Appendix
Distributed Transmission-Distribution Coordinated State Estimation

A.1 Centralized TDCSE Model

A TDCSE model is formulated as

$$\begin{aligned}\min_{x_M, x_B, x_S} & [Z_M - \hbar_M(x_M,\, x_B)]^T W_M [Z_M - \hbar_M(x_M,\, x_B)] \\ & + [Z_B - \hbar_B(x_M,\, x_B,\, x_S)]^T W_B [Z_B - \hbar_B(x_M,\, x_B,\, x_S)] \\ & + [Z_S - \hbar_S(x_B,\, x_S)]^T W_S [Z_S - \hbar_S(x_B,\, x_S)],\end{aligned} \tag{A.1}$$

where the parameters Z_M, Z_B and Z_S are the measurements regarding master, boundary and slave subsystems, respectively; x_M, x_B and x_S are the state variables regarding the subsystems, respectively; $\hbar_M(x_M, x_B)$, $\hbar_B(x_M, x_B, x_S)$ and $\hbar_S(x_B, x_S)$ are the measurement functions associated with the subsystems, respectively, and the function $\hbar_B$ is M-S separable; the parameters W_M, W_B and W_S are weights.

Based on Theorem 2.2, after introducing an auxiliary variable $\upsilon_B = Z_B - \hbar_B(x_M, x_B, x_S)$, we can recast the optimization in (A.1) that has an M-S coupling term into the following model that has an M-S separable objective function:

$$\begin{aligned}\min_{x_M, x_B, x_S, \upsilon_B} & [Z_M - \hbar_M(x_M, x_B)]^T W_M [Z_M - \hbar_M(x_M, x_B)] + \upsilon_B^T W_B \upsilon_B \\ & + [Z_S - \hbar_S(x_B, x_S)]^T W_S [Z_S - \hbar_S(x_B, x_S)] \\ s.t. \quad & \upsilon_B - (Z_B - \hbar_B(x_M, x_B, x_S)) = 0.\end{aligned} \tag{A.2}$$

As for (A.2), let

$$\begin{cases} z_M = [x_M \quad \upsilon_B], z_S = x_S \\ c_M(z_M, x_B) = [Z_M - \hbar_M(x_M, x_B)]^T W_M [Z_M - \hbar_M(x_M, x_B)] + \upsilon_B^T W_B \upsilon_B \\ c_S(x_B, x_S) = [Z_S - \hbar_S(x_B, x_S)]^T W_S [Z_S - \hbar_S(x_B, x_S)] \\ f_B = \upsilon_B - (Z_B - \hbar_B(x_M, x_B, x_S)) \end{cases}. \tag{A.3}$$

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Obviously, f_B is M-S separable, and satisfies

$$\begin{aligned} f_B &= \upsilon_B - (Z_B - \hbar_B(x_M, x_B, x_S)) \\ &= \upsilon_B - (Z_B - (\hbar_{MB}(x_M, x_B) + \hbar_{BS}(x_B, x_S))) \\ &= \underbrace{\upsilon_B - (Z_B - \hbar_{MB}(x_M, x_B))}_{f_{MB}(z_M, x_B)} - \underbrace{(-\hbar_{BS}(x_B, x_S))}_{f_{BS}(x_B, z_S)} \\ &= f_{MB}(z_M, x_B) - f_{BS}(x_B, z_S). \end{aligned} \tag{A.4}$$

Therefore, the optimization in (A.2) is a special instance of the G-TDCM in (2.8) with $f_M = 0, f_S = 0$, $g_M = 0$ and $g_S = 0$. A distributed solution to the TDCSE thus can be derived from the G-MSS theory.

A.2 Distributed TDCSE Solution

A.2.1 Subproblem Formulations

Based on the HGD method developed in Sect. 3.1, the subproblems in the kth iteration for the TDCSE are formulated below:

- Transmission SE (T-SE) Subproblem: given $y_{B,k}^M = \left[f_{BS,k}^M \quad l_{BS,k}^M\right]^T$, solve

$$\begin{aligned} &\min_{z_M, x_B} c_M(z_M, x_B) - {l_{BS,k}^M}^T x_B \\ &s.t. \quad f_{MB}(z_M, x_B) = f_{BS,k}^M \quad \lambda_B^M \end{aligned} \tag{A.5}$$

where λ_B^M is the multiplier to the equality constraint; and

$$f_{BS,k}^M = f_{BS,k}^M\left(x_{B,k-1}, z_{S,k}\right) = -\hbar_{BS}\left(x_{B,k-1}, x_{S,k}\right), \tag{A.6}$$

$$\begin{aligned} l_{BS,k}^M &= l_{BS}\left(\xi_{B,k-1}, \xi_{S,k}\right) = -\left(\frac{\partial c_S(x_B, z_S)}{\partial x_B} + \frac{\partial f_{BS}(x_B, z_S)}{\partial x_B}^T \lambda_B\right)\Bigg|_{\xi_{B,k-1}, z_{S,k}} \\ &= 2\left(H_{SB}^T\left(x_{B,k-1}, x_{S,k}\right) W_S\left[Z_S - \hbar_S\left(x_{B,k-1}, x_{S,k}\right)\right] + H_{BB}''^T\left(x_{B,k-1}, x_{S,k}\right)\lambda_{B,k-1}\right) \end{aligned} \tag{A.7}$$

where $H_{SB} = \dfrac{\partial \hbar_S}{\partial x_B}$, $H_{BB}'' = \dfrac{\partial \hbar_{BS}}{\partial x_B}$.

- Distribution SE (D-SE) Subproblem: Given $\xi^S_{B,k-1} = \begin{bmatrix} x^S_{B,k-1} & \lambda^S_{B,k-1} \end{bmatrix}^T$, solve

$$\min_{x_S} c_S\left(x^S_{B,k-1}, x_S\right) - \lambda^S_{B,k-1}{}^T \hbar_{BS}\left(x^S_{B,k-1}, x_S\right). \tag{A.8}$$

A.2.2 Simplification

The formulations in (A.5) and (A.8) can be further simplified. From the first-order optimality conditions of the Lagrangian of the T-SE in (A.5), it follows that

$$\lambda^M_B = 2W_B \upsilon_B = 2W_B(Z_B - \hbar_B(x_M, x_B, x_S)). \tag{A.9}$$

Therefore, in the kth iteration, we have

$$\lambda^M_{B,k} = 2W_B\left(Z_B - \hbar_{MB}\left(x_{M,k}, x_{B,k}\right) - \hbar_{BS}\left(x_{B,k-1}, x_{S,k}\right)\right), \tag{A.10}$$

from which $\lambda^M_{B,k}$ can be physically interpreted as the residual of measurements in the boundary subsystems.

Next, similar to [1], define $Z''_B = Z_B - \hbar_{MB}(x_M, x_B)$ as virtual measurements in the slave subsystem. Then, (A.10) is recast into

$$\lambda_{B,k} = 2W_B\left(Z''_{B,k} - \hbar_{BS}\left(x_{B,k-1},\ x_{S,k}\right)\right). \tag{A.11}$$

Thus, in the kth iteration, the D-SE subproblem in (A.8) is recast into

$$\min_{x_S} c_S\left(x_{B,k-1}, x_S\right) - 2\left(Z''_{B,k-1} - \hbar_{BS}\left(x_{B,k-2},\ x_{S,k-1}\right)\right)^T W_B \hbar_{BS}\left(x_{B,k-1}, x_S\right). \tag{A.12}$$

Apparently, solving (A.12) is equivalently to solve the equations in (A.13) that are derived from the first-order optimality conditions of (A.12).

$$\begin{bmatrix} H^T_{BS}(x_{B-1}, x_S) & H^T_{SS}\left(x_{B,k-1}, x_S\right) \end{bmatrix} \cdot \begin{bmatrix} W_B\left[Z''_{B,k-1} - \hbar_{BS}\left(x_{B,k-2}, x_{S,k-1}\right)\right] \\ W_S\left[Z_S - \hbar_S\left(x_{B,k-1}, x_S\right)\right] \end{bmatrix} = 0, \tag{A.13}$$

where $H_{BS} = \dfrac{\partial \hbar_{BS}}{\partial x_S}$, $H_{SS} = \dfrac{\partial \hbar_{SS}}{\partial x_S}$. Then, by substituting (A.11) into (A.7), we have

$$\begin{aligned} l_{BS,k} &= 2 \cdot H^T_{SB}\left(x_{B,k-1}, x_{S,k}\right) W_S\left[Z_S - \hbar_S\left(x_{B,k-1}, x_{S,k}\right)\right] \\ &\quad + 2 \cdot H''^T_{BB}\left(x_{B,k-1}, x_{S,k}\right) W_B\left(Z''_{B,k-1} - \hbar_{BS}\left(x_{B,k-2}, x_{S,k-1}\right)\right) \end{aligned} \tag{A.14}$$

where $H_{SB} = \frac{\partial \hbar_S}{\partial x_B}$, $H''_{BB} = \frac{\partial \hbar_{BS}}{\partial x_B}$.

By substituting (A.14) into (A.5), the first-order optimality conditions of the T-SE become

$$\begin{cases} -2H_{MM}^T(x_M, x_B)W_M[Z_M - \hbar_M(x_M, x_B)] - H_{MB}^T(x_M, x_B)\lambda_B = 0 \\ -2H_{MB}^T(x_M, x_B)W_M[Z_M - \hbar_M(x_M, x_B)] - l_{BS,k} - H_{BB}'^T(x_M, x_B)\lambda_B = 0 \end{cases} \tag{A.15}$$

where $H_{MM} = \frac{\partial \hbar_M}{\partial x_M}$, $H_{BM} = \frac{\partial \hbar_{MB}}{\partial x_B}$, $H_{BM} = \frac{\partial \hbar_{MB}}{\partial x_M}$, $H'_{BB} = \frac{\partial \hbar_{MB}}{\partial x_B}$.

In (A.10), define $Z'_{B,k} = Z_B - \hbar_{BS}\left(x_{B,k-1},\, x_{S,k}\right)$ as virtual measurements in the slave subsystem. Then we have

$$\begin{bmatrix} H_{MM}^T(x_M, x_B) & H_{BM}^T(x_M, x_B) \\ H_{MB}^T(x_M, x_B) & H_{BB}'^T(x_M, x_B) \end{bmatrix} \cdot \begin{bmatrix} W_M[Z_M - \hbar_M(x_M, x_B)] \\ W_B\left[Z'_{B,k} - \hbar_{MB}(x_M, x_B)\right] \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} l_{BS,k} \end{bmatrix}. \tag{A.16}$$

where $l_{BS,k}$ is defines as in (A.14).

To summarize, the local optimal solution to the T-SE subproblem in (A.5) can be obtained by solving (A.16); the local optimal solution to the D-SE subproblem in (A.8) can be obtained by solving (A.13).

A.2.3 Procedures

Based on the above results, we obtain the procedures for a distributed TDCSE method:

Procedures for distributed TDCSE (Starting from the D-SE subproblem)

Step 1 (a) Given the maximum iterations K and the tolerance ε.
(b) Initiate the value: $\xi_{B,0} = \left[x_{B,0} \quad \lambda_{B,0}\right]^T$.
(c) Let the iteration counter $k = 1$.

Step 2 (a) For iteration k, solve (A.13) with a given $\xi_{B,k-1}$.
(b) Based on the obtained solution, compute $y_{B,k} = \left[f_{BS,k} \quad l_{BS,k}\right]^T$, where $f_{BS,k} = f_{BS}\left(x_{B,k-1}, x_{S,k}\right)$ and $l_{BS,k} = l_{BS}\left(x_{B,k-1}, x_{S,k}\right)$.

Step 3 (a) Solve (A.16) with a given $y_{B,k}$.
(b) Compute $\xi_{B,k} = \left[x_{B,k} \quad \lambda_{B,k}\right]^T$.

Step 4 If $\left\|\xi_{B,k} - \xi_{B,k-1}\right\| < \varepsilon$, the method is deemed to converge; else:
If $k \geq K$, then terminate the computation;
If $k < K$, return to Step 2 and let $k = k + 1$.

Owing to the mathematical properties of the G-MSS theory, this distributed TDCSE method has guaranteed optimality and convergency properties.

Reference

1. Sun HB, Zhang BM, Xiang ND (1999) Global state estimation for power system including transmission and distribution networks. J Tsinghua Univ (Sci & Tech) 39(7):20–24 (in Chinese)