Conclusions

The accepted classification of the problems of mathematical physics (in the oscillation and wave theory), first of all, draws a sharp distinction between linear and nonlinear models. Such a distinction is caused by understandable mathematical reasons including the inapplicability of the superposition principle in the nonlinear case. However, in-depth physical analysis allows us to introduce another basis for the classification of ordered oscillation problems, focusing on the difference between the stationary (or nonstationary, but non-resonance) and resonance nonstationary processes. In the latter case, the difference between the linear and nonlinear problems is not fundamental, and a specific technique, equally efficient for description in the same degree for description of both linear and nonlinear resonance nonstationary processes, has been developed. The existence of an alternative approach in the framework of linear theory seems unexpected. Really, the superposition principle allows us to find a solution describing arbitrary nonstationary oscillations as a combination of linear normal modes, which correspond to stationary processes. However, in the systems of weakly coupled oscillators, where resonant nonstationary oscillations can occur, another type of fundamental solution exists. It describes strongly modulated nonstationary oscillations characterized by the maximum possible energy exchange between the oscillators or the clusters of the oscillators (effective particles). Such solutions are referred to as limiting phase trajectories (LPTs). This book demonstrates that the LPT concept suggests a unified approach to the study of such physically different processes as strongly nonstationary energy transfer in a wide range of classical oscillatory systems and quantum dynamical systems with both constant and time-varying parameters. Furthermore, this analogy paves the way for simple mechanical simulation of complicated quantum effects. The role of the LPTs in a deeper understanding and the description of resonance highly nonstationary processes is similar to the role of NNMs in the analysis of the stationary and nonstationary non-resonance, regimes. Moreover, the presented technique can be extended to the models with many degrees of freedom. This technique is based on the statement that every periodic process, independently on the class of its smoothness, can be uniquely expressed as a smooth function of non-smooth variables τ and e or as an element of the algebra of hyperbolic numbers with the basis (1,e) (e^2=1, but e does not equal to unit). It is very important that in

L.I. Manevitch et al., *Nonstationary Resonant Dynamics of Oscillatory Chains and Nanostructures*, Foundations of Engineering Mechanics,
DOI 10.1007/978-981-10-4666-7

this case the algebraic operations and (under special conditions) differentiation and integration preserve the structure of a hyperbolic number. This property provides applicability and convenience of the corresponding transformations during the process of solving the differential equations.

Interestingly, the hyperbolic numbers, which are frequently used for a simplest illustration of the Clifford algebra, were known since the middle of the nineteenth century as abstract mathematical objects without any connection with vibration *processes. On the other hand, the elliptic complex numbers with the basis* $\{1, i\}$ $(i^2 = -1)$ *and corresponding trigonometric functions turned out, in essence, the main tool for the description of such processes.*

Finally, we highlight differences between the NNMs and the LPTs that have motivated the introduction and the development of the LPT concept:

NNM	LPT
Represents a stationary process independent of initial conditions	Represents a nonstationary process dependent of initial conditions
Is not involved in energy exchange	Corresponds to maximum possible energy exchange between different parts of the system
Can undergo local bifurcation	Can undergo global bifurcation
Can be localized (stationary localization)	Can be localized (nonstationary localization)
Can become an attractor in an active system (synchronization of a traditional type)	Can become an attractor in an active system (synchronization of a new type)
Corresponds to a steady solution in a system subjected to external periodic excitation	Corresponds to maximum energy transfer from a source of external periodic excitation
Is described by smooth sinusoidal functions	Is described by non-smooth functions
Can be extended to systems with infinite numbers of particles	Cannot be extended to systems with infinite numbers of particles but can be considered as a prototype of a breather