Appendix A
The Operator Notation of Equations of the Theory of Elasticity and Boundary Conditions in Curvilinear Coordinates

The equilibrium equations and elasticity relations at static loading and harmonic vibrations of a three-dimensional body, shell and rod in curvilinear coordinates can be written as

$$D\xi + A\eta - f = 0, \quad D^{*}\eta - B\xi - e = 0, \tag{A.1}$$

where

$$A = C - \lambda^{2}R; \quad \text{and} \quad \lambda \text{ is oscillation frequency.} \tag{A.2}$$

(1) In the case of a three-dimensional body [1, 2] in these equations $\xi = \tau^{ij}$ is the stress tensor; $\eta = u_i$ is the displacement vector; $f = f^{i}$ is the vector of volumetric forces; and $e = e_{ij}$ is the strain tensor that does not satisfy the conditions of continuity (dislocation strain tensor).

The components of tensors and vectors can be attributed to the oblique-angled coordinate system α_i $(i = 1, 2, 3)$, in which the superscript denotes contravariance and the subscript covariance. Further, D and D^{*} are differential operators defined by the formulas

$$D\xi = -\nabla_i \tau^{ij}, \quad D^{*}\eta = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i), \tag{A.3}$$

where ∇_i is the operation of covariant differentiation.

In Eq. (A.1) A and B are algebraic operators such that

© Springer Nature Singapore Pte Ltd. 2018

V. Fridman, *Theory of Elastic Oscillations*, Foundations of Engineering Mechanics, DOI 10.1007/978-981-10-4786-2

$$B\xi = b_{ijkl}\tau^{kl}, \tag{A.4}$$

where b_{ijkl} is the tensor of elastic constants. For an isotropic body

$$b_{ijkl} = -\frac{\nu}{E} g_{ij} g_{kl} + \frac{1+\nu}{2E}(g_{ik}g_{jl} + g_{jk}g_{il}), \tag{A.5}$$

where E is the modulus of elasticity; ν is the ratio of transverse compression; and g_{ij} is the metric tensor.

In formula (A.2) $C = c^{ij}$ is the tensor of coefficients of elasticity of the medium that contains the deformable body

$$R = \rho g^{ij}, \quad \rho \text{ is specific mass.} \tag{A.6}$$

(2) The theory of shells, based on Kirchhoff's hypotheses [3], gives us in Eq. (A.1)

$$\xi = (S, H), \quad \eta = (u, w), \quad f = \begin{pmatrix} t \\ p \end{pmatrix}, \quad e = \begin{pmatrix} g \\ k \end{pmatrix},$$

where $S = S^{\mu\nu}, H = H^{\mu\nu}$ are the symmetric tensors of forces and moments in the system of curvilinear coordinates on the middle surface of the shell $\alpha_\mu (\mu = 1, 2)$; $u = u_\mu$ is the displacement vector in the tangent plane; w is normal displacement; $t = t^\mu$ is the external load vector in the tangent plane; p is normal load; $g = g_{\mu\nu}$ is the dislocation strain tensor of tension and shearing of the middle surface; and $k = k_{\mu\nu}$ is the tensor of dislocation change in the curvature of the middle surface

$$D\xi = \begin{pmatrix} -\nabla_\nu S^{\nu\mu} + & 2b^\mu_\nu \nabla_\kappa H^{\nu\kappa} + H^{\nu\kappa}\nabla_k b^\mu_\nu \\ -b_{\mu\nu}S^{\mu\nu} + & b^\nu_\mu b_{\nu\mu} H^{\mu\kappa} - \nabla_\mu \nabla_\nu H^{\mu\nu} \end{pmatrix}, \tag{A.7}$$

$$D^*\eta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix},$$

$$\theta_{11} = \frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu), \quad \theta_{12} = -b_{\mu\nu} w, \tag{A.8}$$

$$\begin{aligned}
\theta_{21} &= -\frac{1}{2}\Big(b^\kappa_\mu \nabla_\nu u_\kappa + b^\kappa_\nu \nabla_\mu u_\kappa + \nabla_\mu (b^\kappa_\nu u_\kappa) + \nabla_\nu (b^\kappa_\mu u_\kappa)\Big), \\
\theta_{22} &= (b^\kappa_\mu b^\kappa_{\kappa\nu} + b^\kappa_\nu b^\kappa_{\mu\kappa}) w + \nabla_\mu \frac{\partial w}{\partial \alpha^\nu} - \nabla_\nu \frac{\partial w}{\partial \alpha^\mu}.
\end{aligned}$$

For the algebraic operators B, C and R we have

$$B\xi = \begin{pmatrix} h^{-1} b_{\kappa\delta\mu\nu} S^{\kappa\delta} & 0 \\ 0 & 12h^{-3} b_{\kappa\delta\mu\nu} H^{\kappa\delta} \end{pmatrix}, \tag{A.9}$$

$$C\eta = \begin{pmatrix} c^{\mu\nu} u_\nu \\ cw \end{pmatrix}, \quad R = \rho a^{\mu\nu} h,$$

where h is the thickness of the shell;

$$b_{\kappa\delta\mu\nu} = -\frac{\nu}{E} a_{\kappa\delta} a_{\mu\nu} + \frac{1+\nu}{2E} (a_{\kappa\mu} a_{\delta\nu} + a_{\delta\mu} a_{\kappa\nu}); \tag{A.10}$$

$a_{\mu\nu}$ is the metric tensor of the middle surface; ∇_μ is the operation of differentiation on the surface; $b_{\mu\nu}$ are coefficients of the second quadratic form of the middle surface of the shell; $c^{\mu\nu}$ is the tensor of elastic coefficients of foundation at displacement in the tangent plane; c is the coefficient of elasticity of foundation at normal displacement; and ρ is the specific mass of the shell.

(3) The equilibrium equations and elasticity relations for a curvilinear rod can be conveniently recorded in the moving rectangular coordinate system $\zeta_i (i = 1, 2, 3)$, where ζ_1 is the axis tangential to the rod axis, and the axes ζ_2 and ζ_3 coincide with the directions of the principal axes of inertia of the rod [4].

The position of the trihedral vertex can be defined by coordinate α, counting along the axis of the rod. The trihedral angular velocity of rotation at motion of its vertex along the axis of the rod with unit velocity can be denoted by $\omega = \omega_i$.

Equations (A.1) and (A.2) give us

$$\xi = (Q, M), \eta = (u, \vartheta), \quad f = \begin{pmatrix} p \\ m \end{pmatrix}, \quad e = \begin{pmatrix} g \\ k \end{pmatrix},$$

where $Q = Q^i$ is force; $M = M^i$ is cross-sectional moment in the rod; $u = u_i$ is displacement of the point of the rod; $\vartheta = \vartheta_i$ is turning of the cross-sectional plane of the rod under deformation; $p = p^i$ isexternal distributed load on the rod; $m = m^i$ isexternal distributed moment; $g = g_i$ is the dislocation strain of tension and shearing; and $k = k_i$ is the dislocation strain of torsion and bending of the rod.

For curvilinear rods

$$D\xi = \begin{pmatrix} -\frac{dQ^i}{d\alpha} - \varepsilon^{ij}_k \omega_j Q^k \\ \varepsilon^{ij}_k \tau_j Q^k - \frac{dM^i}{d\alpha} - \varepsilon^{ij}_k \omega_j M^k \end{pmatrix}, \tag{A.11}$$

where τ_i is the unit vector of the tangent to the rod axis; and ε^{ij}_k is an absolutely antisymmetric tensor [5].

Further,

$$D^{*}\eta = \begin{pmatrix} \frac{\mathrm{d}u_i}{\mathrm{d}\alpha} + \varepsilon_i^{kj}\omega_j u_k + \varepsilon_i^{kj}\tau_j\vartheta_k \\ \frac{\mathrm{d}\vartheta_i}{\mathrm{d}\alpha} + \varepsilon_i^{kj}\omega_j\vartheta_k \end{pmatrix}, \tag{A.12}$$

$$B\xi = \begin{pmatrix} b_{ij}Q^j \\ \beta_{ij}M^j \end{pmatrix}, \quad C\eta = \begin{pmatrix} c^{ij}u_j \\ \gamma^{ij}\vartheta_j \end{pmatrix}, \quad R\eta = \begin{pmatrix} \rho^{ij}u_j \\ J^{ij}\vartheta_j \end{pmatrix}, \tag{A.13}$$

where b_{ii} are the compliances of a rod at tension ($i = 1$) and at shearing ($i = 2, 3$), $b_{ij} = 0$ ($i \neq j$); β_{ii} are the compliances of a rod at torsion ($i = 1$) and at bending ($i = 2, 3$), $\beta_{ij} = 0$ ($i \neq j$); c^{ii} are the coefficients of elasticity of foundation under longitudinal ($i = 1$) and transversal ($i = 2, 3$) displacement of the point of the axis of the rod, $c^{ij} = 0$ ($i \neq j$); γ^{ii} are the coefficients of elasticity of foundation at turning around the rod axis ($i = 1$) and around the cross-sectional inertia axes ($i = 2, 3$), $\gamma^{ij} = 0$ ($i \neq j$); J^{ii} are moments of inertia per unit length of the rod relative to the axis of the rod ($i = 1$) and to the axes of inertia of the cross-section ($i = 2, 3$), $J^{ij} = 0$ ($i \neq j$), $\rho^{ij} = 0$ ($i \neq j$), $\rho^{ii} = \rho$; and ρ is mass per unit length of the rod.

The specified and unknown force factors, displacements and operators entered in Eq. (A.1) are functions of the point of the domain Ω (occupied by a three-dimensional elastic body, a shell middle surface and a rod axis).

Conditions at the boundary Γ (on the surface of a three-dimensional body, at the edge of a shell, at the ends of the rod) should be added to Eq. (A.1)

$$\begin{array}{lll} X - F = 0 & \text{on} & \Gamma_1(X = N\xi), \\ Y - E = 0 & \text{on} & \Gamma_2(Y = T\eta), \end{array} \quad (\Gamma_1 + \Gamma_2 = \Gamma). \tag{A.14}$$

(1) In the three-dimensional problem X is the stress vector at the site of the body surface with the unit vector of the normal $n = n_j$; Y is the displacement vector of points on the body surface; and F and E are given values for these vectors.

Relations (A.14) in the three-dimensional problem are of the form

$$X^k = t_i^k n_j \tau^{ij}, \quad Y_k = t_k^i u_i, \tag{A.15}$$

where t_i^k, t_k^i are tensors of the components of the basis vectors; and X^k, Y_k are vectors of the stress and displacement relating to the rectangular coordinate system that is associated with the surface point.

(2) In the theory of shells

$$X = (Q, L), \quad Y = (U, \theta), \tag{A.16}$$

where $Q = Q^i$ is the vector of force at the edge of the shell; L is the bending moment at the edge of the shell; $U = U_i$ is the displacement vector at the edge of the shell; and θ is the turning angle of the shell edge.

The projections on the axes of the natural trihedron associated with the edge are

$$\begin{aligned} Q^1 &= \nu_\mu \nu_\nu (S^{\mu\nu} - b^\mu_\kappa H^{\mu\kappa}) - k\nu_\mu t_\nu H^{\mu\nu}, \quad k = t^\mu \nu_\nu H^{\mu\nu}, \\ Q^2 &= \nu_\mu t_\nu (S^{\mu\nu} - b^\nu_\kappa H^{\mu\kappa}) - l\nu_\mu t_\nu H^{\mu\nu}, \quad l = t^\mu t_\nu b^\nu_\mu, \\ Q^3 &= \nu_\mu \nabla_\nu H^{\mu\nu} + \frac{d}{ds}(\nu_\mu t_\nu H^{\mu\nu}), \quad L = \nu_\mu \nu_\nu H^{\mu\nu} \end{aligned} \tag{A.17}$$

where the natural trihedron is formed with the tangential normal, tangent and normal to the middle surface; ν_μ, t_μ are unit vectors of the tangential normal and the tangent; and s is a coordinate counted along the boundary contour. Further

$$U_1 = u_1, \quad U_2 = u_2, \quad U_3 = u_3, \quad \theta = -\nu^\kappa \left(\frac{\partial w}{\partial \alpha^\kappa} + b^\nu_\kappa u_\nu \right). \tag{A.18}$$

Relations (A.16)–(A.18) impart a concrete form to the relationships (A.14).

(3) In the case of a curvilinear rod

$$X = (Q, M), \quad Y = (U, \theta), \tag{A.19}$$

where $Q = Q^i$ is force; $M = M^i$ is moment at the ends of rod; $U = u_i$ is displacement; $\theta = \vartheta_i$ is turning of the rod ends; and F and E are prescribed values of X and Y.

We conclude our consideration of boundary conditions by noting that further reasoning could also be extended to the case when some force factor projections and all generalized displacement projections are given on the part of the boundary Γ. This would require partitioning the border into parts Γ_1 and Γ_2 for each projection separately by just computing integrals over the boundary Γ.

Moreover, for completeness we consider boundary conditions that are more general in nature than (A.14); namely

$$X + KY - F = 0 \quad \text{on} \quad \Gamma_1, \quad Y + PX - E = 0 \quad \text{on} \quad \Gamma_2. \tag{A.20}$$

These conditions apply when a three-dimensional body is surrounded by an elastic layer, the edge of a shell and the ends of a rod (which are elastically supported and elastically clamped). In relations (A.20) the linear bounded operators K and P characterize, respectively, the stiffness and compliance of the elastic layer, elastic support and sealing.

Let us highlight some properties of the operators entering into the system of equations (A.1) and boundary conditions (A.14) and (A.20). To do this, we introduce scalar products:

(1) in the three-dimensional problem

$$\int\limits_{\Omega} \xi e \mathrm{d}\Omega = \int\limits_{\Omega} \tau^{ij} e_{ij} \mathrm{d}\Omega, \int\limits_{\Omega} f \eta \mathrm{d}\Omega = \int\limits_{\Omega} f^i u_i \mathrm{d}\Omega, \\ \int\limits_{\Gamma} XY \mathrm{d}\Gamma = \int\limits_{\Gamma} X^i Y_i \mathrm{d}\Gamma, \tag{A.21}$$

(2) in the shell deformation problem

$$\int\limits_{\Omega} \xi e \mathrm{d}\Omega = \int\limits_{\Omega} (S^{\mu\nu} \varepsilon_{\mu\nu} + H^{\mu\nu} \beta_{\mu\nu}) \mathrm{d}\Omega, \\ \int\limits_{\Omega} f \eta \mathrm{d}\Omega = \int\limits_{\Omega} (t^{\mu} u_{\mu} + pw) \mathrm{d}\Omega, \int\limits_{\Gamma} XY \mathrm{d}\Gamma = \int\limits_{\Gamma} (Q^i U_i + L\theta) \mathrm{d}\Gamma, \tag{A.22}$$

(3) in the curvilinear rod deformation problem

$$\int\limits_{\Omega} \xi e \mathrm{d}\Omega = \int\limits_{\Omega} (Q^i g_i + M^i k_i) \mathrm{d}\Omega, \\ \int\limits_{\Omega} f \eta \mathrm{d}\Omega = \int\limits_{\Omega} (p^i u_i + m^i \vartheta_i) \mathrm{d}\Omega, \int\limits_{\Gamma} XY \mathrm{d}\Gamma = \int\limits_{\Gamma} (Q^i u_i + M^i \vartheta_i) \mathrm{d}\Gamma. \tag{A.23}$$

It can be shown that the operators D and D^* are conjugate in the sense of Lagrange in every case considered [6]

$$\int\limits_{\Omega} D\xi\eta \mathrm{d}\Omega = \int\limits_{\Omega} \xi D^* \eta \mathrm{d}\Omega - \int\limits_{\Gamma} XY \mathrm{d}\Gamma. \tag{A.24}$$

The operators B and R are positive, while the operators C, K and P are nonnegative

$$\int_{\Omega} R\eta\eta \mathrm{d}\Omega > 0(\eta \neq 0), \int_{\Omega} \xi B\xi \mathrm{d}\Omega > 0(\xi \neq 0), \int_{\Omega} C\eta\eta \mathrm{d}\Omega \geq 0,$$
$$\int_{\Gamma_1} KYY \mathrm{d}\Gamma \geq 0, \int_{\Gamma_2} XPX \mathrm{d}\Gamma \geq 0. \tag{A.25}$$

Other scalar products can also be used

$$\int_{\Omega} ee^* \mathrm{d}\Omega = \int_{\Omega} e_{ij}e^{ij} \mathrm{d}\Omega, \quad \int_{\Omega} YY^* \mathrm{d}\Omega = \int_{\Omega} Y_i Y^i \mathrm{d}\Omega,$$

$$\int_{\Omega} ff^* \mathrm{d}\Omega = \int_{\Omega} f^i f_i \mathrm{d}\Omega, \quad \int_{\Omega} XX^* \mathrm{d}\Omega = \int_{\Omega} X^i X_i \mathrm{d}\Omega$$

and so on.

References

1. Lurie, A. I. (2005). *Theory of elasticity* (1050 p). Berlin: Springer.
2. Sneddon, J. N., & Berry, D. S. (1958)*. The classical theory of elasticity* (219 p). Berlin–Gottingen–Heldelberd: Springer.
3. Naghdi, P. M. (1963). Foundations of elastic shell theory. *Progress in solid mechanics* (Vol. 4, 193 p). Amsterdam, North–Holland Publishing Company.
4. Lurie, A. I. (1941). On small deformations of curvilinear rods. In *Proceedings of the Leningrad Polytechnic Institute* (No 3, pp. 38–50) (in Russian).
5. Mc. Connell A. J. (1957). *Application of tensor analysis* (411 p). N.Y.: Dover Publications, Inc.
6. Fridman, V. M., & Chernina, V. S. (1969). *The modified Galerkin-Ritz method, Associated with the mixed variational principle in the theory of elasticity* (No 1, pp. 64–78). The Academy of Sciences of the USSR, Mechanics of Solids.

© Springer Nature Singapore Pte Ltd. 2018

V. Fridman, *Theory of Elastic Oscillations*, Foundations of Engineering Mechanics, DOI 10.1007/978-981-10-4786-2