Major Appendix: On Some Mathematical Additional Materials

Mathematical solutions for individual boundary value problems are given in the previous chapters. Due to the complexity of boundary conditions for realistic problems of physics, we must develop some special techniques, in which the complex function method collaborating conformal mapping and Fourier analysis collaborating dual integral equations are powerful procedures. The use of the procedures has been exhibited in the previous chapters. Though some appendixes in relevant chapters were given in the text for discussing some special problems, we list some necessary additional materials on complex analysis and dual integral equations in separate two parts in Appendices A and B (including Appendix of Chap. 11). It is interesting, in particular, between these two parts, there are some close inherent connections. Probably these materials are not necessary for readers majoring applied mathematics, but which may be helpful for young scholars and postgraduate students who are not majoring applied mathematics.

The discussion of hydrodynamics concerns Poisson bracket method in condensed matter physics, and the Appendix C gives some relevant materials and detailed derivation of equations of motion of hydrodynamics of solid quasicrystals, which are beneficial to learn hydrodynamics of soft-matter quasicrystals as well.

Appendix A: Additional Calculations Related to Complex Analysis

A.1 Additional Derivation of Solution (8.2.19)

In Chap. 8, we emphasized the importance of complex analysis (see [1–3]) and pointed out that the Muskhelishvili's [1] method limited the conformal mapping to be rational functions. However our practice breaks the limitation, e.g. [4, 5]; the calculation is handled as below.

Formula (8.2.19) is obtained from the integral of (8.2.6) in which the conformal mapping $\omega(\zeta)$ is given by (8.2.17). The difficulty of the calculation lies in the

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integral path which is a part of circle rather than whole circle. Substituting (8.2.17) into the right-hand side of (8.2.6) yields

$$F(\zeta) = -\frac{p}{2\pi i}\int_{-1}^{1} \frac{\omega(\sigma)}{(\sigma-\zeta)^2}\mathrm{d}\sigma = -\frac{p}{2\pi i}\int_{-1}^{1}\frac{\omega'(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma$$

where

$$\omega'(\sigma) = -\frac{4H\alpha(1-\beta)}{\pi}\frac{1-\sigma}{1+\sigma}\frac{(1+\sigma)^2}{\left[(1+\sigma)^2+\alpha(1-\sigma^2)^2\right]\left[(1+\sigma)^2+\beta\alpha(1-\sigma)^2\right]}$$

Substituting $\mathrm{e}^{\mathrm{i}\varphi}$ by $(1+ix)/(1-ix)$ yields

$$\frac{1-\sigma}{1+\sigma} = -2x, \quad \mathrm{d}\sigma = \frac{2i}{(1-ix)^2}\mathrm{d}x, \quad (1+\sigma)^2 = \frac{4}{(1-ix)^2},$$
$$\frac{1}{\sigma-\zeta} = \frac{1-ix}{1-\zeta+ix(1+\zeta)} = \frac{(1-ix)[1-\zeta-ix(1+\zeta)]}{(1-\zeta)^2+x^2(1+\zeta)^2} - \frac{1-\zeta-x^2(1+\zeta)-2ix}{(1-\zeta)^2+x^2(1+\zeta)^2}$$

So that

$$\begin{aligned}
F(\zeta) &= -\frac{pH\alpha(1-\beta)}{\pi^2}\int_{-1}^{1}\frac{ix[1-\zeta-x^2(1+\zeta)-2ix]}{(1-\alpha x^2)(1-\beta\alpha x^2)\left[(1-\zeta)^2+x^2(1+\zeta)^2\right]}\mathrm{d}x \\
&= -\frac{4pH\alpha(1-\beta)}{\pi^2}\int_{0}^{1}\frac{x^2}{(1-\alpha x^2)(1-\beta\alpha x^2)\left[(1-\zeta)^2+x^2(1+\zeta)^2\right]}\mathrm{d}x \\
&= \frac{2pH}{\pi^2}\frac{\alpha(1-\beta)(1-\zeta^2)}{\left[\alpha(1-\zeta)^2+(1+\zeta)^2\right]\left[\beta\alpha(1-\zeta)^2+(1+\zeta)^2\right]} \\
&\quad \times\left[\arctan\left(\frac{1+\zeta}{1-\zeta}\right)-\arctan\left(\frac{1+\zeta}{-1-\zeta}\right)\right]-\frac{4pH}{\pi^2}\frac{\sqrt{\alpha}\arctan h\sqrt{\alpha}}{\alpha(1-\zeta)^2+(1+\zeta)^2} \\
&\quad -\frac{\sqrt{\beta\alpha}\arctan h\sqrt{\alpha}}{\beta\alpha(1-\zeta)^2+(1+\zeta)^2}]
\end{aligned} \tag{A.1.1}$$

In the last step, the evaluation is used the ***Mathematica3.0*** [6].
By considering

$$A = \ln \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} = 2 \arctan h\sqrt{\alpha},$$
$$M = \ln \frac{1+\sqrt{\gamma\alpha}}{1-\sqrt{\gamma\alpha}} = 2 \arctan h\sqrt{\gamma\alpha},$$
$$\arctan\left(\frac{1+\zeta}{-1+\zeta}\right) = -2 \arctan\left(\frac{1+\zeta}{1-\zeta}\right) = \frac{i}{2} \ln\left(\frac{i-\zeta}{1-i\zeta}\right)$$

then (A.1.1) is just the formula (8.2.19).

In the calculation, if let $L \to 0$, then the integral (8.2.7) can be obtained.

A.2 Additional Derivation of Solution (11.3.53)

In example 3 of Section 11.3, the calculation is quite lengthy, and here, we provide some details on the evaluation. As an example, we can show the derivation on function $\Phi_4(\zeta)$, which is

$$\Phi_4(\zeta) = d_1(X+iY) \ln \zeta + B\omega(\zeta) + \Phi_4^*(\zeta)$$

in which the first two terms are known (see the text), and the single-valued analytic function $\Phi_4^*(\zeta)$ satisfies the following boundary condition

$$\Phi_4^*(\sigma) + \overline{\Phi_3^*(\sigma)} + \frac{\omega(\sigma)}{\omega'(\sigma)} \cdot \overline{\Phi_4^{*\prime}(\sigma)} = f_0$$

where

$$f_0 = \frac{i}{32c_1} \int (T_x + iT_y) \mathrm{d}s - (d_1 - d_2)(X+iY) \ln \sigma - \frac{\omega(\sigma)}{\omega'(\sigma)} \cdot d_1(X - iY) \cdot \sigma - 2B\omega(\sigma) - (B' - iC')\overline{\omega(\sigma)} \tag{A.1.2}$$

and (referring to Fig. 11.4 of Chap. 11)

$T_x = -p\cos(n,x), T_y = -p\cos(n,y)$ at $\widehat{z_1 M z_2}$

$$(T_x + iT_y)\mathrm{d}s = \begin{cases} ip\mathrm{d}z & \widehat{z_1 M z_2} \\ 0 & \widehat{z_2 N z_1} \end{cases} \tag{A.1.3}$$

$$X + iY = \int (T_x + iT_y)\mathrm{d}s = ip(z_1 - z_2)$$

Multiplying both sides of the above boundary equation by $\frac{1}{2\pi i}\frac{\mathrm{d}\sigma}{\sigma-\zeta}$ and integrating along the unit circle, we have

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_4^*(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma+\frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\Phi_4^{*\prime}(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma+\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Phi_3^*(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma=\frac{1}{2\pi i}\int_{\gamma}\frac{f_0}{\sigma-\zeta}\mathrm{d}\sigma \quad \text{(a)}$$

in which according to the Cauchy's integral formula (referring to formula (11.7.5) in Appendix of Chap. 11), there is

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_4^*(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma=\Phi_4^*(\zeta)$$

and in terms of analytic extension principle and the Cauchy theorem (referring to Appendix of Chap. 11)

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\Phi_4^{*\prime}(\sigma)}}{\sigma-\zeta}=0$$

and according to formula (11.7.9)

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Phi_3^*(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma=\text{const}$$

So that (a) reduces to

$$\Phi_4^*(\zeta)=\frac{1}{2\pi i}\int_{\gamma}\frac{f_0}{\sigma-\zeta}\mathrm{d}\sigma+\text{const}$$

And substituting (A.1.1) into the right-hand side yields

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f_0}{\sigma-\zeta}\mathrm{d}\sigma=\frac{pR_0}{2\pi i c_1}\int_{\sigma_1}^{\sigma_2}\left(\sigma+\frac{m}{\sigma}\right)\frac{\mathrm{d}\sigma}{\sigma-\zeta}+\frac{pz_2}{2\pi i}\int_{\sigma_1}^{\sigma_2}\frac{\mathrm{d}\sigma}{\sigma-\zeta}+\frac{p(z_1-z_2)}{2\pi i}\frac{1}{2\pi i}\int_{\gamma}\frac{\ln\sigma}{\sigma-\zeta}\mathrm{d}\sigma$$

$$-\frac{p(\overline{z_1}-\overline{z_2})}{2\pi i}\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma^2+m}{1-m\sigma^2}\frac{\mathrm{d}\sigma}{\sigma-\zeta}$$

in which

$$\int_{\sigma_1}^{\sigma_2}\left(\sigma+\frac{m}{\sigma}\right)\frac{\mathrm{d}\sigma}{\sigma-\zeta}=\sigma_2-\sigma_1-\frac{m}{\zeta}\ln\frac{\sigma_2}{\sigma_1}+\left(\zeta+\frac{m}{\zeta}\right)\ln\frac{\sigma_2-\zeta}{\sigma_1-\zeta}$$

$$\int_{\sigma_1}^{\sigma_2}\frac{\mathrm{d}\sigma}{\sigma-\zeta}=\ln\frac{\sigma_1-\zeta}{\sigma_2-\zeta}$$

In accordance with the Cauchy's theorem (referring to formula (11.7.4)), we have

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\sigma^2+m}{1-m\sigma^2}\frac{d\sigma}{\sigma-\zeta}=0$$

because the integrand is single-valued analytic function in the region outside the unit circle γ.

The remaining term is

$$I(\zeta)=\frac{1}{2\pi i}\int_{\gamma}\frac{\ln\sigma}{\sigma-\zeta}\mathrm{d}\sigma$$

For calculating it, we consider

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}\zeta}&=\frac{1}{2\pi i}\int_{\gamma}\frac{\ln\sigma}{(\sigma-\zeta)^2}\mathrm{d}\sigma=-\frac{1}{2\pi i}\int_{\gamma}\ln\sigma\mathrm{d}\frac{1}{\sigma-\zeta}\\&=-\frac{1}{2\pi i}\left[\frac{\ln\sigma}{\sigma-\zeta}\right]_{\sigma=\exp(i\varphi_1)}^{\sigma=\exp i(\varphi_1+2\pi)}+\frac{1}{2\pi i}\int_{\gamma}\frac{\mathrm{d}\sigma}{\sigma(\sigma-\zeta)}\\&=-\frac{1}{2\pi i}\frac{1}{\sigma_1-\zeta}\ln\frac{\exp i(\varphi_1+2\pi)}{\exp(i\varphi_1)}-\frac{1}{\zeta}=-\frac{1}{\sigma_1-\zeta}-\frac{1}{\zeta}\end{aligned}$$

So that

$$I(\zeta)=\ln(\sigma_1-\zeta)-\ln\zeta+\mathrm{const}$$

Hence function $\Phi_4^*(\zeta)$ is determined so the function $\Phi_4(\zeta)$in which the constant term is omitted:

$$\begin{aligned}\Phi_4(\zeta)=&\frac{1}{32c_1}\cdot\frac{p}{2\pi i}\cdot\left[-\frac{mR_0}{\zeta}\ln\frac{\sigma_2}{\sigma_1}+z\ln\frac{\sigma_2-\zeta}{\sigma_1-\zeta}+z_1\ln(\sigma_1-\zeta)-z_2\ln(\sigma_2-\zeta)\right]\\&+ip(d_1-d_2)(z_1-z_2)\ln\zeta\end{aligned}$$

which is just the first formula of (11.3.53), where d_1 and d_2 were given by (11.3.34) in the text. The others can be similarly derived. In the derivation, the classical work of Muskhelishvili [1] is referred.

A.3 Detail of Complex Analysis of Solution (14.4.7) of Generalized Cohesive Force Model for Plane Plasticity of Two-Dimensional Point Groups 5m, 10mm and $10, \overline{10}$ Quasicrystals

The elasticity solution (11.3.53) based on complex analysis can be used to solve the present problem.

The generalized Dugdale-Barenblatt model or generalized cohesive model for decagonal quasicrystals makes the plastic problem to be linearized, so the final governing equation is reduced to solve the equation

$$\nabla^2\nabla^2\nabla^2\nabla^2 G = 0 \tag{A.1.4}$$

under boundary conditions

$$\begin{cases} \sigma_{yy} = p, \sigma_{xx} = \sigma_{xy} = 0, H_{xx} = H_{yy} = H_{xy} = H_{yx} = 0 & \sqrt{x^2+y^2} \to \infty \\ \sigma_{yy} = \sigma_{xy} = 0, H_{yy} = H_{yx} = 0 & y = 0, |x| < a \\ \sigma_{yy} = \sigma_c, \sigma_{xy} = 0, H_{yy} = H_{yx} = 0 & y = 0, a < |x| < a+d \end{cases} \tag{A.1.5}$$

which can decomposed into two cases, among them one is

$$\begin{cases} \sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0, H_{xx} = H_{yy} = H_{xy} = H_{yx} = 0 & \sqrt{x^2+y^2} \to \infty \\ \sigma_{yy} = \sigma_{xy} = 0, H_{yy} = H_{yx} = 0 & y = 0, |x| < a \\ \sigma_{yy} = \sigma_c, \sigma_{xy} = 0, H_{yy} = H_{yx} = 0 & a < |x| < a+d \end{cases} \tag{A.1.6}$$

and another

$$\begin{cases} \sigma_{yy} = p, \sigma_{xx} = \sigma_{xy} = 0, H_{xx} = H_{yy} = H_{xy} = H_{yx} = 0 & \sqrt{x^2+y^2} \to \infty \\ \sigma_{yy} = \sigma_{xy} = 0, H_{yy} = H_{yx} = 0 & y = 0, |x| < a+d \end{cases} \tag{A.1.7}$$

The solution of problems (A.1.4) and (A.1.7) can be obtained from (11.3.53); i.e. if we put $m = 1, R_0 = (a+d)/2$, then the elliptic hole reduced to a Griffith crack with half-length (a + d). In Fig. 11.3.3 let $z_1 = (a+d, +0), z_2 = (a, +0)$; from

(11.3.53), we can obtain a solution, similarly put $z_1 = (a+d, -0), z_2 = (a, -0)$, $z_1 = (-a-d, +0), z_2 = (-a, +0)$, and $z_1 = (-a-d, -0), z_2 = (-a, -0)$, respectively; and from (11.3.53), one can find other corresponding three solutions, by superposing which one can obtain solution

$$\begin{cases} \Phi_4^{(1)}(\zeta) = \frac{1}{32c_1} \cdot \frac{\sigma_c(a+d)\varphi_2}{\pi} \cdot \frac{1}{\zeta} - \frac{1}{32c_1} \cdot \frac{\sigma_c}{2\pi i}\left[z\left(\ln\frac{\sigma_2-\zeta}{\overline{\sigma_2}-\zeta} + \ln\frac{\sigma_2+\zeta}{\overline{\sigma_2}+\zeta}\right) - l\ln\frac{(\zeta-\sigma_2)(\zeta+\overline{\sigma_2})}{(\zeta+\sigma_2)(\zeta-\overline{\sigma_2})}\right] \\ \Phi_3^{(1)}(\zeta) = \frac{1}{32c_1} \cdot \frac{\sigma_c(a+d)\varphi_2}{\pi} \cdot \frac{2\zeta}{\zeta^2-1} - \frac{1}{32c_1} \cdot \frac{\sigma_c a}{2\pi i}\ln\frac{(\zeta-\sigma_2)(\zeta+\overline{\sigma_2})}{(\zeta+\sigma_2)(\zeta-\overline{\sigma_2})} \end{cases} \tag{A.1.8}$$

where $\sigma = e^{i\varphi}$ represents the value of ζ at the unit circle in the mapping plane, and $\sigma_2 = e^{i\varphi_2}$, $a = (a+d)\cos\varphi_2$.

And the solution of problems (A.1.4) and (A.1.7), as the solution of Griffith crack problem, is known, i.e.

$$\begin{cases} \Phi_4^{(2)}(\zeta) = -\frac{1}{32c_1}\frac{p}{2}(a+d)\frac{1}{\zeta} \\ \Phi_3^{(2)}(\zeta) = -\frac{p}{32c_1}(a+d)\left[\frac{\zeta}{(\zeta^2-1)}\right] \end{cases} \tag{A.1.9}$$

The superposition of (A.1.8) and (A.1.9) gives the total solution for $\Phi_4(\zeta) = \Phi_4^{(1)}(\zeta) + \Phi_4^{(2)}(\zeta), \Phi_3(\zeta) = \Phi_3^{(1)}(\zeta) + \Phi_3^{(2)}(\zeta)$; for example, the first term of $\Phi_4(\zeta)$ is

$$-\frac{1}{32c_1}\frac{p}{2}(a+d)\frac{1}{\zeta} + \frac{1}{32c_1} \cdot \frac{\sigma_c(a+d)\varphi_2}{\pi} \cdot \frac{1}{\zeta} \tag{A.1.10}$$

and $\Phi_2(\zeta)$ has not been listed here because it is too lengthy, so the stresses and displacements are determined already. In addition, we know that

$$\sigma_{ij}, H_{ij} \sim \Phi'(\zeta)/\omega'(\zeta) \tag{A.1.11}$$

here $\Phi(\zeta)$ means $\Phi_4(\zeta)$ or $\Phi_3(\zeta)$, and $\omega'(\zeta) \sim 1/(1-\zeta^2)$.

From Sect. 14.4 in the text, we know that there is no stress singularity at the generalized Dugdale-Barrenblatt crack tip, and this fact and conjunct with (A.1.10) and (A.1.11) require that the value of formula (A.1.10) must be zero, which leads to (14.4.6) in the text. Considering this the final version of $\Phi_4(\zeta)$ is

$$\Phi_4(\zeta) = -\frac{1}{32c_1} \cdot \frac{\sigma_c}{2\pi i}\left[z\left(\ln\frac{\sigma_2-\zeta}{\overline{\sigma_2}-\zeta} + \ln\frac{\sigma_2+\zeta}{\overline{\sigma_2}+\zeta}\right) - a\ln\frac{(\zeta-\sigma_2)(\zeta+\overline{\sigma_2})}{(\zeta+\sigma_2)(\zeta-\overline{\sigma_2})}\right] \tag{A.1.12}$$

And the displacement at the crack surface presents the form

$$u_y(x,0) = (128c_1c_2 - 64c_3)\mathrm{Im}(\Phi_4(\zeta))_{\zeta=\sigma} \tag{A.1.13}$$

After some calculation, we find that

$$u_y(x,0) = \frac{(4c_1c_2 - 2c_3)}{c_1} \cdot \frac{\sigma_c(a+d)}{2\pi} \cdot \left[\cos\varphi \ln\frac{\sin(\varphi_2 - \varphi)}{\sin(\varphi_2 + \varphi)} - \cos\varphi_2 \ln\frac{(\sin\varphi_2 - \sin\varphi)}{(\sin\varphi_2 + \sin\varphi)}\right] \tag{A.1.14}$$

so the crack tip opening displacement is

$$\delta_t = \mathrm{CTOD} = \lim_{x\to l} 2u_y(x,0) = \lim_{\varphi\to\varphi_2} 2u_y(x,0) = \frac{(8c_1c_2 - 4c_3)\sigma_s a}{c_1\pi} \ln\sec\left(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}\right) \tag{A.1.15}$$

in which the constants c_1, c_2, c_3 are defined in Sect. 11.3, so the solution holds for point groups 5 m and 10 mm as well as point groups $5, \bar{5}, 10, \overline{10}$ quasicrystals. When we assume $R_1 = R$, $R_2 = 0$ in Eq. (A.1.15), δ_t will be the corresponding solution of point groups 5 m and 10 mm quasicrystals, i.e.

$$\delta_t = \mathrm{CTOD} = \frac{2\sigma_s a}{\pi}\left[\frac{1}{L+M} + \frac{K_1}{MK_1 - R^2}\right] \ln\sec\left(\frac{\pi}{2}\frac{\sigma^{(\infty)}}{\sigma_s}\right),$$

which is just the (14.4.7). If let $K_1 = R = 0, L = \lambda, M = \mu$ in above formula, then it exactly reduces to the classical Dugdale-Barenblatt solution holding for engineering material (or structural material) including crystalline material (referring to Sect. 14.4).

The more details can be found in article given by Fan and Fan [7].

A.4 On the Calculation of Integral (9.2.14)

In formula (9.2.14), $y > 0$ results in the integrals being convergent. We let ξ to extend to complex number $\xi = \xi_1 + i\xi_2$, take integration path at complex $\xi-$plane similar to Fig. 11.7; by physical consideration $k(K_1 - K_2) > 0$, $\mu(K_1 - K_2) - R^2 > 0$, and $k = \mu^{(c)}/h$; then we can find the integrand of (9.2.14) in the interior of the region enclosing by the integration path is analytic except poles

$$\xi_1^{(1)} = \frac{k(K_1 - K_2)}{R^2 - \mu(K_1 - K_2)} < 0, \quad \xi_1^{(2)} = -\frac{k(K_1 - K_2)}{R^2 - \mu(K_1 - K_2)} > 0 \tag{A.1.16}$$

at real axis ξ_1; and by a generalized Jordan lemma, the integral along the big half-circle is zero.

If in Fig. 11.7 of Chap. 11 put $\omega = \xi, \omega_1 = \xi_1, \omega_2 = \xi_2, \sqrt{k/m} = \xi_1^{(1)}, -\sqrt{k/m} = \xi_1^{(2)}$according to the additional integration path at complex ξ-plane, through the similar manner for evaluating integral (11.7.18), then we can obtain results (9.2.15) and (9.2.16), respectively where$\xi_1^{(1)}, \xi_1^{(2)}$ are defined by (A.1.16).

A.5 On the Calculation of Integral (8.8.9)

$$\begin{aligned}\phi'(\zeta) &= \frac{p}{2\pi i}\int_{-1}^{1}\frac{2w}{\pi}\cdot\left(\frac{-\sigma\tan\left(\frac{\pi a}{2w}\right)}{\left[1+(1-\sigma^2)\tan^2\left(\frac{\pi a}{2w}\right)\right]\sqrt{1-\sigma^2}}\right)\cdot\frac{1}{\sigma-\zeta}\mathrm{d}\sigma \\ &= -\frac{pw}{\pi^2 i}\tan\left(\frac{\pi a}{2w}\right)\int_{-1}^{1}\left(\frac{\sigma}{\left[1+(1-\sigma^2)\tan^2\left(\frac{\pi a}{2w}\right)\right]\sqrt{1-\sigma^2}}\right)\cdot\frac{1}{\sigma-\zeta}\mathrm{d}\sigma\end{aligned} \tag{A.1.17}$$

Put $m = \tan\left(\frac{\pi a}{2w}\right)$, so $m^2 = \tan^2\left(\frac{\pi a}{2w}\right)$
We calculate

$$\begin{aligned}I(\zeta) &= \int_{-1}^{1}\left(\frac{\sigma}{\left[1+(1-\sigma^2)\tan^2\left(\frac{\pi a}{2w}\right)\right]\sqrt{1-\sigma^2}}\right)\cdot\frac{1}{\sigma-\zeta}\mathrm{d}\sigma \\ &= \int_{-1}^{1}\left(\frac{1}{[1+(1-\sigma^2)m^2]\sqrt{1-\sigma^2}}\right)\mathrm{d}\sigma + \int_{-1}^{1}\left(\frac{\zeta}{[1+(1-\sigma^2)m^2]\sqrt{1-\sigma^2}(\sigma-\zeta)}\right)\mathrm{d}\sigma \\ &= I_1 + I_2\end{aligned} \tag{A.1.18}$$

The calculation of the first integral is easy, and we now calculate the second one. Put $\zeta = b, \sigma = x$, such that

$$I_2 = \int_{-1}^{1}\left(\frac{b}{[1+(1-x^2)m^2]\sqrt{1-x^2}(x-b)}\right)\mathrm{d}\sigma$$

Denote $x = \sin t, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \mathrm{d}x = \cos t\mathrm{d}t$ where $\begin{cases} t = -\dfrac{\pi}{2}\ as\, x = -1 \\ t = \dfrac{\pi}{2}\ as\, x = 1 \end{cases}$

$$
\begin{aligned}
I_2 &= \int_{-\pi/2}^{\pi/2} \frac{b}{[1+m^2\cos t]\cdot(\sin t - b)\cdot\cos t}\cdot\cos t\mathrm{d}\sigma \\
&= \int_{-\pi/2}^{\pi/2} \frac{b}{[1+m^2\cos^2 t](\sin t - b)}\mathrm{d}t \\
&= \int_{0}^{\pi/2} \frac{b}{[1+m^2\cos^2 t](\sin t - b)}\mathrm{d}t + \int_{-\pi/2}^{0} \frac{b}{[1+m^2\cos^2 t](\sin t - b)}\mathrm{d}t
\end{aligned}
\tag{A.1.19}
$$

For the second integral of A.1.19, put $t=-x, \mathrm{d}t=-\mathrm{d}x, t=-\frac{\pi}{2}$ as $x=\frac{\pi}{2}; t=0$ as $x=0$

$$
\int_{-\pi/2}^{0} \frac{b}{[1+m^2\cos^2 t](\sin t - b)}\mathrm{d}t = -\int_{0}^{\pi/2} \frac{b}{[1+m^2\cos^2 x](\sin x + b)}\mathrm{d}x.
$$

So that

$$
\begin{aligned}
\int_{-\pi/2}^{\pi/2} \frac{b}{[1+m^2\cos^2 t](\sin t - b)}\mathrm{d}t &= \int_{0}^{\pi/2} \frac{b}{[1+m^2\cos^2 t](\sin t - b)}\mathrm{d}t \\
&\quad - \int_{0}^{\pi/2} \frac{b}{[1+m^2\cos^2 t](\sin t + b)}\mathrm{d}t \\
&= \int_{0}^{\pi/2} \frac{2b^2}{[1+m^2\cos^2 t](\sin^2 t - b^2)}\mathrm{d}t \\
&= A\cdot\int_{0}^{\pi/2} \frac{1}{[1+m^2\cos^2 t]}\mathrm{d}t + B\cdot\int_{0}^{\pi/2} \frac{1}{(\sin^2 t - b^2)}\mathrm{d}t
\end{aligned}
\tag{A.1.20}
$$

The constants A and B can be determined by

$$
A = \frac{2m^2b^2}{[m^2(1-b^2)+1]}
$$

$$
B = \frac{2b^2}{[m^2(1-b^2)+1]}
$$

Therefore

$$A \cdot \int_0^{\pi/2} \frac{1}{[1+m^2\cos^2 t]}\mathrm{d}t = A \cdot \frac{\pi}{2} \cdot \cos\left(\frac{\pi a}{2w}\right)$$

$$B \cdot \int_0^{\pi/2} \frac{1}{(\sin^2 t - b^2)}\mathrm{d}t = B \cdot \int_0^{\pi/2} \frac{1/2b}{(\sin t - b)}\mathrm{d}t - B \cdot \int_0^{\pi/2} \frac{1/2b}{(\sin t + b)}\mathrm{d}t \quad .$$

$$= B \cdot 1/2b\left[\int_0^{\pi/2} \frac{1}{(\sin t - b)}\mathrm{d}t - \int_0^{\pi/2} \frac{1}{(\sin t + b)}\mathrm{d}t\right]$$

From the integral table, one can find

$$B \cdot \int_0^{\pi/2} \frac{1}{(\sin^2 t - b^2)}\mathrm{d}t = \frac{B}{2b\sqrt{1-b^2}}\left\{\ln\left|\frac{-b+1-\sqrt{1-b^2}}{-b+1+\sqrt{1-b^2}}\right| - \ln\left|\frac{1-\sqrt{1-b^2}}{1+\sqrt{1-b^2}}\right|\right.$$

$$\left. - \ln\left|\frac{b+1-\sqrt{1-b^2}}{b+1+\sqrt{1-b^2}}\right| + \ln\left|\frac{1-\sqrt{1-b^2}}{1+\sqrt{1-b^2}}\right|\right.$$

$$= \frac{B}{2b\sqrt{1-b^2}}\left\{\ln\left|\frac{-b+1-\sqrt{1-b^2}}{-b+1+\sqrt{1-b^2}}\right|\left|\frac{b+1+\sqrt{1-b^2}}{b+1-\sqrt{1-b^2}}\right|\right\}$$

$$= \frac{B}{2b\sqrt{1-b^2}}\left\{\ln\left|\frac{\left(-2b\sqrt{1-b^2}\right)^2}{4b^2(b^2-1)}\right|\right\} = \frac{B}{2b\sqrt{1-b^2}}\ln\left|\frac{1}{b}\right|$$

Therefore

$$\begin{aligned}\phi'(\zeta) = &-\frac{pw}{\pi i}\sin\left(\frac{\pi a}{2w}\right) - \frac{pw}{\pi i}\sin\left(\frac{\pi a}{2w}\right)\frac{\zeta^2\tan^2\left(\frac{\pi a}{2w}\right)}{\left[1+(1-\zeta^{2s})\tan^2\left(\frac{\pi a}{2w}\right)\right]} \\ &+ \frac{pw}{\pi^2 i}\tan\left(\frac{\pi a}{2w}\right)\frac{\ln|\zeta|}{\left[1+(1-\zeta^2)\tan^2\left(\frac{\pi a}{2w}\right)\right]} \cdot \frac{\zeta}{\sqrt{1-\zeta^2}}\mathrm{d}\sigma\end{aligned} \tag{A.1.21}$$

It is easily found that $\phi'(0) = -\frac{pw}{\pi i}\sin\left(\frac{\pi a}{2w}\right)$.

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Appendix B: Dual Integral Equations and Some Additional Calculations

B.1 Dual Integral Equations

It is well known that the Fourier transform or Hankel transform is very useful tool in solving partial differential equations which have been shown in Chaps. 7–9 though the introduction is very limited. For non-harmonic and non-multi-harmonic equations, the complex potential method is not effective, and we have to use the Fourier transform, Hankel transform, Mellin transform, or others. After the transform, the boundary value problems of the dislocations are reduced to some algebraic equations to solve (this is relatively simpler), while those of the cracks are concluded for solving the following dual integral equations

$$\left.\begin{aligned} &\int_0^\infty y^\alpha f(y) J_v(xy)\mathrm{d}y = g(x), \quad 0<x<1 \\ &\int_0^\infty f(y) J_v(xy)\mathrm{d}y = 0, \qquad x>1 \end{aligned}\right\} \tag{B.1.1}$$

or

$$\left.\begin{aligned} &\int_0^\infty y^{\alpha_j} \sum_{k=1}^n a'_{jk} f_j(y) J_{v_j}(xy)\mathrm{d}y = g_j(x), \quad 0<x<1 \\ &\int_0^\infty \sum_{k=1}^n a_{jk} f_j(y) J_{v_j}(xy)\mathrm{d}y = 0, \qquad x>1 \\ &(j=1,2,\ldots,n) \end{aligned}\right\} \tag{B.1.2}$$

or

$$\left.\begin{aligned}\int_0^\infty\int_0^\infty g_1(\xi_1,\xi_2,s,x_1,x_2)f(\xi_1,\xi_2,s)J_\alpha(\xi_1x_1)J_\beta(\xi_2x_2)d\xi_1d\xi_2=h(x_1,x_2,s), \quad (x_1,x_2)\in\Omega_1\\ \int_0^\infty\int_0^\infty g_2(\xi_1,\xi_2,s,x_1,x_2)f(\xi_1,\xi_2,s)J_\alpha(\xi_1x_1)J_\beta(\xi_2x_2)d\xi_1d\xi_2=0, \quad (x_1,x_2)\in\Omega_2\end{aligned}\right\} \tag{B.1.3}$$

Among them, Eqs. (B.1.1) are the simplest ones, which will be discussed in the following only. Eqs. (B.1.2) deal with multi-unknown functions and (B.1.3) are the two-dimensional dual integral equations, and these two kinds of dual integral equations are more complicated.

In Eqs. (B.1.1), $f(x)$ is a unknown function to be determined, $g(x)$ is known one, αv are constants, and $J_v(xy)$ is the first kind Bessel function of v order. Titchmarsh [1] and Busbridge [2] gave the analytic solution of the equations. Various authors [3–11] discussed the solutions with different methods. Here only the procedure of Refs. [1, 2] is introduced. Titchmarsh [1] gave formal solution for the case $\alpha > 0$. Busbridge [2] extended the discussion to the case $\alpha > -2$ and gave proof for the existence of the solution. The solution is given through a complex integral as follows:

$$f(x)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}2^{s-\alpha}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}\alpha-\frac{1}{2}s\right)}\psi(s)x^{-s}\mathrm{d}s \tag{B.1.4}$$

in which $s=\sigma+i\tau$ and

$$\psi(s)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v-\frac{1}{2}\alpha+\frac{1}{2}w\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}w\right)}\cdot\frac{\overline{g}(\alpha+1-w)}{w-s}\mathrm{d}w \tag{B.1.5}$$

where $w=u+i\upsilon$ (in which υ represents the imaginary part of complex variable w, do not confuse with v—the suffix of the Bessel function, which represents the order of the Bessel function) $\sigma<u$ and

$$\overline{g}(\alpha+1-w)=\int_0^1 g(x)x^{\alpha-w}\mathrm{d}x$$

in above formulas $\Gamma(x)$ represent Euler gamma function. The solution of (B.1.4) holds for both $\alpha>0$ and $\alpha>-2$.

For $\alpha>0$, the solution can be expressed by real integral as

$$f(x) = \frac{(2x)^{1-\alpha/2}}{\Gamma(\alpha/2)} \int_0^1 \mu^{1+\alpha/2} J_{v+\alpha/2}(\mu x)\mathrm{d}\mu \int_0^1 g(\rho\mu)\rho^{v+1}(1-\rho^2)^{\alpha/2-1}\mathrm{d}\rho \quad (B.1.4')$$

and for $\alpha > -2$, which is in form

$$f(x) = \frac{2^{-\alpha/2}x^{-\alpha}}{\Gamma(1+\alpha/2)} \left[x^{1+\alpha/2} J_{v+\alpha/2}(x) \int_0^1 y^{v+1}(1-y^2)^{\alpha/2} g(y)\mathrm{d}y \right. \\ \left. + \int_0^1 y^{\alpha+1}(1-y^2)^{\alpha/2} dy \int_0^1 (xu)^{2+\alpha/2} g(yu) J_{v+1+\alpha/2}(xu)\mathrm{d}u \right] \quad (B.1.4'')$$

Theorem *If* $\alpha > -2,\ -v-1 < \alpha - \frac{1}{2} < v+1$, *the Mellin transforms of* $g(x)$ *and* $f(x)$ *exist, the latter is analytic in the strip region* $-v < \mathrm{Re}s = \sigma < \alpha$ *and has the order* $O(|t|^{\sigma-\alpha+\varepsilon})$ $(\varepsilon > 0, t \to \infty)$, *where* $s = \sigma + i\tau$*the Mellin transform parameter, then Eq.* (B.1.1) *have one and only one solution* (B.1.4).

Proof Because the strict proof given by Busbridge [2] is very lengthy, we cannot quote its all details here; instead, only a rough outline of the proof is figured out in the following. One can find that in the proof, a quite lot of complex variable function knowledge is used, and this seems that the theory on dual integral equations presents the inherent connection with complex analysis. So the appendix of Chap. 11 is helpful for the present discussion too.

At first, assume that $0 < \alpha < 2, -v-1 < \alpha - \frac{1}{2} < v+1$, and the Mellin transform of $f(x)$

$$\bar{f}(s) = \int_0^\infty f(x)x^{s-1}\mathrm{d}x \quad s = \sigma + i\tau$$

is analytic in region $-v < \sigma < \alpha$, and assuming as $\varepsilon > 0$ and as $t \to \infty$, it has order $O(|t|^{-\alpha+\varepsilon})$ (in fact this is a lemma, but we omit the proof for simplicity).

According to the definition, the Mellin transform of function $y^\alpha J_v(xy)$ is

$$\bar{J}_\alpha(s) \equiv \int_0^\infty [y^\alpha J_v(xy)]y^{s-1}\mathrm{d}y = \frac{2^{\alpha+s-1}}{x^{\alpha+s}} \frac{\Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}v + \frac{1}{2}s\right)}{\Gamma\left(1 - \frac{1}{2}\alpha + \frac{1}{2}v - \frac{1}{2}s\right)} \quad (B.1.6)$$

Recall that $s = \sigma + i\tau$. By using the notation of relevant Mellin transforms, the left-hand side of the first and second equations in (B.1.1) becomes

$$\int_0^\infty y^\alpha f(y)J_v(xy)\mathrm{d}y = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \bar{f}(s)\bar{J}_\alpha(1-s)\mathrm{d}s$$

$$\int_0^\infty f(y)J_v(xy)\mathrm{d}y = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \bar{f}(s)\bar{J}_0(1-s)\mathrm{d}s$$

and substituting (B.1.6) into the above formulas yields

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{2^{\alpha-s}}{x^{1-s}}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}v-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}v+\frac{1}{2}s\right)}\bar{f}(s)\mathrm{d}s = g(x) \quad 0<x<1$$

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{2^{\alpha-s}\Gamma\left(\frac{1}{2}+\frac{1}{2}v-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}\bar{f}(s)\mathrm{d}s = 0 \quad x>1$$

Put

$$\bar{f}(s) = \frac{2^{\alpha-s}\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}v-\frac{1}{2}s\right)}\psi(s) \tag{B.1.7}$$

Then the above equations reduce to

$$\left.\begin{aligned} &\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v-\frac{1}{2}\alpha+\frac{1}{2}s\right)}\psi(s)x^{s-1-\alpha}\mathrm{d}s = g(x), \quad 0<x<1 \\ &\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}\alpha-\frac{1}{2}s\right)}\psi(s)x^{s-1}\mathrm{d}s = 0, \qquad x>1 \end{aligned}\right\} \tag{B.1.8}$$

Multiply $x^{\alpha-w}$to the first one of (B.1.8), where $w=u+iv$ and $\sigma-u>0$, and then integrate over $(0,1)$ to x and

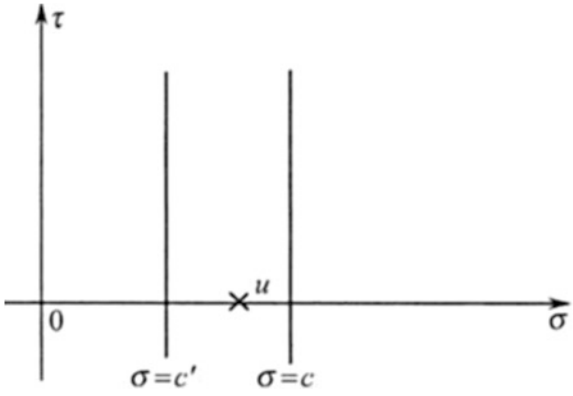
$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v-\frac{1}{2}\alpha+\frac{1}{2}s\right)}\psi(s)\frac{\mathrm{d}s}{s-w} = \bar{g}(\alpha-w+1) \tag{B.1.9}$$

where $u<C$, and

$$\bar{g}(\alpha-w+1) = \int_0^1 g(x)x^{\alpha-w}\mathrm{d}x$$

The left-hand side of Eq. (B.1.9) is analytic everywhere in the strip zone

Fig. B.1 The integration path in $s = \sigma + i\tau$-plane



$$-\nu < \sigma < \alpha$$

except the simple pole $s = w$ and behaves order $O(|t|^{-\alpha+\varepsilon})$. If we move the integration path from $\sigma = C$ to $\sigma = C' < u$, see Fig. B.1, based on the Cauchy's integral formula [referring to formula (11.7.5)].

$$\frac{1}{2\pi i}\int_{C'-i\infty}^{C'+i\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s\right)}\psi(s)\frac{\mathrm{d}s}{s-w}$$
$$= \overline{g}(\alpha - w + 1) - \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}w\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w\right)}\psi(w)$$

This translation of the integration line corresponds to form a closed region, and the value of the integral around the closed region is just equal to the second term of the above formula including the sign of the term. The left-hand side is analytic as $u > C'$, so is the right-hand side. In addition,

$$\psi(w) - \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha\right)}\overline{g}(\alpha - w + 1) \tag{B.1.10}$$

is analytic for the case

$$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w \neq 0, -1, -2\cdots$$

Integrate function (B.1.10) along a big rectangle whose corners are the points

$$C - iT, C + iT, -T + iT, -T - iT \quad (T > |\upsilon|).$$

One can find the absolute values of the integrals

$$\left| \int_{C-iT}^{-T+iT} \right|, \quad \left| \int_{-T+iT}^{-T-iT} \right|, \quad \left| \int_{-T-iT}^{C-iT} \right|$$

have order $O(|T|^{-\alpha/2+\varepsilon})$, and the value of ε can be always taken less than $\alpha/2$, and then they can tend to zero as $T \to \infty$. According to the Cauchy's integral theorem (referring to formula (11.7.4) of Chap. 11),

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \left\{ \psi(s) - \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}v - \frac{1}{2}\alpha + \frac{1}{2}w\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}v + \frac{1}{2}s\right)} \overline{g}(\alpha - s + 1) \right\} \frac{ds}{s - w} = 0 \quad (u < C) \tag{B.1.11}$$

Similarly, multiply $x^{-w}$to the second one of equations (B.1.8), where $\sigma - w < 0$ and then integrate to x over $(1, \infty)$

$$\frac{1}{2\pi i} \int_{C'-i\infty}^{C'+i\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}v - \frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}v + \frac{1}{2}\alpha - \frac{1}{2}s\right)} \psi(s) \frac{ds}{s - w} = 0 \quad (u > C')$$

Move the integration path and find that

$$\psi(w) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \psi(s) \frac{\mathrm{d}s}{s - w} \quad (u < C) \tag{B.1.12}$$

Comparing (B.1.11) and (B.1.12), we can find (B.1.5), so the solution (B.1.4), in which

$$\overline{g}(\alpha - s + 1) = \int_0^1 g(\xi) \xi^{\alpha - s} \mathrm{d}\xi$$

The theorem is proved.

Furthermore the form of real integral of the solution can be obtained as below. In fact

$$\frac{1}{s-w} = \int_0^1 \eta^{s-w-1}\mathrm{d}\eta$$

If we exchange the integration order, then (B.1.5) may be rewritten as

$$\psi(w) = \int_0^1 g(\xi)\xi^{\alpha}\mathrm{d}\xi \int_0^1 \eta^{-w-1}\mathrm{d}\eta \times \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v-\frac{1}{2}\alpha+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}\left(\frac{\xi}{\eta}\right)^{-s}\mathrm{d}s$$

in which the integration is [12, 13]

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v-\frac{1}{2}\alpha+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}\left(\frac{\xi}{\eta}\right)^{-s}\mathrm{d}s$$
$$= \begin{cases} \frac{2}{\Gamma\left(\frac{1}{2}\alpha\right)}\xi^{1+v-\alpha}(\eta^2-\xi^2)^{\alpha/2-1}\eta^{1-v}, & \eta \geq \xi \\ 0, & 0<\eta<\xi \end{cases}$$

So that

$$\psi(w) = \frac{2}{\Gamma(\alpha/2)}\int_0^1 g(\xi)\xi^{1+v}\mathrm{d}\xi \int_{\xi}^1 \eta^{-w-v}(\eta^2-\xi^2)^{\alpha/2-1}\mathrm{d}\eta$$

By exchanging the integration order, we may find that

$$\psi(w) = \frac{2}{\Gamma(\alpha/2)}\int_0^1 \eta^{-w-v}\mathrm{d}\eta \int_0^1 g(\xi)\xi^{1+v}(\eta^2-\xi^2)^{\alpha/2-1}\mathrm{d}\xi$$
$$= \frac{2}{\Gamma(\alpha/2)}\int_0^1 \eta^{\alpha-\omega}\mathrm{d}\eta \int_0^1 g(\xi)\xi^{1+v}(1-\xi^2)^{\alpha/2-1}\mathrm{d}\xi$$

Substituting it into (B.1.4) yields

$$f(x) = \frac{2}{\Gamma(\alpha/2)}\int_0^1 \eta^{\alpha}\mathrm{d}\eta \int_0^1 g(\eta\zeta)\zeta^{1+v}(1-\zeta^2)^{\alpha/2-1}\mathrm{d}\zeta$$
$$\times \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} 2^{s-\alpha}(x\eta)^{-s}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}v+\frac{1}{2}\alpha-\frac{1}{2}s\right)}\mathrm{d}s$$

By utilizing the inversion of the Mellin transform [12, 13],

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} 2^{s-\alpha}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s\right)}(\mathrm{at})^{-s}\mathrm{d}s = 2^{-\alpha/2}(\mathrm{at})^{1-\alpha/2}J_{\nu+\alpha/2}(\mathrm{at})$$

Then one finds (B.1.4').

For the case $\alpha > -2$, the derivation is similar. For some details reader can refer to Busbridge [2].

In the following, some examples are discussed in detail, which are the dual integral equations appeared in Chaps. 8 and 9 respectively, where only the solutions were listed without derivation detail.

B.2 *Additional Derivation on the Solution of Dual Integral Equations (8.3.8) and (9.7.4)*

Equations (8.3.8) in the text are

$$\left.\begin{aligned} &\frac{2}{d_{11}}\int_0^\infty [C(\xi)\xi - 6D(\xi)]\cos(\xi x)\mathrm{d}\xi = -p, && 0<x<a \\ &\int_0^\infty \xi^{-1}[C(\xi)\xi - 6D(\xi)]\cos(\xi x)\mathrm{d}\xi = 0, && x>a \\ &\frac{2}{d_{12}}\int_0^\infty D(\xi)\cos(\xi x)\mathrm{d}\xi = 0, && 0<x<a \\ &\int_0^\infty \xi^{-1}D(\xi)\cos(\xi x)\mathrm{d}\xi = 0, && x>a \end{aligned}\right\} \tag{B.1.13}$$

It is evident that the second pair of dual integral equations (B.1.13) has the zero solution, i.e. $D(\xi) = 0$, and we only consider the first pair in the equations, which is

$$\left.\begin{aligned} &\frac{2}{d_{11}}\int_0^\infty C(\xi)\xi\cos(\xi x)\mathrm{d}\xi = -p, && 0<x<a \\ &\int_0^\infty C(\xi)\cos(\xi x)\mathrm{d}\xi = 0, && x>a \end{aligned}\right\} \tag{B.1.14}$$

and is similar to that of (9.7.4) in Chap. 9. Because of

$$\cos(\xi x) = \left(\frac{\pi\xi x}{2}\right)^{1/2} J_{-1/2}(\xi x)$$

and denoting

$$\xi^{1/2}C(\xi) = f(\xi), \quad \eta = a\xi, \quad \rho = \frac{x}{a}, \quad g(\rho) = a\left(\frac{\pi a d_{11}}{2\rho}\right)^{1/2} p$$

then (B.1.14) is reduced to

$$\left.\begin{aligned} &\frac{2}{d_{11}}\int_0^\infty \eta f(\eta) J_{-1/2}(\eta\rho) d\eta = g(\rho), 0 < \rho < 1 \\ &\int_0^\infty f(\eta) J_{-1/2}(\eta\rho) d\eta = 0, \rho > 1 \end{aligned}\right\} \tag{B.1.14$'$}$$

which becomes one of the standard dual integral equations shown in (B.1.1) with

$$\alpha = 1, \quad \nu = -1/2, \quad g(\rho) = g_0 \rho^{-1/2}, \quad g_0 = \text{const} = a(\pi a d_{11})^{1/2} p$$

In this case, it is very easy to calculate the solution of dual integral equations (B.1.14) (or B.1.14′) by formulas (B.1.4) and (B.1.5), but the key step is the choosing integration path. In the previous introduction on Titchmarsh–Busbridge solution, we mentioned that it must require that $-\nu > k > \alpha, -\nu < C < \alpha$ and $k < C$. At present case, $\nu = -1/2, \alpha = 1$ such that $1/2 < k < 1$ and $1/2 < C < 1$. The concrete calculation is:

$$\overline{g}(\alpha + 1 - t) = \int_0^1 g(\rho)\rho^{\alpha - t} \mathrm{d}\rho = g_0 \int_0^1 \rho^{-1/2}\rho^{1-t} \mathrm{d}\rho = \frac{g_0}{\frac{3}{2} - t}$$

where $t = t_1 + it_2$ represents a complex variable, and requires that $t_1 < 3/2$. Substituting the relevant data and the above result into (B.1.5), we have

$$\psi(s) = g_0 \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} \frac{\Gamma\left(-\frac{1}{4} + \frac{t}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{t}{2}\right)} \frac{1}{t - s} \frac{1}{\frac{3}{2} - t} \mathrm{d}t$$

The integration path is shown in Fig. B.2. In this case, the integrand has only one pole at point $t = 3/2$ of first order. According to the formula for evaluating (11.7.15), the above integral is very easily obtained as

$$\psi(s) = g_0 \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \frac{1}{\frac{3}{2} - s} = g_0 \frac{\sqrt{\pi}}{\frac{3}{2} - s} \tag{B.1.15}$$

whereas $\Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Substituting the result into formula (B.1.4) leads to

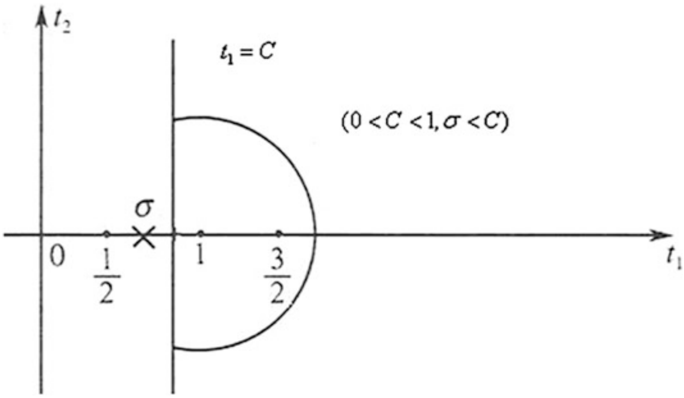


Fig. B.2 The integration path in $t = t_1 + it_2$-plane

$$f(\eta) = g_0\sqrt{\pi}\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{2^{s-\alpha}\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{s}{2}\right)}\eta^{-s}\mathrm{d}s \tag{B.1.16}$$

In terms of the inversion of the Mellin transform [13],

$$\begin{aligned}&\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}2^{s-\lambda}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\lambda-\frac{1}{2}s\right)}(\beta\eta)^{-s}\mathrm{d}s\\&=2^{-\lambda/2}(\beta\eta)^{1-\lambda/2}J_{\mu+\lambda/2}(\beta\eta)\end{aligned} \tag{B.1.17}$$

In formula (B.1.16), $\mu = -1/2, \lambda = 3, \beta = 1$, so that

$$f(\eta) = g_0\left(\frac{\pi}{2\eta}\right)^{1/2}J_1(\eta)$$

and

$$C(\xi) = \xi^{-1/2}f(\xi) = \frac{\pi a d_{11}}{2}\xi^{-1}J_1(a\xi)$$

This is just the result given by (8.3.10) and (9.7.8), and the difference between them lies in a constant factor.

The calculation through (B.1.4′) and (B.1.4″) yields the same result, so the correctness of the result is demonstrated.

B.3 Additional Derivation on the Solution of Dual Integral Equations (9.9.8)

In Sect. 9.9 in the text of Chap. 9, the dual integral equations

$$\begin{aligned} &\int_0^\infty \xi A_i(\xi) J_0(\xi r) d\xi = M_i p_0, \quad 0 < r < a \\ &\int_0^\infty A_i(\xi) J_0(\xi r) d\xi = 0, \quad r > a \end{aligned} \tag{B.1.18}$$

are solved and obtained the solution (9.9.8). We here give the detail for the derivation.

According to the standard type of the equations here $\alpha = 1, v = 0$, $g(\rho) = g_0 = \text{const}$, put $\rho = r/a$, so

$$\overline{g}(\alpha + 1 - t) = \int_0^1 g(\rho)\rho^{\alpha - t} \mathrm{d}\rho = \frac{g_0}{2 - t} \quad (\mathrm{Re}t = t_1 < 2)$$

$$\psi(s) = g_0 \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{t}{2}\right)} \frac{1}{t - s} \frac{1}{2 - t} \mathrm{d}t = g_0 \frac{2}{\sqrt{\pi}} \frac{1}{2 - s}$$

in which $s = \sigma + i\tau$, and the integral is evaluated through the residual of pole $t = 2$ and the integration path is shown in Fig. B.3.

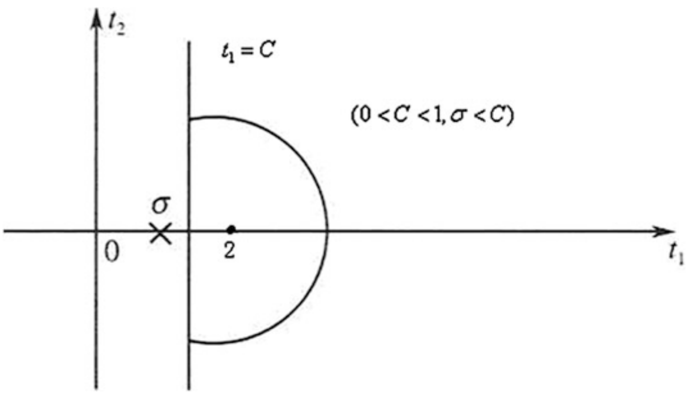


Fig. B.3 Integration path at the $t = t_1 + it_2$-plane

Substituting the result into (B.1.4) yields

$$A_i(\xi) = f(\xi) = g_0 \frac{2}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{2^s \Gamma\left(\frac{1}{2} + \frac{s}{2}\right)}{\Gamma\left(2 - \frac{s}{2}\right)} \xi^{-s} \mathrm{d}s = \frac{2g_0}{\sqrt{2\pi}} \xi^{-1/2} J_{3/2}(\xi)$$

which is just the solution (9.9.8), a little bit difference with that lies where we used the normalized coordinate $\rho = r/a$. In the last step of the calculation, the inversion of the Mellin transform (B.1.17) was used.

The evaluation through formulas (B.1.4′) and (B.1.4″) finds the same result, and this checks the correctness of the above calculation.

The above two subsections demonstrate the effect and simplicity of complex variable function method in evaluating solutions of Titchmarsh–Busbridge dual integral equations.

The system of dual integral equations (B.1.2) and its applications are discussed by Fan [14], and the two-dimensional dual integral equations (B.1.3) are solved approximately by Fan and Sun [15], in which some applications are also given.

Application of integral transforms in elasticity of quasicrystals may be effective and more widely used than that of complex variable function method and has got much analytic solutions, see, e.g. Li [16], Zhou and Fan [17], Zhou [18], and Zhu and Fan [19, 20], and due to the limitation of the space, much results have not been quoted. The application of the method to crack problems often leads to some dual integral equations, so it is helpful for a discussion about this.

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Appendix C: Poisson Brackets in Condensed Matter Physics, Concept of Lie Group and Lie Algebra and Their Applications

In Chap. 16, the equations of motion of solid quasicrystals as a source of those of soft-matter quasicrystals, the latter are put forward as an extension of the former, and the detail of derivation of the equations of solid quasicrystals has not been given in the text, which needs a tool—Poisson brackets in condensed matter physics. In this appendix, we first introduce the method and then give the derivation of those equations.

C.1 Poisson Brackets in Condensed Matter Physics

Due to symmetry breaking, the derivation of some equations of motion of hydrodynamics of some substantive systems cannot be obtained directly by conventional conservations laws. The Poisson brackets in condensed matter physics become a useful method for the derivation, which simplifies the calculation. The method is originated from the Landau and his school in former Soviet Union and Russia (see [1–6]). The physicists Martin et al. [7], and Fleming and Cohen [8] in USA developed the method to hydrodynamics of crystals and liquid crystals, but their derivations were still lengthy. Lubensky et al. [9] further developed the approach in deriving the hydrodynamic equations of quasicrystals, simplified the derivation, and made it arrives in systematization.

Poisson brackets come from the classical analytic mechanics, i.e., for two mechanical quantities f, g and there is the following relation

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{C.1.1}$$

which is the Poisson bracket, where p_i, q_i denote the canonic momentum and canonic coordinate.

According to the terminology of physics, (C.1.1) is named classical Poisson bracket hereafter.

Relative to the classical Poisson bracket (C.1.1), there is a quantum Poisson bracket, which is related to the commutation relation in quantum mechanics

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{C.1.2}$$

in which $\hat{A}, \hat{B}$ represent two operators; e.g. $\hat{A}$ represents coordinate operator x_α and $\hat{B}$ the momentum operator p_β, and then

$$\left[x_\alpha, p_\beta\right] = i\hbar\delta_{\alpha\beta},\ \left[x_\alpha, x_\beta\right] = 0,\ \left[p_\alpha, p_\beta\right] = 0 \tag{C.1.3}$$

where $i = \sqrt{-1}, \hbar = h/2\pi, h$ the Planck constant, $\delta_{\alpha\beta}$ unit tensor. Equation (C.1.3) is named quantum Poisson bracket. In the quantum mechanics, mechanics quantities represent operators. Equation (C.1.3) holds for any operators, in general.

There is inherent connection between the quantum Poisson bracket and classical Poisson bracket, i.e.

$$\lim_{\hbar \to 0} \frac{i\left[\hat{A}\hat{B} - \hat{B}\hat{A}\right]}{\hbar} = \{A, B\} \tag{C.1.4}$$

This is well-known result in the quantum mechanics.

Landau [4] introduced the limit passing over (C.1.4) from quantum Poisson bracket to the classical Poisson bracket in deriving the hydrodynamic equations of superfluid. He takes the expansion of mass density and momentum such as:

$$\hat{\rho}(r) = \sum_\alpha m_\alpha \delta(r_\alpha - r) \tag{C.1.5}$$

$$\hat{g}_k(r) = \sum_\alpha \hat{p}_k^\alpha \delta(r_\alpha - r) \tag{C.1.6}$$

whose quantum Poisson brackets are

$$\begin{aligned} &[\hat{\rho}(r_1), \hat{\rho}(r_2)] = 0 \\ &[\hat{p}_k(r_1), \hat{\rho}(r_2)] = i\hbar\hat{\rho}(r_1)\nabla_k(r_1)\delta(r_1 - r_2) \\ &[\hat{p}_k(r_1), \hat{p}_l(r_2)] = i\hbar(\hat{p}_l(r_1)\nabla_k(r_1) - \hat{p}_k(r_2)\nabla_k(r_2))\delta(r_1 - r_2) \end{aligned} \tag{C.1.7}$$

where $\nabla_k(r_1)$ represents derivative carrying out on coordinate r_1 and $\nabla_l(r_2)$ on coordinate r_2.

By using the limit passing over (C.1.4) from the quantum Poisson to the classical Poisson bracket, from (C.1.7) one can obtain the corresponding classical Poisson brackets:

$$\begin{aligned} \{p_k(r_1), \rho(r_2)\} &= \rho(r_1)\nabla_k(r_1)\delta(r_1 - r_2) \\ \{p_k(r_1), p_l(r_2)\} &= (p_l(r_1)\nabla_k(r_1) - p_k(r_2)\nabla_k(r_2))\delta(r_1 - r_2) \end{aligned} \tag{C.1.8}$$

Lubensky et al. [9] extended the discussion to solid quasicrystals, and they introduced the Landau expansion to phonon field u_i and phason field w_i as below

$$u_k(r) = \sum_{\alpha} u_k^{\alpha} \delta(r_{\alpha} - r) \tag{C.1.9}$$

$$w_k(r) = \sum_{\alpha} w_k^{\alpha} \delta(r_{\alpha} - r) \tag{C.1.10}$$

By using the limit passing over (C.1.4) from the quantum Poisson to the classical Poisson bracket, from (C.1.9) and (C.1.10) one can find whose corresponding classical Poisson brackets as follows:

$$\{u_k(r_1), g_l(r_2)\} = (-\delta_{kl} + \nabla_l(r_1)u_k)\delta(r_1 - r_2) \tag{C.1.11}$$

$$\{w_k(r_1), g_l(r_2)\} = (\nabla_l(r_1)w_k)\delta(r_1 - r_2) \tag{C.1.12}$$

It is evident that (C.1.12) is quite different from (C.1.11), and this leads to the dissipation equations of phasons given in the subsequent discussion which are quite different from those of phonons. The relevant derivations are carried out by Lubensky et al. [9].

C.2 Generalized Langevin Equation and Coarse Graining

Apart from Poisson brackets, it is needed some other basis in the derivation of hydrodynamic equations of quasicrystals, which is related to the Langevin equation or generalized Langevin equation.

It is well known that the conventional Langevin equation stands for

$$\frac{\partial \psi(r,t)}{\partial t} = -\Gamma \psi(r,t) + F_s \tag{C.2.1}$$

in which $\psi(r,t)$ is a mechanics quantity, Γ represents a resistant force, and F_s a stochastic force. The equation describes a stochastic process. Ginzburg and Landau extended it to the case of multi-variables

$$\frac{\partial\psi_\alpha(r,t)}{\partial t} = -\Gamma_{\alpha\beta}\frac{\delta H}{\delta\psi_\beta(r,t)} + (F_s)_\alpha \tag{C.2.2}$$

in which the summation convention is used like that in the previous presentation of this book, where $H = H[\psi(r,t)]$ denotes a energy functional, which can also be named Hamiltonian, $\frac{\delta H}{\delta\psi_\beta(r,t)}$ represents a variation of $H = H[\psi(r,t)]$ to $\psi_\beta(r,t)$,$\Gamma_{\alpha\beta}$ are the elements of resistant matrix (or dissipation kinetic coefficient matrix), and the meanings of definitions of other quantities are the same as before. Equation (C.2.2) is a kind of generalized Langevin equation, which can also be extended in more wide sense. If the macroscopic quantity $\psi_\alpha(r,t)$ may be seen as thermodynamic average of microscopic quantity $\psi_\alpha^\mu(r,\{q^\alpha\},\{p^\alpha\})$, i.e.

$$\psi_\alpha(r,t) = \left\langle\psi_\alpha^\mu(r,\{q^\alpha\},\{p^\alpha\})\right\rangle \tag{C.2.3}$$

this treatment is called coarse graining, in which p^α, q^α are the canonic momentum and canonic coordinate, and the micro-quantities obey the microscopic Liouville equation

$$\frac{\partial\psi_\alpha^\mu}{\partial t} = \left\{H^\mu, \psi_\alpha^\mu\right\} \tag{C.2.4}$$

where $H^\mu(\{q^\alpha\},\{p^\alpha\})$ denote the microscopic Hamiltonian.

In d-dimensional space, the partial derivative of macro-quantity $\psi_\alpha(r,t)$ to time

$$\frac{\partial\psi_\alpha(r,t)}{\partial t}$$

consists of various terms, one among them is

$$-\int\left(\{\psi_\beta(r'),\psi_\alpha(r)\}\frac{\delta H}{\delta\psi_\beta(r',t)}\right)\mathrm{d}^d r' \tag{C.2.5}$$

and the other is

$$\int\left(\frac{\delta\{\psi_\beta(r'),\psi_\alpha(r)\}}{\delta\psi_\beta(r',t)}\right)\mathrm{d}^d r' \tag{C.2.6}$$

Combining (C.2.5), (C.2.6), then (C.2.2) is generalized as

$$\frac{\partial\psi_\alpha(r,t)}{\partial t} = -\int\left(\{\psi_\beta(r'),\psi_\alpha(r)\}\frac{\delta H}{\delta\psi_\beta(r',t)}\right)\mathrm{d}^d r' + \int\left(\frac{\delta\{\psi_\beta(r'),\psi_\alpha(r)\}}{\delta\psi_\beta(r',t)}\right)\mathrm{d}^d r' - \Gamma_{\alpha\beta}\frac{\delta H}{\delta\psi_\beta(r,t)} + (F_s)_\alpha \tag{C.2.7}$$

where $\mathrm{d}^d r' = \mathrm{d}V$ represents volume element of the integral. Based on formulas (C.1.8), (C.1.11) and (C.1.12), Lubensky et al. [9] utilized Eq. (C.2.7) to derive the hydrodynamic equations of quasicrystals. This will be given in the next subsection. In the derivation, the last term in (C.2.7) is omitted.

C.3 Derivation of Hydrodynamic Equations of Solid Quasicrystals

The derivation of equation of mass conservation is very simple, which is the same with that in the conventional fluid, and is omitted here.

At first, consider the derivation of phonon dissipation equations:

Putting $\psi_\alpha(r,t) = u_i(r,t), \psi_\beta(r',t) = g_j(r',t)$ in (C.2.7) and omitting the second and fourth terms in the right-hand side of the equation, then

$$\frac{\partial u_i(r,t)}{\partial t} = -\int\left(\{u_i(r'),g_j(r)\}\frac{\delta H}{\delta g_j(r',t)}\right)\mathrm{d}^d r' - \Gamma_{\mathbf{u}}\frac{\delta H}{\delta u_i(r,t)}$$

Substituting bracket (C.1.11) into the integral of right-hand side yields

$$\begin{aligned}\frac{\partial u_i(r,t)}{\partial t} &= \int\left(-\delta_{ij} + \nabla_j(r)u_i\right)\delta(r-r')\frac{g_j(r')}{\rho(r')}d^d r' + \varGamma_u\frac{\delta H}{\delta u_i(r,t)}\\ &= -V_j\nabla_j(r)u_i - \varGamma_u\frac{\delta H}{\delta u_i(r,t)} + V_i\end{aligned} \tag{C.3.1}$$

where $\varGamma_u$ enotes the phonon dissipation kinematic coefficient, and the Hamiltonian is defined by

$$\begin{aligned}H = H[\psi(r,t)] &= \int\frac{\mathrm{g}^2}{2\rho}\mathrm{d}^d r + \int\left[\frac{1}{2}A\left(\frac{\delta\rho}{\rho_0}\right)^2 + B\left(\frac{\delta\rho}{\rho_0}\right)\nabla\cdot\mathrm{u}\right]\mathrm{d}^d r + F_{\mathrm{el}}\\ &= H_{\mathrm{kin}} + H_{\mathrm{density}} + F_{\mathrm{el}}\\ F_{\mathrm{el}} &= F_u + F_w + F_{uw}, \quad g = \rho V\end{aligned} \tag{C.3.2}$$

and A, B are the constants describe density variation, and the last term of (C.2.7) represents elastic energies, which consist of phonons, phasons and phonon-phason coupling parts, respectively:

$$\begin{aligned} F_{\mathrm{u}} &= \int \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \mathrm{d}^d r \\ F_{\mathrm{w}} &= \int \frac{1}{2} K_{ijkl} w_{ij} w_{kl} \mathrm{d}^d r \\ F_{\mathrm{uw}} &= \int \left(R_{ijkl} \varepsilon_{ij} w_{kl} + R_{klij} w_{ij} \varepsilon_{kl} \right) \mathrm{d}^d r \end{aligned} \tag{C.3.3}$$

C_{ijkl} the phonon elastic constants, K_{ijkl} the phason elastic constants, and R_{ijkl}, R_{klij} the phonon-phason coupling elastic constants, and the strain tensors ε_{ij}, w_{ij} are defined by

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad w_{ij} = \frac{\partial w_i}{\partial x_j} \tag{C.3.4}$$

The associated stress tensors are related through the generalized Hooke's law

$$\left. \begin{aligned} \sigma_{ij} &= \frac{\partial F}{\partial \varepsilon_{ij}} = C_{ijkl} \varepsilon_{kl} + R_{ijkl} w_{kl} \\ H_{ij} &= \frac{\partial F}{\partial w_{ij}} = K_{ijkl} w_{kl} + R_{klij} \varepsilon_{kl} \end{aligned} \right\} \tag{C.3.5}$$

Equation (C.3.1) is just Eq. (16.5.14) in the text of Chap. 16.

Now, consider the derivation of phason dissipation equations.

In (C.2.7), put $\psi_\alpha(r,t) = w_i(r,t), \psi_\beta(r',t) = g_j(r',t)$, neglecting the second and fourth terms in the right-hand side, and then substituting the Poisson bracket (C.2.7) into it leads to

$$\frac{\partial w_i(r,t)}{\partial t} = -\int \left(\left\{ w_i(r'), g_j(r) \right\} \frac{\delta H}{\delta g_j(r',t)} \right) \mathrm{d}^d r' - \Gamma_w \frac{\delta H}{\delta w_i(r,t)}$$

Then

$$\begin{aligned} \frac{\partial w_i(r,t)}{\partial t} &= \int \left(\nabla_j(r) w_i \right) \delta(r - r') \frac{g_j(r')}{\rho(r')} \mathrm{d}^d r' - \Gamma_w \frac{\delta H}{\delta w_i(r,t)} \\ &= - V_j \nabla_j(r) w_i - \Gamma_w \frac{\delta H}{\delta w_i(r,t)} \end{aligned} \tag{C.3.6}$$

which is just Eq. (16.5.16) in the text of Chap. 16, and Γ_w denotes the phason dissipation coefficient, and Hamiltonian is defined by (C.3.2).

By comparing (C.2.7) and (C.2.7), it is found that the physical meanings of phonon and phason in hydrodynamic sense are quite different. According to the explanation of Lubensky et al. [9], the phonon represents wave propagation, while phason represents diffusion.

Of course the other difference between phonon and phason is that they belong to the different irreducible representations of point groups, which has been discussed in Chap. 4.

The derivation of momentum Eq. (16.5.3) is somehow lengthy. The calculation is related with momentum $g_j = \rho V_j$, mass density ρ, phonon u_i and phason w_i, and this means that the integrands in the right-hand side of (C.2.7) need to use simultaneously the Poisson brackets listed in (C.1.8), (C.1.11) and (C.1.12), i.e.

$$\begin{aligned}\frac{\partial g_i(r,t)}{\partial t} = &-\int\left(\{g_i(r),\rho(r')\}\frac{\delta H}{\delta\rho(r',t)}\right)\mathrm{d}^d r' - \int\left(\{g_i(r),g_j(r')\}\frac{\delta H}{\delta g_j(r',t)}\right)\mathrm{d}^d r' \\ &-\int\left(\{g_i(r),u_j(r')\}\frac{\delta H}{\delta u_j(r\prime,t)}\right)\mathrm{d}^d r' - \int\left(\{g_i(r),w_j(r')\}\frac{\delta H}{\delta w_j(r',t)}\right)\mathrm{d}^d r' \\ &\int\left(\frac{\delta\{g_i(r),\psi_\beta(r')\}}{\delta\psi_\beta(r',t)}\right)\mathrm{d}^d r' + \Gamma_g\frac{\delta H}{\delta g_i(r,t)}, \Gamma_g = \eta_{ijkl}\end{aligned} \tag{C.3.7}$$

in which the first integral of the right-hand side can be evaluated as

$$\begin{aligned}\int\left(\{g_i(r),\rho(r')\}\frac{\delta H}{\delta\rho(r',t)}\right)\mathrm{d}^d r' &= \int\left(\rho(r)\nabla_i\delta(r-r')\frac{\delta(H_{\text{kin}}+H_{\text{density}})}{\delta\rho(r',t)}\right)\mathrm{d}^d r' \\ &= \rho(r)\nabla_i\int\left(\delta(r-r')\frac{\delta(H_{\text{kin}}+H_{\text{density}})}{\delta\rho(r',t)}\right)\mathrm{d}^d r' \\ &= \rho(r)\nabla_i\left(\frac{\delta H_{\text{density}}}{\delta\rho}\right) + \rho(r)\nabla_i\left(-\frac{g^2}{2\rho^2}\right) \\ &= \rho(r)\nabla_i\left(\frac{\delta H_{\text{density}}}{\delta\rho}\right) - g_j\nabla_i V_j\end{aligned}$$

Similarly, the second to fifth integrals are evaluated, and the fifth integral is

$$\begin{aligned}\int\left(\frac{\delta\{g_i(r,t),\psi_\beta(r',t)\}}{\delta\psi_\beta(r',t)}\right)\mathrm{d}^d r' = &\int\left(\frac{\delta\{g_i(r,t),\rho(r',t)\}}{\delta\rho(r',t)}\right)\mathrm{d}^d r' + \int\left(\frac{\delta\{g_i(r,t),g_j(r',t)\}}{\delta g_j(r',t)}\right)\mathrm{d}^d r' \\ &+\int\left(\frac{\delta\{g_i(r,t),u_j(r',t)\}}{\delta u_j(r',t)}\right)\mathrm{d}^d r' + \int\left(\frac{\delta\{g_i(r,t),w_j(r',t)\}}{\delta w_j(r',t)}\right)\mathrm{d}^d r'\end{aligned}$$

The right-hand side consists of four terms, and the third one among them results in after some calculation

$$\int \left(\frac{\delta\{g_i(r,t), u_j(r',t)\}}{\delta u_j(r',t)} \right) \mathrm{d}^d r' = \int \left(\frac{\delta\{\delta_{ij} - \nabla_i(r') u_j(r',t)\}}{\delta u_j(r',t)} \delta(r' - r) \right) \mathrm{d}^d r'$$
$$= \int \left(\nabla_i(r') \frac{\delta u_j(r',t)}{\delta u_j(r',t)} \delta(r' - r) \right) d^d r'$$
$$= \int ((\nabla_i(r') 1) \delta(r' - r)) d^d r' = 0$$

Similarly, the fourth term can be evaluated. The sum of the first and second terms is zero. Then making some algebraic manipulations yields

$$\begin{aligned} \frac{\partial g_i(r,t)}{\partial t} &= -\nabla_k(r)(V_k g_i) + \nabla_j(r)\left(\eta_{ijkl} \nabla_k(r) V_l\right) - \left(\delta_{ij} - \nabla_i u_j\right) \frac{\delta H}{\delta u_j(r,t)} \\ &\quad + \left(\nabla_i w_j\right) \frac{\delta H}{\delta w_j(r,t)} - \rho \nabla_i(r) \frac{\delta H}{\delta \rho(r,t)}, \quad g_j = \rho V_j \end{aligned} \tag{C.3.8}$$

in which η_{ijkl} denotes the viscosity coefficient tensor of solid, and the viscous stress tensor is

$$\sigma'_{ij} = \eta_{ijkl} \dot{\xi}_{kl} \tag{C.3.9}$$

with the deformation rate tensor

$$\dot{\xi}_{kl} = \frac{1}{2} \left(\frac{\partial V_k}{\partial x_l} + \frac{\partial V_l}{\partial x_k} \right) \tag{C.3.10}$$

Equations (C.3.1), (C.3.6) and (C.3.8) and mass density conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla_k(\rho V_k) = 0 \tag{C.3.11}$$

are the equations of hydrodynamic equations of icosahedral quaiscrystals, which are obtained by Lubensky et al. [9], in which there are field variables' mass density ρ, velocities V_i (or momentums $g_i = \rho V_i$), phonon displacements u_i and phason displacements w_i.

After the publication of the work since 1985, which are cited by many authors, at meantime there are some discussions [11–13], in which the Ref. [11] suggested some simplifications to the equations, e.g.

$$\frac{\partial \rho}{\partial t} + \nabla_k(\rho V_k) = 0 \tag{C.3.13}$$

$$\frac{\partial g_i(r,t)}{\partial t} = \nabla_j(r)\big(\eta_{ijkl}\nabla_k(r)V_l\big) - \frac{\delta H}{\delta u_i(r,t)} - \rho\nabla_i(r)\frac{\delta H}{\delta\rho(r,t)}, \quad g_j = \rho V_j \tag{C.3.12}$$

$$\frac{\partial u_i(r,t)}{\partial t} = -\Gamma_u\frac{\delta H}{\delta u_i(r,t)} + V_i \tag{C.3.14}$$

$$\frac{\partial w_i(r,t)}{\partial t} = -\Gamma_w\frac{\delta H}{\delta w_i(r,t)} \tag{C.3.15}$$

C.4 Concept of Lie Group and Derivation on Some Formulas

The above derivation indicates that the Poisson brackets (C.1.8), (C.1.11) and (C.1.12) are fundamental, which can also be derived based on the concept of Lie group. In this section, we give an introduction in brief on the derivation.

Lie group is a group like the point groups discussed in the first 15 chapters, which satisfy the axioms of groups, referring to the Appendix of Chap. 1. However there is difference between point group and Lie group which is a kind of continuous group. The momentum operator mentioned previously is a generator of movement group, spin operator is a generator of rotation group in spin space, and the quantum Poisson brackets are connected inherent to the Lie group, so that Ref. [3] suggests the concept of "group Poisson bracket".

Assuming g be an element of group G, it has relation to the m real continuous parameters α_i, i.e.

$$g(\alpha_i) \in G, \quad \alpha_i \in \mathbb{R}, \quad i = 1,2,\ldots,m \tag{C.4.1}$$

$\mathbb{R}$ denotes a real space.

Notion "·" connects two elements, $a(\alpha_i)$ and $b(\beta_i)$, and gives another element $c(\gamma_i) \in G$:

$$c(\gamma_i) = a(\alpha_i)\cdot b(\beta_i), \quad i = 1,2,\ldots,m \tag{C.4.2}$$

For the continuous parameters, there is

$$\gamma_i = \varphi_i(\alpha_1,\alpha_2,\ldots,\alpha_m,\beta_1,\beta_2,\ldots,\beta_m) \tag{C.4.3}$$

If φ_i is an single-valued analytic function of $\alpha_1,\alpha_2,\ldots,\alpha_m,\beta_1,\beta_2,\ldots,\beta_m$, this group is Lie group. The concept on single-valued analytic function can be found in the appendix of Chap. 11 of this book.

People usually take a parameter α_i and identical element E (the concept of identical element E referring to the Appendix of Chap. 1), then $\alpha_i(E) = 0$. The generator of Lie group is taken to be L_i, which can be expressed by the following partial differential

$$L_i = i\frac{\partial a(\ldots, \alpha_i, \ldots)}{\partial \alpha_i}|_{\alpha_i=0} \tag{C.4.4}$$

Group element a can be expressed by the following expansion

$$a(\ldots, \alpha_i, \ldots) = E(\ldots, 0, \ldots) + \alpha_i L_i + O(\alpha_i^2) \tag{C.4.5}$$

The infinitesimal element in the Lie group presents important sense in this kind of groups. Assume matrix $D(A)$ be the representation matrix of element A of Lie group. The parameter α_i of infinitesimal element $A(\alpha)$is an infinitesimal quantity. The matrix $D(A)$ can be expanded as below:

$$D(A) = 1 - i\sum_{j=1}^{N} \alpha_j I_j \tag{C.4.6}$$

and

$$I_j = i\frac{\partial D(A)}{\partial \alpha_j}\bigg|_{\alpha_j=0} \tag{C.4.7}$$

in which N I_j are named generators of representation matrix, and Lie algebra is constituted through the commutation relation between generators

$$[L_i, L_j] = C_{ij}^k L_k, \quad i,j,k = 1, 2, \ldots, m \tag{C.4.7}$$

where C_{ij}^k is called the structure constant. The asymmetry, linearity and Jacobi identity of Lie algebra are as follows:

$$[L_i, L_j] = -[L_j, L_i] \tag{C.4.8}$$

$$\left[\alpha L_i + \beta L_j, L_k\right] = \alpha[L_i, L_k] + \beta\left[L_j, L_k\right], \quad \alpha, \beta \in \mathbb{R} \tag{C.4.9}$$

$$\left[L_i, [L_j, L_k]\right] + \left[L_k, [L_i, L_j]\right] + \left[L_j, [L_k, L_i]\right] = 0 \tag{C.4.10}$$

respectively. In classical continuum mechanics, the coordinate transformation

$$x^k \to x^k + u^k(r) \tag{C.4.11}$$

is often used, which is called translation group or movement group, or infinitesimal movement group. It is interesting that in particular, $u^k(r)$ presents evident physical meaning and represent displacement, or phonon. Note that x^k here is the contravariant vector, while x_i is the covariant vector. We recall that the physical quantities mentioned previously present very close connection to the group algebra, because momentum operator is the generator of movement group, spin operator is the generator of rotation group in spin space, etc. Some correlation between physical quantities $a, b, c, \ldots$ and elements of transformation group $A, B, C, \ldots$

$$\{a, b, c, \ldots\} \rightarrow \{A, B, C, \ldots\} \tag{C.4.12}$$

may be set up. The linear combination of group element$A$ can be given by the following linear expression

$$A = \sum_{g \in G} A(g)g, \quad A(g) \in \mathbb{R} \tag{C.4.13}$$

where $A(g)$can be understood the coefficients of the expansion, but the series is for discrete group only, it should be replaced by integral for continuous group, and in the case, the group elements vary **continuously**.

Assume A can be transformed according to the following version

$$A \rightarrow gAg^{-1} \tag{C.4.14}$$

Suppose δg be an infinitesimal transformation, if $g = 1 + \delta g$, then the linear approximation is

$$A \rightarrow A + \delta A \tag{C.4.15}$$

and

$$\delta A = [\delta g, A] \tag{C.4.16}$$

The infinitesimal transformation δg is of the form

$$\delta g = \frac{i}{\hbar} \int \alpha^k(r) L^k(r) \mathrm{d}^d r \tag{C.4.17}$$

in which $\alpha^k(r)$ is the local infinitesimal "angular", $L^k(r)$ is the generators of local transformation group, $i = \sqrt{-1}, \hbar = h/2\pi$, and h is the Planck constant.

For the movement group, take $\alpha^k(r) = u^k(r)$, and the generator is the momentum, and then from (C.4.15) and (C.4.17),

$$\delta A(r) = \frac{i}{\hbar}\int \alpha^k(r')\left[L^k(r'), A(r)\right]\mathrm{d}^d r' \tag{C.4.18}$$

This equation shows that δA is the linear functional of "angular" $\alpha^k(r)$ of infinitesimal local transformation, and the corresponding variation is

$$\frac{\delta A(r)}{\delta \alpha^k(r')} = \frac{i}{\hbar}\left[L^k(r'), A(r)\right] \tag{C.4.19}$$

The limit passing over from quantum mechanics to classical mechanics is

$$\frac{\delta \hat{A}}{\delta \alpha} = \frac{i}{\hbar}\left[\hat{L}, \hat{A}\right] \to \frac{\delta A}{\delta \alpha} = \{L, A\} \tag{C.4.20}$$

Recall again that $\hat{L}, \hat{A}$ represent operators in quantum mechanics and L, A the field variables in classical mechanics, so that the right-hand side of (C.4.19) may be written as

$$\frac{\delta a}{\delta \alpha} = \{l, a\} \tag{C.4.21}$$

in which a can represent any field variables $a, b, c, \ldots$ of hydrodynamics and l the generator $l^k(r)$ corresponding to the group, so that from (C.4.21)

$$\frac{\delta a(r)}{\delta \alpha^k(r')} = \left\{l^k(r'), a(r)\right\} \tag{C.4.22}$$

Furthermore

$$\frac{\delta l^m(r)}{\delta \alpha^k(r')} = \left\{l^k(r'), l^m(r)\right\}, \{a, a\} = \{a, b\} = \{b, b\} = 0 \tag{C.4.23}$$

At the finite temperature, the Hamiltonian can be expressed by

$$H = \int \varepsilon(\mathrm{p}, \rho, s)\mathrm{d}^d r$$
$$\mathrm{d}\varepsilon = V^k \mathrm{d}p_k + \mu \mathrm{d}\rho + T\mathrm{d}s$$

where ε denotes the energy density, the others are the same before, $\mathrm{p} = (p_x, p_y, p_z)$ and ρ the momentum and mass density, s the entropy, $\mathrm{V} = (V_x, V_y, V_z)$ the velocity, μ the chemical potential, T the absolute temperature, respectively, so

$$\begin{aligned} \delta p_k &= -u^l \nabla_l p_k - p_k \nabla_l u^l - p_k \nabla_l u^l \\ \delta \rho &= -u^l \nabla_l \rho - \rho \nabla_k u^k \\ \delta s &= -u^l \nabla_l s - s \nabla_k u^k \end{aligned} \tag{C.4.24}$$

From (C.4.23) and (C.4.24), one obtains

$$\begin{aligned} \{p_k(r_1), \rho(r_2)\} &= \rho(r_1)\nabla_k(r_1)\delta(r_1 - r_2) \\ \{p_k(r_1), p_l(r_2)\} &= (p_l(r_1)\nabla_k(r_1) - p_k(r_2)\nabla_k(r_2))\delta(r_1 - r_2) \end{aligned} \tag{C.4.25}$$

This is identical to (C.1.8) given by Poisson bracket method of condensed matter physics, which is the result of Ref. [2, 4].

Applying the above results into quasicrystals, there are

$$\{u_k(r_1), g_l(r_2)\} = (-\delta_{kl} + \nabla_l(r_1)u_k)\delta(r_1 - r_2) \tag{C.4.26}$$

$$\{w_k(r_1), g_l(r_2)\} = (\nabla_l(r_1)w_k)\delta(r_1 - r_2) \tag{C.4.27}$$

These are identical to (C.4.23) and (C.4.24) given by Lubensky et al. [9], and they derived directly using the Poisson bracket method.

This description shows the power of Lie group method. Ref. [4] shows further that if introducing the Liouville equation, equations of motion for some complex systems can be obtained, which are identical to those derived Sect. C.3.

Some detailed derivations are given by Fan [14].

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Appendix D: Some Preliminary Introductions on Soft-Matter Quasicrystals

The previous discussions in the text have dealt with solid quasicrystals, including binary and ternary metal alloy quasicrystals and natural quasicrystals observed so far, and focused mainly on their elasticity. Since 2004, the quasicrystals with 12-fold symmetry have been observed in liquid crystals, colloids and polymers. In particular, 18-fold symmetry quasicrystals in colloids were discovered in 2011. These kinds of quasicrystals can be called as soft-matter quasicrystals, which present very interesting and attractive features and have aroused a great deal of attention of researchers in physics and chemistry.

D.1 Soft-Matter Quasicrystals with 12- and 18-Fold Symmetries

D.1.1 The Discovery of Soft-Matter Quasicrystals with 12- and 18-Fold Symmetries

During 2004, Zeng et al. [1] observed the quasicrystals with 12-fold symmetry in liquid crystals. Almost at the same time, in 2005 Takano [2], in 2007 Hayashida et al. [3] discovered the similar structure in polymers. The quasicrystals of 12-fold symmetry were observed also in chalcogenides and organic dendrons.

In 2009, Talapin et al. [4] found the quasicrystals of 12-fold symmetry in complex of binary nanoparticles.

Figure D.1 shows the diffraction pattern of soft-matter quasicrystals with 12-fold symmetry.

More recently, the 12- and 18-fold symmetry quasicrystals are discovered in colloids by Fischer et al. [5], and they observed the structures in PI_{30}-PEO_{120} of one of poly (isoprene-*b*-ethylene oxide) (PI_n-PEO_m) at room temperature, by using X-ray scattering and neutron scattering. The 18-fold symmetry quasicrystal is the

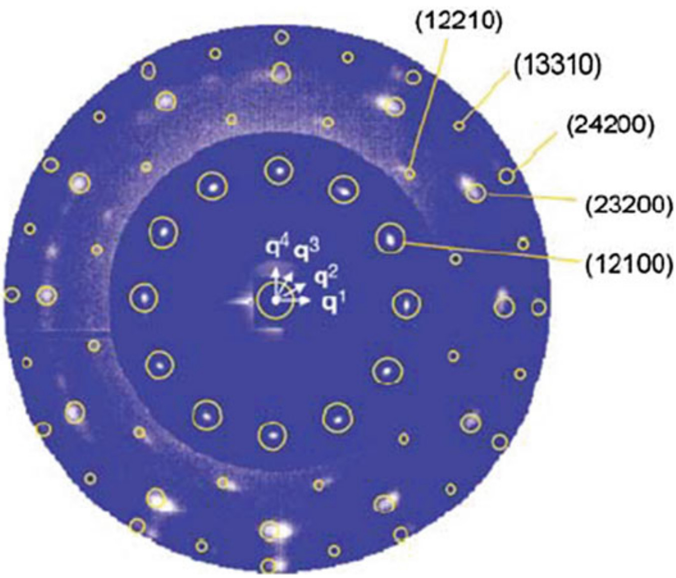

Fig. D.1 Diffraction pattern of 12-fold symmetry quasicrystals in soft matter

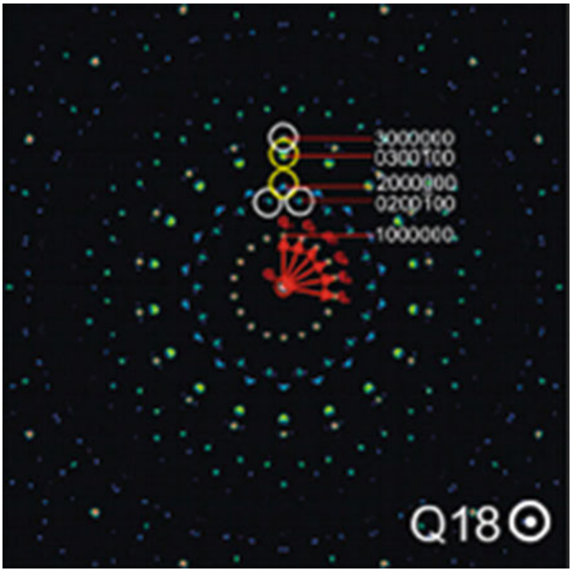

Fig. D.2 Diffraction pattern of soft-matter quasicrystals with 18-fold symmetry

first observed since 1982 in solid and soft-matter quasicrystals, whose diffraction pattern and Penrose tiling are shown in Figs. D.2 and D.3, respectively.

Though the 12-fold symmetry quasicrystals in solids were discussed in Chaps. 6–8, the 18-fold symmetry quasicrystals are studied for the first time to us, which have not

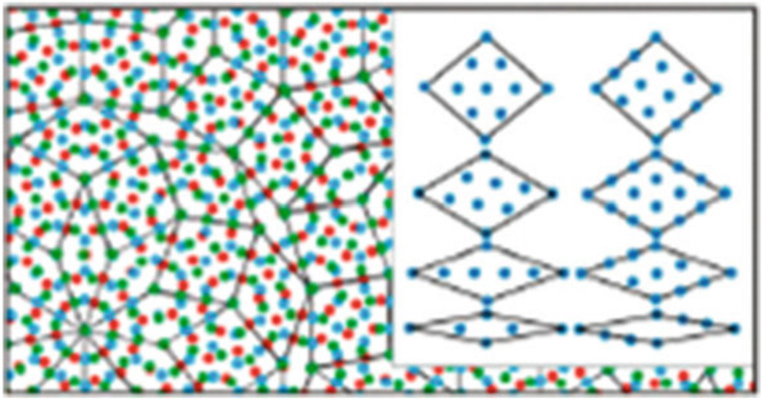

Fig. D.3 The Penrose tiling of quasicrystals with 18-fold symmetry in soft matter

been known previously. This is very new and interesting topics. The 12-fold symmetry quasicrystals in solid are discussed in Chaps. 6–8, but the 18-fold symmetry quasicrystals are total newly phase to the researchers, which have not been discussed in the previous chapters, and we have only very few of understanding for the structure and properties.

These discoveries present highly importance. At first, under certain temperature and density, quasicrystal state in soft matter is stable and this promotes us to understand quasicrystals theoretically. It is well known that quasicrystal state in metallic alloys is formed under rapid cooling condition, which is quite different from that of soft-matter quasicrystals, because these two cases are in quite different thermodynamic environments. The discovery of 18-fold symmetry quasicrystals leads to appearance of new point groups and space groups and promotes the development of symmetry theory and group theory. Of course, the appearance of these new quasicrystals enlarges the scope of the quasicrystal study. Finally, soft-matter quasicrystals may be a class of photon band grasp material, present application meaning. In addition, the self-assembly technique developing in the study is meaningful.

D.1.2 Characters of Soft-Matter Quasicrystals

Based on the experimental results, the soft-matter quasicrystals observed in different kinds of soft matter and their forms and structures are quite different to each other. It is here unable specially and in detail to study soft matter. Our object is only to study the soft-matter quasicrystals, and for this purpose, we have to understand a preliminary and necessary knowledge on soft matter. The nature of soft matter is an intermediate phase between ideal solid and simple fluid, or call is as a complex fluid or structured fluid, which is one of soft condensed matter.

The soft-matter quasicrystals observed so far are two-dimensional. During the process of their formation, it accomplished chemical reactions, and some phase transitions, such as crystal–quasicrystal transition and liquid crystal–quasicrystal transition. In the formation process of quasicrystals coming from colloids, there is connection with electricity, because the particles in colloids have charges. Our understanding to these complex physical–chemical effects is very limited.

The discussion on quasicrystals in soft matter is only an introduction of the subject. The same as we have done in the first 15 chapters for solid quasicrystals, and the main attention here is on mechanical behaviour and continuous mechanics of soft-matter quasicrystals. For example, under action of impact tension with stress amplitude $\sigma_0 = 5$ MPa, the variation of mass density $\delta\rho/\rho_0$ is 10^{-14} for solid quasicrystals, while under action of impact tension with stress amplitude $\sigma_0 = 0.01$ MPa, the variation of mass density $\delta\rho/\rho_0$ is 10^{-3} for soft-matter quasicrystals; the viscosity stress $\sigma'_{yy} = 10^{-19}$ GPa for solid quasicrystals under action of impact tension with stress amplitude $\sigma_0 = 5$ MPa, while the fluid stress $\sigma'_{yy} = 10^{-3}$ GPa for soft-matter quasicrystals under action of impact tension with stress amplitude $\sigma_0 = 0.01$ MPa These show the huge differences of mechanical properties between solid quasicrystals and soft-matter quasicrystals. Of course, for the computation for soft-matter quasicrystals, the equation of state was used, which is needed to be verified by experiments.

The related thermodynamics of soft-matter quasicrystals was done by Lifshitz et al. [7, 8], and they attended the stability of the new phase, which is a very important problem, of course. For studying hydrodynamics of soft-matter quasicrystals, an equation of state is necessary, and Fan and co-workers [9, 10] gave some preliminary discussions, but the model needs experimental verification.

D.2 Mathematical Model of Hydrodynamics of Soft-Matter Quasicrystals

However, the work on the deformation and motion of the new phase has not well been carried out due to the lack of fundamental experimental data to date. In addition, the scope of topic goes beyond elasticity. Hence, one should undertake the research on the relevant hydrodynamics. It is well known that the hydrodynamics in solid quasicrystals is a very difficult subject. Chapter 16, thus, only gave a very brief introduction. Readers might be found that many problems and questions there were left. There are much more principle difficulties in studying hydrodynamics for the new phase in physics and mathematics. At first, some mechanisms of deformation and motion of the matter have not been sufficiently explored owing to the lack of experimental data. Secondly, there is the lack of effective equation of state $p = f(\rho)$ or $\rho = g(p)$ for soft matter, where p denotes fluid pressure and ρ the mass density, respectively. This is a difficulty arising from thermodynamic study of soft matter. The thermodynamics is a more fundamental theory than the hydrodynamics.

The work on thermodynamics of soft-matter quasicrystals has been given by some researchers, as mentioned previously, but has not been well developed. Furthermore, no experimental verification has been made for some proposed models. It is an evident that the study of hydrodynamics for soft-matter quasicrystals is more difficult than that for solid quasicrystals.

In spite of this, the probe on hydrodynamic study of soft-matter quasicrystals is available and significant. The hydrodynamics of solid quasicrystals initiated by Lubensky et al. [6] over the past three decades has accumulated fruit achievements and experience. This is worthwhile providing a mode to develop hydrodynamics of soft-matter quasicrystals by drawing from that of solid quasicrystals.

After the careful consideration and preliminary practice, one can find that the common fundamentals for hydrodynamics, for example the generalized Langevin equation and Poisson bracket method of condensed matter physics, are valid for both solid quasicrystals and soft-matter quasicrystals (a rough address can be referred to Appendix C). In addition, the Hamiltonians in both solid and soft-matter quasicrystals are similar. This fact indicates that the theoretical framework for soft-matter quasicrystals focusing on hydrodynamics may be set up. For simplicity, the study on soft matters here should be confined to the case of small deformations such that the phonon and phason stresses and strains yield the generalized linear Hooke's law, and the fluid stresses and deformation rates follow the generalized linear Newton's law. Under these assumptions, the corresponding hydrodynamic equation system can be deduced. The equations provide a basis for computation at least, which lead to some information and data on displacement, velocity and stress fields in physical time-spatial domain. They can also provide possible comparisons with experiments hereafter. This enables one to explore the physical nature of deformation and motion of soft-matter quasicrystals. Some results are reported in [11–14].

This discussion on soft-matter quasicrystals and their hydrodynamics is beyond the scope of this book, and the relevant contents are given in works [15, 16].

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