbf{0}\,. \tag{A.1.11}$$

If two vectors $\mathbf{V} = \mathbf{V}(t)$ and $\mathbf{W} = \mathbf{W}(t)$ have the same direction, hence the same unit vector $\mathbf{u} = \mathbf{u}(t)$, we may write $\mathbf{V} = V\mathbf{u}$, $\mathbf{W} = W\mathbf{u}$, so that $\mathbf{V} \cdot \mathrm{d}\mathbf{W} = V\mathbf{u} \cdot (\mathbf{u}\mathrm{d}W + W\mathrm{d}\mathbf{u})$; using the previous results, we get

$$\mathbf{V} \cdot \mathrm{d}\mathbf{W} = V\mathrm{d}W\,. \tag{A.1.12}$$

Let be the plane Ox_1x_2 and a point P, specified by the position vector $\mathbf{r} = \mathbf{r}(t)$ (Fig.A.1); let us also consider the unit vector $\mathbf{i}_r(t) = \mathrm{vers}\,\mathbf{r}(t)$ and the angle $\theta(t)$

formed by that vector with the Ox_1-axis. We have $\mathbf{i}_r = \cos\theta\mathbf{i}_1 + \sin\theta\mathbf{i}_2$; we introduce also the unit vector $\mathbf{i}_\theta = -\sin\theta\mathbf{i}_1 + \cos\theta\mathbf{i}_2$, obtained by a positive rotation of right angle of the unit vector $\mathbf{i}_r$.

We may write $\dot{\mathbf{i}}_r = -\sin\theta\dot{\theta}\mathbf{i}_1 + \cos\theta\dot{\theta}\mathbf{i}_2$, wherefrom

$$\dot{\mathbf{i}}_r = \dot{\theta}\mathbf{i}_\theta , \tag{A.1.13}$$

formula which allows to calculate *the derivative of a unit vector*, which is *contained in a fixed plane* (or is parallel to a fixed plane), *passing through a fixed point*. It is obvious that, using the same formula, we may write also

$$\dot{\mathbf{i}}_\theta = -\dot{\theta}\mathbf{i}_r . \tag{A.1.13'}$$

The unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\theta$ define a system of orthogonal co-ordinates, the point P having the polar co-ordinates r and θ; in this system, a vector $\mathbf{V}$ is written in the form

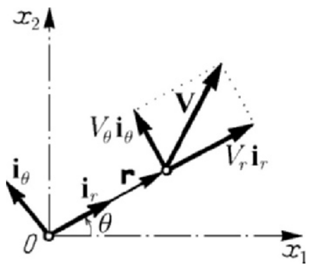

Figure A.1. Polar co-ordinates.

$$\mathbf{V} = V_r\mathbf{i}_r + V_\theta\mathbf{i}_\theta . \tag{A.1.14}$$

We have $\dot{\mathbf{V}} = V_r\dot{\mathbf{i}}_r + \dot{V}_r\mathbf{i}_r + V_\theta\dot{\mathbf{i}}_\theta + \dot{V}_\theta\mathbf{i}_\theta$, wherefrom, taking into account (A.1.13), (A.1.13'), we get

$$\dot{\mathbf{V}} = \left(\dot{V}_r - V_\theta\dot{\theta}\right)\mathbf{i}_r + \left(\dot{V}_\theta + V_r\dot{\theta}\right)\mathbf{i}_\theta . \tag{A.1.14'}$$

In particular, if $V_\theta = 0$, $V_r = \text{const}$, then we may write *the derivative of a vector of constant modulus*, the support of which passes through a fixed point, in the form

$$\dot{\mathbf{V}} = V_r\dot{\theta}\mathbf{i}_\theta . \tag{A.1.14''}$$

Let be an ordered system of several independent variables $q_1, q_2, ..., q_s$; if the point $(q_1, q_2, ..., q_s)$ describes a domain D in the corresponding s-dimensional space, we can define *the function* or *the vector mapping* $(q_1, q_2, ..., q_s) \to \mathbf{V} = \mathbf{V}(q_1, q_2, ..., q_s)$, which may be written also in the canonical form

$$\mathbf{V} = V_j(q_1, q_2, \ldots, q_s)\mathbf{i}_j \,. \tag{A.1.15}$$

We define *the partial derivatives* of the first order by

$$\frac{\partial \mathbf{V}}{\partial q_h} = \frac{\partial V_j}{\partial q_h}\mathbf{i}_j\,, \quad h = 1,2,\ldots,s\,; \tag{A.1.15'}$$

as in the case of vector functions of a single variable, we obtain *the differential*

$$\mathrm{d}\mathbf{V} = \sum_{h=1}^{s} \frac{\partial \mathbf{V}}{\partial q_h}\mathrm{d}q_h = \frac{\partial \mathbf{V}}{\partial q_h}\mathrm{d}q_h\,, \tag{A.1.15''}$$

where we have introduced the summation convention of Einstein in the *s*-dimensional space. Analogously, *partial derivatives as well as differentials of higher order* may be defined.

Considering the mappings $t \to q_h(t)$, $h = 1,2,\ldots,s$, and assuming that the vector $\mathbf{V}$ may depend also explicitly on the variable t, we can write *the total* (or *substantial*) *derivative* of the vector function in the form

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\partial \mathbf{V}}{\partial q_h}\frac{\mathrm{d}q_h}{\mathrm{d}t} + \frac{\partial \mathbf{V}}{\partial t}\,, \tag{A.1.16}$$

the total differential being

$$\mathrm{d}\mathbf{V} = \frac{\partial \mathbf{V}}{\partial q_h}\mathrm{d}q_h = \frac{\partial \mathbf{V}}{\partial t}\mathrm{d}t\,; \tag{A.1.16'}$$

if $q_1, q_2, \ldots, q_s$ are the co-ordinates of a point in the considered *s*-dimensional space, t being the time variable, then we say that

$$\frac{\partial \mathbf{V}}{\partial q_h}\frac{\mathrm{d}q_h}{\mathrm{d}t} = \frac{\partial \mathbf{V}}{\partial q_h}\dot{q}_h \tag{A.1.16''}$$

represents *the space derivative* of the vector function, while

$$\frac{\partial \mathbf{V}}{\partial t} = \dot{\mathbf{V}} \tag{A.1.16'''}$$

represents *the time derivative* of this function (the partial derivative with respect to the time t).

We say that the vector $\mathbf{V}$ is of class $C^n(D)$ with respect to a variable or with respect to a set of variables if its components are of class $C^n(D)$ with respect to that variable or with respect to all variables, respectively. The computation of the mixed derivative of order $m \le n$ of a vector does not depend on the order of differentiation if the components of the vector have this property and that one takes place if the vector is of class $C^n(D)$ with respect to all variables; in particular, we may write

$$\frac{\partial^2 \mathbf{V}}{\partial q_i \partial q_j} = \frac{\partial^2 \mathbf{V}}{\partial q_j \partial q_i}, \quad i, j = 1, 2, ..., s, \tag{A.1.17}$$

in this case (Schwarz's theorem).

1.1.3 Sequences and series of vectors

Let be *the sequence of vectors* $\{\mathbf{V}_n, n \in \mathbb{N}\}$; we say that this sequence tends to the vector $\mathbf{V}$ for $n \to \infty$ and we write

$$\lim_{n \to \infty} \mathbf{V}_n = \mathbf{V} \tag{A.1.18}$$

if

$$\lim_{n \to \infty} V_{ni} = V_i, \quad i = 1, 2, 3. \tag{A.1.18'}$$

As well, let be *the series* of general term

$$\mathbf{V}_n = V_{nj} \mathbf{i}_j; \tag{A.1.19}$$

we say that this series is *convergent* if the series of general terms V_{nj}, $j = 1, 2, 3$, are convergent. Analogously, we may introduce series of vector functions.

Let be given the vector function $t \to \mathbf{V}(t)$, $t \in T$; we may write a development in the neighbourhood of the moment t in the form

$$\mathbf{V}(t + h) = \mathbf{V}(t) + \frac{h}{1!}\dot{\mathbf{V}}(t) + \frac{h^2}{2!}\ddot{\mathbf{V}}(t) + ... + \frac{h^n}{n!}\mathbf{V}^{(n)}(t) + \mathbf{R}_n, \tag{A.1.20}$$

where the rest is given by

$$\mathbf{R}_n = \frac{h^{n+1}}{(n+1)!} V_j^{(n+1)}(t + \tau_j)\mathbf{i}_j, \quad \tau_j \in (0, h), \quad j = 1, 2, 3; \tag{A.1.20'}$$

obviously, this development is equivalent to three developments for the three components of the vector $\mathbf{V}(t)$. If

$$\lim_{n \to \infty} \mathbf{R}_n = \mathbf{0}, \tag{A.1.20''}$$

then we obtain a development into a *Taylor series*. Obviously, we assume that $\mathbf{V}(t)$ is of class $C^{n+1}(D)$ or of class $C^\infty(D)$, respectively. In particular, for $t = 0$, assuming that this moment belongs to the interval of definition, we obtain a development into a *Maclaurin series*

$$\mathbf{V}(h) = \mathbf{V}(0) + \frac{h}{1!}\dot{\mathbf{V}}(0) + \frac{h^2}{2!}\ddot{\mathbf{V}}(0) + ... + \frac{h^n}{n!}\mathbf{V}^{(n)}(0) + ... \tag{A.1.21}$$

In the case of a vector function of several variables one can write analogous developments.

1.1.4 Integration of vectors

Let be the vector function $t \to \mathbf{V}(t)$, $t \in [t', t'']$, and let be $T \equiv [t_0, t_1] \subset [t', t'']$; we say that the function $\mathbf{V}$ is *integrable* on T if its components are integrable functions on T. We may write

$$\int_{t_0}^{t_1} \mathbf{V}(t)\mathrm{d}t = \mathbf{i}_j \int_{t_0}^{t_1} V_j(t)\mathrm{d}t \tag{A.1.22}$$

in this case. In what follows, we consider only *Riemann integrals*; obviously, one may take into consideration also other types of integrals of vector functions. Let now be the vector

$$\mathbf{W}(t) = \int_{t_0}^{t} \mathbf{V}(\tau)\mathrm{d}\tau\,, \quad t_0, t \in T\,; \tag{A.1.23}$$

it results

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} = \mathbf{V}\,. \tag{A.1.23'}$$

The solution of this equation may be written in the form

$$\mathbf{W}(t) = \int \mathbf{V}(t)\mathrm{d}t + \mathbf{C}\,, \quad \mathbf{C} = \overrightarrow{\mathrm{const}}\,, \tag{A.1.23''}$$

where we have introduced *the primitive* of a vector function. We mention following properties:

$$\int_{t_0}^{t_1} \mathbf{V}(t)\mathrm{d}t = \int_{t_0}^{t_2} \mathbf{V}(t)\mathrm{d}t + \int_{t_2}^{t_1} \mathbf{V}(t)\mathrm{d}t\,, \quad t_2 \in T\,, \tag{A.1.24}$$

$$\int_{t_0}^{t_1} \mathbf{V}_1(t)\mathrm{d}t + \int_{t_0}^{t_1} \mathbf{V}_2(t)\mathrm{d}t = \int_{t_0}^{t_1} [\mathbf{V}_1(t) + \mathbf{V}_2(t)]\mathrm{d}t\,, \tag{A.1.24'}$$

$$\int_{t_0}^{t_1} \lambda\mathbf{V}(t)\mathrm{d}t = \lambda\int_{t_0}^{t_1} \mathbf{V}(t)\mathrm{d}t\,, \quad \lambda = \mathrm{const}\,, \tag{A.1.25}$$

$$\int_{t_0}^{t_1} \mathbf{C}\lambda(t)\mathrm{d}t = \mathbf{C}\int_{t_0}^{t_1} \lambda(t)\mathrm{d}t\,, \quad \mathbf{C} = \overrightarrow{\mathrm{const}}\,, \quad \lambda(t) \text{ scalar}, \tag{A.1.25'}$$

$$\int_{t_0}^{t_1} \mathbf{C} \cdot \mathbf{V}(t)\mathrm{d}t = \mathbf{C} \cdot \int_{t_0}^{t_1} \mathbf{V}(t)\mathrm{d}t\,, \quad \mathbf{C} = \overrightarrow{\mathrm{const}}\,, \tag{A.1.26}$$

$$\int_{t_0}^{t_1} \mathbf{C} \times \mathbf{V}(t)\mathrm{d}t = \mathbf{C} \times \int_{t_0}^{t_1} \mathbf{V}(t)\mathrm{d}t\,, \quad \mathbf{C} = \overrightarrow{\mathrm{const}}\,. \tag{A.1.26'}$$

Let be a point P of position vector $\mathbf{r}$; the vector mapping $q \to \mathbf{r}(q)$, $q \in Q \equiv [q', q'']$, of class $C^1(Q)$ determines a curve C, locus of the point P. Let us consider the curvilinear abscissa defined by the function $q \to s(q)$, and two points P_0 and P_1 on the curve C, of curvilinear abscissae $s_0 = s(q_0)$ and $s_1 = s(q_1)$, respectively. We define a vector function $s \to \mathbf{V}(s)$ in any point of the curve C too (Fig.A.2). We introduce thus the *curvilinear vector integral*

$$\int_{\overset{\frown}{P_0P_1}} \mathbf{V}(s)\mathrm{d}s = \int_{s_0}^{s_1} \mathbf{V}(s)\mathrm{d}s = \int_{q_0}^{q_1} \mathbf{V}(s(q))s'(q)\mathrm{d}q\,, \tag{A.1.27}$$

equivalent to three scalar curvilinear integrals, corresponding to the components of the vector $\mathbf{V}(s)$. Noting that the position vector $\mathbf{r}(q) = x_j(q)\mathbf{i}_j$ of the point P has the derivative $\mathbf{r}'(q) = x_j'(q)\mathbf{i}_j$, the latter vector has the same direction as the tangent to the curve C at the point P, and the differential $\mathrm{d}\mathbf{r} = \mathbf{i}_j \mathrm{d}x_j$ has the same property. We introduce the curvilinear integral

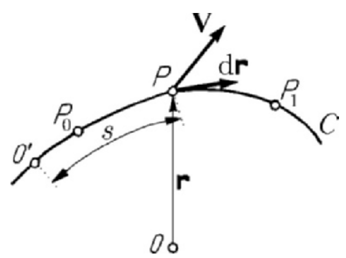

Figure A.2. Curvilinear vector integral.

$$W_{\overset{\frown}{P_0P_1}}(\mathbf{V}) = \int_{\overset{\frown}{P_0P_1}} \mathbf{V}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\overset{\frown}{P_0P_1}} V_j \mathrm{d}x_j = \int_{q_0}^{q_1} V_j(q)x_j'(q)\mathrm{d}q\,, \tag{A.1.28}$$

which represents *the work* of the vector $\mathbf{V} = \mathbf{V}(\mathbf{r})$ along the curve C, between the points P_0 and P_1; obviously, the direction of travelling through that curve is from P_0 to P_1. The work of a vector is a scalar quantity. We denote by

$$\mathrm{d}W = \mathbf{V}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \tag{A.1.28'}$$

the elementary work, which – in general – is not an exact differential. We notice that the work of the sum of n vectors applied at the same point is equal to the sum of the works of those vectors; this result is obvious, taking into account the property of distributivity of the scalar product with respect to the addition of vectors.

In the case of a closed curve C, we consider the curvilinear vector integral

$$\oint_C \mathbf{V}(s)\mathrm{d}s = \oint_C \mathbf{V}(s(q))s'(q)\mathrm{d}q \tag{A.1.29}$$

too, the direction of travelling through being that indicated (the counterclockwise). Analogously, we may also consider the work of the vector $\mathbf{V}$ along the closed curve C, in the form

$$W_C(\mathbf{V}) = \oint_C \mathbf{V}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \oint_C V_j \mathrm{d}x_j \;; \tag{A.1.29'}$$

this work is called *the circulation of the vector* $\mathbf{V}$ on the closed curve C. We mention that the curvilinear vector integrals along a closed curve do not depend on the point from which the travelling through of the curve begins.

Let be a surface Σ, which is represented in a parametric form by $\mathbf{r} = \mathbf{r}(u,v)$, $(u,v) \in D$, as well as the vector function $(u,v) \to \mathbf{V}(u,v)$, defined at the point P, of position vector $\mathbf{r}$. If $S \subset \Sigma$ and if the vector function $\mathbf{V}(u,v)$ is integrable on S, then we may introduce *the surface vector integral* in the form

$$\iint_S \mathbf{V}(P)\mathrm{d}S = \mathbf{i}_j \iint_S V_j(P)\mathrm{d}S\,, \tag{A.1.30}$$

where $\mathrm{d}S$ is the element of area; obviously, the vector function $\mathbf{V}(u,v)$ is integrable on S if its components have the same property. We may express the surface integral by means of the variables u and v too. As well, we can consider also the surface integrals for which S is a closed surface.

Let be a domain $D \subset \mathbb{R}^3$ and let be the vector mapping $\mathbf{r} \to \mathbf{V}(\mathbf{r})$, defined for $P \in D$, where $\mathbf{r}$ represents the position vector of the point P; we say that the vector function $\mathbf{V}(\mathbf{r})$ is integrable if its components are integrable functions. In this case, we may introduce *the volume vector integral*

$$\iiint_D \mathbf{V}(\mathbf{r})\mathrm{d}\tau = \mathbf{i}_j \iiint_D V_j(\mathbf{r})\mathrm{d}\tau\,, \tag{A.1.31}$$

where $\mathrm{d}\tau = \mathrm{d}x_1 x_2 x_3$ is the volume element.

1.1.5 Curvilinear co-ordinates

Let us consider, in what follows, the vector mapping $(q_1, q_2, q_3) \to \mathbf{V}(q_1, q_2, q_3)$, $(q_1, q_2, q_3) \in D \subset \mathbb{R}^3$, and the point P of position vector $\mathbf{r}$, defined by (Fig.A.3)

$$\mathbf{r}(q_1, q_2, q_3) = x_j(q_1, q_2, q_3)\mathbf{i}_j\,; \tag{A.1.32}$$

if the point (q_1, q_2, q_3) describes the domain D, then the point P describes a domain V. Through each point of the domain V may pass three co-ordinate lines, that is the curves $q_2, q_3 = \mathrm{const}$, $q_3, q_1 = \mathrm{const}$ and $q_1, q_2 = \mathrm{const}$; the co-ordinates on these co-ordinate lines are called *curvilinear co-ordinates*. The link between the Cartesian and the curvilinear co-ordinates will be expressed in the form

$$x_j = x_j(q_1, q_2, q_3), \quad j = 1,2,3\,, \tag{A.1.33}$$

where $x_j \in C^1(D)$; the transformation (A.1.33) is locally reversible only if the functional determinant J does not vanish

$$J = \det\left[\frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)}\right] \neq 0 . \quad \text{(A.1.34)}$$

We assume that this transformation is one-to-one, that is to a curvilinear system of co-ordinates (q_1, q_2, q_3) corresponds a single point P and reciprocally.

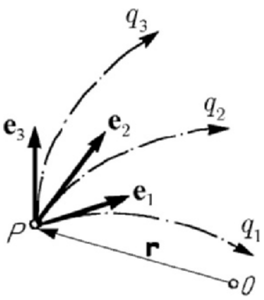

Figure A.3. Curvilinear co-ordinates.

If we consider the mappings $t \to q_i(t)$, $i = 1,2,3$, $t \in [t_0, t_1]$, then the point P describes a curve C, the tangent at that point being specified by

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q_i} \mathrm{d}q_i ; \quad \text{(A.1.35)}$$

the vectors

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial q_i}, \quad i = 1,2,3 , \quad \text{(A.1.36)}$$

are tangent to the co-ordinate curves and form a *local basis*, because

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = J \neq 0 . \quad \text{(A.1.34')}$$

The arc element $\mathrm{d}s = |\mathrm{d}\mathbf{r}|$ on the curve C is given by

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{r}^2 = g_{ij} \mathrm{d}q_i \mathrm{d}q_j , \quad \text{(A.1.37)}$$

where

$$g_{ij} = g_{ji} = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} = \mathbf{e}_i \cdot \mathbf{e}_j , \quad i, j = 1,2,3 ; \quad \text{(A.1.38)}$$

the metrics of the considered *Euclidean space* is thus defined. The volume element, that is the volume of the curvilinear parallelepipedon built up with the vectors $\mathbf{e}_1 \mathrm{d} q_1$, $\mathbf{e}_2 \mathrm{d} q_2$, $\mathbf{e}_3 \mathrm{d} q_3$ is given by

$$\mathrm{d} V = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \mathrm{d} q_1 \mathrm{d} q_2 \mathrm{d} q_3 = J \mathrm{d} q_1 \mathrm{d} q_2 \mathrm{d} q_3 , \tag{A.1.39}$$

assuming that we have to do with a positive basis. Using Gramm's determinant (2.1.42'), we may write

$$g = \det\left[g_{ij} \right] = \det\left[\mathbf{e}_i \cdot \mathbf{e}_j \right] = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^2 = J^2 \tag{A.1.34''}$$

too, so that

$$\mathrm{d} V = \sqrt{g} \mathrm{d} q_1 \mathrm{d} q_2 \mathrm{d} q_3 . \tag{A.1.39'}$$

In the case of a system of *orthogonal curvilinear co-ordinates* we have $g_{ij} = 0$, $i \neq j$, and

$$g_{11} = \mathbf{e}_1^2 = \left(\frac{\partial \mathbf{r}}{\partial q_1} \right)^2 = H_1^2 = \frac{1}{h_1^2}, \quad g_{22} = \mathbf{e}_2^2 = \left(\frac{\partial \mathbf{r}}{\partial q_2} \right)^2 = H_2^2 = \frac{1}{h_2^2}, \\ g_{33} = \mathbf{e}_3^2 = \left(\frac{\partial \mathbf{r}}{\partial q_3} \right)^2 = H_3^2 = \frac{1}{h_3^2}, \tag{A.1.40}$$

H_1, H_2, H_3 being *Lamé's coefficients*, while h_1, h_2, h_3 are *differential parameters of first order*; it results

$$g = g_{11} g_{22} g_{33} = (H_1 H_2 H_3)^2 = \frac{1}{(h_1 h_2 h_3)^2} . \tag{A.1.40'}$$

The element of arc is given by

$$\mathrm{d} s^2 = \mathrm{d} s_1^2 + \mathrm{d} s_2^2 + \mathrm{d} s_3^2 = (H_1 \mathrm{d} q_1)^2 + (H_2 \mathrm{d} q_2)^2 + (H_3 \mathrm{d} q_3)^2 \\ = \left(\frac{\mathrm{d} q_1}{h_1} \right)^2 + \left(\frac{\mathrm{d} q_2}{h_2} \right)^2 + \left(\frac{\mathrm{d} q_3}{h_3} \right)^2 . \tag{A.1.40''}$$

and the element of volume reads

$$\mathrm{d} V = H_1 H_2 H_3 \mathrm{d} q_1 \mathrm{d} q_2 \mathrm{d} q_3 = \frac{\mathrm{d} q_1 \mathrm{d} q_2 \mathrm{d} q_3}{h_1 h_2 h_3} . \tag{A.1.40'''}$$

A system of *spherical co-ordinates* (r, θ, φ) is linked to the orthogonal Cartesian co-ordinates (see Fig.1.5,c) by the relations

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta, \quad r \geq 0, \\ 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi; \tag{A.1.41}$$

the element of arc is expressed in the form ($H_1 = 1$, $H_2 = r$, $H_3 = r\sin\theta$)

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2 = \mathrm{d}r^2 + \mathrm{d}s_\theta^2 + \mathrm{d}s_\varphi^2, \tag{A.1.41'}$$

while the element of volume is given by ($g = r^4\sin^2\theta$)

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi. \tag{A.1.41''}$$

The functional determinant fulfils the condition $J = r^2\sin\theta \neq 0$ if $r > 0$, $0 < \theta < \pi$.

The system of *cylindrical co-ordinates* (r, θ, z) is linked to the orthogonal Cartesian co-ordinates (see Fig.1.5,b) by the relations

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z, \quad r \geq 0, \quad 0 \leq \theta < 2\pi, \\ -\infty < z < \infty; \tag{A.1.42}$$

the element of arc is given by ($H_1 = H_3 = 1$, $H_2 = r$)

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + \mathrm{d}s_\theta^2 + \mathrm{d}z^2 \tag{A.1.42'}$$

and the element of volume is expressed in the form ($g = r^2$)

$$\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z. \tag{A.1.42''}$$

As well, to have $J = r \neq 0$, it is necessary that $r > 0$.

Differentiating the formula (A.1.38) with respect of the variable q_k, we may write

$$\mathbf{e}_{i,k} \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_{j,k} = g_{ij,k}, \quad i, j, k = 1,2,3, \tag{A.1.43}$$

where the index at the right to the comma indicates the differentiation with respect to the corresponding variable; we write again this relation by circular permutations

$$\mathbf{e}_{j,i} \cdot \mathbf{e}_k + \mathbf{e}_j \cdot \mathbf{e}_{k,i} = g_{jk,i}, \quad \mathbf{e}_{k,j} \cdot \mathbf{e}_i + \mathbf{e}_k \cdot \mathbf{e}_{i,j} = g_{ki,j}, \quad i, j, k = 1,2,3. \tag{A.1.43'}$$

Summing the relations (A.1.43') and subtracting the relation (A.1.43), we may express *Christoffel's symbols of first species* in the form

$$[ij, k] = \begin{bmatrix} i\ j \\ k \end{bmatrix} = \Gamma_{kij} = \mathbf{e}_{i,j} \cdot \mathbf{e}_k = \frac{1}{2}\left(-g_{ij,k} + g_{jk,i} + g_{ki,j}\right), \\ i, j, k = 1,2,3, \tag{A.1.44}$$

where we have introduced the most used notations; to obtain this result, we have taken into consideration that $\mathbf{e}_{i,j} = \mathbf{e}_{j,i}$, $i, j = 1,2,3$, due to the relation of definition (A.1.38) and to the property of the mixed derivatives of second order of the position vectors $\mathbf{r} \in C^2(D)$ of not depending on the order of differentiation. *The Christoffel symbols of second species* are defined in the form

$$\begin{Bmatrix} k \\ i\ j \end{Bmatrix} = \Gamma_{ij}^{k} = g^{kl}[ij,l], \quad i,j,k = 1,2,3, \tag{A.1.45}$$

where g^{ij} is *the normalized algebraic complement* (the algebraic complement divided by g) of the element g_{ij} of $\det[g_{ij}]$; we have $g^{ij} = g^{ji}$ because $g_{ij} = g_{ji}$. We notice that the relations

$$g_{ik}g^{kj} = \delta_i^j, \quad g^{ik}g_{kj} = \delta_j^i, \quad i,j = 1,2,3, \quad \det\left|g^{ij}\right| = \frac{1}{g} \tag{A.1.43''}$$

take place. Christoffel's symbols are symmetric with respect to the indices i and j, so that

$$[ij,k] = [ji,k], \quad \begin{Bmatrix} k \\ i\ j \end{Bmatrix} = \begin{Bmatrix} k \\ j\ i \end{Bmatrix}, \quad i,j,k = 1,2,3; \tag{A.1.46}$$

hence, there are 18 distinct symbols of each species. Multiplying the relation (A.1.45) by g_{km} and taking into account (A.1.43''), we get

$$[ij,k] = g_{kl}\begin{Bmatrix} l \\ i\ j \end{Bmatrix}, \quad i,j,k = 1,2,3. \tag{A.1.45'}$$

Christoffel's symbols are defined by the relations (A.1.44), (A.1.45) in the case of a linear space L_n too.

1.2 Exterior differential calculus

In what follows, we introduce the external product of vectors as well as differential forms of various orders; in connection with these forms, we put then in evidence the operator of exterior differentiation.

1.2.1 External product of vectors

A (free) *n-dimensional vector* is a mathematical entity characterized by an ordered set of n numbers V^i, $i = 1,2,...,n$; using the way indicated in Chap. 1, Subsec. 1.1.2, we may set up an *n-dimensional vector space* (*the linear space* L_n, which has the same properties as the linear space L_3). *There exist*, in this space, *at the most* n *independent* *linear vectors*; an ordered set of n independent linear vectors $\{\mathbf{e}_i, i = 1,2,...,n\}$ forms a *basis*, and an arbitrary vector $\mathbf{V}$ may be written in the form

$$\mathbf{V} = \sum_{i=1}^{n} V^i \mathbf{e}_i \ , \tag{A.1.47}$$

where V^i are *the components of the vector* in this basis (the contravariant components, but – as till now – we do not use the notions of contravariance and covariance). *The external product* or *the bivector* $\mathbf{V}_1 \wedge \mathbf{V}_2$ (it coincides with the vector product for $n = 3$; we replace the sign "$\times$" by the sign "$\wedge$") of two vectors $\mathbf{V}_1, \mathbf{V}_2$ is defined by the properties ($\mathbf{V}, \mathbf{V}_1, \mathbf{V}_2$ are vectors; λ_1, λ_2 are scalars):

i) $(\lambda_1 \mathbf{V}_1 + \lambda_2 \mathbf{V}_2) \wedge \mathbf{V} = \lambda_1 (\mathbf{V}_1 \wedge \mathbf{V}) + \lambda_2 (\mathbf{V}_2 \wedge \mathbf{V})$
$\mathbf{V} \wedge (\lambda_1 \mathbf{V}_1 + \lambda_2 \mathbf{V}_2) = \lambda_1 (\mathbf{V} \wedge \mathbf{V}_1) + \lambda_2 (\mathbf{V} \wedge \mathbf{V}_2)$
(distributivity with respect to addition of vectors);

ii) $\mathbf{V} \wedge \mathbf{V} = \mathbf{0}$;

iii) $\mathbf{V}_1 \wedge \mathbf{V}_2 + \mathbf{V}_2 \wedge \mathbf{V}_1 = \mathbf{0}$ (anticommutativity).

We may write

$$\mathbf{V}_1 \wedge \mathbf{V}_2 = \sum_{i=1}^{n} V_1^i \mathbf{e}_i \wedge \sum_{j=1}^{n} V_2^j \mathbf{e}_j = \sum_{i=1}^{n} \sum_{j=1}^{n} V_1^i V_2^j \mathbf{e}_i \wedge \mathbf{e}_j \ ; \tag{A.1.48}$$

inverting i by j and taking into account the property iii), it results

$$\mathbf{V}_1 \wedge \mathbf{V}_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(V_1^i V_2^j - V_1^j V_2^i \right) \mathbf{e}_i \wedge \mathbf{e}_j \ . \tag{A.1.48'}$$

The properties ii) and iii) lead to the relation

$$\mathbf{V}_1 \wedge \mathbf{V}_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(V_1^i V_2^j - V_1^j V_2^i \right) \mathbf{e}_i \wedge \mathbf{e}_j \ , \quad i < j \ . \tag{A.1.48''}$$

We denote by $\wedge^2 L_n$ the vector space of the bivectors defined on L_n (corresponding to this notation $\wedge^1 L_n = L_n$); noting that the set $\{\mathbf{e}_i \wedge \mathbf{e}_j, 1 \le i < j \le n\}$ forms a basis in this space, its dimension is given by

$$\dim \wedge^2 L_n = 1 + 2 + ... + (n-1) = \frac{n(n-1)}{2} = C_n^2 \ . \tag{A.1.49}$$

In general, a *p-vector* $\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge ... \wedge \mathbf{V}_p$ in L_n is defined by the properties ($\mathbf{W}_1, \mathbf{W}_2, \mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_p$ are vectors; λ_1, λ_2 are scalars):

i) $(\lambda_1 \mathbf{W}_1 + \lambda_2 \mathbf{W}_2) \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p = \lambda_1 (\mathbf{W}_1 \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p)$
$+ \lambda_2 (\mathbf{W}_2 \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p)$
(and analogous relations, obtained by replacing the other vectors $\mathbf{V}_i$, $i = 2,3,\ldots,p$, which form the p-vector by linear combinations);

ii) $\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p = \mathbf{0}$ if and only if $\mathbf{V}_i = \mathbf{V}_j$, $i \neq j$;

iii) The external product $\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p$ changes its sign if two factors of it permute.

Let be $\wedge^p L_n$ the vector space formed by p-vectors defined on L_n, $2 \leq p \leq n$. The set $\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \ldots \wedge \mathbf{e}_{i_p}, 1 \leq i_1 < i_2 < \ldots < i_p \leq n\}$ forms a basis in this space, so that

$$\dim \wedge^p L_n = C_n^p, \tag{A.1.49'}$$

where we have introduced the combination symbol of n things p at a time; in particular, $\dim \wedge^n L_n = C_n^n = 1$.

In the case of a *trivector* $\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3$ we may write

$$\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} V_1^i V_2^j V_3^k \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k ; \tag{A.1.50}$$

inverting two upper indices and taking into account the property iii), we find $3! = 6$ different representations of the external product (obtained by permuting the indices i, j and k and by introducing the sign minus in the case of an odd permutation). Summing these six representations, we obtain a representation of the form

$$\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \mathbf{V}_3 = \frac{1}{3!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} V^{ijk} \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k , \tag{A.1.50'}$$

where the scalar V^{ijk} is totally skew-symmetric with respect to the upper indices i, j, k. In general, a p-vector $\boldsymbol{\varphi}$ may be represented by

$$\boldsymbol{\varphi} \equiv \mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p = \frac{1}{p!} \sum_{i_k = 1}^{n} V^{i_1 i_2 \ldots i_p} \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \ldots \wedge \mathbf{e}_{i_p} , \tag{A.1.51}$$

where the scalar $V^{i_1 i_2 \ldots i_p}$ is totally skew-symmetric with respect to the upper indices i_k, $k = 1,2,\ldots,p$ (the summation takes place for all upper indices). For $p = 2$ one obtains the representation (A.1.48').

One can define a p-vector $\boldsymbol{\varphi} \equiv \mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p$ and a q-vector $\boldsymbol{\psi} \equiv \mathbf{W}_1 \wedge \mathbf{W}_2 \wedge \ldots \wedge \mathbf{W}_q$ on L_n; their external product is a $p+q$-vector, $p + q \leq n$, defined in the form

$$\boldsymbol{\varphi} \wedge \boldsymbol{\psi} \equiv (\mathbf{V}_1 \wedge \mathbf{V}_2 \wedge \ldots \wedge \mathbf{V}_p) \wedge (\mathbf{W}_1 \wedge \mathbf{W}_2 \wedge \ldots \wedge \mathbf{W}_q)$$
$$= \mathbf{V}_1 \wedge \ldots \wedge \mathbf{V}_p \wedge \mathbf{W}_1 \wedge \ldots \wedge \mathbf{W}_q \,. \tag{A.1.52}$$

We mention following properties:

i) $\boldsymbol{\varphi} \wedge (\lambda_1 \boldsymbol{\psi}_1 + \lambda_2 \boldsymbol{\psi}_2) = \lambda_1 \boldsymbol{\varphi} \wedge \boldsymbol{\psi}_1 + \lambda_2 \boldsymbol{\varphi} \wedge \boldsymbol{\psi}_2$ (*distributivity with respect to addition*; $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2$ are *p*-vectors, λ_1, λ_2 are scalars);

ii) $(\boldsymbol{\varphi} \wedge \boldsymbol{\psi}) \wedge \boldsymbol{\chi} = \boldsymbol{\varphi} \wedge (\boldsymbol{\psi} \wedge \boldsymbol{\chi})$ (*associativity*; $\boldsymbol{\chi}$ is *r*-vector; $p + q + r \leq n$);

iii) $\boldsymbol{\varphi} \wedge \boldsymbol{\psi} = (-1)^{pq} \boldsymbol{\psi} \wedge \boldsymbol{\varphi}$.

If one of the numbers p, q is even, then the external product is commutative, otherwise it is anticommutative.

1.2.2 Differential forms. Exterior derivative

We call *differential form of first degree* in $x \equiv (x_1, x_2, \ldots, x_n) \in E_n$ the expression

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x_i \,, \quad a_i = \mathrm{const}\,. \tag{A.1.53}$$

If between the basis $\{\mathbf{e}_i\}$ of the linear space L_n and $\{\mathrm{d}x_i\}$, $i = 1,2,\ldots,n$, is established an isomorphism, then ω is an element of the space $\wedge^1 L_n$, introduced in the preceding subsection. Analogously, a *form of pth degree* in x is an element of the space $\wedge^p L_n$, being expressed by

$$\omega = \sum a_{i_1 i_2 \ldots i_p} \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_p} \,, \quad a_{i_1 i_2 \ldots i_p} = \mathrm{const}\,. \tag{A.1.53'}$$

In the case of a form of *p*th degree on $D \subset E_n$, $a_i = a_i(x_1, x_2, \ldots, x_n)$ may be smooth functions (differentiable as much as it is necessary) on D. We denote by $F^p(D)$ the set of the forms of *p*th degree on D; in this case, $F^0(D)$ represents the set of smooth functions.

Let be the forms of first degree

$$\omega_1 = a_i(x_1, x_2, x_3)\mathrm{d}x_i \,, \quad \omega_2 = b_j(x_1, x_2, x_3)\mathrm{d}x_j \,; \tag{A.1.54}$$

the *external product* introduced in the previous subsection is calculated in the form ($\mathrm{d}x_i$ play the rôle of vectors $\mathbf{e}_i$)

$$\omega_1 \wedge \omega_2 = (a_2 b_3 - a_3 b_2)\mathrm{d}x_2 \wedge \mathrm{d}x_3 + (a_3 b_1 - a_1 b_3)\mathrm{d}x_3 \wedge \mathrm{d}x_1$$
$$+ (a_1 b_2 - a_2 b_1)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \,. \tag{A.1.54'}$$

One obtains thus a *form of second degree*.

The operator $\mathrm{d} : F^p(D) \to F^{p+1}(D)$, called (after Cartan) *operator of exterior differentiation*, exists and is unique, being defined by the properties (ω, ω_1 and ω_2 are p-forms):

i) $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$;

ii) $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{\mathrm{grad}\,\omega_1}\,\omega_1 \wedge \mathrm{d}\omega_2$;

iii) $\mathrm{d}(\mathrm{d}\omega) = 0$ (Poincaré's lemma);

iv) $\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\mathrm{d}x_j$ for any function f.

These properties are independent of the system of co-ordinates. One may show that the exterior derivative of the form (A.1.53') is expressed by

$$\mathrm{d}\omega = \sum \frac{\partial a_{i_1 i_2 \dots i_p}}{\partial x_j}\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \dots \wedge \mathrm{d}x_{i_p}\ . \tag{A.1.55}$$

In the particular case $n = 3$ one may introduce *the gradient* operator in the form

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \frac{\partial f}{\partial x_3}\mathrm{d}x_3\,, \tag{A.1.56}$$

where f is a function defined on $D \subset E_3$. For a form of first degree on D

$$\omega = a_1(x_1, x_2, x_3)\mathrm{d}x_1 + a_2(x_1, x_2, x_3)\mathrm{d}x_2 + a_3(x_1, x_2, x_3)\mathrm{d}x_3 \tag{A.1.57}$$

the exterior derivative

$$\mathrm{d}\omega = \sum \frac{\partial a_i}{\partial x_j}\mathrm{d}x_j \wedge \mathrm{d}x_i = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathrm{d}x_2 \wedge \mathrm{d}x_3$$
$$+\left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathrm{d}x_3 \wedge \mathrm{d}x_1 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{A.1.57'}$$

allows to introduce *the* operator *curl*. As well, the exterior derivative

$$\mathrm{d}\omega = \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3\,, \tag{A.1.58}$$

corresponding to the form of second degree on D

$$\omega = a_1\mathrm{d}x_2 \wedge \mathrm{d}x_3 + a_2\mathrm{d}x_3 \wedge \mathrm{d}x_1 + a_3\mathrm{d}x_1 \wedge \mathrm{d}x_2\,, \tag{A.1.58'}$$

introduces *the* operator *divergence*.

2. Notions of field theory

In what follows, we deal with conservative vectors, with the operator gradient, as well as with the introduction of the curl and the divergence of a vector; differential operators of second order are considered too and some integral formulae are given. We mention also the absolute and the relative derivatives.

2.1 Conservative vectors. Gradient

Let be a point $P(x_1, x_2, x_3) \in D \subset \mathbb{R}^3$. In what follows, the vector mapping $(x_1, x_2, x_3) \to \mathbf{V}(x_1, x_2, x_3)$ defines a *vector field* ($\mathbf{V} : D \to L$); the respective vectors are bound vectors, the points P being their points of application. This field is a *steady* one; if $\mathbf{V} = \mathbf{V}(x_1, x_2, x_3; t)$, then the vector field defined on $D \times [t_0, t_1]$ is *non-stationary*. We express the vector $\mathbf{V} = \mathbf{V}(x_1, x_2, x_3) = \mathbf{V}(\mathbf{r})$ in the form

$$\mathbf{V}(x_1, x_2, x_3) = V_j(x_1, x_2, x_3)\mathbf{i}_j, \tag{A.2.1}$$

where $V_j \in C^1(D)$, $j = 1,2,3$; let us introduce the vector fields

$$\frac{\partial \mathbf{V}}{\partial x_i} = \mathbf{V}_{,i} = \frac{\partial V_j}{\partial x_i}\mathbf{i}_j = V_{j,i}\mathbf{i}_j, \quad i = 1,2,3, \tag{A.2.2}$$

and the differential

$$\mathrm{d}\mathbf{V} = \mathbf{V}_{,i}\mathrm{d}x_i, \tag{A.2.3}$$

where the index at the right of the comma specifies the derivative with respect to the corresponding variable. The curves for which the tangents at each point P are directed along the vectors $\mathbf{V} = \mathbf{V}(\mathbf{r})$ of the field are called *vector lines* (or *field lines*); the lines form a congruence of curves. Because the differential $\mathrm{d}\mathbf{r}$ is tangent to these lines, their vector equation is of the form

$$\mathbf{V}(\mathbf{r}) \times \mathrm{d}\mathbf{r} = \mathbf{0}; \tag{A.2.4}$$

scalarly, these lines are given by a system of differential equations of first order

$$\frac{\mathrm{d}x_1}{V_1} = \frac{\mathrm{d}x_2}{V_2} = \frac{\mathrm{d}x_3}{V_3}. \tag{A.2.4'}$$

Let be C a curve which is not a field line; on the basis of the theorem of existence and uniqueness for a system of differential equations of the form (A.2.4'), through each point of the curve C passes a field line (the integral curve of the system (A.2.4')). The surface generated by these curves is called *field surface*.

In the case of a non-steady field, *the differential* is of the form

$$\mathrm{d}\mathbf{V} = \mathbf{V}_{,i}\mathrm{d}x_i + \dot{\mathbf{V}}\mathrm{d}t, \tag{A.2.3'}$$

where $\dot{V} = \partial V / \partial t$, and *the total derivative* is given by (we consider the mapping $t \to \mathbf{r}(t)$)

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{V}_{,i}\dot{x}_i + \dot{\mathbf{V}}, \tag{A.2.3''}$$

being the sum of the *space* and *time derivatives*; in the case of a steady field remains only the space derivative.

In what follows we introduce some particular fields of vectors.

2.1.1 Conservative vectors. The nabla operator

Let us consider a scalar function $(x_1, x_2, x_3) \to U(x_1, x_2, x_3)$, $(x_1, x_2, x_3) \in D \subset \mathbb{R}^3$; the function $U \in C^1(D)$ defines a scalar field ($U : D \to \mathbb{R}^3$), because to each point P, which is of position vector $\mathbf{r}(x_1, x_2, x_3)$ one can associate the scalar $U = U(P)$. This scalar field is *steady*; we may consider also *non-steady* fields of the form $U(x_1, x_2, x_3; t)$, defined on $D \times [t_0, t_1]$. If it is necessary, the function U may have also continuous derivatives of higher order. Let be a vector field $\mathbf{V}$, defined by the relations

$$V_i = U_{,i}, \quad i = 1,2,3. \tag{A.2.5}$$

Considering a unit vector $\mathbf{n}(n_i)$, we notice that

$$\mathbf{V} \cdot \mathbf{n} = U_{,i} n_i = \frac{\partial U}{\partial n}; \tag{A.2.6}$$

hence, the components of $\mathbf{V}$ with respect to a new three-orthogonal trihedron of reference $Ox_1'x_2'x_3'$ are $\partial U / \partial x_i'$, $i = 1,2,3$. Thus, the definition given to the vector field does not depend on the chosen co-ordinate system. Such a field is called a *conservative field* (which derives from the potential U); the corresponding vectors are called *conservative vectors* and the field U is called *potential*. Assuming that $U = U(\mathbf{r}; t)$, one obtains a vector field defined by the same formulae (A.2.5); this is a *quasi-conservative field* and the corresponding vectors are *quasi-conservative vectors*. Analogously, the function U is called *quasi-potential*.

We notice that one can formally write $\mathbf{V} = U_{,j}\mathbf{i}_j = (\mathbf{i}_j \partial / \partial x_j) U$; applying the vector differential operator

$$\nabla = \mathbf{i}_j \frac{\partial}{\partial x_j} = \mathbf{i}_j \partial_j, \tag{A.2.7}$$

which is called *nabla* (or *del*) and was introduced by Hamilton, we get

$$\mathbf{V} = \nabla U. \tag{A.2.8}$$

In the case of a quasi-potential, *the differential* is of the form

$$\mathrm{d}U = U_{,j} \mathrm{d}x_j + \dot{U} \mathrm{d}t = \nabla U \cdot \mathrm{d}\mathbf{r} + \dot{U} \mathrm{d}t, \tag{A.2.9}$$

where $\dot{U} = \partial U / \partial t$, while *the total* (*substantial*) *derivative* is given by (we consider the mapping $t \to \mathbf{r}(t)$)

$$\frac{\mathrm{d}U}{\mathrm{d}t} = U_{,j}\dot{x}_j + \dot{U} = \nabla U \cdot \dot{\mathbf{r}} + \dot{U} \tag{A.2.9'}$$

and is the sum of the *space derivative* and the *time derivative*; in the case of a potential remains only the space derivative.

We observe also that, in the case of a conservative vector field, *the elementary work* of a conservative vector is given by

$$\mathrm{d}W = \nabla U \cdot \mathrm{d}\mathbf{r} = \mathrm{d}U\,, \tag{A.2.10}$$

so that it is a total differential; it results

$$W_{\widehat{P_0P_1}} = U(P_1) - U(P_0)\,, \tag{A.2.10'}$$

where we started from the formula (A.1.28). Hence, in the case of a conservative vector, the work between two points does not depend on the path, but only on the values of the potential at its extremities; analogously, using the formula (A.1.29'), we notice that the work of a conservative vector on a closed curve (*circulation of a conservative vector*) vanishes if the corresponding domain D is simply connected.

2.1.2 Equipotential surfaces. Gradient

Let be

$$U(x_1, x_2, x_3) = C\,, \quad C = \text{const} \tag{A.2.11}$$

the equation of a surface, which is the locus of the points for which the scalar potential is constant; we assume that $U \in C^1$. This surface is called an *equipotential surface*; if $U(\mathbf{r})$ defines an arbitrary scalar field, then the surface (A.2.11) is called also a *level surface*. If $U = U(x_1, x_2, x_3; t)$, then we get an *equiquasi-potential surface*

$$U(x_1, x_2, x_3; t) = C\,, \quad C = \text{const}\,, \tag{A.2.11'}$$

which is a variable surface in time. Let us suppose that through the point $\left(x_1^0, x_2^0, x_3^0\right)$ pass two equipotential surfaces, so that for that point we may write $U\left(x_1^0, x_2^0, x_3^0\right) = C_1$, $U\left(x_1^0, x_2^0, x_3^0\right) = C_2$, $C_1, C_2 = \text{const}$; it follows that $0 = C_1 - C_2$. Hence, the two equipotential surfaces coincide, admitting that the function U is uniform. If two equipotential surfaces do not coincide, then they have not common points.

Applying the operator ∇ to a scalar function, one obtains a vector function which is called *the gradient* of the scalar function. Hence,

$$\operatorname{grad} U = \nabla U\,, \tag{A.2.12}$$

so that the operator ∇ transforms the scalar field in a vector one. Thus, the gradient allows to appreciate the variation of a scalar function, obtaining also its derivatives of first order. We observe thus that a field of conservative vectors is a field of gradients. We mention the properties:

i) $\operatorname{grad}(U_1 + U_2) = \operatorname{grad} U_1 + \operatorname{grad} U_2$ (distributivity with respect to addition of scalars);

ii) $\operatorname{grad} CU = C \operatorname{grad} U$, $C = \mathrm{const}$;

iii) $\operatorname{grad} C = \mathbf{0}$, $C = \mathrm{const}$.

The relation (A.2.6) may be written in the form

$$\mathbf{n} \cdot \operatorname{grad} U = \frac{\partial U}{\partial n}, \tag{A.2.6'}$$

so that the gradient forms a vector field, independent on the chosen system of co-ordinate axes; $\partial U / \partial n$ is the derivative of the scalar field U in the direction of the unit vector $\mathbf{n}$. Because

$$\operatorname{grad} U = U_{,j} \mathbf{i}_j = \partial_j U \mathbf{i}_j, \tag{A.2.12'}$$

it results that the gradient of the function U is normal to the equipotential surface $U(x_1, x_2, x_3) - C = 0$. As well, if we take $\mathbf{n} = \operatorname{vers} \operatorname{grad} U$ in the relation (A.2.12'), then the gradient of the scalar function U is a vector directed in the same direction for which the value of the function is increasing. The congruence of gradient lines is thus normal to the family of corresponding equipotential surfaces, the direction of travelling through being that in which the value of the scalar function U is increasing. These properties hold also for a quasi-conservative scalar field. If $\mathrm{d}\mathbf{r}$ is along the tangent to a curve C, which is travelled through by a point $P(x_1, x_2, x_3)$, then it results – by equating to zero the expression (A.2.9) – that, for a quasi-potential scalar field, the curve C cannot stay on an equiquasi-potential surface; but – in exchange – for a potential scalar field, the curve C may belong to a corresponding equipotential surface. We can verify the above mentioned properties, e.g., in the case of the scalar potential

$$U(x_1, x_2, x_3) = x_i x_i = x_1^2 + x_2^2 + x_3^2 \tag{A.2.13}$$

and in the case of the quasi-potential scalar

$$U(x_1, x_2, x_3; t) = x_i x_i - \frac{1}{c^2} t^2 = x_1^2 + x_2^2 + x_3^2 - \frac{1}{c^2} t^2, \quad c = \mathrm{const}. \tag{A.2.13'}$$

The necessary and sufficient conditions for the scalar functions

$$V_i = V_i(x_1, x_2, x_3; t), \quad i = 1, 2, 3 \tag{A.2.14}$$

to be the components of a quasi-conservative vector are of the form

$$\in_{ijk} \partial_j V_k = \in_{ijk} V_{k,j} = 0, \quad i = 1, 2, 3; \tag{A.2.15}$$

they result from the relation (A.2.5) and from the condition that the mixed derivatives of second order of the quasi-potential $U = U(x_1, x_2, x_3; t)$ be immaterial on the order of differentiation (we assume that $U \in C^2(D)$, $(x_1, x_2, x_3) \in D \subset \mathbb{R}^3$, with respect to the space variables).

To a determined potential $U = U(x_1, x_2, x_3)$ one may add an arbitrary constant, without any change of its properties. We mention also the properties:

iv) $\operatorname{grad}(U_1 U_2) = U_1 \operatorname{grad} U_2 + U_2 \operatorname{grad} U_1$;

v) $\operatorname{grad} f(U) = f'(U) \operatorname{grad} U$, $f \in C^1$;

vi) $\operatorname{grad} f(U_1, U_2) = f'_{U_1} \operatorname{grad} U_1 + f'_{U_2} \operatorname{grad} U_2$;

vii) $f(U) \operatorname{grad} U = \operatorname{grad} \int f(U) \mathrm{d} U$, f integrable.

These properties hold for a quasi-potential field too. Concerning the position vector $\mathbf{r}$, we may write

$$\operatorname{grad} r = \frac{1}{r}\mathbf{r}, \quad \operatorname{grad} f(r) = \frac{f'(r)}{r}\mathbf{r}, \quad f \in C^1, \tag{A.2.16}$$

$$\operatorname{grad}(\mathbf{C} \cdot \mathbf{r}) = \mathbf{C}, \quad \mathbf{C} = \overrightarrow{\text{const}}. \tag{A.2.16'}$$

In the case of a potential $U = U(x_1, x_2, x_3) = U(\mathbf{r})$, the formula (A.2.9) allows to write the differential

$$\mathrm{d} U = \operatorname{grad} U \cdot \mathrm{d}\mathbf{r}. \tag{A.2.17}$$

Introducing the mapping $s \to \mathbf{r}(s)$, which defines a curve C, it results that $U = U(s)$, so that

$$\frac{\partial U}{\partial s} = \frac{\mathrm{d} U}{\mathrm{d} s} = \operatorname{grad} U \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d} s} = \operatorname{grad} U \cdot \boldsymbol{\tau}, \tag{A.2.18}$$

obtaining thus the derivative on the direction of the unit vector $\boldsymbol{\tau}$ of the tangent to this curve; the partial derivative is, in this case, equal to the total derivative. By means of the above introduced operator ∇, one may conceive the symbol d of the total differential as an operator, in the form of a scalar product

$$\mathrm{d} = \mathrm{d}\mathbf{r} \cdot \nabla = \mathrm{d}\mathbf{r} \cdot \operatorname{grad}. \tag{A.2.19}$$

The formula (A.2.6') leads to the operator derivative in the direction of the unit vector $\mathbf{n}$, which may be expressed in the form

$$\frac{\partial}{\partial n} = \mathbf{n} \cdot \nabla = \mathbf{n} \cdot \operatorname{grad}; \tag{A.2.20}$$

if $\mathbf{r} = \mathbf{r}(s)$ and $\mathbf{n} = \boldsymbol{\tau}$, then we get the operator

$$\frac{\partial}{\partial s} = \frac{\mathrm{d}}{\mathrm{d}s} = \mathbf{r}'(s) \cdot \nabla = \boldsymbol{\tau} \cdot \mathrm{grad}\,. \tag{A.2.20'}$$

Thus, one introduces the linear scalar differential operator

$$\mathbf{A} \cdot \mathrm{grad} = \mathbf{A} \cdot \nabla = A_i \frac{\partial}{\partial x_i} = A_i \partial_i\,, \tag{A.2.21}$$

where $\mathbf{A}$ is a constant or variable given vector. Applying this operator to the scalar field U, one obtains the scalar

$$(\mathbf{A} \cdot \mathrm{grad})U = (A_i \partial_i)U = A_i U_{,i} \tag{A.2.21'}$$

as well, but applying the operator to the vector field $\mathbf{V}$, it results the vector

$$(\mathbf{A} \cdot \mathrm{grad})\mathbf{V} = (A_j \partial_j)(V_k \mathbf{i}_k) = A_j V_{k,j} \mathbf{i}_k\,. \tag{A.2.21''}$$

In particular, we get the total differential of a vector field $\mathbf{V} = \mathbf{V}(x_1, x_2, x_3)$ in the form

$$\mathrm{d}\mathbf{V} = (\mathrm{d}\mathbf{r} \cdot \mathrm{grad})\mathbf{V}\,, \tag{A.2.22}$$

and the derivative in the direction of the unit vector $\mathbf{n}$ is given by

$$\frac{\partial \mathbf{V}}{\partial n} = (\mathbf{n} \cdot \mathrm{grad})\mathbf{V}\,; \tag{A.2.23}$$

for $\mathbf{r} = \mathbf{r}(s)$ and $\mathbf{n} = \boldsymbol{\tau}$ one obtains

$$\frac{\partial \mathbf{V}}{\partial s} = \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}s} = (\boldsymbol{\tau} \cdot \mathrm{grad})\mathbf{V}\,. \tag{A.2.23'}$$

In the system of curvilinear co-ordinates (q_1, q_2, q_3), introduced in Subsec. 1.1.5, the moduli of the gradients of the surfaces of co-ordinates $q_i = \mathrm{const}$, $i = 1,2,3$, are given by the differential parameters of the first order

$$|\mathrm{grad}\, q_i| = \sqrt{g^{ii}} = h_i \text{ (!)}, \quad i = 1,2,3\,. \tag{A.2.24}$$

In orthogonal curvilinear co-ordinates, the gradient operator has the form

$$\mathrm{grad} = \frac{1}{H_1}\mathbf{i}_1 \frac{\partial}{\partial q_1} + \frac{1}{H_2}\mathbf{i}_2 \frac{\partial}{\partial q_2} + \frac{1}{H_3}\mathbf{i}_3 \frac{\partial}{\partial q_3}\,. \tag{A.2.25}$$

In spherical co-ordinates, we obtain

$$\operatorname{grad} = \mathbf{i}_r \frac{\partial}{\partial s_r} + \mathbf{i}_\theta \frac{\partial}{\partial s_\theta} + \mathbf{i}_\varphi \frac{\partial}{\partial s_\varphi} = \mathbf{i}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{i}_\theta \frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\mathbf{i}_\varphi \frac{\partial}{\partial \varphi}, \qquad \text{(A.2.25')}$$

while in cylindrical co-ordinates we may write

$$\operatorname{grad} = \mathbf{i}_r \frac{\partial}{\partial s_r} + \mathbf{i}_\theta \frac{\partial}{\partial s_\theta} + \mathbf{i}_z \frac{\partial}{\partial s_z} = \mathbf{i}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{i}_\theta \frac{\partial}{\partial \theta} + \mathbf{i}_z \frac{\partial}{\partial z}. \qquad \text{(A.2.25'')}$$

2.2 Differential operators of first and second order

One may set up vector differential operators of first order and scalar differential operators of second order by means of the vector differential operator ∇; such operators may appear in problems of mechanics.

2.2.1 Vector differential operators of first order

Besides the scalar differential operator $\mathbf{A} \cdot \nabla$, we introduce also *the vector differential operator*

$$\mathbf{A} \times \nabla = \in_{jkl} A_j \mathbf{i}_l \partial_k = \in_{jkl} A_j \mathbf{i}_l \frac{\partial}{\partial x_k}, \qquad \text{(A.2.26)}$$

where $\mathbf{A}$ is a constant or variable given vector; applying this operator to the scalar field U, one obtains the vector

$$(\mathbf{A} \times \nabla)U = \in_{jkl} A_j U_{,k} \mathbf{i}_l. \qquad \text{(A.2.26')}$$

The differential operator "$\nabla \cdot$" applied to a vector $\mathbf{V}$ defines *the divergence* of the vector; we have thus

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{\partial V_i}{\partial x_i} = \partial_i V_i = V_{i,i}. \qquad \text{(A.2.27)}$$

This is a scalar quantity, hence invariant to a change of co-ordinate axes. We mention the properties:

i) $\operatorname{div}(\mathbf{V}_1 + \mathbf{V}_2) = \operatorname{div}\mathbf{V}_1 + \operatorname{div}\mathbf{V}_2$;

ii) $\operatorname{div}(\lambda\mathbf{V}) = \lambda \operatorname{div}\mathbf{V} + \mathbf{V} \cdot \operatorname{grad}\lambda$, $\lambda = \lambda(\mathbf{r})$ scalar.

In orthogonal curvilinear co-ordinates, we obtain

$$\operatorname{div}\mathbf{V} = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial}{\partial q_1}(H_2 H_3 V_1) + \frac{\partial}{\partial q_2}(H_3 H_1 V_2) + \frac{\partial}{\partial q_3}(H_1 H_2 V_3)\right]; \qquad \text{(A.2.28)}$$

in particular, in spherical co-ordinates, we have

$$\operatorname{div}\mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}, \qquad \text{(A.2.28')}$$

while in cylindrical co-ordinates we may write

$$\operatorname{div}\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}. \qquad \text{(A.2.28")}$$

The differential operator "$\nabla\times$" applied to a vector $\mathbf{V}$ leads to the *curl* of this vector; we may write

$$\operatorname{curl}\mathbf{V} = \nabla\times\mathbf{V} = \in_{jkl}\frac{\partial V_k}{\partial x_j}\mathbf{i}_l = \in_{jkl}\partial_j V_k\mathbf{i}_l = \in_{jkl} V_{k,j}\mathbf{i}_l, \qquad \text{(A.2.29)}$$

this definition being immaterial of the co-ordinate axes too. We mention the properties:

i) $\operatorname{curl}(\mathbf{V}_1 + \mathbf{V}_2) = \operatorname{curl}\mathbf{V}_1 + \operatorname{curl}\mathbf{V}_2$;

ii) $\operatorname{curl}(\lambda\mathbf{V}) = \lambda\operatorname{curl}\mathbf{V} + \mathbf{V}\times\operatorname{grad}\lambda$, $\lambda = \lambda(\mathbf{r})$ scalar.

In orthogonal curvilinear co-ordinates we have

$$\operatorname{curl}\mathbf{V} = \frac{1}{H_2H_3}\left[\frac{\partial}{\partial q_2}(H_3V_3) - \frac{\partial}{\partial q_3}(H_2V_2)\right]\mathbf{i}_1 + \frac{1}{H_3H_1}\left[\frac{\partial}{\partial q_3}(H_1V_1) - \frac{\partial}{\partial q_1}(H_3V_3)\right]\mathbf{i}_2$$
$$+\frac{1}{H_1H_2}\left[\frac{\partial}{\partial q_1}(H_2V_2) - \frac{\partial}{\partial q_2}(H_1V_1)\right]\mathbf{i}_3; \qquad \text{(A.2.30)}$$

in particular, in spherical co-ordinates we may write

$$\operatorname{curl}\mathbf{V} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta V_\varphi) - \frac{\partial V_\theta}{\partial\varphi}\right]\mathbf{i}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}(rV_\varphi)\right]\mathbf{i}_\theta$$
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right]\mathbf{i}_\varphi, \qquad \text{(A.2.30')}$$

while in cylindrical co-ordinates we get

$$\operatorname{curl}\mathbf{V} = \left(\frac{1}{r}\frac{\partial V_z}{\partial\theta} - \frac{\partial V_\theta}{\partial z}\right)\mathbf{i}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\mathbf{i}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right]\mathbf{i}_z. \qquad \text{(A.2.30")}$$

Concerning the operators grad, div and curl we mention the formulae

$$\operatorname{grad}(\mathbf{V}_1\cdot\mathbf{V}_2) = (\mathbf{V}_2\cdot\nabla)\mathbf{V}_1 + (\mathbf{V}_1\cdot\nabla)\mathbf{V}_2 + \mathbf{V}_1\times\operatorname{curl}\mathbf{V}_2 + \mathbf{V}_2\times\operatorname{curl}\mathbf{V}_1, \qquad \text{(A.2.31)}$$
$$\operatorname{div}(\mathbf{V}_1\times\mathbf{V}_2) = \mathbf{V}_2\cdot\operatorname{curl}\mathbf{V}_1 - \mathbf{V}_1\cdot\operatorname{curl}\mathbf{V}_2, \qquad \text{(A.2.31')}$$
$$\operatorname{curl}(\mathbf{V}_1\times\mathbf{V}_2) = (\mathbf{V}_2\cdot\nabla)\mathbf{V}_1 - (\mathbf{V}_1\cdot\nabla)\mathbf{V}_2 + \mathbf{V}_1\operatorname{div}\mathbf{V}_2 - \mathbf{V}_2\operatorname{div}\mathbf{V}_1. \qquad \text{(A.2.31")}$$

In particular, the formula (A.2.31) leads to

$$\frac{1}{2}\operatorname{grad}\mathbf{V}^2 = (\mathbf{V}\cdot\nabla)\mathbf{V} + \mathbf{V}\times\operatorname{curl}\mathbf{V}. \qquad \text{(A.2.32)}$$

These results have been obtained by the methods of vector algebra, effecting formal calculations by means of the vector operator ∇.

A vector field for which

$$\operatorname{curl}\mathbf{V} = \nabla \times \mathbf{V} = \mathbf{0} \tag{A.2.33}$$

is called *irrotational*. We notice that a field of gradients

$$\mathbf{V} = \operatorname{grad} U = \nabla U \tag{A.2.33'}$$

is irrotational ($\operatorname{curl}\operatorname{grad} U = \mathbf{0}$); hence, the fields of quasi-conservative vectors (in particular, conservative) are irrotational, being the only ones which have the mentioned property. A vector field for which

$$\operatorname{div}\mathbf{V} = \nabla \cdot \mathbf{V} = 0 \tag{A.2.34}$$

is called *solenoidal*. It is easy to see that a field of curls

$$\mathbf{V} = \operatorname{curl}\mathbf{W} = \nabla \times \mathbf{W} \tag{A.2.34'}$$

is solenoidal ($\operatorname{div}\operatorname{curl}\mathbf{W} = 0$); one may show that this is the only vector field which has the property (A.2.34).

2.2.2 Absolute and relative derivatives

Let $\mathbf{V}$ be a vector expressed in the form $V_j'\mathbf{i}_j'$ with respect to a fixed orthonormed frame of reference $O'x_1'x_2'x_3'$ and in the canonical form $V_j\mathbf{i}_j$ with respect to a movable orthonormed frame $Ox_1x_2x_3$. The mobility of the latter frame is characterized by the mappings $t \to \mathbf{i}_j(t)$, $t \in [t_0, t_1]$, $i = 1,2,3$; as well the vector V defines a vector field by the mapping $t \to \mathbf{V}(t)$. The independent variable t may be – eventually – the time. Differentiating with respect to t, one obtains

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\mathrm{d}V_j}{\mathrm{d}t}\mathbf{i}_j + V_j\frac{\mathrm{d}\mathbf{i}_j}{\mathrm{d}t} = \dot{V}_j\mathbf{i}_j + V_j\dot{\mathbf{i}}_j .$$

We denote

$$\frac{\partial\mathbf{V}}{\partial t} = \dot{V}_j\mathbf{i}_j , \tag{A.2.35}$$

that is *the relative derivative* of the vector $\mathbf{V}$ with respect to the movable frame. Let be the vector

$$\boldsymbol{\omega} = \frac{1}{2}\in_{jkl}\left(\dot{\mathbf{i}}_j \cdot \mathbf{i}_k\right)\mathbf{i}_l ; \tag{A.2.36}$$

we notice that

$$\boldsymbol{\omega} \times \mathbf{V} = \in_{lmn} \omega_l V_m \mathbf{i}_n = \frac{1}{2} \in_{ljk} \in_{lmn} V_m \left(\dot{\mathbf{i}}_j \cdot \mathbf{i}_k \right) \mathbf{i}_n = V_j \left(\dot{\mathbf{i}}_j \cdot \mathbf{i}_k \right) \mathbf{i}_k = V_j \dot{\mathbf{i}}_j ,$$

where we took into account a formula of the form (2.1.46') and the relation $\dot{\mathbf{i}}_j \cdot \mathbf{i}_k + \mathbf{i}_j \cdot \dot{\mathbf{i}}_k = 0$ (consequence of the relation (2.1.14)). Finally, we obtain the relation

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{\omega} \times \mathbf{V} , \tag{A.2.37}$$

where $\mathrm{d}\mathbf{V} / \mathrm{d}t$ is *the absolute derivative* of the vector $\mathbf{V}$ with respect to the fixed frame of reference. In particular, we obtain *Poisson's formulae* (the unit vectors $\mathbf{i}_k$ are fixed with respect to the movable frame, so that $\partial \mathbf{i}_k / \partial t = \mathbf{0}$)

$$\dot{\mathbf{i}}_k = \boldsymbol{\omega} \times \mathbf{i}_k , \quad k = 1,2,3 . \tag{A.2.38}$$

From the formula (A.2.37), which links the absolute derivative of a vector to its relative one, one sees that these derivatives are – in general – not equal. The two derivatives are equal only in the case in which $\boldsymbol{\omega} \times \mathbf{V} = \mathbf{0}$, hence if the vectors $\mathbf{V}$ and $\boldsymbol{\omega}$ are collinear or if $\boldsymbol{\omega} = \mathbf{0}$; this happens if $\dot{\mathbf{i}}_j = \mathbf{0}$, $j = 1,2,3$, hence if the movable frame moves without any rotation (the unit vectors $\mathbf{i}_j$, $j = 1,2,3$, are of constant direction). We state also that $\mathrm{d}\boldsymbol{\omega} / \mathrm{d}t = \partial \boldsymbol{\omega} / \partial t = \dot{\boldsymbol{\omega}}$.

2.2.3 Scalar differential operators of second order

We introduce *the scalar differential operator of second order*

$$\Delta = \nabla^2 = \operatorname{div} \operatorname{grad} = \partial_i \partial_i = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{A.2.39}$$

called *the operator of Laplace* (*Laplacian*). If it is applied to a scalar $U = U(x_1, x_2, x_3)$, then one obtains

$$\Delta U = \nabla^2 U = \partial_i \partial_i U = U_{,ii} , \tag{A.2.39'}$$

while it is applied to a vector, then we get

$$\Delta \mathbf{V} = \Delta V_j \mathbf{i}_j = V_{j,kk} \mathbf{i}_j . \tag{A.2.39''}$$

A scalar function $U \in C^2(D)$, which verifies the equation

$$\Delta U = 0 \tag{A.2.40}$$

in the domain D, is called *harmonic function* in this domain; analogously, a scalar function $U \in C^4(D)$, which verifies the equation

$$\varDelta^2 U = \nabla^4 U = 0 \tag{A.2.41}$$

in the domain D, is called *biharmonic function* in this domain. One may introduce – on this way – *polyharmonic functions* too. In orthogonal curvilinear co-ordinates it results

$$\varDelta = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2 H_3}{H_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_3 H_1}{H_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{H_1 H_2}{H_3}\frac{\partial}{\partial q_3}\right)\right]; \tag{A.2.42}$$

in particular, in spherical co-ordinates, we get

$$\varDelta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \tag{A.2.42'}$$

while, in cylindrical co-ordinates, one has

$$\varDelta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}. \tag{A.2.42''}$$

The operator of Laplace is of *elliptic type*. Analogously, we introduce *the operator of d'Alembert of hyperbolic type* (*d'Alembertian*) in the form

$$\Box_c = \varDelta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \quad c = \text{const}; \tag{A.2.43}$$

assuming that $U = U(x_1, x_2, x_3; t)$, the equation

$$\Box_c U = 0, \tag{A.2.44}$$

where $U \in C^2(D)$, is called *the wave equation*, while c is the propagation velocity of these waves. Analogously, the equation

$$\Box_c \mathbf{V} = \mathbf{0}, \tag{A.2.44'}$$

where $\mathbf{V} = \mathbf{V}(x_1, x_2, x_3; t)$, corresponds to three scalar equations of wave propagation. If $U \in C^4(D)$, then we may introduce *the double wave equation*

$$\Box_1 \Box_2 U = 0 \tag{A.2.45}$$

too, where the operators

$$\square_i = \Delta - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}, \quad c_i = \text{const}, \quad i = 1,2, \tag{A.2.45'}$$

correspond to two simple wave equations. Analogously, one may introduce functions which verify a *poly-wave equation*.

We introduce also *Nicolescu's caloric operator of parabolic type*

$$\underline{\cap} = \Delta - \frac{1}{a}\frac{\partial}{\partial t}, \quad a = \text{const}, \tag{A.2.46}$$

where a is *the thermic diffusivity*; assuming that $U = U(x_1, x_2, x_3; t)$, the equation

$$\underline{\cap}\, U = 0, \tag{A.2.47}$$

where $U \in C^2(D)$, is called *the caloric equation*; analogously, one may introduce *polycaloric functions*.

Between the differential operators grad, div, curl and Δ takes place the relation

$$\operatorname{curl}\operatorname{curl}\mathbf{V} = \operatorname{grad}\operatorname{div}\mathbf{V} - \Delta\mathbf{V}. \tag{A.2.48}$$

We notice the relations

$$\Delta(x_i U) = 2U_{,i} + x_i \Delta U, \quad i = 1,2,3, \tag{A.2.49}$$

$$\Delta(r^2 U) = 6U + 4\mathbf{r}\cdot\operatorname{grad}U + r^2\Delta U, \tag{A.2.49'}$$

which, if U is a harmonic function, become

$$\Delta(x_i U) = 2U_{,i}, \quad i = 1,2,3, \tag{A.2.50}$$

$$\Delta(r^2 U) = 6U + 4\mathbf{r}\cdot\operatorname{grad}U; \tag{A.2.50'}$$

we mention the formula

$$\Delta(\mathbf{r}\cdot\mathbf{V}) = 2\operatorname{div}\mathbf{V} + \mathbf{r}\cdot\Delta\mathbf{V}, \tag{A.2.51}$$

which, if the vector $\mathbf{V}$ is harmonic, becomes

$$\Delta(\mathbf{r}\cdot\mathbf{V}) = 2\operatorname{div}\mathbf{V}. \tag{A.2.51'}$$

Also, one verifies the relations

$$\square_i(\mathbf{r}\cdot\mathbf{V}) = 2\operatorname{div}\mathbf{V} + \mathbf{r}\cdot\square_i\mathbf{V}, \quad i = 1,2, \tag{A.2.52}$$

analogous to the formula (A.2.51), which – in the case when the vector $\mathbf{V}$ satisfies a simple wave equation – becomes

$$\square_i(\mathbf{r}\cdot\mathbf{V}) = 2\operatorname{div}\mathbf{V}, \quad i = 1,2. \tag{A.2.52'}$$

If

$$\mathbf{V} = \operatorname{curl}\mathbf{W}, \tag{A.2.53}$$

then the formulae (A.2.51) and (A.2.52) take the form

$$\Delta(\mathbf{r}\cdot\operatorname{curl}\mathbf{W}) = \mathbf{r}\cdot\Delta\operatorname{curl}\mathbf{W} = \mathbf{r}\cdot\operatorname{curl}\Delta\mathbf{W}, \tag{A.2.54}$$

$$\square_i(\mathbf{r}\cdot\operatorname{curl}\mathbf{W}) = \mathbf{r}\cdot\square_i\operatorname{curl}\mathbf{W} = \mathbf{r}\cdot\operatorname{curl}\square_i\mathbf{W}, \quad i = 1,2. \tag{A.2.54'}$$

2.2.4 Theorems of Almansi and Boggio type

Sometimes, the study of certain partial derivative equations may be reduced to the study of several equations of the same type but of a smaller order. We can thus state

Theorem A.2.1 (*of Almansi type*). *If $\mathscr{D}$ is a differential operator with constant coefficients, of order m, in s variables $q_1, q_2, ..., q_s$, and if we assume that $U = U(q_1, q_2, ..., q_s)$, then the solution of the equation*

$$\mathscr{D}^n U = 0, \tag{A.2.55}$$

where $U \in C^{mn}(D)$, may be written in the form

$$U = U_0 + q_1 U_1 + q_1^2 U_2 + ... + q_1^{n-1} U_{n-1}, \tag{A.2.55'}$$

where $U_0, U_1, ..., U_{n-1}$ are functions of the same variables, which verify the equations

$$\mathscr{D} U_i = 0, \quad i = 0,1,2,...,n-1; \tag{A.2.55''}$$

obviously, the variable q_1 may be replaced by anyone of the other $n-1$ independent variables and the functions U_i can be of the class C^p, $p < mn$.

In particular, let be the biharmonic equation (A.2.41) and two harmonic functions U_1 and U_2 in the domain D, which satisfy the equation (A.2.40). The biharmonic function U may be expressed univocally by means of Almansi's formula

$$U = U_1 + x_1 U_2; \tag{A.2.56}$$

hence, a biharmonic function is – in a certain manner – equivalent to two harmonic functions. We notice that one may replace the variable x_1 by one of the variables x_2 or x_3; as well, we may write

$$U = U_1 + r^2 U_2. \tag{A.2.56'}$$

Analogously, let $\mathscr{D}_i$, $i = 1,2,...,p$, be differential operators of order m_i in the variables $q_1, q_2, ..., q_s$ and let be a function $U = U(q_1, q_2, ..., q_s)$; we may state

Theorem A.2.2 (*of Boggio type*). *If* $\mathscr{D}_i$ *are permutable differential operators, prime two by two,*

$$\mathscr{D}_i \mathscr{D}_j = \mathscr{D}_j \mathscr{D}_i , \quad i, j = 1,2,\ldots, p , \tag{A.2.57}$$

then the solution of the equation

$$\mathscr{D}_1 \mathscr{D}_2 \ldots \mathscr{D}_p U = 0 , \tag{A.2.57'}$$

where $U \in C^{m_1 + m_2 + \ldots + m_p}(D)$, *may be written in the form*

$$U = U_1 + U_2 + \ldots + U_p , \tag{A.2.57''}$$

$U_1, U_2, \ldots, U_p$ *being functions of the same variables and the same class, which satisfy the equations*

$$\mathscr{D}_i U_i = 0 , \quad i = 1,2,\ldots, p . \tag{A.2.57'''}$$

The condition (A.2.57) is fulfilled, for instance, in the case of operators with constant coefficients. In particular, observing that Laplace's operator (A.2.39) can be written in the form $\Delta = (x_1 + \mathrm{i} x_2)(x_1 - \mathrm{i} x_2)$, $\mathrm{i} = \sqrt{-1}$, in the two-dimensional case, a harmonic function can be written in the form

$$U = U_1 + U_2 , \; U_1 = U_1(x_1 + \mathrm{i} x_2) , \; U_2 = U_2(x_1 - \mathrm{i} x_2) , \tag{A.2.40'}$$

obtained by a change of variable of the form $z = x_1 + \mathrm{i} x_2$, $\bar{z} = x_1 - \mathrm{i} x_2$, so that one gets the equations $\partial U_1 / \partial \bar{z} = 0$ and $\partial U_2 / \partial z = 0$, respectively.

In the case of the double wave equation (A.2.45) we obtain

$$U = U_1 + U_2 , \tag{A.2.58}$$

where U_1 and U_2 verify the equations

$$\Box_1 U_1 = 0 , \quad \Box_2 U_2 = 0 , \tag{A.2.58'}$$

d'Alembert's operators being given by (A.2.45').

2.3 Integral formulae

We have considered discrete systems of bound and sliding vectors in Chap. 2, Sec. 2.2; in what follows, we will deal with continuous systems of such vectors. Besides, a continuous system of bound vectors is a field of vectors. We show thus the form taken in this case by the results in Chap. 2, Subsec. 2.2; we introduce also some integral formulae useful in practice.

2.3.1 Continuous systems of vectors

Let be a *continuous system of vectors* $\{\mathbf{V}\}$, which may be bound or sliding ones; assuming that at any point P, of position vector $\mathbf{r}$, belonging to a domain D, is applied a vector $\mathbf{V}\mathrm{d}\tau$, we may consider the vector field $\mathbf{V} = \mathbf{V}(\mathbf{r})$, proportional to the volume element $\mathrm{d}\tau$ at any point of the domain. We introduce *the resultant* $\mathbf{R}$ of the system of vectors in the form of a free vector, given by

$$\mathbf{R} = \iiint_D \mathbf{V}(\mathbf{r})\mathrm{d}\tau \tag{A.2.59}$$

and *the resultant moment*, as a bound vector, applied at the pole O and given by

$$\mathbf{M}_O = \iiint_D \mathbf{r} \times \mathbf{V}(\mathbf{r})\mathrm{d}\tau\,; \tag{A.2.60}$$

thus, we obtain *the torsor* $\tau_O\{\mathbf{V}\} = \{\mathbf{R}, \mathbf{M}_O\}$ of the continuous system of vectors. The components of the resultant $\mathbf{R}$ are given by

$$R_i = \iiint_D V_i(\mathbf{r})\mathrm{d}\tau\,, \quad i = 1,2,3\,, \tag{A.2.59'}$$

while the components of the resultant moment $\mathbf{M}_O$ are written in the form

$$M_{Ox_i} = \in_{ijk} \iiint_D x_j V_k(\mathbf{r})\mathrm{d}\tau\,, \quad i = 1,2,3\,. \tag{A.2.60'}$$

A continuous system of sliding vectors $\{\mathbf{V}\}$ is equivalent to zero if and only if its torsor with respect to an arbitrary pole is equal to zero (condition (2.2.25)); this condition is only necessary in the case of a continuous system of bound vectors. Taking into account (A.2.59') and (A.2.60'), it results that this condition is equivalent to six equations of projection on the three axes of co-ordinates. As well, two continuous systems of sliding vectors $\{\mathbf{V}\}$ and $\{\mathbf{V}'\}$ are equivalent if and only if their torsors with respect to the same arbitrary pole are equal (condition (2.2.26)); this condition is only necessary in the case of a continuous system of bound vectors.

Analogous considerations can be made, in particular, for a continuous system of parallel or coplanar vectors. Let be, for instance, a *continuous system of parallel vectors*, the direction of which is specified by the unit vector $\mathbf{u}$; we may write

$$\mathbf{V}(\mathbf{r}) = V(\mathbf{r})\mathbf{u}\,, \tag{A.2.61}$$

where $V(\mathbf{r})$ represents the component of the vector $\mathbf{V}(\mathbf{r})$ along the given direction. If

$$V = \iiint_D V(\mathbf{r})\mathrm{d}\tau\,, \tag{A.2.62}$$

then we obtain the resultant

$$\mathbf{R} = V\mathbf{u}\,; \tag{A.2.62'}$$

if $V \neq 0$, then we put in evidence the centre C of the continuous system of parallel vectors, specified by the position vector

$$\boldsymbol{\rho} = \frac{1}{V}\iiint_D V(\mathbf{r})\mathbf{r}\mathrm{d}\tau\,. \tag{A.2.63}$$

The properties emphasized for the point C remain valid for a continuous system of vectors too.

2.3.2 Stokes' formula

Let be a closed curve C, situated on the sufficiently smooth surface Σ, limiting on it a simply connected domain S (reducible – by continuous deformation – to a point, without leaving the surface Σ) (Fig.A.4,a). The surface Σ is oriented by means of the unit vector $\mathbf{n}$ of the normal to it; as well, we assume a direction of travelling through the curve C. Let us consider the vector mapping $(x_1, x_2, x_3) \to \mathbf{V}(x_1, x_2, x_3)$, with $\mathbf{V} \in C^1(D)$, where D is a domain which includes $S + C$. One may prove *Stokes' formula*

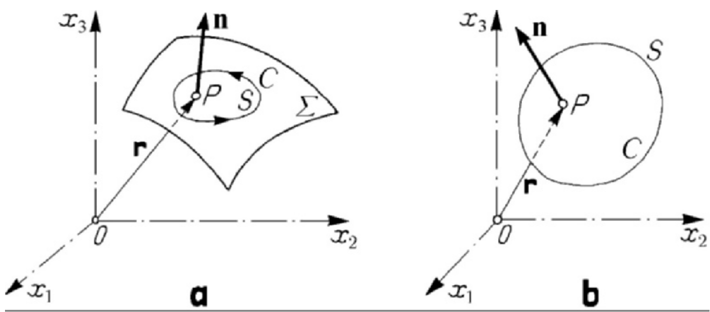

Figure A.4. The Stokes formula (a). The Gauss-Ostrogradskiĭ formula (b).

$$\oint_C \mathbf{V}\cdot\mathrm{d}\mathbf{r} = \iint_S \mathbf{n}\cdot\operatorname{curl}\mathbf{V}\mathrm{d}S\,; \tag{A.2.64}$$

because the left member of the formula depends only on the curve C, we may replace the surface S by any other surface $S_0 \subset D$, which satisfies analogous conditions.

As we have seen in Subsec. 1.1.4, the curvilinear integral is called circulation (it represents the work of a field of vectors), while the surface integral represents the flux of a field of curls; in the above mentioned conditions, it results that the circulation of a field of vectors along a closed curve is equal to the flux of the curl of the very same field of vectors through a sufficiently smooth arbitrary surface, bounded by the given curve. We also mention that the circulation of a field of irrotational vectors, hence of a field of conservative vectors, vanishes. With respect to the orthonormed frame of reference $Ox_1x_2x_3$, we may write

$$\oint_C V_i \mathrm{d}x_i = \in_{ijk} \iint_S n_i V_{k,j} \mathrm{d}S \,; \tag{A.2.64'}$$

in particular, if $V_1 = F \in C^1(D)$, $V_2 = V_3 = 0$, then it results

$$\oint_C F \mathrm{d}x_1 = \iint_S \left(n_2 F_{,3} - n_3 F_{,2} \right) \mathrm{d}S \,. \tag{A.2.65}$$

If in Stokes' formula (A.2.64) we concentrate all the surface at a point of position vector $\mathbf{r}$, then we get

$$\mathrm{pr}_{\mathbf{n}} \operatorname{curl} \mathbf{V} = \mathbf{n} \cdot \operatorname{curl} \mathbf{V} = \lim_{S \to 0} \frac{\oint_C \mathbf{V} \cdot \mathrm{d}\mathbf{r}}{S} \,. \tag{A.2.66}$$

Hence, we may determine the projection of the vector $\operatorname{curl} \mathbf{V}$ on an arbitrary axis of unit vector $\mathbf{n}$ (hence the very same vector), without any reference to a frame; thus, the definition given to the curl in Subsec. 2.2.1 has an intrinsic value, being immaterial on the frame.

2.3.3 Gauss-Ostrogradskiĭ formula

Let be a sufficiently smooth closed surface S, limiting a domain D, and a field of vectors $\mathbf{V} = \mathbf{V}(\mathbf{r})$ (Fig.A.4,b); we assume that $\mathbf{V} \in C^1(D + S)$. We may write *the Gauss-Ostrogradskiĭ formula* in the form

$$\iint_S \mathbf{V} \cdot \mathbf{n} \mathrm{d}S = \iiint_D \operatorname{div} \mathbf{V} \mathrm{d}\tau \,, \tag{A.2.67}$$

where $\mathbf{n}$ is the unit vector of the external normal to the surface; the surface integral represents the flux of the field of vectors through the surface S, so that the formula is called *the flux-divergence formula* too. The flux of a solenoidal vector vanishes. In the orthonormed frame $Ox_1x_2x_3$ we obtain

$$\iint_S V_j n_j \mathrm{d}S = \iiint_D V_{j,j} \mathrm{d}\tau \,. \tag{A.2.67'}$$

In particular, if we consider the component $V_j = F \in C^1(D + S)$, the other components vanishing, we may write

$$\iint_S F n_j \mathrm{d}S = \iiint_D F_{,j} \mathrm{d}\tau \,, \quad j = 1,2,3 \,; \tag{A.2.68}$$

multiplying by the unit vector $\mathbf{i}_j$ and summing, we get

$$\iint_S F \mathbf{n} \mathrm{d}S = \iiint_D \operatorname{grad} F \mathrm{d}\tau \,. \tag{A.2.68'}$$

If we take F of the form $\in_{ijk} V_j$ in (A.2.68), we obtain

$$\iint_S \in_{ijk} \; n_j V_k \mathrm{d}S = \iiint_D \in_{ijk} \; V_{k,j} \mathrm{d}\tau \,,$$

so that

$$\iint_S \mathbf{n} \times \mathbf{V} \mathrm{d}S = \iiint_D \operatorname{curl} \mathbf{V} \mathrm{d}\tau \,. \tag{A.2.69}$$

Let be a domain $D' \subset D$, reducible by continuous deformation to a point of position vector $\mathbf{r}$ and S' the surface (sufficiently smooth) which bounds it. Starting from the Gauss-Ostrogradskiĭ formula (A.2.67), we may represent the divergence of the vector $\mathbf{V}$ in the form

$$\operatorname{div} \mathbf{V} = \lim_{D' \to 0} \frac{\iint_{S'} \mathbf{V} \cdot \mathbf{n} \mathrm{d}S}{D'} \,; \tag{A.2.70}$$

hence, the definition given in Subsec. 2.2.1 is immaterial of the frame and has an intrinsic value.

2.3.4 Green's formulae

The formula (A.2.67) leads to

$$\iint_S \frac{\partial U}{\partial n} \mathrm{d}S = \iiint_D \Delta U \mathrm{d}\tau \tag{A.2.71}$$

for a field of conservative vectors of the form (A.2.33'), where we took into account (A.2.6') and the definition of Laplace's operator. If $\mathbf{V} = \psi \mathbf{W}$, $\psi = \psi(x_1, x_2, x_3)$ scalar, then we may write

$$\iint_S \psi \mathbf{W} \cdot \mathbf{n} \mathrm{d}S = \iiint_D (\psi \operatorname{div} \mathbf{W} + \mathbf{W} \cdot \operatorname{grad} \psi) \mathrm{d}\tau \,, \tag{A.2.72}$$

where we used the property ii) of the divergence. If, after this, we take $\mathbf{W} = \operatorname{grad} \varphi$, $\varphi = \varphi(x_1, x_2, x_3)$ scalar, then we get

$$\iint_S \psi \frac{\partial \varphi}{\partial n} \mathrm{d}S = \iiint_D (\psi \Delta \varphi + \operatorname{grad} \varphi \cdot \operatorname{grad} \psi) \mathrm{d}\tau \,; \tag{A.2.73}$$

inverting φ and ψ and subtracting the relation thus obtained from (A.2.73), it results

$$\iint_S \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) \mathrm{d}S = \iiint_D (\psi \Delta \varphi - \varphi \Delta \psi) \mathrm{d}\tau \,. \tag{A.2.74}$$

The formulae (A.2.71), (A.2.73) and (A.2.74) are known as *Green's formulae*.

2.3.5 The differentiation formula for integrals which depend on a parameter

We consider, in what follows, integrals of functions which depend on a parameter t, very useful in applications. Let thus be the curvilinear integral on the curve C

$$I_C \equiv \int_C F_i(x_1, x_2, x_3; t)\mathrm{d}x_i , \quad \text{(A.2.75)}$$

where F_i, $i = 1,2,3$, are functions of class C^1 on the domain of definition, with respect both to the variables x_j, $j = 1,2,3$, and to the time t; we have

$$\frac{\mathrm{d}I_C}{\mathrm{d}t} = \int_C \frac{\mathrm{d}}{\mathrm{d}t} F_i(x_1, x_2, x_3; t)\mathrm{d}x_i . \quad \text{(A.2.75')}$$

If $x_j = x_j(t)$, $j = 1,2,3$, then we may calculate the total derivative under the integral operator.

Analogously, for the surface integral

$$I_S \equiv \iint_S F(x_1, x_2, x_3; t)\mathrm{d}S \quad \text{(A.2.76)}$$

we may write

$$\frac{\mathrm{d}I_S}{\mathrm{d}t} = \iint_S \frac{\mathrm{d}}{\mathrm{d}t} F(x_1, x_2, x_3; t)\mathrm{d}S . \quad \text{(A.2.76')}$$

We notice that both the curve C and the surface S may be open or closed.

For the volume integral

$$I_D \equiv \iiint_D F(x_1, x_2, x_3; t)\mathrm{d}\tau \quad \text{(A.2.77)}$$

we obtain, as well,

$$\frac{\mathrm{d}I_D}{\mathrm{d}t} = \iiint_D \frac{\mathrm{d}}{\mathrm{d}t} F(x_1, x_2, x_3; t)\mathrm{d}\tau . \quad \text{(A.2.77')}$$

We obtain analogous results if $\mathbf{I}_S$ or $\mathbf{I}_D$ are vectors, the functions under the integral operator being vector functions.

If the curve C, the surface S or the volume D depend on the parameter t too, being variable quantities, then these formulae must be completed, taking into account the respective limits. Let thus be *the volume integral*

$$I_{D(t)} \equiv \iiint_{D(t)} F(x_1, x_2, x_3; t)\mathrm{d}\tau , \quad \text{(A.2.78)}$$

where $x_j = x_j(t)$, $j = 1,2,3$ and $\mathrm{d}\tau = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$. By a change of variable of the form $x_i = x_i\left(x_1^0, x_2^0, x_3^0; t\right)$, $i = 1,2,3$, including the condition that the Jacobian of the transformation be non-zero ($J \equiv \det\left[\mathrm{d}x_i / \mathrm{d}x_j^0\right] \neq 0$), we obtain

$$\frac{\mathrm{d}I_{D(t)}}{\mathrm{d}t} = \iiint_{D(t)} \left(\frac{\mathrm{d}F}{\mathrm{d}t} + F \operatorname{div} \mathbf{v} \right) \mathrm{d}\tau = \iiint_{D(t)} \left[\frac{\partial F}{\partial t} + \operatorname{div}(F\mathbf{v}) \right] \mathrm{d}\tau , \qquad (A.2.79)$$

where we have introduced the velocity $\mathbf{v} = \mathrm{d}\mathbf{r} / \mathrm{d}t$, we used the total derivative (A.2.9'), and we took into account the property ii) of the divergence. With the aid of the Gauss-Ostrogradskiĭ formula (A.2.67), we may also write

$$\frac{\mathrm{d}I_{D(t)}}{\mathrm{d}t} = \iiint_{D(t)} \frac{\partial F}{\partial t} \mathrm{d}\tau + \iint_{S(t)} (\mathbf{v} \cdot \mathbf{n}) F \mathrm{d}S , \qquad (A.2.79')$$

$\mathbf{n}$ being the external normal to the surface $S(t)$. In the case of a vector field

$$\mathbf{I}_{D(t)} \equiv \iiint_{D(t)} \mathbf{V}(\mathbf{r};t) \mathrm{d}\tau , \qquad (A.2.80)$$

we obtain, analogously,

$$\frac{\mathrm{d}\mathbf{I}_{D(t)}}{\mathrm{d}t} = \iiint_{D(t)} \left(\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \mathbf{V} \operatorname{div} \mathbf{v} \right) \mathrm{d}\tau = \iiint_{D(t)} \frac{\partial \mathbf{V}}{\partial t} \mathrm{d}\tau + \iint_{S(t)} (\mathbf{v} \cdot \mathbf{n}) \mathbf{V} \mathrm{d}S . \qquad (A.2.80')$$

For *the surface integral*

$$I_{S(t)} \equiv \iint_{S(t)} \mathbf{V}(\mathbf{r};t) \cdot \mathbf{n} \mathrm{d}S \qquad (A.2.81)$$

one may show that

$$\frac{\mathrm{d}I_{S(t)}}{\mathrm{d}t} = \iint_{S(t)} \left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{v} \operatorname{div} \mathbf{V} + \operatorname{curl}(\mathbf{V} \times \mathbf{v}) \right] \cdot \mathbf{n} \mathrm{d}S ; \qquad (A.2.81')$$

by means of Stokes' formula (A.2.64), we can write

$$\frac{\mathrm{d}I_{S(t)}}{\mathrm{d}t} = \iint_{S(t)} \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{v} \operatorname{div} \mathbf{V} \right) \cdot \mathbf{n} \mathrm{d}S + \oint_{C(t)} (\mathbf{V} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{r} . \qquad (A.2.81'')$$

Let be *the curvilinear integral*

$$I_{C(t)} \equiv \iiint_{C(t)} F(\mathbf{r};t) \mathrm{d}s \qquad (A.2.82)$$

too, where $\mathbf{r} = \mathbf{r}(\lambda;t)$, $\lambda \in [\lambda_0, \lambda_1]$, is the parametric equation of the curve $C(t)$; we get

$$\frac{\mathrm{d}I_{C(t)}}{\mathrm{d}t} = \int_{C(t)} \left[\frac{\partial F}{\partial t} + \mathbf{v} \cdot \operatorname{grad} F + \frac{\dfrac{\partial \mathbf{r}}{\partial \lambda} \cdot \dfrac{\partial \mathbf{v}}{\partial \lambda}}{\left(\dfrac{\partial \mathbf{r}}{\partial \lambda} \right)^2} \right] \mathrm{d}s . \qquad (A.2.82')$$

2.3.6 Basic integral formula. Newtonian potentials

The third formula of Green (A.2.74) allows a study of Poisson's equation. To this goal, we write the respective formula for a function $u \in C^2(D)$ and for a function

$$v = \frac{1}{4\pi R}, \quad R = |\mathbf{r} - \boldsymbol{\xi}|, \tag{A.2.83}$$

which satisfies the equation of Laplace (excepting the point $\mathbf{r} = \boldsymbol{\xi}$, which is a singular one) and represents the basic solution in the sense of the theory of distributions of Poisson's equation; the presence of the factor $1/4\pi$ is due to the fact that – in distributions – we have

$$\Delta v = -\delta(\mathbf{r}), \tag{A.2.84}$$

where δ is Dirac's distribution. To calculate the integrals in the formula (A.2.74), we isolate the singular point $\mathbf{r} = \boldsymbol{\xi}$ by a sphere containing this point; the volume integral is calculated for the domain D without the interior of this sphere, while the surface integral is calculated for the sphere too. If the radius of the sphere, centred at the singular point, tends to zero, then we obtain *the basic integral formula*

$$u(\mathbf{r}) = -\frac{1}{4\pi}\iiint_D \frac{\Delta u}{R}\mathrm{d}\tau + \frac{1}{4\pi}\iint_S \left[\frac{1}{R}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right]\mathrm{d}S\,. \tag{A.2.85}$$

This formula remains valid also for infinite domains if the function u is regular at infinity. It puts in evidence properties of Laplace's operator, to which it associates the identical operator and the normal derivative operator.

The formula (A.2.85) introduces three Newtonian potentials, linked by the basic formula, i.e.: *the volume potential*

$$F_1(\mathbf{r}) = \iiint_D \frac{f_1(\xi)}{R}\mathrm{d}\tau, \tag{A.2.86}$$

the potential of simple stratum

$$F_2(\mathbf{r}) = \iint_S \frac{f_2(\xi)}{R}\mathrm{d}S \tag{A.2.87}$$

and *the potential of double stratum*

$$F_3(\mathbf{r}) = -\iint_S f_3(\xi)\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\mathrm{d}S\,, \tag{A.2.87'}$$

f_1, f_2, f_3 being the respective densities. The study of those potentials allows to find the solution of Poisson's equation if the value of the function or of its normal derivative is given on the frontier.

3. Elements of theory of distributions

As it was shown in Chap. 1, Subsec. 1.1.7, in the study of discontinuous phenomena and for their representation in a unitary form, together with the continuous ones, it is necessary to use some notions of the theory of distributions. In what follows, we give some results concerning the composition of distributions and the integral transforms in distributions; as well, we introduce the notion of basic solution of a differential equation in the sense of the theory of distributions. These notions acquire thus a larger interest.

3.1 Composition of distributions

In general, the product of two distributions has no meaning; we have seen that the product by a function of class C^∞ has sense. That is why we will define products of a special type (composition of distributions). We introduce thus the direct (or tensor) product and the convolution product.

3.1.1 Direct product of two distributions

Let $x \equiv (x_1, x_2, ..., x_n)$ be a point of the *n*-dimensional Euclidean space X^n and $y \equiv (y_1, y_2, ..., y_m)$ a point of the *m*-dimensional Euclidean space Y^m; by direct Cartesian product $X^n \times Y^m$ of the two Euclidean spaces we mean a new *n*+*m*-dimensional Euclidean space, built up of the points $(x, y) \equiv (x_1, x_2, ..., x_n, y_1, y_2, ..., y_m)$, where – obviously – we have put in evidence the co-ordinates of a point of this space, in the order in which they have been written.

The direct product $f(x) \times g(y)$ of two distributions $f(x)$ and $g(y)$, defined on the basic spaces $K_x\,(x \in X^n)$ and $K_y\,(y \in Y^m)$, respectively, is given by the relation

$$(f(x) \times g(y), \varphi(x, y)) = (f(x), (g(y), \varphi(x, y))), \qquad \text{(A.3.1)}$$

where $\varphi(x, y)$ is a basic function defined on $X^n \times Y^m$; this product is a distribution defined on the basic space $K_x \times K_y$. In the case of usual functions, this product coincides with their usual product. We mention the properties:

i) $f(x) \times g(y) = g(y) \times f(x)$ (*commutativity*);

ii) $[f(x) \times g(y)] \times h(z) = f(x) \times [g(y) \times h(z)] = f(x) \times g(y) \times h(z)$ (*associativity*).

The first of these properties allows to write the definition relation (A.3.1) also in the form

$$(f(x) \times g(y), \varphi(x, y)) = (g(y), (f(x), \varphi(x, y))). \qquad \text{(A.3.1')}$$

The second property takes into account the fact that the direct product may be defined for an arbitrary finite number of distributions.

Let D_{x_1} and D_{x_2} be two differential operators with respect to the variables x_1 and x_2, respectively; we may write the relation

$$D_{x_1} D_{x_2} \left[f(x_1) \times g(x_2) \right] = D_{x_1} f(x_1) \times D_{x_2} g(x_2) . \tag{A.3.2}$$

In particular, we get

$$\frac{\partial^2}{\partial x_1 \partial x_2} [\theta(x_1) \times \theta(x_2)] = \frac{\mathrm{d}\theta(x_1)}{\mathrm{d}x_1} \times \frac{\mathrm{d}\theta(x_2)}{\mathrm{d}x_2} = \delta(x_1) \times \delta(x_2) = \delta(x_1, x_2) . \tag{A.3.3}$$

3.1.2 The convolution product of two distributions

Let $f(x)$ and $g(x)$ be locally integrable functions of x; their convolution product is the function defined by

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\xi) g(x - \xi) \mathrm{d}\xi ; \tag{A.3.4}$$

obviously, the definition remains valid for $x \in \mathbb{R}^n$. If the functions $f(x)$ and $g(x)$ are continuous, then their convolution product is a continuous function too. In order that the convolution product may exist, it is necessary that the functions $f(x)$ and $g(x)$ should satisfy certain conditions; thus, a sufficient condition in this respect is that the supports of the two functions $f(x)$ and $g(x)$ be compact.

If $f(x)$ and $g(x)$ are two distributions on $\mathbb{R}^n$, then their *convolution product* $f(x) * g(x)$ represents a new distribution on $\mathbb{R}^n$, defined by the formula

$$\begin{aligned} &(f(x) * g(x), \varphi(x)) = (f(x) \times g(y), \varphi(x + y)) \\ &= (f(x), (g(y), \varphi(x + y))) = (g(y), (f(x), \varphi(x + y))) ; \end{aligned} \tag{A.3.5}$$

this definition is reduced to the first one in case of usual functions. We may show that the convolution product has a meaning if one of the following conditions is satisfied:

i) one of the distributions $f(x)$, $g(x)$ has a compact support;

ii) the distributions $f(x)$ and $g(x)$ have the supports bounded on the same side.

Thus, if $f(x) = 0$ for $x < a$ and $g(x) = 0$ for $x < b$, then the supports of the two distributions are bounded at the left.

We remark that the convolution product may be defined for an arbitrary finite number of distributions.

Under the conditions required for the existence of the convolution product one may prove the properties:

i) $f(x) * g(x) = g(x) * f(x)$ (*commutativity*);

ii) $[f(x) * g(x)] * h(x) = f(x) * [g(x) * h(x)] = f(x) * g(x) * h(x)$ (*associativity*).

We notice that

$$\delta(x) * f(x) = f(x) * \delta(x) = f(x); \tag{A.3.6}$$

hence, Dirac's distribution is a *unit element* for the product of convolution. We have, as well,

$$\delta(x-a) * f(x) = f(x) * \delta(x-a) = f(x-a). \tag{A.3.6'}$$

If D is an arbitrary differential operator, then we may write

$$D[f(x) * g(x)] = Df(x) * g(x) = f(x) * Dg(x). \tag{A.3.7}$$

3.2 Integral transforms in distributions

A strong tool for the integration of differential equations is the method of integral transforms. We give, in the following, some general results concerning Fourier and Laplace transforms.

3.2.1 Fourier transform of a distribution

If $f(x)$ is a real or complex function of the real variable $x \in \mathbb{R}$, which satisfies Dirichlet's conditions (it is bounded, piecewise monotone and has at the most a finite number of points of discontinuity of the first species) and is absolutely integrable, then the function

$$\mathrm{F}(u) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{\mathrm{i}ux} \mathrm{d}x, \quad \mathrm{i} = \sqrt{-1}, \tag{A.3.8}$$

exists and is called *the Fourier transform of the function* $f(x)$; we shall write

$$\mathrm{F}[f(x)] = \mathrm{F}(u) = f(\tilde{u}), \tag{A.3.8'}$$

noting that the variable u is real. In general, *the image function* $\mathrm{F}(u)$ is complex, although the function $f(x)$ may be a real function. Assuming that the function $\mathrm{F}(u)$ is given, the equality (A.3.8) may be considered as an integral equation with respect to the unknown function $f(x)$ under the integral symbol; the solution of this integral equation is written in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{F}(u) \mathrm{e}^{-\mathrm{i}ux} \mathrm{d}u. \tag{A.3.9}$$

The function $f(x)$ is called *the inverse Fourier transform of the function* $\mathrm{F}(u)$; we have

$$f(x) = \mathrm{F}^{-1}[\mathrm{F}[f(x)]] = \mathrm{F}^{-1}[\mathrm{F}(u)]. \tag{A.3.9'}$$

Let $\varphi(x)$ be a basic complex function of a real variable x; hence, $\varphi(x) \in C^{\infty}$ and has a compact support (e.g., $|x| \leq a$). By the formula (A.3.8), the Fourier transform of the basic function is

$$\mathrm{F}[\varphi(x)] = \psi(u) = \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{\mathrm{i}ux} \mathrm{d}x . \tag{A.3.10}$$

The function $\psi(u)$ may be defined also for complex values $s = u + \mathrm{i}v$, namely

$$\psi(s) = \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{\mathrm{i}sx} \mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{-vx} \mathrm{e}^{\mathrm{i}ux} \mathrm{d}x . \tag{A.3.10'}$$

The set of functions $\psi(s) = \mathrm{F}[\varphi(x)]$, where the support of the basic functions is included in the segment $[-a, a]$, forms the vector space $Z(a)$. We denote by

$$Z = \bigcup_a Z(a), \quad K = \bigcup_a K(a) \tag{A.3.11}$$

the new complex linear space; then, Z' is the set of linear and continuous functionals defined on Z (ultradistributions).

If $\mathrm{F}(s)$ is a distribution defined on Z and $f(x)$ is a distribution defined on K, then the functional $\mathrm{F}(s) \in Z'$, specified by the equality of the Parseval type

$$(\mathrm{F}(s), \psi(s)) = 2\pi (f(x), \varphi(x)), \tag{A.3.12}$$

is called *the Fourier transform of the distribution* $f(x)$ and is denoted by

$$\mathrm{F}(s) = \mathrm{F}[f(x)]. \tag{A.3.13}$$

We can also write

$$(\mathrm{F}[f(x)], \mathrm{F}[\varphi(x)]) = 2\pi (f(x), \varphi(x)). \tag{A.3.12'}$$

Analogously, one may introduce the Fourier transform of a distribution of several variables.

The classical properties of the Fourier transform are maintained in the form:

i) $P\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)\mathrm{F}(s) = P\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)\mathrm{F}[f(x)] = \mathrm{F}[P(\mathrm{i}x) f(x)]$, P polynomial;

ii) $\mathrm{F}\left[P\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x)\right] = P(-\mathrm{i}s)\mathrm{F}[f(x)] = P(-\mathrm{i}s)\mathrm{F}(s)$;

iii) $\mathrm{F}^{-1}[\mathrm{F}[f(x)]] = f(x)$;

iv) $\mathrm{F}[\mathrm{F}[f(x)]] = 2\pi f(-x)$.

We denote by F^{-1} *the inverse operator* defined on Z'.

We can prove the relation

$$\mathrm{F}[f(x) \times g(y)] = \mathrm{F}[f(x)] \times \mathrm{F}[g(y)] \tag{A.3.14}$$

for the direct product of two distributions. As well, in connection with the convolution product, one may show the relations

$$\mathrm{F}[f(x) * g(x)] = \mathrm{F}[f(x)]\mathrm{F}[g(x)], \tag{A.3.15}$$

$$\mathrm{F}[f(x)g(x)] = \mathrm{F}[f(x)] * \mathrm{F}[g(x)]. \tag{A.3.15'}$$

The first relation is valid if $f(x) \in S'$ and $g(x)$ is a distribution with bounded support; the second relation is valid if $f(x) \in S'$ and the function $g(x) \in C^\infty$ is such that $f(x)g(x) \in S'$ and the support of its Fourier transform is bounded.

We mention the Fourier transforms:

$$\mathrm{F}[\delta(x)] = 1, \tag{A.3.16}$$

$$\mathrm{F}[\delta(x_1, x_2, ..., x_n)] = 1, \tag{A.3.16'}$$

$$\mathrm{F}[\delta(x-a)] = \mathrm{e}^{\mathrm{i}ua}, \tag{A.3.16''}$$

$$\mathrm{F}[\theta(x)] = \pi\delta(u) + \frac{\mathrm{i}}{u}, \tag{A.3.17}$$

$$\mathrm{F}[x_+] = -\frac{1}{u^2} - \mathrm{i}\pi\delta'(u), \tag{A.3.17'}$$

$$\mathrm{F}[1(x)] = 2\pi\delta(u). \tag{A.3.17''}$$

3.2.2 Laplace transform of a distribution

Let $f(x)$ be a complex function of a real variable, which satisfies the conditions:

i) $f(x) = 0$ for $x < 0$;

ii) $f(x)$ is piecewise differentiable;

iii) $|f(x)| \leq M\mathrm{e}^{ax}$, where M is a positive constant, while the non-negative constant a represents the incremental ratio of the function.

Then the function $\mathrm{L}(p)$ of the complex variable $p = u + \mathrm{i}v$, defined by the expression

$$\mathrm{L}(p) = \int_0^\infty f(x)\mathrm{e}^{-px}\mathrm{d}x \tag{A.3.18}$$

is called *the Laplace transform of the function* $f(x)$ and is denoted by

$$\mathrm{L}[f(x)] = \mathrm{L}(p). \tag{A.3.18'}$$

The function $f(x)$ is also called *the original function* and the function $\mathrm{L}(p)$ *the image function*. To the Laplace transform thus defined there corresponds an inverse Laplace transform, given by

$$\mathrm{L}^{-1}\left[\mathrm{L}(p)\right] = f(x) = \frac{1}{2\pi\mathrm{i}}\int_{u-\mathrm{i}\infty}^{u+\mathrm{i}\infty}\mathrm{L}(p)\mathrm{e}^{px}\mathrm{d}p\,, \quad u > a\,. \tag{A.3.19}$$

If $f(x)$ is a distribution having its support on the half-line $x \geq 0$ and is such that the distribution $f(x)\mathrm{e}^{-px}$ is a temperate distribution, then

$$\mathrm{L}[\,f(x)] = (\,f(x), \mathrm{e}^{-px}\,) \tag{A.3.20}$$

represents *the Laplace transform of that distribution*. It is obvious that the relation (A.3.20) generalizes the relation (A.3.18).

We mention the properties:

i) $\mathrm{L}[\,f(x-a)] = \mathrm{e}^{-pa}\mathrm{L}[\,f(x)]$ (*the delay theorem*);

ii) $\mathrm{L}[\,f(kx)] = \dfrac{1}{k}\mathrm{L}\left(\dfrac{p}{k}\right), \quad k > 0$ (*the theorem of similitude*);

iii) $\mathrm{L}[\,f(x)\mathrm{e}^{qx}\,] = \mathrm{L}(p-q)$ (*the theorem of translation*; *the damping theorem*).

One may give analogous results for the distributions of several variables.

In the case of a derivative of a distribution one may write

$$\mathrm{L}[\,f'(x)] = p\mathrm{L}[\,f(x)]\,. \tag{A.3.21}$$

For a convolution product it results

$$\mathrm{L}[\,f(x) * g(x)] = \mathrm{L}[\,f(x)]\mathrm{L}[\,g(x)]\,. \tag{A.3.22}$$

We may write the Laplace transforms:

$$\mathrm{L}[\,\delta(x)] = 1\,, \tag{A.3.23}$$

$$\mathrm{L}[\,\delta(x_1, x_2, \ldots, x_n)] = 1\,, \tag{A.3.23'}$$

$$\mathrm{L}\left[\,\delta^{(m)}(x)\right] = p^m\,, \quad m = 0,1,2,\ldots. \tag{A.3.23''}$$

3.3 Applications to the study of differential equations. Basic solutions

The theory of distributions is particularly useful in the study of ordinary or partial differential equations, as well as in the case of various boundary value problems. We shall give first some general results concerning the basic solutions and then we shall deal with the problem of obtaining them for some particular differential equations.

3.3.1 Ordinary differential equations

Let be the *linear ordinary differential equation* with constant coefficients

$$\mathrm{D}y(x) \equiv y^{(n)}(x) + a_1 y^{(n-1)}(x) + \ldots + a_n y(x) = f(x)\,, \tag{A.3.24}$$

where $f(x)$ is a distribution. The distribution $E(x)$ which satisfies the equation

$$\mathrm{D}E(x) = \delta(x) \tag{A.3.25}$$

is called *the basic solution* of the equation (A.3.24) and is of the form

$$E(x) = Y(x) + E_+(x), \tag{A.3.26}$$

where $Y(x)$ is *the general solution of the homogeneous equation*

$$\mathrm{D}Y(x) = 0, \tag{A.3.27}$$

while $E_+(x)$ is a *particular basic solution* (corresponding to the non-homogeneous equation (A.3.25)). We shall give a simple method for determining this solution; to this end, we determine first the solution $Y(x)$ which satisfies the initial conditions

$$Y(0) = Y'(0) = Y''(0) = \ldots = Y^{(n-2)}(0) = 0, \quad Y^{(n-1)}(0) = 1; \tag{A.3.28}$$

one can prove that a basic particular solution is, in this case, given by

$$E_+(x) = Y(x)\theta(x). \tag{A.3.28'}$$

The basic solution of a differential equation is useful to determine its general solution; thus, *the general solution* of the equation (A.3.24) is given by

$$y(x) = E(x) * f(x). \tag{A.3.29}$$

Let now be again the equation (A.3.24) with $x > 0$, $f(x)$ being a continuous function having the support in $[0,\infty)$; in the case of *initial conditions of Cauchy type*

$$y^{(k)}(0) = y_k, \quad k = 0,1,2,\ldots,n-1, \tag{A.3.30}$$

the solution of the equation (A.3.24) is expressed in the form

$$y(x) = E_+(x) * f(x)\theta(x) + \sum_{k=0}^{n-1} h_k \frac{\tilde{\mathrm{d}}^k}{\mathrm{d}x^k} E_+(x), \tag{A.3.30'}$$

where the coefficients h_k, $k = 0,1,2,\ldots,n-1$, are given by

$$h_k = y_{n-k-1} + a_1 y_{n-k-2} + \ldots + a_{n-k-1} y_0. \tag{A.3.30''}$$

The solution of the homogeneous equation

$$\mathrm{D}y(x) = 0 \tag{A.3.31}$$

for $x \geq 0$, with the initial conditions of Cauchy type (A.3.30), is given by

$$y(x) = \sum_{k=0}^{n-1} h_k \frac{\tilde{\mathrm{d}}^k}{\mathrm{d}x^k} E_+(x) . \tag{A.3.31'}$$

In particular, in the case of the differential equation

$$y^{(n)}(x) = f(x), \tag{A.3.32}$$

with initial conditions of the form (A.3.30), we obtain the basic particular solution

$$E_+(x) = \theta(x) \frac{x^{n-1}}{(n-1)!} = \frac{1}{(n-1)!} x_+^{n-1} , \quad x \in \mathbb{R} ; \tag{A.3.33}$$

the solution of the boundary value problem is given by

$$y(x) = y_0 + y_1 x + y_2 \frac{x^2}{2!} + \ldots + y_{n-1} \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi) \mathrm{d}\xi , \tag{A.3.32'}$$

where the latter integral, which represents the convolution product, is known as *the Cauchy formula*; for $n = 2$, we get

$$E_+(x) = x_+ . \tag{A.3.33'}$$

The above ideas may be extended to *systems of ordinary differential equations with constant coefficients*.

3.3.2 General considerations on partial differential equations

Problems similar to those in the preceding subsection may be put in the case of *partial differential equations*. Let thus be

$$P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m} ; \frac{\partial}{\partial t} \right) u(x_1, x_2, \ldots, x_m ; t) = 0 \tag{A.3.34}$$

a homogeneous linear partial differential equation of *n*th order with respect to the variable t , with constant coefficients. For example, *the Cauchy problem* for this equation consists in the determination of the function $u(x_1, x_2, \ldots, x_m ; t)$ which satisfies the equation (A.3.34) and the initial conditions

$$\begin{aligned} u(x_1, x_2, \ldots, x_m ; t_0) &= u_0(x_1, x_2, \ldots, x_m) , \\ \frac{\partial}{\partial t} u(x_1, x_2, \ldots, x_m ; t_0) &= u_1(x_1, x_2, \ldots, x_m) , \\ &\cdots\cdots\cdots\cdots \\ \frac{\partial^{n-1}}{\partial t^{n-1}} u(x_1, x_2, \ldots, x_m ; t_0) &= u_{n-1}(x_1, x_2, \ldots, x_m) . \end{aligned} \tag{A.3.34'}$$

To solve this problem, we consider the function

$$\overline{u}(x_1, x_2, ..., x_m; t) = u(x_1, x_2, ..., x_m; t)\theta(t - t_0), \tag{A.3.35}$$

as well as the corresponding regular distribution; taking into account the formula which links the derivative in the sense of the theory of distributions to its derivative in the usual sense and using the initial conditions (A.3.34'), it results

$$\begin{aligned}
\frac{\partial}{\partial t}\overline{u}(x_1, x_2, ..., x_m; t) &= \frac{\tilde{\partial}}{\partial t}\overline{u}(x_1, x_2, ..., x_m; t) + u_0(x_1, x_2, ..., x_m)\delta(t - t_0), \\
\frac{\partial^2}{\partial t^2}\overline{u}(x_1, x_2, ..., x_m; t) &= \frac{\tilde{\partial}^2}{\partial t^2}\overline{u}(x_1, x_2, ..., x_m; t) + u_1(x_1, x_2, ..., x_m)\delta(t - t_0) \\
&\quad + u_0(x_1, x_2, ..., x_m)\dot{\delta}(t - t_0), \qquad \text{(A.3.36)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial^n}{\partial t^n}\overline{u}(x_1, x_2, ..., x_m; t) &= \frac{\tilde{\partial}^n}{\partial t^n}\overline{u}(x_1, x_2, ..., x_m; t) + u_{n-1}(x_1, x_2, ..., x_m)\delta(t - t_0) \\
&\quad + u_{n-2}(x_1, x_2, ..., x_m)\dot{\delta}(t - t_0) + ... + u_0(x_1, x_2, ..., x_m)\delta^{(n-1)}(t - t_0).
\end{aligned}$$

Noting that the derivatives in the usual sense with respect to the variable t of the function $\overline{u}(x_1, x_2, ..., x_m; t)$ are equal to the corresponding derivatives of the function $u(x_1, x_2, ..., x_m; t)$ for $t \geq t_0$, the equation (A.3.34) takes the form

$$P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_m}; \frac{\partial}{\partial t}\right)\overline{u}(x_1, x_2, ..., x_m; t) = f(x_1, x_2, ..., x_m; t) \tag{A.3.37}$$

in distributions, where $f(x_1, x_2, ..., x_m; t)$ is a given distribution, which contains the initial conditions considered above.

Thus, we call basic solution of the equation (A.3.37) the distribution $E(x_1, x_2, ..., x_m; t)$ which satisfies the equation

$$P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_m}; \frac{\partial}{\partial t}\right)E(x_1, x_2, ..., x_m; t) = \delta(x_1, x_2, ..., x_m; t). \tag{A.3.38}$$

The solution of the above Cauchy problem is given by (A.3.35), where

$$\overline{u}(x_1, x_2, ..., x_m; t) = E(x_1, x_2, ..., x_m; t) * f(x_1, x_2, ..., x_m; t), \tag{A.3.37'}$$

the convolution product corresponding to all $m + 1$ variables.

It should be noted that some equations of mathematical physics cannot be always deduced directly in the space of distribution, owing to the difficulties encountered in modelling physical phenomena. In general, the equations which describe such phenomena are obtained first by classical methods. Next, an extension is effected, where the unknown functions take zero values, so that they be defined on the whole space; the derivatives, considered in the usual sense, are expressed by relations which connect derivatives in the sense of the theory of distributions to the derivatives in the usual sense of a distribution corresponding to an almost everywhere continuous function, having a finite number of discontinuities of the first species. In this way, the unknowns of the problem will be regular distributions; then it will be assumed that these unknowns may be arbitrary distributions. Another possibility, which is frequently used is to suppose, from the very beginning, that the unknowns of the problem are arbitrary distributions, assuming the same form in distributions for the differential equation obtained by classical methods (obviously, these ones are no longer valid for the whole space). However, there are not general methods for passing to differential equations in distributions.

3.3.3 Equations of elliptic type

Let be *Poisson's equation*

$$\Delta u(x_1, x_2, x_3) = f(x_1, x_2, x_3), \tag{A.3.39}$$

where $f(x_1, x_2, x_3)$ is a given distribution; the basic solution is of the form

$$E(x_1, x_2, x_3) = -\frac{1}{4\pi r}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}\,. \tag{A.3.39'}$$

Analogously, for the equation

$$\Delta\Delta u(x_1, x_2, x_3) = f(x_1, x_2, x_3) \tag{A.3.40}$$

we have

$$E(x_1, x_2, x_3) = -\frac{1}{8\pi} r\,. \tag{A.3.40'}$$

In case of the equation

$$\Delta u(x_1, x_2, x_3) + k^2 u(x_1, x_2, x_3) = f(x_1, x_2, x_3) \tag{A.3.41}$$

we may use the basic solution

$$E(x_1, x_2, x_3) = -\frac{1}{4\pi r} \cos kr\,; \tag{A.3.41'}$$

we also notice that an integral of *the homogeneous Helmholtz equation*

$$\Delta u(x_1, x_2, x_3) + k^2 u(x_1, x_2, x_3) = 0 \tag{A.3.42}$$

is given by

$$u(x_1, x_2, x_3) = \frac{1}{r}\sin kr\,. \tag{A.3.42'}$$

Replacing k by $\pm \mathrm{i}k$, $\mathrm{i} = \sqrt{-1}$, we find the basic solution

$$E(x_1, x_2, x_3) = -\frac{1}{4\pi r}\cosh kr \tag{A.3.43}$$

for the equation

$$\Delta u(x_1, x_2, x_3) - k^2 u(x_1, x_2, x_3) = f(x_1, x_2, x_3)\,; \tag{A.3.43'}$$

analogously, we notice that

$$u(x_1, x_2, x_3) = \frac{1}{r}\sinh kr \tag{A.3.44}$$

is an integral of the homogeneous equation

$$\Delta u(x_1, x_2, x_3) - k^2 u(x_1, x_2, x_3) = 0\,. \tag{A.3.44'}$$

In case of *Poisson's equation in two variables*

$$\Delta u(x_1, x_2) = f(x_1, x_2)\,, \tag{A.3.45}$$

where $f(x_1, x_2)$ is a given distribution, a basic solution is

$$E(x_1, x_2) = -\frac{1}{2\pi}\ln\frac{1}{r}\,, \quad r = \sqrt{x_1^2 + x_2^2}\,. \tag{A.3.45'}$$

As well, we get the basic solution

$$E(x_1, x_2) = -\frac{1}{8}r^2 \ln\frac{1}{r} \tag{A.3.46}$$

for the equation

$$\Delta\Delta u(x_1, x_2) = f(x_1, x_2)\,. \tag{A.3.46'}$$

3.3.4 Equations of hyperbolic type

We consider *the wave equations*

$$\Box_i\, u(x_1, x_2, x_3; t) = f_i(x_1, x_2, x_3; t)\,, \quad i = 1,2\,, \tag{A.3.47}$$

where $f_i(x_1, x_2, x_3; t)$ are given distributions; a basic solution of these equations is of the form

$$E_i(x_1, x_2, x_3; t) = -\frac{1}{4\pi r}\delta\left(t - \frac{r}{c_i}\right), \quad i = 1,2. \tag{A.3.47'}$$

Analogously, for the equations

$$\Box_i u(x_1, x_2, x_3; t) + 4\pi\varkappa(t) \times \delta(x_1, x_2, x_3) = 0, \quad i = 1,2, \tag{A.3.48}$$

where $\varkappa(t)$ is a distribution, one obtains

$$u(x_1, x_2, x_3; t) = \frac{1}{r}\varkappa\left(t - \frac{r}{c_i}\right), \quad i = 1,2. \tag{A.3.48'}$$

The solution of the equation

$$\Box_1 \Box_2 u(x_1, x_2, x_3; t) + 4\pi\varkappa(t) \times \delta(x_1, x_2, x_3) = 0 \tag{A.3.49}$$

reads

$$u(x_1, x_2, x_3; t) = -\frac{c_1^2 c_2^2}{c_1^2 - c_2^2}\left[\left(t - \frac{r}{c_1}\right)_+ - \left(t - \frac{r}{c_2}\right)_+\right] \underset{(t)}{*} \varkappa(t), \tag{A.3.49'}$$

where the convolution product concerns only the time variable. The basic solution of the equation

$$\Box_1 \Box_2 u(x_1, x_2, x_3; t) = f(x_1, x_2, x_3) \tag{A.3.50}$$

is of the form

$$E(x_1, x_2, x_3; t) = \frac{c_1^2 c_2^2}{4\pi\left(c_1^2 - c_2^2\right)}\left[\left(t - \frac{r}{c_1}\right)_+ - \left(t - \frac{r}{c_2}\right)_+\right]. \tag{A.3.50'}$$

In the case of only two space variables, we consider the equations

$$\Box_i u(x_1, x_2; t) = f_i(x_1, x_2; t), \quad i = 1,2, \tag{A.3.51}$$

$$\Box_1 \Box_2 u(x_1, x_2; t) = f(x_1, x_2; t), \tag{A.3.52}$$

where $f_i(x_1, x_2; t)$ and $f(x_1, x_2; t)$ are given distributions. Introducing the distribution defined by the function

$$f_0\left(t; \frac{r}{c_i}\right) = \mathrm{L}^{-1}\left[K_0\left(p\frac{r}{c_i}\right)\right] = \frac{\theta\left(t - \dfrac{r}{c_i}\right)}{\sqrt{t^2 - \dfrac{r^2}{c_i^2}}}, \quad t > 0, \quad i = 1,2, \tag{A.3.53}$$

where K_0 is *the modified Bessel function of order zero*, we obtain the basic solutions

$$E(x_1, x_2; t) = -\frac{1}{2\pi} f_0\left(t; \frac{r}{c_i}\right), \quad i = 1,2, \tag{A.3.51'}$$

for the equations (A.3.51). As well, if we introduce the distribution defined by the function

$$f_{-2}\left(t; \frac{r}{c_i}\right) = \mathrm{L}^{-1}\left[\frac{1}{p^2} K_0\left(p\frac{r}{c_i}\right)\right] = \theta\left(t - \frac{r}{c_i}\right)\left[t \ln\left(t + \sqrt{t^2 - \frac{r^2}{c_i^2}}\right) - \sqrt{t^2 - \frac{r^2}{c_i^2}}\right], \quad t > 0, \quad i = 1,2, \tag{A.3.54}$$

then we get the basic solution

$$E(x_1, x_2; t) = \frac{c_1^2 c_2^2}{2\pi\left(c_1^2 - c_2^2\right)}\left[f_{-2}\left(t; \frac{r}{c_1}\right) - f_{-2}\left(t; \frac{r}{c_2}\right)\right], \tag{A.3.52'}$$

corresponding to the equation (A.3.52).

3.3.5 Equations of parabolic type

Let be *the caloric equation*

$$\square\, u(x_1, x_2, x_3; t) = f(x_1, x_2, x_3; t), \tag{A.3.55}$$

where $f(x_1, x_2, x_3; t)$ is a given distribution; a basic solution is of the form

$$E(x_1, x_2, x_3; t) = \frac{1}{8\pi at\sqrt{\pi at}}\theta(t)\mathrm{e}^{-\frac{r^2}{4at}}, \quad t \in (-\infty, \infty), \tag{A.3.55'}$$

or of the form

$$E(x_1, x_2, x_3; t) = \frac{1}{8\pi at\sqrt{\pi at}}\mathrm{e}^{-\frac{r^2}{4at}}, \quad t \geq 0. \tag{A.3.55''}$$

In case of two space variables, let be the equation

$$\square\, u(x_1, x_2; t) = f(x_1, x_2; t), \tag{A.3.56}$$

where $f(x_1, x_2; t)$ is a given distribution; a basic solution is of the form

$$E(x_1, x_2; t) = \frac{1}{4\pi at}\theta(t)\mathrm{e}^{-\frac{r^2}{4at}}, \quad t \in (-\infty, \infty), \tag{A.3.56'}$$

which can be expressed also in the form

$$E(x_1, x_2; t) = \frac{1}{4\pi a t} \mathrm{e}^{-\frac{r^2}{4at}}, \quad t \geq 0. \tag{A.3.56''}$$

Analogously, we can consider also other equations which occur in the study of mechanical phenomena.

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