Appendix A
Expressions for the Operators in the Stability Equation

$$
\left.\begin{aligned}
\mathbf{A}(1,1) &= \frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + k\frac{v}{h_1} - im\omega \\
\mathbf{A}(1,2) &= \frac{1}{h_1}\frac{\partial \rho}{\partial x} \\
\mathbf{A}(1,3) &= \frac{\partial \rho}{\partial y} + k\frac{\rho}{h_1} \\
\mathbf{A}(1,4) &= in\beta\rho \\
\mathbf{A}(1,5) &= 0
\end{aligned}\right\} \tag{A.1}
$$

$$
\left.\begin{aligned}
\mathbf{A}(2,1) &= \frac{u}{h_1}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + k\frac{uv}{h_1} + \frac{1}{\gamma \mathrm{Ma}^2}\frac{1}{h_1}\frac{\partial T}{\partial x} \\
\mathbf{A}(2,2) &= \frac{\rho}{h_1}\frac{\partial u}{\partial x} + k\frac{\rho v}{h_1} + \frac{k}{h_1}\frac{d\mu}{dT}\frac{\partial T}{\partial y}\frac{1}{\mathrm{Re}_0} + 27\frac{k^2}{h_1^2}\frac{\mu}{Re_0} \\
&\quad - im\omega\rho - n^2\beta^2\frac{\mu}{Re_0} \\
\mathbf{A}(2,3) &= \rho\frac{\partial u}{\partial y} + k\frac{\rho u}{h_1} - \frac{4}{3}\frac{k}{h_1^2}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{1}{Re_0} \\
\mathbf{A}(2,4) &= (in\beta)\frac{2}{3}\frac{1}{h_1}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{1}{Re_0} \\
\mathbf{A}(2,5) &= -\frac{1}{h_1}\frac{d^2\mu}{dT^2}\frac{\partial T}{\partial x}\frac{1}{Re_0}\left(\frac{4}{3}\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{4}{3}\frac{v}{h_1}k - \frac{2}{3}\frac{\partial v}{\partial y}\right) \\
&\quad - \frac{1}{h_1}\frac{d^2\mu}{dT^2}\frac{\partial T}{\partial y}\frac{1}{Re_0}\left(h_1\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - uk\right) \\
&\quad - \frac{d\mu}{dT}\frac{1}{Re_0}\frac{k}{h_1}\left(\frac{\partial u}{\partial y} + \frac{1}{h_1}\frac{\partial v}{\partial x} - k\frac{u}{h_1}\right) - \frac{1}{h_1}\frac{d\mu}{dT}\frac{1}{Re_0}\left(h_1\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) \\
&\quad + \frac{1}{\gamma Ma^2}\frac{1}{h_1}\frac{\partial \rho}{\partial x} - \frac{1}{h_1}\frac{d\mu}{dT}\frac{1}{Re_0}\left(\frac{4}{3}\frac{1}{h_1}\frac{\partial^2 u}{\partial x^2} + \frac{4}{3}\frac{\partial v}{\partial x}\frac{k}{h_1} - \frac{2}{3}\frac{\partial^2 v}{\partial x \partial y}\right)
\end{aligned}\right\} \tag{A.2}
$$

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$$\left.\begin{aligned}
\mathbf{A}(3,1) &= \frac{u}{h_1}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - k\frac{uu}{h_1} + \frac{1}{\gamma Ma^2}\frac{\partial T}{\partial y} \\
\mathbf{A}(3,2) &= \frac{\rho}{h_1}\frac{\partial v}{\partial x} - 2k\rho\frac{u}{h_1} + \frac{k}{{h_1}^2}\frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial x} \\
\mathbf{A}(3,3) &= \rho\frac{\partial v}{\partial y} + \frac{2k^2}{{h_1}^2}\frac{\mu}{Re_0} + \frac{2}{3}\frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial y}\frac{k}{h_1} - \frac{2}{3}\frac{1}{Re_0}\mu\frac{k^2}{h_1^2} \\
&\quad - im\omega\rho - n^2\beta^2\frac{\mu}{Re_0} \\
\mathbf{A}(3,4) &= (in\beta)\frac{2}{3}\frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial y} \\
\mathbf{A}(3,5) &= \frac{1}{\gamma Ma^2}\frac{\partial \rho}{\partial y} - \frac{1}{h_1}\frac{1}{Re_0}\frac{d^2\mu}{dT^2}\frac{\partial T}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{1}{h_1}\frac{\partial v}{\partial x} - k\frac{u}{h_1}\right) \\
&\quad - \frac{1}{h_1}\frac{1}{Re_0}\frac{d\mu}{dT}\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{1}{h_1}\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial x}\frac{k}{h_1}\right) \\
&\quad - \frac{2k}{h_1}\frac{1}{Re_0}\frac{d\mu}{dT}\left(\frac{\partial v}{\partial y} - \frac{1}{h_1}\frac{\partial u}{\partial x} - k\frac{v}{h_1}\right) \\
&\quad + \frac{1}{Re_0}\frac{d^2\mu}{dT^2}\frac{\partial T}{\partial y}\left(\frac{2}{3}\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{2k}{3}\frac{v}{h_1} - \frac{4}{3}\frac{\partial v}{\partial y}\right) \\
&\quad - \frac{1}{Re_0}\frac{d\mu}{dT}\left(-\frac{2}{3}\frac{1}{h_1}\frac{\partial^2 u}{\partial x\partial y} - \frac{2}{3}\frac{k}{h_1}\frac{\partial v}{\partial y}\right) \\
&\quad - \frac{1}{Re_0}\frac{d\mu}{dT}\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2} + \frac{2}{3}\frac{k}{{h_1}^2}\frac{\partial u}{\partial x} + \frac{2}{3}\frac{k^2 v}{h_1^2}\right)
\end{aligned}\right\} \tag{A.3}$$

$$\left.\begin{aligned}
\mathbf{A}(4,1) &= in\beta\frac{T}{\gamma Ma^2} \\
\mathbf{A}(4,2) &= -in\beta\frac{1}{h_1}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{1}{Re_0} \\
\mathbf{A}(4,3) &= in\beta\left(-\frac{1}{3}\mu\frac{1}{Re_0}\frac{1}{h_1}k - \frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial y}\right) \\
\mathbf{A}(4,4) &= -im\omega\rho - (n^2\beta^2)\frac{3}{4}\frac{\mu}{Re_0} \\
\mathbf{A}(4,5) &= in\beta\frac{1}{\gamma Ma^2}\rho \\
&\quad - in\beta\frac{\partial \mu}{\partial T}\left(-\frac{2}{3}\frac{1}{Re_0}\frac{1}{h_1}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{1}{Re_0}\frac{vk}{h_1} - \frac{2}{3}\frac{1}{Re_0}\frac{\partial v}{\partial y}\right)
\end{aligned}\right\} \tag{A.4}$$

$$\left.\begin{aligned}
\mathbf{A}(5,1) &= \frac{u}{h_1}\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{\gamma-1}{\gamma}\frac{u}{h_1}\frac{\partial T}{\partial x} - \frac{\gamma-1}{\gamma}v\frac{\partial T}{\partial y} + im\omega\frac{\gamma-1}{\gamma}T \\
\mathbf{A}(5,2) &= \frac{\rho}{h_1}\frac{\partial T}{\partial x} - \frac{\gamma-1}{\gamma}\left(\frac{1}{h_1}\rho\frac{\partial T}{\partial x} + \frac{1}{h_1}T\frac{\partial \rho}{\partial x}\right) \\
&\quad + 2\frac{(\gamma-1)\,Ma^2}{Re_0}\mu\left(-\frac{k^2}{{h_1}^2}u + \frac{k}{{h_1}^2}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{k}{h_1}\right) \\
\mathbf{A}(5,3) &= \rho\frac{\partial T}{\partial y} - \frac{\gamma-1}{\gamma}\left(\rho\frac{\partial T}{\partial y} + T\frac{\partial \rho}{\partial y}\right) \\
&\quad - 2\frac{(\gamma-1)Ma^2}{Re_0}\mu\left(\frac{4}{3}\frac{k}{{h_1}^2}\frac{\partial u}{\partial x} + \frac{4}{3}\frac{k^2 v}{{h_1}^2} - \frac{2}{3}\frac{k}{h_1}\frac{\partial v}{\partial y}\right) \\
\mathbf{A}(5,4) &= (in\beta)\frac{4}{3}\frac{(\gamma-1)\,Ma^2}{Re_0}\mu\left(\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{vk}{h_1} + \frac{\partial v}{\partial y}\right) \\
\mathbf{A}(5,5) &= -\frac{\gamma-1}{\gamma}\frac{u}{h_1}\frac{\partial \rho}{\partial x} - \frac{\gamma-1}{\gamma}v\frac{\partial \rho}{\partial y} - \frac{1}{Re_0\,\mathrm{Pr}}\frac{1}{{h_1}^2}\frac{d^2\kappa}{dT^2}\frac{\partial T}{\partial x}\frac{\partial T}{\partial x} \\
&\quad - \frac{1}{Re_0\,\mathrm{Pr}}\frac{1}{{h_1}^2}\frac{d\kappa}{dT}\frac{\partial^2 T}{\partial x^2} - \frac{1}{Re_0\,\mathrm{Pr}}\frac{d^2\kappa}{dT^2}\frac{\partial T}{\partial y}\frac{\partial T}{\partial y} \\
&\quad - \frac{1}{Re_0\,\mathrm{Pr}}\frac{k}{h_1}\frac{d\kappa}{dT}\frac{\partial T}{\partial y} - \frac{1}{Re_0\,\mathrm{Pr}}\frac{d\kappa}{dT}\frac{\partial^2 T}{\partial y^2} \\
&\quad - \frac{(\gamma-1)\,Ma^2}{Re_0}\frac{d\mu}{dT}\left(\frac{4}{3}\left(\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{vk}{h_1}\right)^2 + \frac{4}{3}\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right) \\
&\quad + \frac{4}{3}\frac{(\gamma-1)\,Ma^2}{Re_0}\frac{d\mu}{dT}\left(\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{vk}{h_1}\right)\left(\frac{\partial v}{\partial y}\right) \\
&\quad - \frac{(\gamma-1)\,Ma^2}{Re_0}\frac{d\mu}{dT}\left(\frac{1}{h_1}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} - \frac{uk}{h_1}\right)^2 \\
&\quad - im\omega(\frac{\rho}{\gamma}) - n^2\beta^2\frac{\kappa}{Re_0\,\mathrm{Pr}}
\end{aligned}\right\} \tag{A.5}$$

$$\left.\begin{aligned}
\mathbf{B}(1,1) &= v \\
\mathbf{B}(1,2) &= 0 \\
\mathbf{B}(1,3) &= \rho \\
\mathbf{B}(1,4) &= 0 \\
\mathbf{B}(1,5) &= 0
\end{aligned}\right\} \tag{A.6}$$

$$\left.\begin{aligned}
&\mathbf{B}(2,1) = 0\\
&\mathbf{B}(2,2) = \rho v - \frac{d\mu}{dT}\frac{\partial T}{\partial y}\frac{1}{Re_0} - \frac{\mu}{Re_0}\frac{k}{h_1}\\
&\mathbf{B}(2,3) = \frac{2}{3}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{1}{Re_0}\frac{1}{h_1}\\
&\mathbf{B}(2,4) = 0\\
&\mathbf{B}(2,5) = -\frac{1}{h_1}\frac{d\mu}{dT}\frac{1}{Re_0}\left(h_1\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - uk\right)
\end{aligned}\right\} \tag{A.7}$$

$$\left.\begin{aligned}
&\mathbf{B}(3,1) = \frac{T}{\gamma Ma^2}\\
&\mathbf{B}(3,2) = -\frac{1}{h_1}\frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\\
&\mathbf{B}(3,3) = \rho v - \frac{2k}{h_1}\frac{\mu}{Re_0} - \frac{4}{3}\frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial y} + \frac{2}{3}\frac{\mu}{Re_0}\frac{k}{h_1}\\
&\mathbf{B}(3,4) = -in\beta\frac{1}{3}\frac{\mu}{Re_0}\\
&\mathbf{B}(3,5) = \frac{\rho}{\gamma Ma^2} + \frac{1}{Re_0}\frac{d\mu}{dT}\left(\frac{2}{3}\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{2}{3}\frac{vk}{h_1} - \frac{4}{3}\frac{\partial v}{\partial y}\right)
\end{aligned}\right\} \tag{A.8}$$

$$\left.\begin{aligned}
&\mathbf{B}(4,1) = 0\\
&\mathbf{B}(4,2) = 0\\
&\mathbf{B}(4,3) = -in\beta\frac{1}{3}\frac{\mu}{Re_0}\\
&\mathbf{B}(4,4) = \rho v - \frac{k}{h_1}\frac{\mu}{Re_0} - \frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial y}\\
&\mathbf{B}(4,5) = 0
\end{aligned}\right\} \tag{A.9}$$

$$\left.\begin{aligned}
&\mathbf{B}(5,1) = -\frac{\gamma - 1}{\gamma}Tv\\
&\mathbf{B}(5,2) = -2\frac{(\gamma - 1)\,Ma^2}{Re_0}\mu\left(\frac{\partial u}{\partial y} + \frac{1}{h_1}\frac{\partial v}{\partial x} - \frac{uk}{h_1}\right)\\
&\mathbf{B}(5,3) = \frac{4}{3}\frac{(\gamma - 1)\,Ma^2}{Re_0}\mu\left(-2\frac{\partial v}{\partial y} + \frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{vk}{h_1}\right)\\
&\mathbf{B}(5,4) = 0\\
&\mathbf{B}(5,5) = \frac{\rho v}{\gamma} - \frac{2}{Re_0\,\mathrm{Pr}}\frac{d\kappa}{dT}\frac{\partial T}{\partial y} - \frac{1}{Re_0\,\mathrm{Pr}}\frac{k}{h_1}\kappa
\end{aligned}\right\} \tag{A.10}$$

$$\left.\begin{aligned} &\mathbf{C}(2,2) = \mathcal{L}_5(4,4) = -\frac{\mu}{Re_0} \\ &\mathbf{C}(3,3) = -\frac{4}{3}\frac{\mu}{Re_0} \\ &\mathbf{C}(5,5) = -\frac{\kappa}{Re_0\,\mathrm{Pr}} \end{aligned}\right\}\text{only none-zeros elemetns in } \mathbf{C} \text{ are shown} \tag{A.11}$$

$$\left.\begin{aligned} &\mathbf{D}(1,1) = \frac{u}{h_1} \\ &\mathbf{D}(1,2) = \frac{\rho}{h_1} \\ &\mathbf{D}(1,3) = 0 \\ &\mathbf{D}(1,4) = 0 \\ &\mathbf{D}(1,5) = 0 \end{aligned}\right\} \tag{A.12}$$

$$\left.\begin{aligned} &\mathbf{D}(2,1) = \frac{1}{\gamma Ma^2}\frac{1}{h_1}T \\ &\mathbf{D}(2,2) = \rho\frac{u}{h_1} - \frac{1}{{h_1}^2}\frac{4}{3}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{1}{Re_0} \\ &\mathbf{D}(2,3) = -\frac{4}{3}\frac{k}{{h_1}^2}\frac{\mu}{Re_0} - \frac{1}{h_1}\frac{d\mu}{dT}\frac{\partial T}{\partial y}\frac{1}{Re_0} - \frac{k}{{h_1}^2}\frac{\mu}{Re_0} \\ &\mathbf{D}(2,4) = -(in\beta)\frac{1}{3}\frac{1}{h_1}\frac{\mu}{Re_0} \\ &\mathbf{D}(2,5) = -\frac{1}{h_1}\frac{d\mu}{dT}\frac{1}{Re_0}\left(\frac{4}{3}\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{4}{3}\frac{vk}{h_1} - \frac{2}{3}\frac{\partial v}{\partial y}\right) + \frac{1}{\gamma Ma^2}\frac{\rho}{h_1} \end{aligned}\right\} \tag{A.13}$$

$$\left.\begin{aligned} &\mathbf{D}(3,1) = 0 \\ &\mathbf{D}(3,2) = \frac{3}{{h_1}^2}k\frac{1}{Re_0}\mu + \frac{2}{3}\frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial y}\frac{1}{h_1} - \frac{2}{3}k\frac{\mu}{Re_0}\frac{1}{{h_1}^2} \\ &\mathbf{D}(3,3) = \rho\frac{u}{h_1} - \frac{1}{{h_1}^2}\frac{1}{Re_0}\frac{d\mu}{dT}\frac{\partial T}{\partial x} \\ &\mathbf{D}(3,4) = 0 \\ &\mathbf{D}(3,5) = -\frac{1}{h_1}\frac{1}{Re_0}\frac{d\mu}{dT}\left(\frac{\partial u}{\partial y} + \frac{1}{h_1}\frac{\partial v}{\partial x} - \frac{u}{h_1}k\right) \end{aligned}\right\} \tag{A.14}$$

$$\left.\begin{aligned} \mathbf{D}(4,1) &= 0 \\ \mathbf{D}(4,2) &= -(in\beta)\frac{1}{3}\frac{1}{h_1}\frac{\mu}{Re_0} \\ \mathbf{D}(4,3) &= 0 \\ \mathbf{D}(4,4) &= \rho\frac{u}{{h_1}^2} - \frac{1}{h_1}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{1}{Re_0} \\ \mathbf{D}(4,5) &= 0 \end{aligned}\right\} \tag{A.15}$$

$$\left.\begin{aligned} \mathbf{D}(5,1) &= -\frac{\gamma-1}{\gamma}T\frac{u}{h_1} \\ \mathbf{D}(5,2) &= \frac{4}{3}\frac{(\gamma-1)\,Ma^2}{Re_0}\mu\left(\frac{1}{h_1}\frac{\partial v}{\partial y} - \frac{2}{{h_1}^2}\frac{\partial u}{\partial x} - \frac{2vk}{{h_1}^2}\right) \\ \mathbf{D}(5,3) &= -2\frac{(\gamma-1)\,Ma^2}{Re_0}\mu\left(\frac{1}{{h_1}^2}\frac{\partial v}{\partial x} + \frac{1}{h_1}\frac{\partial u}{\partial y} - \frac{uk}{{h_1}^2}\right) \\ \mathbf{D}(5,4) &= 0 \\ \mathbf{D}(5,5) &= \frac{\rho}{\gamma}\frac{u}{h_1} - \frac{2}{\mathrm{Re}_0\,\mathrm{Pr}}\frac{1}{{h_1}^2}\frac{d\kappa}{dT}\frac{\partial T}{\partial x} \end{aligned}\right\} \tag{A.16}$$