# Appendix A
# Notation and Identities for Differential Operations

Notation conventions, definitions and identities employed for the operations of calculus of variations used in this study are hereby synoptically collected. These are first summarized in Sect. A.1 for single scalar real-valued functions and for the related functionals. Section A.2 reports conventions and identities for multiple variable functions and for the related functionals and functional operators. All mathematical hypotheses necessary for the definitions and operations herein reported to make sense are given for granted.

## A.1 Single Variable Functions

Symbol $\varphi$ is used to refer to a functional operating from a space of real valued functions $\mathscr{F}$ into $\mathbb{R}$, such that $\alpha = \varphi(f)$ with $\alpha \in \mathbb{R}$ and $f \in \mathscr{F}$, being $f$ a real valued function such that $y = f(x)$ with $x,\ y \in \mathbb{R}$.

An engineering notation is used to denote variations, employing the prefixes $\delta$ or $d$ for differentials. In particular, in the body of the paper prefix $d$ is used to denote *real variations* in linearized theories, while $\delta$ is used to denote *virtual* variations when expressing stationarity conditions; henceforth in this appendix only prefix $\delta$ is used. An infinitesimal variation $\delta f$ of function $f$ is the function $x \rightarrow \delta y$ with $\delta y \in \mathbb{R}$ being the variation in the codomain variable $y$.

An engineering notation is also adopted to indicate the result of the variation, $\partial_f \varphi$, of a functional $\varphi$ with respect to its argument function $f$. This variation is written in the form of a linear relation among the infinitesimal variation of the argument function $\delta f$, reported under square brackets, and the increment $\delta\alpha$ of $\alpha$ with the following notation:

$$\delta\alpha = \partial_f \varphi\,[\delta f]\,, \tag{A.1}$$



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and $\partial_f \varphi$ is such that, for any $\delta f$

$$\partial_f \varphi\,[\delta f] = \lim_{\lambda \to 0} \frac{\varphi(f + \lambda \delta f) - \varphi(f)}{\lambda}. \tag{A.2}$$

*Variation of function composition*

Let $a : (f, x) \to y$ be a functional application relating a real function $f \in \mathscr{F}$ and $x \in \mathbb{R}$ to a scalar $y \in \mathbb{R}$ and let $b : (f, y) \to z$ be a second functional application relating a real function $f \in \mathscr{F}$ and $y \in \mathbb{R}$ to a scalar $z \in \mathbb{R}$. Consider the following composition of these two functional applications $c = b \circ a$ with:

$$c(f, x) = b(f, a(f, x)). \tag{A.3}$$

Computation of the total variation with respect to $f$ by application of the chain rule yields

$$\partial_f c\,[\delta f] = \partial_f b\,[\delta f] \circ a + \frac{\partial b}{\partial y} \partial_f a\,[\delta f]\,. \tag{A.4}$$

In particular, when $b$ is independent from $f$ so that $\partial_f b\,[\delta f] = 0$, one obtains as a special case of (A.4)

$$\partial_f c\,[\delta f] = \frac{\partial b}{\partial y} \partial_f a\,[\delta f]\,. \tag{A.5}$$

*Variation of the application relating functions to their inverses*

By choosing in (A.4), for $a$, the application such that $a(f, x) = f(x)$, and, for $b$, the application such that $b(f, y) = f^{-1}(y)$, with $f \in \mathscr{F}$ being $\mathscr{F}$ a set of invertible functions, then the resulting composed application $c$ turns out to be

$$c(f, x) = f^{-1}(f(x)) = x \tag{A.6}$$

This particular choice makes $c$ an application no longer dependent on $f$, with trivially

$$\partial_f c\,(f, x)\,[\delta f] = 0 \qquad \forall f \in \mathscr{F},\ x \in \mathbb{R},\ \delta f \in \delta\mathscr{F}. \tag{A.7}$$

Furthermore, the chosen application $a$ is already linear in the argument function $f$, so that its linearization in $f$ coincides with $a$ itself and:

$$\partial_f a\,(f, x)\,[\delta f] = \partial_f f(x)\,[\delta f] = \delta f\,(x)\,. \tag{A.8}$$

According to the previous identity, and using the following notation for the application of $\partial_f b$ to $\delta f$

$$\partial_f b\,(f, y)\,[\delta f] = \partial_f f^{-1}(y)\,[\delta f]\,, \tag{A.9}$$

the application of (A.4) can be written as follows

$$\partial_f f^{-1}(y)\,[\delta f] \circ a + \frac{\partial f^{-1}}{\partial y}\delta f(x) = 0. \tag{A.10}$$

Since $\left.\frac{\partial f^{-1}}{\partial y}\right|_y = \left(\left.\frac{\partial f}{\partial x}\right|_x\right)^{-1}$, with $y = f(x)$, and since

$$\partial_f f^{-1}(y)\,[\delta f] \circ a = \partial_f f^{-1}\,(f,\ x)\,[\delta f]\,, \tag{A.11}$$

one computes the following formula for the variation of the inverse function:

$$\partial_f f^{-1}\,(f,\ x)\,[\delta f] = -\left(\left.\frac{\partial f}{\partial x}\right|_x\right)^{-1}\delta f(x). \tag{A.12}$$

## A.2 Multi-variable functions

Consider a functional $\varphi$ operating from a space of three-dimensional invertible mappings $\mathscr{M}$ into $\mathbb{R}$, such that $\alpha = \varphi(\boldsymbol{\chi})$ with $\alpha \in \mathbb{R}$ and $\boldsymbol{\chi} \in \mathscr{M}$, being $\boldsymbol{\chi}$ a three-dimensional mapping. Specifically $\boldsymbol{\chi}$ defines the correspondence $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ with $\mathbf{X},\ \mathbf{x} \in \mathbb{R}^3$ being points of the reference and current configurations, respectively.

An infinitesimal variation $\delta\boldsymbol{\chi}$ of mapping $\boldsymbol{\chi}$ is the function $\mathbf{X} \to \delta\mathbf{x}$ where $\delta\mathbf{x} \in \mathbb{R}^3$ is the variation of the codomain point $\mathbf{x}$. The variation of functional $\varphi$ with respect to the argument mapping $\boldsymbol{\chi}$ is indicated by $\partial_{\boldsymbol{\chi}}\varphi$. Its components $\partial_{\chi_i}\varphi$ are the linear functionals such that, for a given variation function $\delta\chi_i$ of the $i$-th component of mapping $\delta\boldsymbol{\chi}$, the limit condition holds

$$\partial_{\chi_i}\varphi\,[\delta\chi_i] = \lim_{\lambda\to 0}\frac{\varphi(\boldsymbol{\chi} + \lambda\delta\chi_i\mathbf{e}^{(i)}) - \varphi(\boldsymbol{\chi})}{\lambda}, \tag{A.13}$$

being $\mathbf{e}^{(i)}$ the unit vector of the $i$-th Cartesian director.

The relation between the scalar variation $\delta\alpha \in \mathbb{R}$ and $\delta\boldsymbol{\chi} = \sum_{i=1}^{3}\delta\chi_i\mathbf{e}^{(i)}$ is notated as usual by placing $\delta\boldsymbol{\chi}$ under square brackets:

$$\delta\alpha = \partial_{\boldsymbol{\chi}}\varphi\,[\delta\boldsymbol{\chi}] = \sum_{i=1}^{3}\partial_{\chi_i}\varphi\,[\delta\chi_i]\,. \tag{A.14}$$

*Variation of composition of functional applications*

Let $\mathbf{A}^{(a)}:\ (\boldsymbol{\chi},\ \mathbf{X}) \to \mathbf{x}$ be a functional application relating a real function $\boldsymbol{\chi} \in \mathscr{M}$ and a point $\mathbf{X} \in \mathbb{R}^3$ to points $\mathbf{Y} \in \mathbb{R}^3$, viz.:

$$\mathbf{Y} = \mathbf{A}^{(a)}(\boldsymbol{\chi}, \mathbf{X}), \tag{A.15}$$

and let $\mathbf{A}^{(a)} : (\boldsymbol{\chi}, \mathbf{Y}) \to \mathbf{Z}$ be a second functional application relating a mapping $\boldsymbol{\chi} \in \mathscr{M}$ and a point $\mathbf{Y} \in \mathbb{R}^3$ to points $\mathbf{Z} \in \mathbb{R}^3$:

$$\mathbf{Z} = \mathbf{A}^{(b)}(\boldsymbol{\chi}, \mathbf{Y}). \tag{A.16}$$

Consider the composition of such two functional applications $\mathbf{A}^{(c)} = \mathbf{A}^{(b)} \circ \mathbf{A}^{(a)}$:

$$\mathbf{A}^{(c)}(\boldsymbol{\chi}, \mathbf{X}) = \mathbf{A}^{(b)}\left(\boldsymbol{\chi}, \mathbf{A}^{(a)}(\boldsymbol{\chi}, \mathbf{X})\right). \tag{A.17}$$

Computation of the total variation of $\mathbf{A}^{(c)}$ with respect to $\boldsymbol{\chi}$, $\delta\mathbf{Z} = \partial_{\boldsymbol{\chi}}\mathbf{A}^{(c)}(\boldsymbol{\chi}, \mathbf{X})$ $[\delta\boldsymbol{\chi}]$, by application of the chain rule yields

$$\begin{aligned}
\delta Z_i &= \partial_{\boldsymbol{\chi}} A_i^{(c)}(\boldsymbol{\chi}, \mathbf{X})[\delta\boldsymbol{\chi}] \\
&= \sum_{j=1}^{3} \partial_{\chi_j} A_i^{(c)}(\boldsymbol{\chi}, \mathbf{X})\left[\delta\chi_j\right] \\
&= \sum_{j=1}^{3} \partial_{\chi_j} A_i^{(b)}(\boldsymbol{\chi}, \mathbf{X})\left[\delta\chi_j\right] \circ \mathbf{A}^{(a)}(\boldsymbol{\chi}, \mathbf{X}) \\
&\quad + \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial A_i^{(b)}}{\partial Y_k} \partial_{\chi_j} A_k^{(a)}(\boldsymbol{\chi}, \mathbf{X})\left[\delta\chi_j\right].
\end{aligned} \tag{A.18}$$

Special cases and variants of (A.18) are now examined. A first special case of (A.18) is obtained by replacing $\mathbf{A}^{(b)}$ with an application having a scalar codomain with images $Z$ in place of $\mathbf{Z}$:

$$\begin{aligned}
\delta Z &= \sum_{j=1}^{3} \partial_{\chi_j} A^{(b)}(\boldsymbol{\chi}, \mathbf{X})\left[\delta\chi_j\right] \circ \mathbf{A}^{(a)}(\boldsymbol{\chi}, \mathbf{X}) \\
&\quad + \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial A^{(b)}}{\partial Y_k} \partial_{\chi_j} A_k^{(a)}(\boldsymbol{\chi}, \mathbf{X})\left[\delta\chi_j\right].
\end{aligned} \tag{A.19}$$

A second simple special case of (A.18) is retrieved when $\mathbf{A}^{(b)}$ is independent from $\boldsymbol{\chi}$, so that $\partial_{\boldsymbol{\chi}}\mathbf{A}^{(b)}[\delta\boldsymbol{\chi}] = \mathbf{o}$. We have:

$$\delta Z_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial A_i^{(b)}}{\partial Y_k} \partial_{\chi_j} A_k^{(a)}(\boldsymbol{\chi}, \mathbf{X})\left[\delta\chi_j\right], \tag{A.20}$$

and, in case the external function has scalar codomain, the further specialization holds:

$$\delta Z = \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial A^{(b)}}{\partial Y_k} \partial_{\chi_j} A_k^{(a)} (\boldsymbol{\chi}, \mathbf{X}) \left[\delta \chi_j\right]. \tag{A.21}$$

A variant of (A.20) is obtained when $\mathbf{Z}$ is replaced with a scalar function with images $Z$ and when $\mathbf{A}^{(a)}$ is function of several independent mappings $\boldsymbol{\chi}^{(1)}$, $\boldsymbol{\chi}^{(2)}$, $\ldots$, $\boldsymbol{\chi}^{(N_m)}$, and of several additional scalar functions $\varphi^{(1)}$, $\varphi^{(2)}$, $\ldots$, $\varphi^{(N_s)}$:

$$\mathbf{A}^{(a)} = \mathbf{A}^{(a)} \left(\boldsymbol{\chi}^{(1)}, \boldsymbol{\chi}^{(2)}, \ldots, \boldsymbol{\chi}^{(N_m)}, \varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(N_s)}\right). \tag{A.22}$$

In this case one has:

$$\delta Z = \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial A^{(b)}}{\partial Y_k} \left( \sum_{h=1}^{N_m} \partial_{\chi_j^{(h)}} A_k^{(a)} (\boldsymbol{\chi}, \mathbf{X}) \left[\delta \chi_j^{(h)}\right] + \sum_{h=1}^{N_s} \partial_{\varphi^{(h)}} A_k^{(a)} (\boldsymbol{\chi}, \mathbf{X}) \left[\delta \varphi^{(h)}\right] \right). \tag{A.23}$$

For the composition of a scalar function $A^{(b)}(\mathbf{T})$ with codomain in $\mathbb{R}$ defined in a domain of second order tensors $\mathbf{T}$, with components $T_{ij}$, and a functional application over three-dimensional mappings $\boldsymbol{\chi}$, $\mathbf{A}^{(a)}(\boldsymbol{\chi})$, with codomain in the space of second order tensors, developments analogous to those leading to (A.20) yield for the variation of the composed functional application $A^{(c)}(\boldsymbol{\chi}) = A^{(b)}\left(\mathbf{A}^{(a)}(\boldsymbol{\chi})\right)$:

$$\delta Z = \partial_{\boldsymbol{\chi}} A^{(c)} (\boldsymbol{\chi}) [\delta \boldsymbol{\chi}] = \sum_{j=1}^{3} \sum_{h=1}^{3} \sum_{k=1}^{3} \frac{\partial A^{(b)}}{\partial T_{hk}} \partial_{\chi_j} A_{hk}^{(a)} (\boldsymbol{\chi}) \left[\delta \chi_j\right]. \tag{A.24}$$

For the composition $A^{(c)}(t) = A^{(b)}(\boldsymbol{\chi}(t))$ of a functional $A^{(b)}(\boldsymbol{\chi})$, with codomain in $\mathbb{R}$ defined in a domain of 3D mappings $\boldsymbol{\chi} \in \mathscr{M}$, with a 3D motion, defined as a correspondence between $t \in \mathbb{R} \rightarrow \boldsymbol{\chi}$, the total time derivative $\dot{A}^{(c)}$ is computed by the chain rule written as it follows:

$$\dot{A}^{(c)} = \sum_{j=1}^{3} \partial_{\chi_j} A^{(b)} (\boldsymbol{\chi}) \left[\dot{\chi}_j\right] \tag{A.25}$$

*Variation of the inverse of a 3D mapping*

By choosing in (A.18), for $\mathbf{A}^{(a)}$, the application such that $\mathbf{A}^{(a)}(\boldsymbol{\chi}, \mathbf{X}) = \boldsymbol{\chi}(\mathbf{X})$, and, for $\mathbf{A}^{(b)}$, the application such that $\mathbf{A}^{(b)}(\boldsymbol{\chi}, \mathbf{x}) = \boldsymbol{\chi}^{-1}(\mathbf{x})$, with $\boldsymbol{\chi} \in \mathscr{M}$, being $\mathscr{M}$ a set of invertible mappings, then the resulting composed application $\mathbf{A}^{(c)}$ turns out to be:

$$\mathbf{A}^{(c)}(\boldsymbol{\chi}, \mathbf{X}) = \boldsymbol{\chi}^{-1}(\boldsymbol{\chi}(\mathbf{X})) = \mathbf{X}. \tag{A.26}$$

Since the images of $\mathbf{A}^{(c)}$ are no longer dependent on $\boldsymbol{\chi}$, one has:

$$\delta \mathbf{Z} = \partial_{\boldsymbol{\chi}} \mathbf{A}^{(c)} (\boldsymbol{\chi}, \mathbf{X}) [\delta \boldsymbol{\chi}] = \mathbf{o} \qquad \forall \boldsymbol{\chi} \in \mathscr{M}, \mathbf{X} \in \mathbb{R}^3, \delta \boldsymbol{\chi} \in \delta \mathscr{M}, \tag{A.27}$$

where $\delta\mathscr{M}$ is the tangent space to set $\mathscr{M}$. Moreover, considering that the selected application $\mathbf{A}^{(a)}$ is already linear in in the argument $\boldsymbol{\chi}$, it turns out to be:

$$\partial_{\boldsymbol{\chi}}\mathbf{A}^{(a)}(\boldsymbol{\chi},\,\mathbf{X})\,[\delta\boldsymbol{\chi}] = \partial_{\boldsymbol{\chi}}\boldsymbol{\chi}(\mathbf{X})\,[\delta\boldsymbol{\chi}] = \delta\boldsymbol{\chi}\,(\mathbf{X}) \tag{A.28}$$

which in components reads:

$$\sum_{j=1}^{3}\partial_{\chi_j}A_i^{(a)}(\boldsymbol{\chi},\,\mathbf{X})\left[\delta\chi_j\right] = \delta_{ij}\delta\chi_j = \delta\chi_i. \tag{A.29}$$

According to the previous identity, and denoting the variation of $\mathbf{A}^{(b)}$ with respect to $\boldsymbol{\chi}$ applied to $\delta\boldsymbol{\chi}$ as:

$$\partial_{\boldsymbol{\chi}}\mathbf{A}^{(b)}(\boldsymbol{\chi},\,\mathbf{x})\,[\delta\boldsymbol{\chi}] = \partial_{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1}(\mathbf{x})\,[\delta\boldsymbol{\chi}], \tag{A.30}$$

application of (A.18) reads:

$$\partial_{\chi_j}\chi_I^{-1}(\mathbf{x})\left[\delta\chi_j\right]\circ\mathbf{A}^{(a)}(\boldsymbol{\chi},\,\mathbf{X}) + \sum_{j=1}^{3}\frac{\partial\chi_I^{-1}}{\partial x_j}(\mathbf{X})\,\delta\chi_j = 0. \tag{A.31}$$

Recalling that $\left.\frac{\partial\chi_I^{-1}}{\partial x_j}\right|_{\mathbf{x}} = \left(\left.\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}\right|_{\mathbf{X}}\right)^{-1}_{Ij}$ with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, the formula of the variation of the inverse mapping is computed:

$$\partial_{\chi_j}\chi_I^{-1}(\boldsymbol{\chi},\,\mathbf{X})\left[\delta\chi_j\right] = -\sum_{j=1}^{3}\left.\frac{\partial\chi_I^{-1}}{\partial x_j}\right|_{\mathbf{x}}\delta\chi_j\,(\mathbf{X}) = -\sum_{j=1}^{3}\left(\left.\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}\right|_{\mathbf{X}}\right)^{-1}_{Ij}\delta\chi_j\,(\mathbf{X})\,. \tag{A.32}$$

Henceforth, and throughout the monograph in order to achieve an abbreviated notation, summation symbols are omitted and the summation convention over repeated indices is applied.

*Variation of the Jacobian of a mapping*

With the previously introduced notation and identities at hand, the variation of the determinant of an invertible mapping $\boldsymbol{\chi}$ is computed from (A.24)

$$\partial_{\chi_i}\bar{J}\,[\delta\chi_i] = \partial_{\frac{\partial\chi_h}{\partial X_J}}\det\left(\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}\right)\partial_{\chi_i}\left(\frac{\partial\chi_h}{\partial X_J}\right)[\delta\chi_i] = \bar{J}\frac{\partial\,(\chi)_J^{-1}}{\partial x_i}\frac{\partial\delta\chi_i}{\partial X_J} = \bar{J}\frac{\partial\delta\chi_i}{\partial x_i}. \tag{A.33}$$

*Time rate of the Jacobian of a mapping*

Application of (A.25) and account of (A.33) provide

$$\dot{\bar{J}} = \partial_{\chi_j}\bar{J}\,(\chi)\left[\dot{\chi}_j\right] = \bar{J}\frac{\partial\,(\chi)_J^{-1}}{\partial x_i}\frac{\partial^2\chi_i}{\partial t\,\partial X_J} = \bar{J}\frac{\partial\dot{\chi}_i}{\partial x_i}. \tag{A.34}$$

According to (A.34) the variation with respect to the deformation rate $\dot{\boldsymbol{\chi}}$ of the time rate of the Jacobian of a mapping simply turns out to be

$$\partial_{\dot{\chi}_i} \dot{\bar{J}} \left[\delta \chi_i\right] = \bar{J} \frac{\partial \left(\chi\right)_J^{-1}}{\partial x_i} \frac{\partial \delta \chi_i}{\partial X_J}. \tag{A.35}$$

## A.3 Euler-Lagrange Equations

With the notation of Sect. A.2 at hand, the developments yielding the strong form Euler-Lagrange equations from the generic least-Action condition are hereby recalled for a continuum system whose state is defined by one scalar continuum field $\varphi : \Omega_0^{(M)} \times [t_0, t_f] \to \varphi \in \mathbb{R}$. All mathematical hypotheses necessary for the developments below reported to make sense are given for granted.

The generic statement of the least-Action condition for a Lagrange function $L_0^{(M)}$ is written as follows

$$\delta \int_{t_0}^{t_f} L_0^{(M)} \left(\varphi(t), \dot{\varphi}(t)\right) dt = 0, \tag{A.36}$$

with $\varphi(t_0)$ and $\varphi(t_f)$ being given initial and final states, respectively. Account of the dependence $L_0^{(M)} = L_0^{(M)} (\varphi, \dot{\varphi})$ up to first-order time derivatives of $\varphi$ and application of (A.14) with $\chi_1 = \varphi$ and $\chi_2 = \dot{\varphi}$ yields

$$\int_{t_0}^{t_f} \left\{ \partial_\varphi L_0^{(M)}[\delta\varphi] + \partial_{\dot{\varphi}} L_0^{(M)}[\delta\dot{\varphi}] \right\} dt = 0. \tag{A.37}$$

Considering the property $\delta\dot{\varphi} = \frac{d\delta\varphi}{dt}$ and that for a linear operator $A^{(l)}$ over a variable $\delta\varphi$ the identity holds $\frac{d}{dt}\left(A^{(l)}\left[\delta\varphi\right]\right) = \left(\frac{d}{dt}A^{(l)}\right)[\delta\varphi] + A^{(l)}\left[\frac{d\delta\varphi}{dt}\right]$, relation (A.37) can be equated to:

$$\int_{t_0}^{t_f} \left\{ \partial_\varphi L_0^{(M)}[\delta\varphi] + \frac{d}{dt}\left(\partial_{\dot{\varphi}} L_0^{(M)}\left[\delta\varphi\right]\right) - \left(\frac{d}{dt}\partial_{\dot{\varphi}} L_0^{(M)}\right)[\delta\varphi] \right\} dt = 0, \tag{A.38}$$

and hence to

$$\begin{aligned} &\int_{t_0}^{t_f} \left\{ \partial_\varphi L_0^{(M)}[\delta\varphi] - \left(\frac{d}{dt}\partial_{\dot{\varphi}} L_0^{(M)}\right)[\delta\varphi] \right\} dt + \\ &\qquad + \partial_{\dot{\varphi}} L_0^{(M)}\left(t_f\right)\left[\delta\varphi\left(t_f\right)\right] - \partial_{\dot{\varphi}} L_0^{(M)}\left(t_0\right)\left[\delta\varphi\left(t_0\right)\right] = 0. \end{aligned} \tag{A.39}$$

Taking into account that $\varphi(t_0)$ and $\varphi(t_f)$ are given, so that their variations $\delta\varphi\left(t_f\right)$ and $\delta\varphi\left(t_0\right)$ are null, the following generic format of the Euler-Lagrange equation associated with field $\varphi$ is inferred from (A.38):

$$\left(\frac{d}{dt}\partial_{\dot{\varphi}}L_0^{(M)} - \partial_{\varphi}L_0^{(M)}\right)[\delta\varphi] = 0. \tag{A.40}$$

Since the previous relation must hold for any function $\delta\varphi$ one infers the strong-form equation:

$$\frac{d}{dt}\partial_{\dot{\varphi}}L_0^{(M)} - \partial_{\varphi}L_0^{(M)} = 0 \tag{A.41}$$

which is complemented by constraints for field $\varphi$ and its variations $\delta\varphi$ over the space-time boundary $\partial\Omega_0^{(M)} \times [t_0, t_f]$.

For mechanical systems depending on a finite number of continuum fields $\varphi_1, \varphi_2, \ldots, \varphi_N$, the weak statement provided by (A.40) generalizes to:

$$\sum_{k=1}^{N}\left(\frac{d}{dt}\partial_{\dot{\varphi}_k}L_0^{(M)} - \partial_{\varphi_k}L_0^{(M)}\right)[\delta\varphi_k] = 0, \tag{A.42}$$

and the derivation of the system of strong-form PDE is subordinated to the specification of the space-time boundary constraints for fields $\varphi_1, \varphi_2, \ldots, \varphi_N$ (with the associated constraints for their variations $\delta\varphi_1, \delta\varphi_2, \ldots, \delta\varphi_N$) over $\partial\Omega_0^{(M)} \times [t_0, t_f]$.

## A.4 Parallel with Moiseiwitsch's Notation

This appendix is intended to draw a parallel between the notation conventions employed in this monograph for denoting operations of calculus of variations and the notation used in classical books of variational continuum mechanics. To this end, the correspondence between the symbols used in this work to denote variations and the symbols and notation conventions used by Moiseiwitsch [1] is clarified.

The correspondence among the symbols of this monograph that denote the seven primary scalar kinematic descriptor fields $\bar{\chi}_1^{(s)}, \bar{\chi}_2^{(s)}, \bar{\chi}_3^{(s)}, \bar{\chi}_1^{(f)}, \bar{\chi}_2^{(f)}, \bar{\chi}_3^{(f)}, \hat{J}^{(s)}$, the related variations and space derivatives, and their counterparts, as they would be denoted according to Moiseiwitsch's notation, is detailed in Table A.1.

The following notation is employed in this monograph to denote variations:

$$\partial_{\varphi}A\,[\delta\varphi]\,, \tag{A.43}$$

where $\varphi$ is a generic scalar field and $A$ is a functional, or a functional application, depending on the context in which the notation (A.43) is applied. For instance such notation is adopted in Chap. 2 to indicate the variation of the potential energy of the solid with respect to the placement of the solid phase (considering the summation convention over repeated indices):

$$\partial_{\bar{\chi}_i^{(s)}}U^{(s)}\left[\delta\bar{\chi}_i^{(s)}\right]. \tag{A.44}$$

**Table A.1** Relation with symbols in Moiseiwitsch [1]

| Description | Moiseiwitsch's notation | Present notation |
|---|---|---|
| Kinematic scalar field | $\psi_1$ | $\bar{\chi}_1^{(s)}$ |
| ” | $\psi_2$ | $\bar{\chi}_2^{(s)}$ |
| ” | $\psi_3$ | $\bar{\chi}_3^{(s)}$ |
| ” | $\psi_4$ | $\bar{\chi}_1^{(f)}$ |
| ” | $\psi_5$ | $\bar{\chi}_2^{(f)}$ |
| ” | $\psi_6$ | $\bar{\chi}_3^{(f)}$ |
| ” | $\psi_7$ | $\hat{J}^{(s)}$ |
| Variation of kinematic scalar field | $\eta_1$ | $\delta\bar{\chi}_1^{(s)}$ |
| ” | $\eta_2$ | $\delta\bar{\chi}_2^{(s)}$ |
| ” | $\eta_3$ | $\delta\bar{\chi}_3^{(s)}$ |
| ” | $\eta_4$ | $\delta\bar{\chi}_1^{(f)}$ |
| ” | $\eta_5$ | $\delta\bar{\chi}_2^{(f)}$ |
| ” | $\eta_6$ | $\delta\bar{\chi}_3^{(f)}$ |
| ” | $\eta_7$ | $\delta\hat{J}^{(s)}$ |
| Derivatives of variations of kinematic fields | $\dfrac{\partial\eta_\sigma}{\partial x_\mu}$ | $\dfrac{\partial\delta\bar{\chi}_i^{(s)}}{\partial X_J}, \dfrac{\partial\delta\bar{\chi}_i^{(f)}}{\partial X_J}, \dfrac{\partial\delta\hat{J}^{(s)}}{\partial X_J}$ |

The notation in (A.43) is directly correspondent with the term in curly brackets on the right hand side of Moisewitsch's formula (3.5). To explain this correspondence we use the symbol $\mathscr{L}$ to refer to the *Lagrangian density* appearing in (3.25) of [1] (denoted in [1] by an 'L' in italic font) which is a functional application whose images are scalar fields defined in the physical space domain $\Omega$. $\mathscr{L}$ is representative, for instance, of the potential density fields of the solid and fluid phases denoted instead in this monograph by the symbols $\bar{\psi}^{(s)}$ and $\bar{\psi}^{(f)}$ (with a bar accent), and is linked to the kinematic descriptors by a relation which, in Moisewitsch's notation, reads:

$$\mathscr{L}\left(\psi_1, \ldots, \psi_7, \frac{\partial\psi_1}{\partial x_1}, \frac{\partial\psi_1}{\partial x_2}, \frac{\partial\psi_1}{\partial x_3}, \ldots, \frac{\partial\psi_7}{\partial x_1}, \frac{\partial\psi_7}{\partial x_2}, \frac{\partial\psi_7}{\partial x_3}\right). \tag{A.45}$$

Based on the relationship between symbols reported in Table A.1, according to which $\bar{\chi}_2^{(f)} \equiv \psi_5$, and $\delta\bar{\chi}_2^{(f)} \equiv \partial\eta_5$, the two following notations are equivalent:

$$\partial_{\bar{\chi}_2^{(f)}}\mathscr{L}[\delta\bar{\chi}_2^{(f)}] \equiv \frac{\partial\mathscr{L}}{\partial\psi_5}\eta_5 + \sum_{\mu=1}^{3}\frac{\partial\mathscr{L}}{\partial\left(\partial\psi_5/\partial x_\mu\right)}\frac{\partial\eta_5}{\partial x_\mu}. \tag{A.46}$$

It is thus clarified that the notation (A.43) is *not* directly linked with the *functional derivatives* used in relations (3.23)–(3.25) in [1].

In particular, the notation (A.44) used in this manuscript underlies the convention of summation over repeated indices, i.e.:

$$\partial_{\bar{\chi}_i^{(s)}} U^{(s)} \left[ \delta \bar{\chi}_i^{(s)} \right] = \sum_{i=1}^{3} \partial_{\bar{\chi}_i^{(s)}} U^{(s)} \left[ \delta \bar{\chi}_i^{(s)} \right] = \\ = \partial_{\bar{\chi}_1^{(s)}} U^{(s)} \left[ \delta \bar{\chi}_1^{(s)} \right] + \partial_{\bar{\chi}_2^{(s)}} U^{(s)} \left[ \delta \bar{\chi}_2^{(s)} \right] + \partial_{\bar{\chi}_3^{(s)}} U^{(s)} \left[ \delta \bar{\chi}_3^{(s)} \right] \tag{A.47}$$

Moreover, recalling that the symbol $U^{(s)}$ indicates the potential energy of the solid, so that the following relation applies

$$U^{(s)} \left( \psi_1, \ldots, \psi_7, \frac{\partial \psi_1}{\partial x_1}, \frac{\partial \psi_1}{\partial x_2}, \frac{\partial \psi_1}{\partial x_3}, \ldots, \frac{\partial \psi_7}{\partial x_1}, \frac{\partial \psi_7}{\partial x_2}, \frac{\partial \psi_7}{\partial x_3} \right), \tag{A.48}$$

and considering the correspondence among the symbols reported in the table, it can be seen that (A.47), rewritten in Moiseiwitsch's notation, corresponds to:

$$\partial_{\bar{\chi}_i^{(s)}} U^{(s)} \left[ \delta \bar{\chi}_i^{(s)} \right] \equiv \sum_{k=1}^{3} \left( \frac{\partial U^{(s)}}{\partial \psi_k} \eta_k + \sum_{\mu=1}^{3} \frac{\partial U^{(s)}}{\partial \left( \partial \psi_k / \partial x_\mu \right)} \frac{\partial \eta_k}{\partial x_\mu} \right). \tag{A.49}$$

It is remarked that notation (A.43) is adopted in (A.46) to indicate the variation of a functional application (whose images are the Lagrangian density fields defined over $\Omega$). However, when used in (A.49), it indicates the variation of a functional (with codomain over $\mathbb{R}$).

# Appendix B
# Variation of Individual Terms in Lagrange Function

Hereby the computation of the variations of the individual terms $\delta U^{(s)}$, $\delta U_0^{(f)}$, $\delta T^{(s)}$ $\delta T_0^{(f)}$ required to obtain the explicit form of (2.65) and (2.66) is reported. In the developments below, it is accounted for the property that the domain $\Omega_0^{(M)}$ is independent from kinematic descriptors $\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\hat{J}^{(s)}$, and that, consequently, all variations can be directly transferred to the integrand functions inside the integrals $\int_{\Omega_0^{(M)}} (\cdot)\, dV_0$.

*Computation of* $\delta U^{(s)}$

Application of (A.14) to (2.34) accounting for (2.26) and (2.44) yields:

$$\begin{aligned}\delta U^{(s)} &= \partial_{\bar{\chi}_i^{(s)}} U^{(s)} \left[\delta \bar{\chi}_i^{(s)}\right] + \partial_{\hat{J}^{(s)}} U^{(s)} \left[\delta \hat{J}^{(s)}\right] = \\ &= \int_{\Omega_0^{(M)}} \partial_{\bar{\chi}_i^{(s)}} \bar{\psi}_0^{(s)} \left[\delta \bar{\chi}_i^{(s)}\right] dV_0 + \int_{\Omega_0^{(M)}} \partial_{\hat{J}^{(s)}} \bar{\psi}_0^{(s)} \left[\delta \hat{J}^{(s)}\right] dV_0 \\ &= \int_{\Omega_0^{(M)}} \check{P}_{kJ}^{(s)} \frac{\partial \delta \bar{\chi}_k^{(s)}}{\partial X_J}\, dV_0 - \int_{\Omega_0^{(M)}} \hat{\Pi}^{(s)} \delta \hat{J}^{(s)}\, dV_0,\end{aligned} \tag{B.1}$$

where the following trivial identities have been taken into account:

$$\partial_{\bar{\chi}_i^{(s)}} \bar{\psi}_0^{(s)} \left[\delta \bar{\chi}_i^{(s)}\right] = \frac{\partial \bar{\psi}_0^{(s)}}{\partial \bar{F}_{kJ}^{(s)}} \partial_{\bar{\chi}_i^{(s)}} \bar{F}_{kJ}^{(s)} \left[\delta \bar{\chi}_i^{(s)}\right], \tag{B.2}$$

and

$$\partial_{\bar{\chi}_i^{(s)}} \bar{F}_{kJ}^{(s)} \left[\delta \bar{\chi}_i^{(s)}\right] = \frac{\partial}{\partial X_J} \partial_{\bar{\chi}_i^{(s)}} \bar{\chi}_k^{(s)} \left[\delta \bar{\chi}_i^{(s)}\right] = \frac{\partial}{\partial X_J} \delta_{ik} \left[\delta \bar{\chi}_i^{(s)}\right] = \frac{\partial \delta \bar{\chi}_k^{(s)}}{\partial X_J}. \tag{B.3}$$

*Computation of* $\delta U_0^{(f)}$

Due to the independence of domain $\Omega_0^{(M)}$ and field $\Phi_0^{(f)}$ from the kinematic descriptors, the variation of (2.35) turns out to be



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$$\delta U_0^{(f)} = \int_{\Omega_0^{(M)}} \Phi_0^{(f)} \delta \left( \hat{\psi}_0^{(f)} \circ \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} \right) dV_0. \tag{B.4}$$

Applying the chain rule in the form provided by (A.23) to the variation term in the integrand of (B.4), accounting for (2.24) and recalling also (2.47), one obtains

$$\delta \left( \hat{\psi}_0^{(f)} \circ \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} \right) = \frac{\partial \hat{\psi}_0^{(f)}}{\partial \hat{J}^{(f)}} \delta \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} = -\hat{\hat{\Pi}}^{(f)} \delta \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}}, \tag{B.5}$$

with

$$\delta \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} = \partial_{\bar{\chi}_i^{(s)}} \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} \left[ \delta \bar{\chi}_i^{(s)} \right] + \partial_{\bar{\chi}_i^{(f)}} \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} \left[ \delta \bar{\chi}_i^{(f)} \right] + \partial_{\hat{J}^{(s)}} \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} \left[ \delta \hat{J}^{(s)} \right]. \tag{B.6}$$

The addends in the RHS of (B.6) are now separately computed based on the definition of $\hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}}$ provided by (2.23) and (2.24). For the first term it is computed

$$\partial_{\bar{\chi}_i^{(s)}} \hat{J}^{(f)}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}} = -\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}} \partial_{\bar{\chi}_i^{(s)}} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(f)}}. \tag{B.7}$$

The variation $\partial_{\bar{\boldsymbol{\chi}}^{(s)}} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(f)}}$ in (B.7) has a complex kinematic meaning since it represents the variation of solid volume fraction experimented in a point which follows the macroscopic motion of the fluid phase, as it is induced by an infinitesimal isochoric variation of the solid phase deformation. Such term is however straightforwardly computed based on formulas (A.19) and (A.32). Actually, application of (A.19) to (2.23) yields

$$\begin{aligned}
\partial_{\bar{\chi}_i^{(s)}} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(f)}} \left[ \delta \bar{\chi}_i^{(s)} \right] &= \partial_{\bar{\chi}_i^{(s)}} \left( \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \circ \left( \bar{\boldsymbol{\chi}}^{(s)} \right)^{-1} \circ \bar{\boldsymbol{\chi}}^{(f)} \right) \left[ \delta \bar{\chi}_i^{(s)} \right] \\
&= \left( \partial_{\bar{\chi}_i^{(s)}} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \left[ \delta \bar{\chi}_i^{(s)} \right] \right) \circ \left( \bar{\boldsymbol{\chi}}^{(s)} \right)^{-1} \circ \bar{\boldsymbol{\chi}}^{(f)} \\
&\quad + \frac{\partial \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}}{\partial X_J} \left\{ \partial_{\bar{\chi}_i^{(s)}} \left( \bar{\chi}^{(s)} \right)^{-1}_J \left[ \delta \bar{\chi}_i^{(s)} \right] \right\} \circ \bar{\boldsymbol{\chi}}^{(f)}.
\end{aligned} \tag{B.8}$$

Recalling (2.22) and (A.33), the variation in the second row of (B.8) is recognized to be equal to

$$\begin{aligned}
\partial_{\bar{\chi}_i^{(s)}} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \left[ \delta \bar{\chi}_i^{(s)} \right] &= -\Phi_0^{(s)} \frac{\hat{J}^{(s)}}{\left( \bar{J}^{(s)} \right)^2} \partial_{\bar{\chi}_i^{(s)}} \bar{J}^{(s)} \left[ \delta \bar{\chi}_i^{(s)} \right] \\
&= -\Phi_0^{(s)} \frac{\hat{J}^{(s)}}{\left( \bar{J}^{(s)} \right)^2} \bar{J}^{(s)} \frac{\partial \delta \bar{\chi}_i^{(s)}}{\partial x_i} \left[ \delta \bar{\chi}_i^{(s)} \right] \\
&= -\phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \frac{\partial \delta \bar{\chi}_i^{(s)}}{\partial x_i} \left[ \delta \bar{\chi}_i^{(s)} \right].
\end{aligned} \tag{B.9}$$

The variation in the third row of (B.8) is computed invoking (A.32)

$$\partial_{\bar{\chi}_i^{(s)}} \left(\bar{\boldsymbol{\chi}}^{(s)}\right)^{-1} \left[\delta \bar{\chi}_i^{(s)}\right] = -\frac{\partial \left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i} \delta \bar{\chi}_i^{(s)}. \tag{B.10}$$

Accordingly, one obtains for (B.8)

$$\partial_{\bar{\chi}_i^{(s)}} \phi_{\bar{\boldsymbol{\chi}}^{(f)}}^{(s)} \left[\delta \bar{\chi}_i^{(s)}\right] = -\phi_{\bar{\boldsymbol{\chi}}^{(s)}}^{(s)} \frac{\partial \delta \bar{\chi}_i^{(s)}}{\partial x_i} \circ \left(\bar{\boldsymbol{\chi}}^{(s)}\right)^{-1} \circ \bar{\boldsymbol{\chi}}^{(f)} - \frac{\partial \phi_{\bar{\boldsymbol{\chi}}^{(s)}}^{(s)}}{\partial X_J} \frac{\partial \left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i} \delta \bar{\chi}_i^{(s)} \circ \bar{\boldsymbol{\chi}}^{(f)}. \tag{B.11}$$

As a side comment, the minus sign appearing in the first addend in (B.11) is explained considering that a macroscopic isochoric volumetric dilatation induces a decrease in the solid volume fraction. The second minus sign is justified by the consideration that a variation of solid displacement determines in a fixed space point a convected variation of porosity. This convected porosity variation is negative when directed along the porosity gradient.

For the second addend in the RHS of (B.6) one computes from (2.24)

$$\partial_{\bar{\chi}_i^{(f)}} \hat{J}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}}^{(f)} \left[\delta \bar{\chi}_i^{(f)}\right] = \frac{\phi_{\bar{\boldsymbol{\chi}}^{(f)}}^{(f)}}{\Phi_0^{(f)}} \partial_{\bar{\chi}_i^{(f)}} \bar{J}^{(f)} \left[\delta \bar{\chi}_i^{(f)}\right] - \frac{\bar{J}^{(f)}}{\Phi_0^{(f)}} \partial_{\bar{\chi}_i^{(f)}} \phi_{\bar{\boldsymbol{\chi}}^{(f)}}^{(s)} \left[\delta \bar{\chi}_i^{(f)}\right]. \tag{B.12}$$

The variation of the first term in the RHS of (B.12) is provided by (A.33)

$$\partial_{\bar{\chi}_i^{(f)}} \bar{J} \left[\delta \bar{\chi}_i^{(f)}\right] = \bar{J}^{(f)} \frac{\partial \delta \bar{\chi}_i^{(f)}}{\partial x_i}. \tag{B.13}$$

The variation in the second term in the RHS of (B.12) contains the variation of solid volume fraction measured in a point which follows the macroscopic motion of the fluid phase, as the effect of the application of an infinitesimal variation to the fluid placement. It is computed by application of (A.21) to the composition of functions $A^{(b)} = \phi_{\bar{\boldsymbol{\chi}}^{(s)}}^{(s)} \circ \left(\bar{\boldsymbol{\chi}}^{(s)}\right)^{-1}$ and $\mathbf{A}^{(a)} = \bar{\boldsymbol{\chi}}^{(f)}$ appearing in (2.23), and turns out to be:

$$\partial_{\bar{\chi}_i^{(f)}} \phi_{\bar{\boldsymbol{\chi}}^{(f)}}^{(s)} \left[\delta \bar{\chi}_i^{(f)}\right] = \frac{\partial \phi_{\bar{\boldsymbol{\chi}}^{(f)}}^{(s)}}{\partial X_J} \frac{\partial \left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i} \delta \bar{\chi}_i^{(f)}. \tag{B.14}$$

The variation of the third term in the RHS of (B.6) is provided again by the chain rule

$$\partial_{\hat{J}^{(s)}} \hat{J}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}}^{(f)} \left[\delta \hat{J}^{(s)}\right] = \frac{\partial \hat{J}_{sat\,\bar{\boldsymbol{\chi}}^{(f)}}^{(f)}}{\partial \phi_{\bar{\boldsymbol{\chi}}^{(f)}}^{(s)}} \frac{\partial \phi_{\bar{\boldsymbol{\chi}}^{(f)}}^{(s)}}{\partial \hat{J}^{(s)}} \left[\delta \hat{J}^{(s)}\right] = -\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}} \frac{\Phi_0^{(s)}}{\bar{J}^{(s)}} \left[\delta \hat{J}^{(s)}\right]. \tag{B.15}$$

Addition of all the above computed variation terms appearing in the RHS of (B.6) provides the explicit expression for $\delta U_0^{(f)}$ shown by Eq. (2.67) in the body of the monograph. Moreover, composition of (2.67) with (B.4) and (B.5) provides the sought explicit expression for $\delta U_0^{(f)}$ shown by Eq. (2.73) in the body of the monograph.

*Variation of $\delta U^{ext}_{\Omega 0}$*

The variation $\delta U^{ext}_{\Omega 0}$ is computed from (2.36), (2.37) and (2.42), and turns out to be

$$\begin{aligned} \delta U^{ext}_{\Omega 0} &= \partial_{\bar{\chi}_i^{(s)}} U^{ext}_{\Omega 0}\left[\delta \bar{\chi}_i^{(s)}\right] + \partial_{\bar{\chi}_i^{(f)}} U^{ext}_{\Omega 0}\left[\delta \bar{\chi}_i^{(f)}\right] \\ &= -\int_{\Omega_0^{(M)}} \bar{b}_{0i}^{(s)} \delta \bar{\chi}_i^{(s)}\, dV_0 - \int_{\Omega_0^{(M)}} \bar{b}_{0i}^{(f,ext)} \delta \bar{\chi}_i^{(f)}\, dV_0. \end{aligned} \tag{B.16}$$

*Variation terms associated with kinetic energy*

The variation terms associated with $T^{(s)}$ and $T_0^{(f)}$, appearing in (2.66), are also computed applying the chain rules of Sect. A.2 to Eqs. (2.29)–(2.31) and (2.40)–(2.41). Specifically, one computes for the terms associated with $T^{(s)}$

$$\begin{aligned} \frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(s)}} T^{(s)}\left[\delta \bar{\chi}_i^{(s)}\right] = &\int_{\Omega_0^{(M)}} \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} \delta \bar{\chi}_i^{(s)}\, dV_0 \\ &+ \int_{\Omega_0^{(M)}} \frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(s)}} \bar{\kappa}^{(s)}_{0\,add}\left[\delta \bar{\chi}_i^{(s)}\right] dV_0. \end{aligned} \tag{B.17}$$

Computation of the microinertia term with $\bar{\kappa}^{(s)}_{0\,add}$ is carried out accounting for (2.29) with the aid of (A.34) and (A.35), according to which it results:

$$\begin{aligned} \frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(s)}} \bar{\kappa}^{(s)}_{0\,add}\left[\delta \bar{\chi}_i^{(s)}\right] &= \bar{\rho}^{(s)}_{add.0}\frac{d}{dt}\left(\dot{\bar{J}}^{(s)}\partial_{\dot{\bar{\chi}}_i^{(s)}} \dot{\bar{J}}^{(s)}\left[\delta \bar{\chi}_i^{(s)}\right]\right) \\ &= \bar{\rho}^{(s)}_{add.0}\frac{d}{dt}\left(\dot{\bar{J}}^{(s)}\frac{\partial \dot{\bar{J}}^{(s)}}{\partial \dot{\bar{J}}^{(s)}}\partial_{\dot{\bar{\chi}}_i^{(s)}} \dot{\bar{J}}^{(s)}\left[\delta \bar{\chi}_i^{(s)}\right]\right) \\ &= \bar{\rho}^{(s)}_{add.0}\frac{d}{dt}\left(\bar{J}^{(s)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\dot{\bar{J}}^{(s)}\right)\frac{\partial \delta \bar{\chi}_i^{(s)}}{\partial X_J}. \end{aligned} \tag{B.18}$$

Change of index $s$ to $f$ in (B.18) provides its counterpart associated with $T_0^{(f)}$.

A similar computation of the variation associated with the rate of the intrinsic solid strain yields

$$\begin{aligned} &\frac{d}{dt}\partial_{\dot{\hat{J}}^{(s)}} L_0^{(M)}\left[\delta \hat{J}^{(s)}\right] = \\ &= \frac{d}{dt}\int_{\Omega_0^{(M)}}\left(\frac{\partial \bar{\kappa}^{(s)}_{0\,add}}{\partial \dot{\bar{J}}^{(s)}}\frac{\partial \dot{\bar{J}}^{(s)}}{\partial \dot{\hat{J}}^{(s)}}\partial_{\dot{\hat{J}}^{(s)}}\dot{\hat{J}}^{(s)}\left[\delta \hat{J}^{(s)}\right] + \frac{\partial \bar{\kappa}^{(f)}_{0\,add}}{\partial \dot{\bar{J}}^{(f)}}\frac{\partial \dot{\bar{J}}^{(f)}}{\partial \dot{\hat{J}}^{(f)}}\partial_{\dot{\hat{J}}^{(s)}}\dot{\hat{J}}^{(f)}_{sat\,\bar{\chi}^{(f)}J}\left[\delta \hat{J}^{(s)}\right]\right) dV_0 \\ &= -\int_{\Omega_0^{(M)}}\left[\bar{\rho}^{(s)}_{add.0}\frac{d}{dt}\left(\dot{\bar{J}}^{(s)} - \dot{\hat{J}}^{(s)}\right) - \frac{d}{dt}\left(\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\frac{\Phi_0^{(s)}}{\bar{J}^{(s)}}\bar{\rho}^{(f)}_{add.0}\left(\dot{\bar{J}}^{(f)} - \dot{\hat{J}}^{(f)}\right)\right)\right]\delta \hat{J}^{(s)}\, dV_0 \end{aligned} \tag{B.19}$$

where, in the computation of (B.19), account has been taken of the property $\partial_{\dot{g}} \dot{f} = \partial_g f$.

For simplicity, in the above computed variations associated with microinertia terms $\bar{\kappa}^{(s)}_{0\,add}$ and $\bar{\kappa}^{(s)}_{0\,add}$, the terms containing second time rates are considered to be dominant with respect to terms containing only first time derivatives so that the latter are ruled out. Accordingly, the following inertia terms are finally computed for (2.66):

$$
\begin{aligned}
\frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(s)}} L_0^{(M)}\left[\delta\bar{\chi}_i^{(s)}\right] &= \int_{\Omega_0^{(M)}} \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} \delta\bar{\chi}_i^{(s)}\, dV_0 \\
&\quad + \int_{\Omega_0^{(M)}} \bar{\rho}^{(s)}_{add.0} \bar{J}^{(s)} \frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \frac{\partial\delta\bar{\chi}_i^{(s)}}{\partial X_J}\left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)}\right) dV_0 \\
\frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(f)}} L_0^{(M)}\left[\delta\bar{\chi}_i^{(f)}\right] &= \int_{\Omega_0^{(M)}} \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(f)} \delta\bar{\chi}_i^{(f)}\, dV_0 \\
&\quad + \int_{\Omega_0^{(M)}} \bar{\rho}^{(f)}_{add.0} \bar{J}^{(f)} \frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \frac{\partial\delta\bar{\chi}_i^{(s)}}{\partial X_J}\left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)}\right) dV_0 \\
\frac{d}{dt}\partial_{\dot{\hat{J}}^{(s)}} L_0^{(M)}\left[\delta\hat{J}^{(s)}\right] &= -\int_{\Omega_0^{(M)}} \left[\bar{\rho}^{(s)}_{add.0}\left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)}\right) - \frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\frac{\Phi_0^{(s)}}{\bar{J}^{(s)}}\bar{\rho}^{(f)}_{add.0}\left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)}\right)\right]\delta\hat{J}^{(s)}\, dV_0.
\end{aligned}
\tag{B.20}
$$

To facilitate the assembly of all the above computed variations of the individual terms of $\delta U^{(s)}$, $\delta U_0^{(f)}$, $\delta T^{(s)}$ $\delta T_0^{(f)}$ in (2.62), as an intermediate step towards the computation of (2.74) and (2.75), these individual terms are grouped in the order appearing in the first three rows of (2.62). We also report for each group the application of integration by parts and of the divergence theorem, required to transform (2.74) in (2.75). Accordingly, the first row of (2.62) turns out to be equal to

$$
\begin{aligned}
&\tfrac{d}{dt}\partial_{\dot{\bar{\boldsymbol{\chi}}}^{(s)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(s)}\right] - \partial_{\bar{\boldsymbol{\chi}}^{(s)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(s)}\right] = \\
&\quad = \int_{\Omega_0^{(M)}} \left[\check{P}_{iJ} - \bar{J}^{(f)}\phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}\hat{\bar{\Pi}}^{(f)} \tfrac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} + \bar{\rho}^{(s)}_{add.0}\bar{J}^{(s)}\tfrac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)}\right)\right]\tfrac{\partial\delta\bar{\chi}_i^{(s)}}{\partial X_J^{(s)}}\, dV_0 \\
&\quad + \int_{\Omega_0^{(M)}}\left(-\hat{\bar{\Pi}}^{(f)}\bar{J}^{(f)}\tfrac{\partial\phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}}{\partial X_J}\tfrac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} - \bar{b}_{0i}^{(s,ext)} + \bar{\rho}_0^{(s)}\ddot{\bar{\chi}}_i^{(s)}\right)\delta\bar{\chi}_i^{(s)}\, dV_0.
\end{aligned}
\tag{B.21}
$$

Applying integration by parts and the divergence theorem and accounting for the identity

$$
\bar{J}^{(f)}\phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}\hat{\bar{\Pi}}^{(f)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} - \hat{\bar{\Pi}}^{(f)}\bar{J}^{(f)}\frac{\partial\phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}}{\partial X_J}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} = \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}\frac{\partial}{\partial X_J}\left(\hat{\bar{\Pi}}^{(f)}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\right),
\tag{B.22}
$$

one finally computes:

$$\frac{d}{dt}\partial_{\dot{\bar{\boldsymbol{\chi}}}^{(s)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(s)}\right] - \partial_{\bar{\boldsymbol{\chi}}^{(s)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(s)}\right] =$$
$$= \int_{\partial\Omega_0^{(M)}} \left[\check{P}_{iJ} - \bar{J}^{(f)}\phi^{(s)}_{\bar{\chi}^{(s)}}\hat{\hat{\Pi}}^{(f)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} + \bar{\rho}^{(s)}_{add.0}\bar{J}^{(s)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)}\right)\right] N_J\,\delta\bar{\chi}_i^{(s)}\,dV_0 +$$
$$- \int_{\Omega_0^{(M)}} \left[\frac{\partial\check{P}_{iJ}}{\partial X_J^{(s)}} - \phi^{(s)}_{\bar{\chi}^{(s)}}\frac{\partial}{\partial X_J}\left(\hat{\hat{\Pi}}^{(f)}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\right)\right]\delta\bar{\chi}_i^{(s)}\,dV_0 +$$
$$+ \int_{\Omega_0^{(M)}} \left[-\bar{b}_{0i}^{(s,ext)} + \bar{\rho}_0^{(s)}\ddot{\bar{\chi}}_i^{(s)} - \frac{\partial}{\partial X_J^{(s)}}\left(\bar{\rho}^{(s)}_{add.0}\bar{J}^{(s)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)}\right)\right)\right]\delta\bar{\chi}_i^{(s)}\,dV_0. \tag{B.23}$$

In a similar way, collection of terms appearing in the second row of (2.62) yields:

$$\frac{d}{dt}\partial_{\dot{\bar{\boldsymbol{\chi}}}^{(f)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(f)}\right] - \partial_{\bar{\boldsymbol{\chi}}^{(f)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(f)}\right] =$$
$$= \int_{\Omega_0^{(M)}} \left[-\hat{\hat{\Pi}}^{(f)}\phi^{(f)}_{\bar{\chi}^{(f)}}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} + \bar{\rho}^{(f)}_{add.0}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)}\right)\right]\frac{\partial\delta\bar{\chi}_i^{(f)}}{\partial X_J}\,dV_0 +$$
$$+ \int_{\Omega_0^{(M)}} \left[-\hat{\hat{\Pi}}^{(f)}\bar{J}^{(f)}\frac{\partial\phi^{(f)}_{\bar{\chi}^{(f)}}}{\partial X_J}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} - \bar{b}_{0i}^{(f,ext)} + \bar{\rho}_0^{(f)}\ddot{\bar{\chi}}_i^{(f)}\right]\delta\bar{\chi}_i^{(f)}\,dV_0. \tag{B.24}$$

Applying derivation by parts and the divergence theorem one obtains:

$$\frac{d}{dt}\partial_{\dot{\bar{\boldsymbol{\chi}}}^{(f)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(f)}\right] - \partial_{\bar{\boldsymbol{\chi}}^{(f)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(f)}\right] =$$
$$= \int_{\partial\Omega_0^{(M)}} \left[-\hat{\hat{\Pi}}^{(f)}\phi^{(f)}_{\bar{\chi}^{(f)}}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} + \bar{\rho}^{(f)}_{add.0}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)}\right)\right] N_J\,\delta\bar{\chi}_i^{(f)}\,dV_0$$
$$+ \int_{\Omega_0^{(M)}} \left[\phi^{(f)}_{\bar{\chi}^{(f)}}\frac{\partial}{\partial X_J}\left(\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\hat{\hat{\Pi}}^{(f)}\right) - \bar{\rho}^{(f)}_{add.0}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)}\right)\right]\delta\bar{\chi}_i^{(f)}\,dV_0$$
$$+ \int_{\Omega_0^{(M)}} \left[-\bar{b}_{0i}^{(f,ext)} + \bar{\rho}_0^{(f)}\ddot{\bar{\chi}}_i^{(f)}\right]\delta\bar{\chi}_i^{(f)}\,dV_0. \tag{B.25}$$

Finally, the third row of (2.62) turns out to be:

$$\frac{d}{dt}\partial_{\dot{\hat{J}}^{(s)}} L_0^{(M)}\left[\delta\hat{J}^{(s)}\right] - \partial_{\hat{J}^{(s)}} L_0^{(M)}\left[\delta\hat{J}^{(s)}\right] =$$
$$= \int_{\Omega_0^{(M)}} \left[-\hat{\Pi}^{(s)} + \frac{\bar{J}^{(f)}}{\bar{J}^{(s)}}\Phi_0^{(s)}\hat{\hat{\Pi}}^{(f)}\delta\hat{J}^{(s)} - \bar{\rho}^{(s)}_{add.0}\left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)}\right) + \frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\frac{\Phi_0^{(s)}}{\bar{J}^{(s)}}\bar{\rho}^{(f)}_{add.0}\left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)}\right)\right]$$
$$\delta\hat{J}^{(s)}\,dV_0 = 0. \tag{B.26}$$

## Reference

1. Moiseiwitsch, B.L.: Variational principles. Courier Corporation (2013)