Appendix A
Characterizations of Several Covering Properties

This appendix is prepared for the convenience of text reading. We mainly introduce characterizations and relevant mapping theorems of five covering properties used in the text: paracompactness, metacompactness, subparacompactness, submetacompactness and meta-Lindelöfness. For a systematic introduction of covering properties, we recommend reading “Covering Properties” [83] or “Selected Topics in General Topology” [212].

In this appendix, no separation axiom is assumed for spaces except it is specially mentioned and all mappings are assumed to be continuous and onto. We first recall some basic terminologies. Suppose X is a topological space and $\mathscr{U} = \{U_\alpha\}_{\alpha<\gamma}$ is a cover of X, where γ is an ordinal number. $\mathscr{U}$ is called a *well-monotone cover* of X if $U_\alpha \subset U_\beta$ for every $\alpha < \beta < \gamma$. $\mathscr{U}$ is called a *directed cover* of X if for every $\alpha, \beta < \gamma$, there is $\delta < \gamma$ such that $U_\alpha \cup U_\beta \subset U_\delta$. For families $\mathscr{V}$ and $\mathscr{W}$ of subsets of X, $\mathscr{V}$ is said to be a *partial refinement* of $\mathscr{W}$ if for each $V \in \mathscr{V}$, there is $W \in \mathscr{W}$ such that $V \subset W$. $\mathscr{V}$ said to be a *refinement* of $\mathscr{W}$ if $\mathscr{V}$ covers X and $\mathscr{V}$ is a partial refinement of $\mathscr{W}$. For other notations and terminologies, please refer to Sect. 1.1 of the text.

A.1 Paracompact Spaces

Definition A.1.1 A space X is a called a *paracompact space* [112] if every open cover of X has a locally finite open refinement. X is called a *countably paracompact space* [116, 224] if every countable open cover of X has a locally finite open refinement.

The following characterizations of paracompact spaces given by A.H. Stone and Michael are well-known. Let $\mathscr{U}$, $\mathscr{V}$ be covers of a space X. $\mathscr{V}$ is a *star-refinement* of $\mathscr{U}$ if $\{\mathrm{st}(V, \mathscr{V}) : V \in \mathscr{V}\}$ is a refinement of $\mathscr{U}$.

S. Lin and Z. Yun, *Generalized Metric Spaces and Mappings*,
Atlantis Studies in Mathematics 6, DOI 10.2991/978-94-6239-216-8

Theorem A.1.2 *For every regular space X, the following are equivalent:*

(1) X is a paracompact space.
(2) Every open cover of X has an open star-refinement [440].
(3) Every open cover of X has a σ-discrete open refinement [325].
(4) Every open cover of X has a σ-cushioned open refinement [327].
(5) Every open cover of X has a closure-preserving closed refinement [326].

Corollary A.1.3 ([326]) (The Michael theorem) T_2 *paracompactness is invariant under closed mappings.*

Proposition A.1.4 ([173]) *Every perfect preimage of a paracompact space is a paracompact space.*

Proof Suppose $f : X \to Y$ is a perfect mapping and Y is a paracompact space. For each open cover $\mathscr{U}$ of X and each $y \in Y$, there is $\mathscr{U}_y \in \mathscr{U}^{<\omega}$ such that $f^{-1}(y) \subset \cup\mathscr{U}_y$. Take an open neighborhood V_y of y for each $y \in Y$ such that $f^{-1}(V_y) \subset \cup\mathscr{U}_y$. Then the open cover $\{V_y : y \in Y\}$ of Y has a locally finite open refinement $\{W_y : y \in Y\}$ such that $W_y \subset V_y$. So $\{f^{-1}(W_y) \cap U : y \in Y,\ U \in \mathscr{U}_y\}$ is a locally finite open refinement of $\mathscr{U}$. Thus X is a paracompact space. ∎

Definition A.1.5 Suppose $\mathscr{U}, \mathscr{V}$ are covers of a space X. $\mathscr{V}$ is called a *local star-refinement* of $\mathscr{U}$ [219] if, for each $x \in X$, there are an open neighborhood G of x and $U \in \mathscr{U}$ such that $\mathrm{st}(G, \mathscr{V}) \subset U$. $\mathscr{V}$ is called a *pointwise W-refinement* of $\mathscr{U}$ [479] if, for each $x \in X$, there is $\mathscr{U}' \in \mathscr{U}^{<\omega}$ such that $(\mathscr{V})_x$ is a partial refinement of $\mathscr{U}'$. $\mathscr{V}$ is called a *local W-refinement* of $\mathscr{U}$ [480] if, for each $x \in X$, there exist an open neighborhood G of x and $\mathscr{U}' \in \mathscr{U}^{<\omega}$ such that $(\mathscr{V})_G$ is a partial refinement of $\mathscr{U}'$.

Lemma A.1.6 ([219]) *Suppose $\mathscr{U}$ is an open cover of a space X. Consider the following conditions:*

(1) $\mathscr{U}^F$ has a closure-preserving closed refinement $\mathscr{F}$ such that $\mathscr{F}^\circ$ covers X.
(2) $\mathscr{U}^F$ has an interior-preserving open local star-refinement.
(3) $\mathscr{U}^F$ has an interior-preserving open local W-refinement.

Then (3) $\Rightarrow$ (2) $\Rightarrow$ (1). If $\mathscr{U}$ is an interior-preserving open cover of X, then (1) $\Rightarrow$ (3).

Proof (3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Suppose $\mathscr{V}$ is an interior-preserving open local star-refinement of $\mathscr{U}^F$. Let

$$F(\mathscr{U}') = \{x \in X : \mathrm{st}(x, \mathscr{V}) \subset \cup\mathscr{U}'\},\ \mathscr{U}' \in \mathscr{U}^{<\omega};$$
$$\mathscr{F} = \{F(\mathscr{U}') : \mathscr{U}' \in \mathscr{U}^{<\omega}\}.$$

Then $\mathscr{F}$ is a closure-preserving closed refinement of $\mathscr{U}^F$ and $\mathscr{F}^\circ$ covers X.

Now suppose $\mathscr{U}$ is an interior-preserving open cover of X. Let $\mathscr{F}$ be a closure-preserving closed refinement of $\mathscr{U}^F$ such that $\mathscr{F}^\circ$ covers X. Define

$$W_x = (\cap(\mathscr{U})_x) \cap (X - \cup(\mathscr{F} - (\mathscr{F})_x)),\ x \in X;$$
$$\mathscr{W} = \{W_x : x \in X\}.$$

Then $\mathscr{W}$ is an interior-preserving open cover of X. For each $x \in X$, there is $F \in \mathscr{F}$ such that $x \in F^\circ$, then there is $\mathscr{U}' \in \mathscr{U}^{<\omega}$ such that $F \subset \cup\mathscr{U}'$, and hence $(\mathscr{W})_{F^\circ}$ is a partial refinement of $\mathscr{U}'$. Thus $\mathscr{W}$ is a local W-refinement of $\mathscr{U}$. ■

Lemma A.1.7 *Suppose $\mathscr{F}$ is a σ-closure-preserving family of closed sets in a countably paracompact space X. If $\mathscr{F}^\circ$ covers X, then $\mathscr{F}$ has a closure-preserving closed refinement $\mathscr{H}$ such that $\mathscr{H}^\circ$ covers X.*

Proof Denote $\mathscr{F} = \bigcup_{n\in\mathbb{N}} \mathscr{F}_n$, where each $\mathscr{F}_n$ is closure-preserving family of closed sets in X. For each $n \in \mathbb{N}$, let

$$P_n = \cup\mathscr{F}_n^\circ \text{ and } \mathscr{P} = \{P_n : n \in \mathbb{N}\}.$$

Then $\mathscr{P}$ has a locally finite open refinement $\mathscr{R} = \{R_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, take $n_\alpha \in \mathbb{N}$ such that $R_\alpha \subset P_{n_\alpha}$. Let

$$\mathscr{H}_\alpha = \{F \cap \overline{R}_\alpha : F \in \mathscr{F}_{n_\alpha}\},$$
$$\mathscr{H} = \cup\{\mathscr{H}_\alpha : \alpha \in \Lambda\}.$$

Then $\mathscr{H}$ is a closure-preserving closed refinement of $\mathscr{F}$ such that $\mathscr{H}^\circ$ cover X. ■

Lemma A.1.8 ([219]) *Suppose $\{\mathscr{U}_n\}$ is a sequence of open covers of a space X. If each $\mathscr{U}_{n+1}$ is a pointwise (resp. local) W-refinement of $\mathscr{U}_n$, then $\mathscr{U}_1$ has a σ-point-finite (resp. σ-locally finite) open refinement.*

Proof Denote $\mathscr{U}_1 = \{U_\alpha : \alpha < \gamma\}$. For each $U \in \bigcup_{n\in\mathbb{N}} \mathscr{U}_n$, let

$$\alpha(U) = \min\{\alpha < \gamma : U \subset U_\alpha\}.$$

For every $n > 1$ and $U \in \mathscr{U}_n$, $\mathscr{U}_{n-1}$ is said to have the property $\Phi(U)$ if $\alpha(U') = \alpha(U)$ whenever $U \subset U' \in \mathscr{U}_{n-1}$. Define

$$\mathscr{W}_n = \{U \in \mathscr{U}_n : \mathscr{U}_{n-1} \text{ has the property } \Phi(U)\}.$$

(8.1) $\bigcup_{n>1} \mathscr{W}_n$ covers X.

For every $n > 1$ and $x \in X$, define $\alpha_n = \sup\{\alpha(U) : x \in U \in \mathscr{U}_n\}$. Since $\mathscr{U}_{n+1}$ is a pointwise W-refinement of $\mathscr{U}_n$, $\alpha_{n+1} \leqslant \alpha_n < \gamma$, so there exist $\beta < \gamma$ and $k \geqslant 3$ such that $\alpha_n = \beta$ when $n \geqslant k-1$. Since $\mathscr{U}_{k+1}$ is a pointwise W-refinement of $\mathscr{U}_k$, there is $\mathscr{U} \in (\mathscr{U}_k)_x^{<\omega}$ such that $(\mathscr{U}_{k+1})_x$ is a partial refinement of $\mathscr{U}$. Pick $U \in \mathscr{U}$ such that $\alpha(U') \leqslant \alpha(U)$ whenever $U' \in \mathscr{U}$. Since $(\mathscr{U}_{k+1})_x^{<\omega}$ is a partial refinement of $\mathscr{U}$, $\alpha_{k+1} \leqslant \alpha(U)$ and $\alpha(U) \leqslant \alpha_k$, so $\alpha(U) = \beta$. Because $\alpha_{k-1} = \beta$, for every $U' \in (\mathscr{U}_{k-1})_x$, $\alpha(U') \leqslant \beta = \alpha(U)$, and if $U \subset U'$, then $\alpha(U') \geqslant \alpha(U)$, so $\alpha(U') = \alpha(U)$, it follows that $\mathscr{U}_{k-1}$ has the property $\Phi(U)$, and $x \in U \in \mathscr{W}_k$.

Now, for each $n > 1$, define

$$V_{n,\alpha} = \cup\{W \in \mathscr{W}_n : \alpha(U) = \alpha\},\ \alpha < \gamma;$$
$$\mathscr{V}_n = \{V_{n,\alpha} : \alpha < \gamma\}.$$

Obviously, $\bigcup_{n>1} \mathscr{V}_n$ is an open refinement of $\mathscr{U}_1$.

(8.2) $\mathscr{V}_n$ is a point-finite (resp. locally finite) family.

For each $x \in X$, take $G = \{x\}$ (resp. take an open neighborhood G of x) and $\mathscr{U}' \in \mathscr{U}_{n-1}^{<\omega}$ such that $(\mathscr{U}_n)_G$ partially refines $\mathscr{U}'$. Let

$$\Gamma = \{\alpha(U') : U' \in \mathscr{U}'\},$$
$$\Lambda = \{\alpha < \gamma : G \cap V_{n,\alpha} \neq \varnothing\}.$$

Then $\Lambda \subset \Gamma$. Because in fact, if $\beta \in \Lambda$, then there is $U \in \mathscr{W}_n$ such that $\alpha(U) = \beta$ and $U \cap G \neq \varnothing$, so $U \in (\mathscr{U}_n)_G$, and hence there is $U' \in \mathscr{U}'$ such that $U \subset U'$, it follows that $\alpha(U) = \alpha(U')$, consequently $\beta \in \Gamma$. Thus $\mathscr{V}_n$ is a point-finite (resp. locally finite) family in X.

In summary, $\bigcup_{n>1} \mathscr{V}_n$ is a σ-point-finite (resp. σ-locally finite) open refinement of $\mathscr{U}_1$. ■

Theorem A.1.9 ([219, 317]) *For every space X, the following are equivalent:*

(1) X is a paracompact space.
(2) Every well-monotone open cover of X has a locally finite open refinement.
(3) Every interior-preserving directed open cover of X has an interior-preserving open local star-refinement.
(4) Every interior-preserving directed open cover of X has a σ-closure-preserving closed refinement $\mathscr{F}$ such that $\mathscr{F}^\circ$ covers X.
(5) Every directed open cover of X has a closure-preserving closed refinement $\mathscr{F}$ such that $\mathscr{F}^\circ$ covers X.

Proof (1) ⇒ (5). Suppose $\mathscr{U}$ is a directed open cover of X. Let $\mathscr{V}$ be a locally finite open refinement of $\mathscr{U}$. Then $\mathscr{V}$ is an interior-preserving open local W-refinement of $\mathscr{U}$. By Lemma A.1.6, $\mathscr{U}^F$ has a closure-preserving closed refinement $\mathscr{F}$ such that $\mathscr{F}^\circ$ covers X. Since $\mathscr{U}$ is a directed cover, $\mathscr{F}$ is also a refinement of $\mathscr{U}$.

(5) ⇒ (4) is obvious.

(4) ⇒ (3). By Lemmas A.1.7 and A.1.6, we only need to prove that X is a countably paracompact space. Suppose $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ is an open cover of X. Then the interior-preserving directed open cover $\{\bigcup_{k\leqslant n} U_k : n \in \mathbb{N}\}$ of X has a closed refinement $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$, such that, $\mathscr{F}_n$ is a closure-preserving family and $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n^\circ$ covers X. Define

$$R_0 = \varnothing;$$
$$R_n = \cup\left\{F \in \bigcup_{k\leqslant n} \mathscr{F}_k : F \subset \bigcup_{k\leqslant n} U_k\right\},\ n \in \mathbb{N}.$$

Then $\{U_n - R_{n-1} : n \in \mathbb{N}\}$ is a locally finite open refinement of $\mathscr{U}$. Thus X is a countably paracompact space.

(3) $\Rightarrow$ (2). Suppose $\mathscr{U}$ is a well-monotone open cover of X. By Lemma A.1.6, there is a sequence $\{\mathscr{U}_n\}$ of interior-preserving open covers of X such that $\mathscr{U}_1 = \mathscr{U}$ and $\mathscr{U}_{n+1}$ is a local W-refinement of $\mathscr{U}_n$. By Lemma A.1.8, $\mathscr{U}$ has an open refinement $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ such that each $\mathscr{V}_n$ is locally finite in X. By Lemma A.1.6, X satisfies (4), so X is a countably paracompact space, and hence $\{\cup\mathscr{V}_n : n \in \mathbb{N}\}$ has a locally finite open refinement $\{W_n : n \in \mathbb{N}\}$ such that $W_n \subset \cup\mathscr{V}_n$. Then $\bigcup_{n\in\mathbb{N}}\{W_n \cap V : V \in \mathscr{V}_n\}$ is a locally finite open refinement of $\mathscr{U}$.

(2) $\Rightarrow$ (1). For each cardinal number γ, denote the following assumption by $P(\gamma)$: every open cover of X with cardinality γ has a locally finite open refinement. Suppose γ is an infinite cardinal number and for each $\lambda < \gamma$, $P(\lambda)$ is true. If $\mathscr{U}$ is an open cover of X with cardinality γ, denote $\mathscr{U} = \{U_\alpha : \alpha < \gamma\}$, then $\{\bigcup_{\beta\leqslant\alpha} U_\beta : \alpha < \gamma\}$ is a well-monotone open cover of X, and hence it has a locally finite open refinement $\mathscr{W}$. For each $W \in \mathscr{W}$, take $\alpha(W) < \gamma$ such that $W \subset \cup\{U_\beta : \beta \leqslant \alpha(W)\}$. For each $\alpha < \gamma$, let

$$P_\alpha = \cup\{W \in \mathscr{W} : \alpha(W) > \alpha\},$$
$$\mathscr{P}_\alpha = \{P_\alpha\} \cup \{U_\beta : \beta \leqslant \alpha\}.$$

By the induction hypothesis, $\mathscr{P}_\alpha$ has a locally finite open refinement $\mathscr{H}_\alpha$. Define

$$\mathscr{R}(W) = \{W \cap H : H \in \mathscr{H}_{\alpha(W)} \text{ and } H \subset U_\alpha \text{ for some } \alpha \leqslant \alpha(W)\},\ W \in \mathscr{W};$$
$$\mathscr{R} = \cup\{\mathscr{R}(W) : W \in \mathscr{W}\}.$$

Then $\mathscr{R}$ is a locally finite family of open sets in X, and for each $x \in X$, there is $W \in (\mathscr{W})_x$ such that $\alpha(W') \leqslant \alpha(W)$ whenever $W' \in (\mathscr{W})_x$. Let $\beta = \alpha(W)$ and take $H \in \mathscr{H}_\beta$ such that $x \in H$. Since $x \notin P_\beta$ and $\mathscr{H}_\beta$ refines $\mathscr{P}_\beta$, there is $\alpha \leqslant \beta$ such that $H \subset U_\alpha$, so $x \in W \cap H \in \mathscr{R}(W) \subset \mathscr{R}$, and hence $\mathscr{U}$ has a locally finite open refinement $\mathscr{R}$. Thus $P(\gamma)$ holds. ■

In the following, we investigate the relationships between paracompactness and normality.

Definition A.1.10 A space X is said to be a *collectionwise normal space* [62] if, for every discrete family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed sets in X, there is a disjoint family $\{U_\alpha\}_{\alpha\in\Lambda}$ of open sets in X such that $F_\alpha \subset U_\alpha$. X is called an *expandable space* [225] if, for every locally finite family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed sets in X, there is a locally finite family $\{U_\alpha\}_{\alpha\in\Lambda}$ of open sets in X such that $F_\alpha \subset U_\alpha$.

Collectionwise normality can be simply described as every discrete family of closed sets has a disjoint open expansion. Also, expandability can be simply described as every locally finite family of closed sets has a locally finite open expansion.

Theorem A.1.11 *Every T_2 paracompact space is a both a collectionwise normal space and an expandable space [225].*

Proof Suppose X is a T_2 paracompact space. Let $\mathscr{F} = \{F_\alpha\}_{\alpha\in\Lambda}$ be a discrete family of closed sets in X. For each $\alpha \in \Lambda$, let $V_\alpha = X - \cup(\mathscr{F} - \{F_\alpha\})$. Then the open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X has an open star-refinement $\mathscr{V}$. Let $U_\alpha = \mathrm{st}(F_\alpha, \mathscr{V})$. Then $\{U_\alpha\}_{\alpha\in\Lambda}$ is a disjoint open expansion of $\mathscr{F}$. Thus X is a collectionwise normal space.

Let $\mathscr{F} = \{F_\alpha\}_{\alpha\in\Lambda}$ be a locally finite family of closed sets in X. For each $\Gamma \in \Lambda^{<\omega}$, define

$$V(\Gamma) = X - \cup\{F_\alpha : \alpha \in \Lambda - \Gamma\}.$$

For each $x \in X$, let $\Gamma_x = \{\alpha \in \Lambda : x \in F_\alpha\}$. Then $\Gamma_x \in \Lambda^{<\omega}$ and $x \in V(\Gamma_x)$. Let $\mathscr{V}$ be a locally finite open refinement of the open cover $\{V(\Gamma) : \Gamma \in \Lambda^{<\omega}\}$ of X. For each $\alpha \in \Lambda$, let $U_\alpha = \mathrm{st}(F_\alpha, \mathscr{V})$. Then $F_\alpha \subset U_\alpha$ and for each $x \in X$, there exist an open neighborhood V of x and $n \in \mathbb{N}$ such that $(\mathscr{V})_V = \{V_i : i \leqslant n\}$. For each $i \leqslant n$, take $\Gamma_i \in \Lambda^{<\omega}$ such that $V_i \subset V(\Gamma_i)$. Let $\Gamma = \{\alpha \in \Lambda : V \cap U_\alpha \neq \varnothing\}$. Then $\Gamma \subset \bigcup_{i\leqslant n} \Gamma_i$, so $\Gamma \in \Lambda^{<\omega}$. Hence $\{U_\alpha\}_{\alpha\in\Lambda}$ is a locally finite family of open sets in X. Thus X is an expandable space. ■

We did not use the axiom of T_2 separation in the proof for expandability above. Therefore, for every locally finite family $\{F_n\}_{n\in\mathbb{N}}$ of closed sets in a countably paracompact space X, there is a locally finite family $\{U_n\}_{n\in\mathbb{N}}$ of open sets in X such that $F_n \subset U_n$.

Proposition A.1.12 *Suppose $\{F_\alpha\}_{\alpha\in\Lambda}$ is a discrete family of closed sets in a collectionwise normal space X. Then there is a discrete family $\{G_\alpha\}_{\alpha\in\Lambda}$ of open sets in X such that $F_\alpha \subset G_\alpha$.*

Proof By the collectionwise normality of X, there is a family $\{U_\alpha\}_{\alpha\in\Lambda}$ of disjoint open sets in X such that $F_\alpha \subset U_\alpha$. Let $F = \bigcup_{\alpha\in\Lambda} F_\alpha$, $U = \bigcup_{\alpha\in\Lambda} U_\alpha$. By the normality of X, there is a continuous function $f : X \to \mathbb{I}$ such that $f(F) \subset \{1\}$ and $f(X - U) \subset \{0\}$. For each $\alpha \in \Lambda$, define $G_\alpha = U_\alpha \cap \{x \in X : f(x) > 1/2\}$. Then $\{G_\alpha\}_{\alpha\in\Lambda}$ is a discrete family of open sets in X and $F_\alpha \subset G_\alpha$. ■

Proposition A.1.13 ([204]) *A space X is a countably paracompact space if and only if for every increasing open cover $\{G_n\}_{n\in\mathbb{N}}$ of X, there is a sequence $\{H_n\}$ of closed sets in X such that $H_n \subset G_n$ and $X = \bigcup_{n\in\mathbb{N}} H_n^\circ$.*

Proof Suppose $\{G_n\}_{n\in\mathbb{N}}$ is an increasing open cover of the countably paracompact space X. Then there is a locally finite open cover $\{V_n\}_{n\in\mathbb{N}}$ of X such that $V_n \subset G_n$. For each $n \in \mathbb{N}$, let $H_n = X - \bigcup_{m>n} V_m$. Then H_n is a closed in X and $H_n \subset \bigcup_{m\leqslant n} V_m \subset G_n$. For each $x \in X$, there is an open neighborhood U_x of x which only meets finitely many V_n. Let $k = \max\{n \in \mathbb{N} : U_x \cap V_n \neq \varnothing\}$. Then $U_x \subset H_k$. So $X = \bigcup_{n\in\mathbb{N}} H_n^\circ$.

On the other hand, suppose $\mathscr{U} = \{U_n\}_{n\in\mathbb{N}}$ is an open cover of X. For each $n \in \mathbb{N}$, let $G_n = \bigcup_{m\leqslant n} U_m$. Then $\{G_n\}_{n\in\mathbb{N}}$ is an increasing open cover of X, and hence there is a sequence $\{H_n\}$ of closed sets in X such that $H_n \subset G_n$ and $X = \bigcup_{n\in\mathbb{N}} H_n^\circ$. Let $V_n = U_n - \bigcup_{m<n} H_m$. Then $\{V_n\}_{n\in\mathbb{N}}$ is an open refinement of $\mathscr{U}$. For each $x \in X$, take $n \in \mathbb{N}$ such that $x \in H_n^\circ$. Then $H_n^\circ \cap V_m = \varnothing$ whenever $m > n$. So $\{V_n\}_{n\in\mathbb{N}}$ is locally finite, and hence X is a countably paracompact space. ■

A.2 Metacompact Spaces

Definition A.2.1 A space X is called a *metacompact space* (or *weakly paracompact space*) [20] if every open cover of X has a point-finite open refinement. X is called a *countably metacompact space* [155] if every countable open cover of X has a point-finite open refinement.

Using the same method as Proposition A.1.4 and Lemma A.1.6, we can prove the following Proposition A.2.2 and Lemma A.2.3 respectively.

Proposition A.2.2 *Every perfect preimage of a metacompact space is a metacompact space.*

Lemma A.2.3 *Suppose $\mathscr{U}$ is an interior-preserving open cover of a space X. Then $\mathscr{U}^F$ has a closure-preserving closed refinement if and only if $\mathscr{U}$ has an interior-preserving open pointwise W-refinement.*

Theorem A.2.4 ([219, 420]) *For every space X, the following are equivalent:*

(1) X is a metacompact space.
(2) Every well-monotone open cover of X has a point-finite open refinement.
(3) For every open cover $\mathscr{U}$ of X, $\mathscr{U}^F$ has a closure-preserving closed refinement.

Proof By Lemma A.2.3 we get (1) ⇒ (3). By an argument same as that in the proof of (2) ⇒ (1) in Theorem A.1.9 and replacing "locally finite" with "point-finite", we can prove (2) ⇒ (1). Now we prove (3) ⇒ (2).

Suppose $\mathscr{U}$ is a well-monotone open cover of X. By Lemma A.2.3, there is a sequence $\{\mathscr{U}_n\}$ of interior-preserving open covers of X such that $\mathscr{U}_1 = \mathscr{U}$ and $\mathscr{U}_{n+1}$ is a pointwise W-refinement of $\mathscr{U}_n$. By Lemma A.1.8, $\mathscr{U}$ has an open refinement $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ such that each $\mathscr{V}_n$ is a point-finite family of subsets in X. Then the directed open cover $\{\bigcup_{k\leqslant n}(\cup\mathscr{V}_k) : n \in \mathbb{N}\}$ of X has a closure-preserving closed refinement $\{F_n : n \in \mathbb{N}\}$ such that $F_n \subset \bigcup_{k\leqslant n}(\cup\mathscr{V}_k)$. So $\bigcup_{n\in\mathbb{N}}\{V - \bigcup_{k<n} F_k : V \in \mathscr{V}_n\}$ is a point-finite open refinement of $\mathscr{U}$. ■

By (1) ⇔ (3) of Theorem A.2.4, we have the following mapping theorem for metacompact spaces.

Corollary A.2.5 ([479] (The Worrell theorem) *Metacompactness is invariant under closed mappings.*

Theorem A.2.6 ([218, 479]) *For every space X, the following are equivalent:*

(1) X is a metacompact space.
(2) Every open cover of X has a point-finite refinement $\mathscr{H}$ such that $x \in st(x, \mathscr{H})^\circ$ for each $x \in X$.
(3) Every open cover of X has an open pointwise W-refinement.

Proof (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Suppose each open cover $\mathscr{U}$ of X has a point-finite refinement $\mathscr{H}$ satisfying the assumption of (2). For each $H \in \mathscr{H}$, take $U_H \in \mathscr{U}$ such that $H \subset U_H$. For each $x \in X$, define

$$V_x = \mathrm{st}(x, \mathscr{H})^\circ \cap (\cap\{U_H : H \in (\mathscr{H})_x\}).$$

Then $\{V_x : x \in X\}$ is an open pointwise W-refinement of $\mathscr{U}$.

(3) $\Rightarrow$ (1). Suppose $\mathscr{U}$ is an open cover of X. Take a sequence $\{\mathscr{U}_n\}$ of open covers of X such that $\mathscr{U}_1 = \mathscr{U}$ and $\mathscr{U}_{n+1}$ is a pointwise W-refinement of $\mathscr{U}_n$. By Lemma A.1.8, $\mathscr{U}$ has an open refinement $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ such that each $\mathscr{V}_n$ is point-finite in X. Let $\mathscr{W}$ be an open pointwise W-refinement of $\{\cup\mathscr{V}_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let

$$F_n = \{x \in X : \mathrm{st}(x, \mathscr{W}) \subset \cup\{\cup\mathscr{V}_i : i \leqslant n\}\}.$$

Then $\overline{F}_n \subset \bigcup_{i\leqslant n}(\cup\mathscr{V}_i)$ and $X = \bigcup_{n\in\mathbb{N}} \overline{F}_n$. Thus $\bigcup_{n\in\mathbb{N}}\{V - \bigcup_{i<n} \overline{F}_i : V \in \mathscr{V}_n\}$ is a point-finite open refinement of $\mathscr{U}$. ■

Corollary A.2.7 ([219] *Every pseudo-open compact image of a paracompact space is a metacompact space.*

Proof Suppose $f : X \to Y$ is a pseudo-open compact mapping and X is a paracompact space. If $\mathscr{U}$ is an open cover of Y, then the open cover $f^{-1}(\mathscr{U})$ of X has a locally finite open refinement $\mathscr{V}$, so $f(\mathscr{V})$ is a point-finite refinement of $\mathscr{U}$ satisfying $y \in \mathrm{st}(y, f(\mathscr{V}))^\circ$ for each $y \in Y$. It follows from Theorem A.2.6 Y is a metacompact space. ■

Similar to the proof of Proposition A.1.13, we can prove the following Proposition A.2.8.

Proposition A.2.8 ([155, 204]) *A space X is a countably metacompact space if and only if for each decreasing sequence $\{F_n\}$ of closed sets in X with $\bigcap_{n\in\mathbb{N}} F_n = \varnothing$, there is a sequence $\{G_n\}$ of open sets in X such that $F_n \subset G_n$ and $\bigcap_{n\in\mathbb{N}} G_n = \varnothing$.*

A.3 Subparacompact Spaces

Definition A.3.1 [75] A space X is a *subparacompact space* if every open cover of X has a σ-discrete closed refinement.

Proposition A.3.2 ([74]) *Suppose $f : X \to Y$ is a perfect mapping and X is a T_2 space. If Y is a subparacompact space, then X is a subparacompact space.*

Proof Suppose $\mathscr{U}$ is an open cover of X. For each $y \in Y$, $f^{-1}(y)$ is a compact set in X. By Lemma 3.2.8 in the text, there is $O_y \in \tau(X)$ such that $f^{-1}(y) \subset O_y$ and $\mathscr{U}_{|O_y}$ has a finite closed refinement in O_y. Since f is a closed mapping, there is $H_y \in \tau(Y)$ such that $y \in H_y$ and $f^{-1}(H_y) \subset O_y$. Then $\mathscr{U}_{|f^{-1}(H_y)}$ has a finite closed refinement $\mathscr{F}_y$ in $f^{-1}(H_y)$. Let $\mathscr{H} = \{H_y : y \in Y\}$. Since Y is a subparacompact space, $\mathscr{H}$ has a refinement $\bigcup_{i\in\mathbb{N}} \mathscr{W}_i$ such that each $\mathscr{W}_i = \{W_{i,y} : y \in Y\}$ is a discrete family of closed sets and $W_{i,y} \subset H_y$. For each $i \in \mathbb{N}$, let $\mathscr{P}_i = \{f^{-1}(W_{i,y}) \cap F : y \in Y,\ F \in \mathscr{F}_y\}$. We prove $\mathscr{P} = \bigcup_{i\in\mathbb{N}} \mathscr{P}_i$ is a σ-discrete closed refinement of $\mathscr{U}$. Obviously, $\mathscr{P}$ is a refinement of $\mathscr{U}$. For each $i \in \mathbb{N}$, since $f^{-1}(\mathscr{W}_i)$ is discrete and $\mathscr{F}_y$ is finite for each $y \in Y$, $\mathscr{P}_i$ is σ-discrete. For every $y \in Y$, $F \in \mathscr{F}_y$ and $x \in X - (f^{-1}(W_{i,y}) \cap F)$, define

$$L_x = \begin{cases} X - f^{-1}(W_{i,y}), & x \in X - f^{-1}(W_{i,y}), \\ f^{-1}(H_y) - F, & x \in f^{-1}(W_{i,y}) - F. \end{cases}$$

Then L_x is an open neighborhood of x in X and $L_x \cap (f^{-1}(W_{i,y}) \cap F) = \varnothing$. Thus $f^{-1}(W_{i,y}) \cap F$ is closed, and hence X is a subparacompact space. ■

Theorem A.3.3 ([75, 217]) *For every space X, the following are equivalent:*

(1) X is a subparacompact space.
(2) Every open cover $\mathscr{U}$ of X has a sequence $\{\mathscr{U}_n\}$ of open refinements such that for each $x \in X$, there exist $n \in \mathbb{N}$ and $U \in \mathscr{U}$ with $st(x, \mathscr{U}_n) \subset U$.
(3) Every open cover of X has a σ-locally finite closed refinement.
(4) Every open cover of X has a σ-closure-preserving closed refinement.
(5) Every open cover of X has a σ-cushioned refinement.

Proof It is obvious that (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), so we only need to prove (1) $\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose $\mathscr{U}$ is an open cover of X. Let $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ be a closed refinement of $\mathscr{U}$ such that each $\mathscr{F}_n$ is discrete. For every $n \in \mathbb{N}$ and $F \in \mathscr{F}_n$, take $U_F \in \mathscr{U}$ such that $F \subset U_F$. Define

$$E_n = \cup\mathscr{F}_n,$$
$$V_F = U_F - (E_n - F),$$
$$\mathscr{V}_n = \{V_F : F \in \mathscr{F}_n\} \cup \{U - E_n : U \in \mathscr{U}\}.$$

Then the sequence $\{\mathscr{V}_n\}$ of open covers of X satisfies the requirement of (2).

(2) $\Rightarrow$ (5). Suppose $\mathscr{U}$ is an open cover of X. Let $\{\mathscr{U}_n\}$ be a sequence of open refinements of $\mathscr{U}$ satisfying the assumption of (2). Denote $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$. For every $n \in \mathbb{N}$ and $\alpha \in \Lambda$, define

$$F(n,\alpha) = \{x \in X : \mathrm{st}(x, \mathscr{U}_n) \subset U_\alpha\},$$
$$\mathscr{F}_n = \{F(n,\alpha) : \alpha \in \Lambda\}.$$

Then $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ is a refinement of $\mathscr{U}$. For each $\Lambda' \subset \Lambda$, suppose $x \in X - \bigcup_{\alpha\in\Lambda'} U_\alpha$. If $n \in \mathbb{N}, \alpha \in \Lambda'$ and $y \in F(n,\alpha)$, then $x \notin \mathrm{st}(y, \mathscr{U}_n)$, so $y \notin \mathrm{st}(x, \mathscr{U}_n)$, and hence $x \notin \overline{\bigcup_{\alpha\in\Lambda'} F(n,\alpha)}$. Thus, $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ is a σ-cushioned refinement of $\mathscr{U}$.

(5) $\Rightarrow$ (1). For the narrative brevity, we use the following notation: for every $n, k \in \mathbb{N}$ and $s = (i_1, i_2, \ldots, i_k) \in \mathbb{N}^k$, denote $s \oplus n = (i_1, i_2, \ldots, i_k, n)$.

Let $\mathscr{U} = \{U_\alpha : \alpha < \gamma\}$ be an open cover of X. If $k \in \mathbb{N}$ and $s \in \mathbb{N}^k$, then we make an induction on k as follows: take an open refinement $\mathscr{U}(s)$ of $\mathscr{U}$ and a σ-cushioned refinement $\mathscr{F}(s)$ of $\mathscr{U}(s)$. For each $t \in \mathbb{N}$, define

$$V_\alpha(t) = W_\alpha(t) = U_\alpha,\ \alpha < \gamma;$$
$$\mathscr{U}(t) = \{V_\alpha(t) : \alpha < \gamma\} \cup \{W_\alpha(t) : \alpha < \gamma\}.$$

Now assume that $\mathscr{U}(s)$ is an open refinement of $\mathscr{U}$ with the form

$$\mathscr{U}(s) = \{V_\alpha(s) : \alpha < \gamma\} \cup \{W_\alpha(s) : \alpha < \gamma\}.$$

Let $\mathscr{F}(s)$ be a σ-cushioned refinement of $\mathscr{U}(s)$ with the form

$$\mathscr{F}(s) = \{H_\alpha(s \oplus n) : \alpha < \gamma,\ n \in \mathbb{N}\} \cup \{K_\alpha(s \oplus n) : \alpha < \gamma,\ n \in \mathbb{N}\},$$

where $\{H_\alpha(s \oplus n) : \alpha < \gamma\}$ and $\{K_\alpha(s \oplus n) : \alpha < \gamma\}$ are cushioned in $\{V_\alpha(s) : \alpha < \gamma\}$ and $\{W_\alpha(s) : \alpha < \gamma\}$ respectively. Let $U_\alpha(s) = V_\alpha(s) \cup W_\alpha(s)$. Define

$$\begin{aligned} V_\alpha(s \oplus n) &= U_\alpha(s) - \overline{\cup\{H_\beta(s \oplus n) \cup K_\beta(s \oplus n) : \alpha \neq \beta < \gamma\}}, \\ W_\alpha(s \oplus n) &= U_\alpha(s) \cap (\cup\{U_\beta(s) : \alpha < \beta < \gamma\}) \\ &\quad - \overline{\cup\{H_\beta(s \oplus n) \cup K_\beta(s \oplus n) : \beta < \alpha\}}, \\ \mathscr{U}(s \oplus n) &= \{V_\alpha(s \oplus n) : \alpha < \gamma\} \cup \{W_\alpha(s \oplus n) : \alpha < \gamma\}. \end{aligned}$$

(3.1) $\mathscr{U}(s \oplus n)$ is a cover of X.

For each $x \in X - \bigcup_{\alpha<\gamma} V_\alpha(s \oplus n)$, let $\delta = \min\{\alpha < \gamma : x \in U_\alpha(s)\}$. Then

$$x \in \overline{\bigcup_{\delta<\beta<\gamma} H_\beta(s \oplus n) \cup K_\beta(s \oplus n)},$$

so there is $\beta > \delta$ such that $x \in U_\beta(\delta)$, and hence $x \in W_\delta(s \oplus n)$. Thus, $\mathscr{U}(s \oplus n)$ covers X.

For every $s \in \mathbb{N}^k$, $n \in \mathbb{N}$, and $\alpha < \gamma$, define

$$T_\alpha(s \oplus n) = \overline{H_\alpha(s \oplus n)} - \cup\{V_\beta(s) : \alpha \neq \beta < \gamma\}.$$

Since $\{H_\alpha(s \oplus n) : \alpha < \gamma\}$ is cushioned in $\{V_\alpha(s) : \alpha < \gamma\}$, $\{T_\alpha(s \oplus n) : \alpha < \gamma\}$ is a discrete family of closed sets in X and $T_\alpha(s \oplus n) \subset U_\alpha$.

(3.2) $\{T_\alpha(s \oplus n) : \alpha < \gamma,\ s \in \bigcup_{k \in \mathbb{N}} \mathbb{N}^k,\ n \in \mathbb{N}\}$ is a cover of X.

For each $x \in X$, let $\delta = \min\{\beta < \gamma : x \in H_\beta(h) \cup K_\beta(h),\ h \in \bigcup_{k \in \mathbb{N}} \mathbb{N}^k\}$. Then there exist $t \in \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$ and $n \in \mathbb{N}$ such that $x \in H_\delta(t \oplus n) \cup K_\delta(t \oplus n)$. Take $m \in \mathbb{N}$ and $\sigma < \gamma$ such that

$$x \in H_\sigma(t \oplus n \oplus m) \cup K_\sigma(t \oplus n \oplus m) \subset V_\sigma(t \oplus n) \cup W_\sigma(t \oplus n).$$

Then $x \notin V_\sigma(t \oplus n) \cup W_\sigma(t \oplus n)$ whenever $\alpha > \delta$, so $\sigma = \delta$. While it follows from $x \notin \bigcup_{\beta > \delta} U_\beta(t \oplus n)$ that $x \notin W_\delta(t \oplus n \oplus m)$, and hence $x \in H_\delta(t \oplus n \oplus m)$. Since $x \in H_\delta(t \oplus n) \cup K_\delta(t \oplus n)$, $x \notin V_\beta(t \oplus n)$ whenever $\beta \neq \delta$, so

$$x \in H_\delta(t \oplus n \oplus m) - \bigcup_{\beta \neq \delta} V_\beta(t \oplus n) \subset T_\delta(t \oplus n \oplus m).$$

In summary, $\mathscr{U}$ has a σ-discrete closed refinement, thus X is a subparacompact space. ■

Corollary A.3.4 ([75]) *Subparacompactness is invariant under closed mappings.*

A.4 Submetacompact Spaces

Definition A.4.1 A space X is called a *submetacompact space* [217] (or *θ-refinable space* [481]) if, every open cover of X has a sequence $\{\mathscr{U}_n\}$ of open refinements satisfying that for each $x \in X$, there is $n \in \mathbb{N}$ such that x is only in finitely many elements of $\mathscr{U}_n$. The sequence $\{\mathscr{U}_n\}$ is said to be an open *θ-refinable sequence* of this cover.

A sequence $\{\mathscr{U}_n\}$ of covers of X is said to be a *pointwise W-refining sequence* of a cover $\mathscr{U}$ of X if for each $x \in X$, there exist $\mathscr{U}' \in \mathscr{U}^{<\omega}$ and $n \in \mathbb{N}$ such that $(\mathscr{U}_n)_x$ is a partial refinement of $\mathscr{U}'$.

Theorem A.4.2 ([481]) *Every submetacompact countably compact space is a compact space.*

Proof Suppose X is a submetacompact countably compact space. Let $\mathscr{U}$ be an open cover of X. Then $\mathscr{U}$ has an open θ-refinable sequence $\{\mathscr{U}_n\}$. For each $n \in \mathbb{N}$, denote $\mathscr{U}_n = \{U_\alpha : \alpha \in \Lambda_n\}$. For different elements $\alpha_1, \ldots, \alpha_k$ in Λ_n, let

$$F_n(\alpha_1, \ldots, \alpha_k) = \cap\{U_{\alpha_i} : i \leqslant k\} - \cup\{U_\alpha : \alpha \in \Lambda_n - \{\alpha_1, \ldots, \alpha_k\}\}.$$

Since $\{F_n(\alpha) : \alpha \in \Lambda_n\}$ is a discrete family in X, it is a finite set, so there is a finite subfamily $\mathscr{U}_{n,1}$ of $\mathscr{U}_n$ covering $\{F_n(\alpha) : \alpha \in \Lambda_n\}$. By the same reason, there is a finite subfamily $\mathscr{U}_{n,2}$ of $\mathscr{U}_n$ covering $\cup\{F_n(\alpha_1, \alpha_2) - \cup\mathscr{U}_{n,1} : \alpha_1, \alpha_2 \in \Lambda_n\}$, and continue in this way, we can obtain a sequence $\{\mathscr{U}_{n,k}\}$ consisting of finite subfamilies of $\mathscr{U}_n$ such that $\bigcup_{k\in\mathbb{N}} \mathscr{U}_{n,k}$ covers $\cup\{F_n(\alpha_1, \ldots, \alpha_k) : \alpha_i \in \Lambda_n, i \leqslant k \in \mathbb{N}\}$. Then $\bigcup_{n,k\in\mathbb{N}} \mathscr{U}_{n,k}$ is a countable subcover of $\bigcup_{n\in\mathbb{N}} \mathscr{U}_n$, and hence $\mathscr{U}$ has a finite subcover. Thus X is a compact space. ■

In the following, we introduce the characterizations of submetacompact spaces obtained by Junnila.

Lemma A.4.3 *If every (interior-preserving) open cover of a space X has an (interior-preserving) open pointwise W-refining sequence, then every (interior-preserving) open cover of X has a sequence $\{\mathscr{U}_n\}$ of open refinements such that*

(∗) for each $x \in X$, there is a strictly increasing sequence $\{n_i\}$ in $\mathbb{N}$ such that $(\mathscr{U}_{n_{i+1}})_x$ is a partial refinement of $\mathscr{U}_{n_i}$.

Proof For an (interior-preserving) open cover $\mathscr{U}$ of X, there is an (interior-preserving) open pointwise W-refining sequence $\{\mathscr{H}_{1,n}\}$, and we denote it by $\mathscr{U} \to \{\mathscr{H}_{1,n}\}_{n\geqslant 1}$. By the same reason, we have, for each $m \in \mathbb{N}$,

$$\mathscr{H}_{m,m} \wedge \left(\bigwedge_{i\leqslant m} \mathscr{H}_{i,m+1}\right) \to \{\mathscr{H}_{m+1,n}\}_{n\geqslant m+1}.$$

Let $\mathscr{U}_n = \mathscr{H}_{n,n}$. Then for every $x \in X$ and $m \in \mathbb{N}$, there is $n > m$ such that $(\mathscr{U}_n)_x$ is a partial refinement of $\mathscr{U}_m$. Thus the (interior-preserving) open cover $\mathscr{U}$ of X has a sequence $\{\mathscr{U}_n\}$ of (interior-preserving) open refinements satisfying (∗). ■

Lemma A.4.4 *If a cover $\mathscr{B}$ of a space X has a sequence $\{\mathscr{U}_n\}$ of open refinements satisfying (∗), then $\mathscr{B}$ has an open θ-refinable sequence.*

Proof Let $\mathscr{B} = \{B_\alpha : \alpha < \gamma\}$, where γ is a cardinal number. For each $U \in \bigcup_{k\in\mathbb{N}} \mathscr{U}^k$, define $\alpha(U) = \min\{\beta < \gamma : U \subset B_\beta\}$. For every $n \in \mathbb{N}$ and $V \in \bigcup_{k\in\mathbb{N}} \mathscr{U}^k$, we say that $\mathscr{U}_n$ has the property $\Phi(V)$ if $\{\alpha(U) : V \subset U \in \mathscr{U}_n\} \subset \{\alpha(V)\}$. For every $n, k \in \mathbb{N}$, define

$$\mathscr{V}_{n,k} = \{V \in \mathscr{U}_k : \mathscr{U}_n \text{ has the property } \Phi(V)\},$$
$$L_{n,k} = \{x \in X : (\mathscr{U}_k)_x \text{ is a partial refinement of } \mathscr{U}_n\}.$$

For every $n > 1$ and $s \in \mathbb{N}^n$, let

$$L_s = \bigcap_{i<n} L_{s(i),s(i+1)}, \text{ where } s(i) \text{ is the } i\text{-th coordinate of } s(i);$$

$$H_s = \left\{x \in L_s : \mathrm{st}(x, \mathscr{U}_{s(n)}) \subset \cup\left(\bigcup_{i<n} \mathscr{V}_{s(i),s(i+1)}\right)\right\}.$$

(4.1) $\{H_s : s \in \bigcup_{n>1} \mathbb{N}^n\}$ is a cover of X.

For each $x \in X$, there exist a strictly increasing sequence $\{t_n\}$ in $\mathbb{N}$ and a sequence $\{\mathscr{F}_n\}$, where each $\mathscr{F}_n$ is a finite family of sets in X, such that, $\mathscr{F}_n \subset (\mathscr{U}_{t_n})_x$ and $(\mathscr{U}_{t_{n+1}})_x$ partially refines $\mathscr{U}_{t_n}$. For each $n > 1$, let

$$\mathscr{P}_n = \{F \in \mathscr{F}_n : F \not\subset \cup(\cup\{\mathscr{V}_{t_i,t_{i+1}} : i < n\})\}.$$

Then $\mathscr{P}_n = \varnothing$ for some $n > 1$. Because otherwise, for each $n > 1$, let $\alpha_n = \max\{\alpha(P) : P \in \mathscr{P}_n\}$. Since $\mathscr{P}_{n+1}$ partially refines $\mathscr{P}_n$, $\alpha_{n+1} \leqslant \alpha_n$, so there is $k > 3$ such that $\alpha_k = \alpha_{k-1} = \alpha_{k-2}$. Take $P \in \mathscr{P}_k$ such that $\alpha(P) = \alpha_k$. We prove that $\mathscr{U}_{t_{k-1}}$ has the property $\Phi(P)$. In fact, if $P \subset U \in \mathscr{U}_{t_{k-1}}$, then $U \in (\mathscr{U}_{t_{k-1}})_x$, so there is $F \in \mathscr{F}_{k-2}$ such that $U \subset F$, and hence $P \subset U \subset F$, it follows that $\alpha_k = \alpha(P) \leqslant \alpha(U) \leqslant \alpha(F) \leqslant \alpha_{k-2}$, as a consequence, $\alpha(U) = \alpha(P)$, thus $\mathscr{U}_{t_{k-1}}$ has the property $\Phi(P)$. It follows that $P \in \mathscr{V}_{t_{k-1},t_k}$, which contradicts the fact that $P \in \mathscr{P}_k$. So there is $n > 1$ such that $\mathscr{P}_n = \varnothing$. Let $t = (t_1, t_2, \ldots, t_{n+1})$. Then $x \in H_t$ and hence (4.1) is proved.

For each $k \in \mathbb{N}$, define

$$W_{n,k,\alpha} = \cup\{V \in \mathscr{V}_{n,k} : \alpha(V) = \alpha\},\ \alpha < \gamma;$$
$$\mathscr{W}_{n,k} = \{W_{n,k,\alpha} : \alpha < \gamma\}.$$

(4.2) $\mathscr{W}_{n,k}$ is a point-finite family in $L_{n,k}$.

For each $x \in L_{n,k}$, there is $\mathscr{F} \in \mathscr{U}_n^{<\omega}$ such that $(\mathscr{U}_k)_x$ refines $\mathscr{F}$. Let $\Lambda = \{\alpha(F) : F \in \mathscr{F}\}$. To prove $(\mathscr{W}_{n,k})_x$ is finite, we only need to verify $\{\alpha < \gamma : x \in W_{n,k,\alpha}\} \subset \Lambda$. Suppose $\alpha < \gamma$ and $x \in W_{n,k,\alpha}$. Then there is $V \in \mathscr{V}_{n,k}$ such that $x \in V$ and $\alpha(V) = \alpha$, so $V \in (\mathscr{U}_k)_x$, and hence there is $F \in \mathscr{F}$ such that $V \subset F$. Since $\mathscr{U}_n$ has the property $\Phi(V)$, $\alpha(V) = \alpha(F)$, and hence $\alpha \in \Lambda$. Thus, (4.2) is proved.

For every $n > 1$ and $s \in \mathbb{N}^n$, define

$$\mathscr{W}_s = \bigcup_{i<n} \mathscr{W}_{s(i),s(i+1)},$$
$$\mathscr{U}_s = \{U \in \mathscr{U}_{s(n)} : U \not\subset \cup\mathscr{W}_s\},$$
$$\mathscr{H}_s = \mathscr{W}_s \cup \mathscr{U}_s.$$

Then $\mathscr{H}_s$ is an open refinement of $\mathscr{B}$ and

(4.3) $\mathscr{H}_s$ is a point-finite family in H_s.

For each $x \in H_s$, $\mathrm{st}(x, \mathscr{U}_{s(n)}) \subset \cup(\bigcup_{i<n} \mathscr{V}_{s(i),s(i+1)}) = \cup\mathscr{W}_s$, so $x \notin \cup\mathscr{U}_s$, and hence $(\mathscr{H}_s)_x = (\mathscr{W}_s)_x = \bigcup_{i<n}(\mathscr{W}_{s(i),s(i+1)})_x$. By (4.2), $(\mathscr{H}_s)_x$ is finite.

By a combination of (4.1) and (4.3), $\{\mathscr{H}_s : s \in \bigcup_{n>1} \mathbb{N}^n\}$ is an open θ-refinable sequence of $\mathscr{B}$. ■

Lemma A.4.5 *If every well-monotone open cover of a space X has an open θ-refinable sequence, then every open cover of X has an open pointwise W-refining sequence.*

Proof For each cardinal number γ, the following proposition is expressed as $P(\gamma)$: every open cover of X with cardinality γ has an open pointwise W-refining sequence. We prove $P(\gamma)$ is true for every cardinal number γ by a method of transfinite induction.

Suppose γ is an infinite cardinal number and for each $\lambda < \gamma$, $P(\lambda)$ is true. If $\mathscr{U}$ is an open cover of X with cardinality γ, denote $\mathscr{U} = \{U_\alpha : \alpha < \gamma\}$. Then $\{\bigcup_{\beta \leqslant \alpha} U_\beta : \alpha < \gamma\}$ is a well-monotone open cover of X, and hence it has an open θ-refinable sequence $\{\mathscr{V}_n\}$. For $V \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n$, take $\alpha(V) < \gamma$ such that $V \subset \cup\{U_\beta : \beta \leqslant \alpha(V)\}$. For every $\alpha < \gamma$, $n \in \mathbb{N}$, let

$$P_{\alpha,n} = \cup\{V \in \mathscr{V}_n : \alpha(V) > \alpha\},$$
$$\mathscr{P}_{\alpha,n} = \{P_{\alpha,n}\} \cup \{U_\beta : \beta \leqslant \alpha\}.$$

By the induction hypothesis, $\mathscr{P}_{\alpha,n}$ has an open pointwise W-refining sequence $\{\mathscr{W}_{\alpha,n,k}\}$. For every $n, i \in \mathbb{N}$, define

$$F_{n,i} = \{x \in X : |(\mathscr{V}_n)_x| \leqslant i\},$$
$$G_n = \bigcup_{i\in\mathbb{N}} F_{n,i}.$$

Then $F_{n,i}$ is a closed set in X and $X = \bigcup_{n\in\mathbb{N}} G_n$. For each $x \in G_n$, let

$$\alpha(x, n) = \max\{\alpha(V) : V \in (\mathscr{V}_n)_x\}.$$

Take $W_{n,k}(x) \in (\mathscr{W}_{\alpha(x,n),n,k})_x$, and let

$$H_{n,k}(x) = \left(\bigcap_{i\leqslant k} W_{n,i}(x)\right) \cap (\cap(\mathscr{V}_n)_x),$$
$$\mathscr{H}_{n,k} = \{H_{n,k}(x) : x \in G_n\} \cup \{X - F_{n,k}\}.$$

(5.1) $\mathscr{H}_{n,k}$ is a cover of X.
Because for each $x \in X$, $x \in H_{n,k}(x)$ whenever $x \in F_{n,k}$.
(5.2) $\{\mathscr{H}_{n,k}\}$ is a pointwise W-refining sequence of $\mathscr{U}$.
For each $x \in X$, take $n \in \mathbb{N}$ such that $x \in G_n$. Let

$$\Lambda = \{\alpha(V) : V \in (\mathscr{V}_n)_x\}.$$

Then for each $\alpha \in \Lambda$, there exist $k_\alpha \in \mathbb{N}$ and $\mathscr{F}_\alpha \in \mathscr{P}_{\alpha,n}^{<\omega}$ such that $(\mathscr{W}_{\alpha,n,k_\alpha})_x$ is a partial refinement of $\mathscr{F}_\alpha$. Define

$$\mathscr{F} = \bigcup_{\alpha \in \Lambda} \mathscr{F}_\alpha \cap \mathscr{U},$$
$$k = \max\{k_\alpha : \alpha \in \Lambda\} + |(\mathscr{V}_n)_x|.$$

Then $(\mathscr{H}_{n,k})_x$ is a partial refinement of $\mathscr{F}$. In fact, let $H \in (\mathscr{H}_{n,k})_x$. Then $x \in F_{n,k}$, so there is $y \in G_n$ such that $H = H_{n,k}(y)$. Denote $\alpha = \alpha(y, n)$ and take $V \in (\mathscr{V}_n)_y$ such that $\alpha(V) = \alpha$. Then $x \in H_{n,k}(y) \subset V$ and $\alpha \in \Lambda$. Let $j = k_\alpha$. Since $j \leqslant k$, $H_{n,k}(y) \subset W_{n,j}(y)$, so there is $F \in \mathscr{F}_\alpha$ such that $W_{n,j}(y) \subset F$. Since $y \in F - P_{\alpha,n}$, $F \in \mathscr{U}$, and hence $H \subset F \in \mathscr{F}$. Since $\mathscr{F}$ is finite, $\{\mathscr{H}_{n,k}\}$ is a pointwise W-refining sequence of $\mathscr{U}$.

Thus, every open cover of X has an open pointwise W-refining sequence. ■

Lemma A.4.6 *If every interior-preserving directed open cover of a space X has a σ-closure-preserving closed refinement, then every interior-preserving open cover of X has an open θ-refinable sequence.*

Proof Suppose $\mathscr{U}$ is an interior-preserving open cover of X. Then $\mathscr{U}^F$ is an interior-preserving directed open refinement of X. So $\mathscr{U}^F$ has a closed refinement $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ such that each $\mathscr{F}_n$ is closure-preserving. For every $x \in X$ and $n \in \mathbb{N}$, let

$$V_n(x) = \cap(\mathscr{U})_x - \cup\{F \in \mathscr{F}_n : x \notin F\},$$
$$\mathscr{V}_n = \{V_n(x) : x \in X\}.$$

Then $\mathscr{V}_n$ is an interior-preserving open refinement of $\mathscr{U}$. We prove $\{\mathscr{V}_n\}$ is a pointwise W-refining sequence of $\mathscr{U}$. For each $x \in X$, take $n \in \mathbb{N}$ and $F \in \mathscr{F}_n$ such that $x \in F$, then there is $\mathscr{U}' \in \mathscr{U}^{<\omega}$ such that $F \subset \cup\mathscr{U}'$. Let $y \in X$ and $x \in V_n(y)$. Then $y \in F$. Thus there is $U \in \mathscr{U}'$ such that $y \in U$, so $V_n(y) \subset U$, and hence $(\mathscr{V}_n)_x$ is a partial refinement of $\mathscr{U}'$. By Lemmas A.4.3 and A.4.4, $\mathscr{U}$ has an open θ-refinable sequence. ■

Lemma A.4.7 *If an open cover $\mathscr{U}$ of a space X has an open θ-refinable sequence, then there is a σ-closure-preserving closed cover $\mathscr{F}$ of X satisfying that for each $x \in X$, there exist $F \in \mathscr{F}$ and $\mathscr{U}' \in (\mathscr{U})_x^{<\omega}$ such that $x \in F \subset \cup\mathscr{U}'$.*

Proof Suppose $\{\mathscr{V}_n\}$ is an open θ-refinable sequence of $\mathscr{U}$. For every $n, k \in \mathbb{N}$, let

$$F_{n,k} = \{x \in X : |(\mathscr{V}_n)_x| \leqslant k\}.$$

Then $F_{n,k}$ is a closed set in X. Define

$$F_n(\mathscr{V}) = X - \cup(\mathscr{V}_n - \mathscr{V}),\ \mathscr{V} \in \mathscr{V}_n^{<\omega};$$
$$\mathscr{F}_n = \{F_n(\mathscr{V}) : \mathscr{V} \in \mathscr{V}_n^{<\omega}\};$$
$$\mathscr{F}_{n,k} = \mathscr{F}_{n|F_{n,k}}.$$

Since $\mathscr{V}_{n|F_{n,k}}$ is point-finite, it is an interior-preserving open cover for the subspace $F_{n,k}$ of X, and hence $\mathscr{F}_{n,k}$ is a closure-preserving family of closed sets in X. Let

$$\mathscr{F} = \cup\{\mathscr{F}_{n,k} : n, k \in \mathbb{N}\}.$$

For each $x \in X$, take $n, k \in \mathbb{N}$ such that $x \in F_{n,k}$. Further take $\mathscr{U}' \in (\mathscr{U})_x^{<\omega}$ such that $(\mathscr{V}_n)_x$ is a partial refinement of $\mathscr{U}'$. Then $F = F_n((\mathscr{V}_n)_x) \cap F_{n,k} \in \mathscr{F}$ and $x \in F \subset \cup\mathscr{U}'$. ∎

Theorem A.4.8 ([217]) *For every space X, the following are equivalent:*

(1) X is a submetacompact space.
(2) Every open cover of X has an open pointwise W-refining sequence.
(3) Every well-monotone open cover of X has an open θ-refinable sequence.
(4) Every interior-preserving directed open cover of X has a σ-closure-preserving closed refinement.
(5) Every directed open cover of X has a σ-closure-preserving closed refinement.
(6) For every open cover $\mathscr{U}$ of X, there is a σ-closure-preserving family $\mathscr{F}$ of closed sets in X satisfying that for each $x \in X$, there exist $F \in \mathscr{F}$ and $\mathscr{U}' \in (\mathscr{U})_x^{<\omega}$ such that $x \in F \subset \cup\mathscr{U}'$.

Proof By Lemma A.4.7, we get (1) ⇒ (6). (6) ⇒ (5) ⇒ (4) is obvious. (4) ⇒ (3) can be obtain by Lemma A.4.6, and (3) ⇒ (2) follows from Lemma A.4.5. By Lemmas A.4.3 and A.4.4, (2) ⇒ (1) holds. ∎

Corollary A.4.9 ([217] The Junnila theorem) *Submetacompactness is invariant under closed mappings.*

Theorem A.4.10 ([431]) *A space X is a paracompact space if and only if X is a submetacompact expandable space.*

Proof We only need to prove the sufficiency. Suppose X is a submetacompact expandable space. Since every expandable space is a countably paracompact space, to prove X is a paracompact space, we only need to prove every open cover of X has a σ-locally finite open refinement. Suppose $\mathscr{U}$ is an open cover of X. Let $\{\mathscr{U}_i\}$ be an open θ-refinable sequence of $\mathscr{U}$. For each $i \in \mathbb{N}$, denote $\mathscr{U}_i = \{U_\alpha : \alpha \in \Lambda_i\}$. We inductively construct a sequence $\{\mathscr{V}_{i,j}\}$ of families of open sets in X such that

(1) $\mathscr{V}_{i,j}$ is a locally finite partial refinement of $\mathscr{U}_i$;
(2) if $|(\mathscr{U}_i)_x| \leqslant m$ for some $x \in X$, then $x \in \bigcup_{j \leqslant m} V_{i,j}$, where $V_{i,j} = \cup\mathscr{V}_{i,j}$;
(3) if $x \in V_{i,j}$, then $|(\mathscr{U}_i)_x| \geqslant j$.

First, let $\mathscr{V}_{i,0} = \varnothing$. Suppose for $0 \leqslant j \leqslant n$, we have constructed $\mathscr{V}_{i,j}$ satisfying conditions (1)–(3). Let $\mathscr{A} = \{A \subset \Lambda_i : |A| = n + 1\}$. For each $A \in \mathscr{A}$, define

$$F_A = \left(X - \bigcup_{j \leqslant n} V_{i,j}\right) \cap (X - \cup\{U_\alpha : \alpha \in \Lambda_i - A\}).$$

Then $\mathscr{F} = \{F_A : A \in \mathscr{A}\}$ is a family of closed sets in X. For each $x \in X$, if $|(\mathscr{U}_i)_x| < n+1$, then the neighborhood $\bigcup_{j \leqslant n} V_{i,j}$ of x disjoints with any element of

$\mathscr{F}$; if $|(\mathscr{U}_i)_x| \geqslant n+1$, take $A \in \mathscr{A}$ such that $x \in \cap\{U_\alpha : \alpha \in A\}$, then $\cap\{U_\alpha : \alpha \in A\}$ is only possible to meet the element F_A of $\mathscr{F}$. So $\mathscr{F}$ is a discrete family of closed sets in X. By the expandability of X, there is a locally finite family $\{G_A : A \in \mathscr{A}\}$ of open sets in X such that $F_A \subset G_A$. Let

$$\mathscr{V}_{i,n+1} = \{G_A \cap (\cap\{U_\alpha : \alpha \in A\}) : A \in \mathscr{A}\}.$$

Then $\mathscr{V}_{i,n+1}$ is a family satisfying conditions (1)–(3).

Since $\{\mathscr{U}_i\}$ is an open θ-refinable sequence of $\mathscr{U}$, $\bigcup_{i,j\in\mathbb{N}} \mathscr{V}_{i,j}$ is a σ-locally finite open refinement of $\mathscr{U}$, and hence X is a paracompact space. ■

Proposition A.4.11 ([155]) *Every submetacompact space is a countably metacompact space.*

Proof Suppose $\mathscr{U} = \{U_n\}_{n\in\mathbb{N}}$ is a countable open cover of a submetacompact space X. Let $\{\mathscr{U}_n\}$ be an open θ-refinable sequence of $\mathscr{U}$. Let

$$V_1 = U_1, \text{ and } V_n = \mathrm{st}(X - \bigcup_{i<n} U_i, \bigwedge_{i<n} \mathscr{U}_i), \ n > 1.$$

Then $\{V_n\}_{n\in\mathbb{N}}$ is a point-finite open refinement of $\mathscr{U}$, and hence X is a countably metacompact space. ■

A.5 Meta-Lindelöf Spaces

Definition A.5.1 [18] A space X is called a *meta-Lindelöf space* if every open cover of X has a point-countable open refinement. If every subspace of X is a meta-Lindelöf space, then X is said to be a *hereditarily meta-Lindelöf space*.

A cover $\mathscr{U}$ of a space X is called a *well-ordered cover* of X if $\mathscr{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ and Λ is a well-ordered set. For a well-ordered cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of X, denote $\alpha_x = \min\{\alpha \in \Lambda : x \in U_\alpha\}$ for each $x \in X$. If $\alpha \in \Lambda$, then denote $\widetilde{U}_\alpha = \{x \in X : \alpha_x = \alpha\}$. Obviously, $\widetilde{U}_\alpha \subset U_\alpha$.

Theorem A.5.2 ([167]) *For every space X, the following are equivalent:*

(1) X is a hereditarily meta-Lindelöf space.
(2) Every well-ordered open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X has a point-countable open refinement $\mathscr{V}$ satisfying that for each $x \in X$, there is $V \in \mathscr{V}$ such that $x \in V \subset U_{\alpha_x}$.
(3) Every well-ordered open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X has a point-countable open refinement $\{V_\alpha\}_{\alpha\in\Lambda}$ such that $\widetilde{U}_\alpha \subset V_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.

Proof (1) ⇒ (2). Suppose X is a hereditarily meta-Lindelöf space, and there exist an ordinal number γ and a well-ordered open cover $\{U_\alpha\}_{\alpha<\gamma}$ of X which has no point-countable open refinement satisfying the requirement of (2). We may assume

γ is the least ordinal number satisfying the above property. For each $\alpha < \gamma$, the well-ordered open cover $\{U_\beta\}_{\beta<\alpha}$ of the subspace $\bigcup_{\beta<\alpha} U_\beta$ has a point-countable open refinement $\mathscr{V}_\alpha$ satisfying the requirement of (2). If there is an ordinal number α such that $\gamma = \alpha + 1$, then $\mathscr{V}_\alpha \cup \{U_\alpha\}$ is a point-countable open refinement of $\{U_\beta\}_{\beta<\gamma}$ satisfying the requirement of (2), and hence γ must be a limit ordinal number. Suppose $\mathscr{W}$ is a point-countable open refinement of the open cover $\{U_\alpha\}_{\alpha<\gamma}$ of X. For each $W \in \mathscr{W}$, pick $\alpha(W) < \gamma$ such that $W \subset U_{\alpha(W)}$. Define

$$\mathscr{V} = \{W \cap V : W \in \mathscr{W},\ V \in \mathscr{V}_{\alpha(W)+1}\}.$$

Then $\mathscr{V}$ is a point-countable family of open sets in X. For each $x \in X$, take $W \in (\mathscr{W})_x$. Then $x \in U_{\alpha(W)} \subset \cup\{U_\beta : \beta < \alpha(W) + 1\}$, so there is $V \in \mathscr{V}_{\alpha(W)+1}$ such that $x \in V \subset U_{\alpha_x}$, it follows that $W \cap V \in \mathscr{V}$ and $x \in W \cap V \subset U_{\alpha_x}$. Thus, $\mathscr{V}$ is a point-countable open refinement of $\{U_\alpha\}_{\alpha<\gamma}$ satisfying the requirement of (2), a contradiction.

(2) $\Rightarrow$ (3). Suppose $\{U_\alpha\}_{\alpha\in\Lambda}$ is a well-ordered open cover of X. Let $\mathscr{V}$ be a point-countable open refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$ satisfying the requirement of (2). For each $\alpha \in \Lambda$, let

$$V_\alpha = \cup\{V \in \mathscr{V} : V \subset U_\alpha;\ \text{if}\ \beta < \alpha,\ \text{then}\ V \not\subset U_\beta\}.$$

Then $\{V_\alpha\}_{\alpha\in\Lambda}$ is a point-countable open cover of X satisfy the requirement of (3).

(3) $\Rightarrow$ (1). Suppose Y is a subspace of X and $\mathscr{W} = \{W_\alpha\}_{\alpha<\gamma}$ is a well-ordered open cover of Y. Take a well-ordered open cover $\{U_\alpha\}_{\alpha\leqslant\gamma}$ of X such that $U_\gamma = X$ and $U_\alpha \cap Y = W_\alpha$ when $\alpha < \gamma$. Let $\{V_\alpha\}_{\alpha\leqslant\gamma}$ be a point-countable open refinement of $\{U_\alpha\}_{\alpha\leqslant\gamma}$ satisfying the requirement of (3). If $y \in Y$, then there is $\alpha < \gamma$ such that $y \in \widetilde{U}_\alpha \subset V_\alpha$, so $\{V_\alpha \cap Y\}_{\alpha<\gamma}$ is a point-countable open refinement of $\mathscr{W}$. Thus Y is a meta-Lindelöf space, and hence X is a hereditarily meta-Lindelöf space. ∎

Corollary A.5.3 ([167] *Hereditarily meta-Lindelöf spaces are preserved by pseudo-open s-mappings.*

Proof Suppose $f : X \to Y$ is a pseudo-open s-mapping and X is a hereditarily meta-Lindelöf space. If Z is a subspace of Y, then $f_{|f^{-1}(Z)} : f^{-1}(Z) \to Z$ is also a pseudo-open s-mapping, and hence we only need to prove Y is a meta-Lindelöf space. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a well-ordered open cover of Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ is a well-ordered open cover of X, and hence it has a point-countable open refinement $\{V_\alpha\}_{\alpha\in\Lambda}$ such that $f^{-1}(U_\alpha)^\sim \subset V_\alpha \subset f^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$. Then $f(V_\alpha) \subset U_\alpha$ and $f^{-1}(\widetilde{U}_\alpha) \subset f^{-1}(U_\alpha)^\sim \subset V_\alpha$. Let $W_\alpha = f(V_\alpha)^\circ$ for each $\alpha \in \Lambda$. Since f is a pseudo-open s-mapping, $\{W_\alpha\}_{\alpha\in\Lambda}$ is a point-countable open refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$ such that $\widetilde{U}_\alpha \subset W_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$. Thus Y is a meta-Lindelöf space. ∎

Appendix B
The Formation of the Theory of Generalized Metric Spaces

In the 1900 Paris International Congress of Mathematicians, Hilbert [187] pointed out at his famous speech titled "Mathematical problems" that as long as a branch of science can make a lot of problems, it is full of vitality, while the lack of problems indicates that the decline or suspension of independent development. The development of general topology in the 20th century confirmed the Hilbert quotes.

In 1944, Dieudonné [112] introduced the concept of paracompactness. This is an important symbol of general topology into the heyday. Subsequently, the theory of topological spaces developed rapidly. The form of its representation is a variety of topological properties found to adapt for different purposes, and the basic direction is to give various generalizations of paracompactness and metrizability for solving all kinds of problems. In virtue of these researches, the theory of generalized metric spaces became an important topic of general topology since the 1960s. The book named "Handbook of the History of General Topology" edited by Aull and Lowen [46–48] has made detailed description on the history of general topology. In this book, the chapter of "A history of generalized metrizable spaces" [193] written by Hodel focused on clarifying the main achievements of the theory of generalized metric spaces during the period of 1950–1980. This appendix try to clarify the formation process and some development of this theory by taking problems as the clue. Due to space limitations, we only focus on the description of the background of some kinds of generalized metric properties introduced during the period of 1944–1976. It should be noticed that the division of every period here is completely in order to illustrate the sequence of events of the problems. As for the modern development of the theory of generalized metric spaces and related topics, in addition to this book, readers can refer to the reference literature listed at the end of this book, especially the book called "Encyclopedia of General Topology" [175] edited by Hart, Nagata and Vaughan.

S. Lin and Z. Yun, *Generalized Metric Spaces and Mappings*,
Atlantis Studies in Mathematics 6, DOI 10.2991/978-94-6239-216-8

B.1 A Historical Review

The central topic of general topology in early study is problems about metrization and compactness [4].

As pointed out by Rudin [411], "The most fundamental theorems, those which are part of every mathematician's background, were proved at this time". A lot of fundamental work has laid a solid foundation for general topology, and made it become an independent branch of mathematics and played a positive role in promoting the development of other disciplines of mathematics. The most important results are as follows.

Theorem B.1.1 (The Urysohn–Tychonoff metrization theorem, 1925) *Every regular space with a countable base is a metrizable space.*

Theorem B.1.2 (The Urysohn extension theorem, 1925) *A space X is a normal space if and only if every real valued continuous function from a closed subspace of X is continuously extendable over X.*

Theorem B.1.3 (The Tychonoff theorem, 1935) *Every Cartesian product of compact spaces is compact.*

Theorem B.1.4 (The Tychonoff compact extension theorem, 1935) *A space is completely regular if and only if there is a compact extension of it.*

There are four classes of spaces involving in above theorems and they are metrizable spaces, compact spaces, normal spaces and completely regular spaces. Due to the widespread applications, the four classes of spaces became the main object concerned by topologists at that time and thus achieved fruitful results. It also produced a series of problems to be solved. The reasons for this are mainly from two aspects: First, there are larger gaps among metrizable spaces, compact spaces and normal spaces. It is necessary to find the classes of spaces with nice properties and between metrizable spaces, compact spaces and normal spaces. Second, the main topics concerned by topologists in this period are certain finiteness or countability of families of sets. In which ways some uncountable situation should be discussed? For example, in a paper published in 1925, Urysohn raised a question on seeking metrization theorems for general spaces such that the Urysohn–Tychonoff metrization theorem is a natural corollary of them.

The significance of the main early achievements in general topology is to serve as a model for future development of general topology.[1] The concepts defined in that period or more early, such as the topological sum of topological spaces (Tietze 1923), box products (Tietze 1923), Cartesian products (Tychonoff 1930), limits of inverse systems (Lefschetz 1931), adjunction operator (Borsuk 1937), different kinds of compact extensions [compactifications (Carathéodory 1913), Alexandroff compactification (1924), Čech-Stone compactification (1937), Wallman compactification (1938)],

[1] Data mainly come from R. Engelking [119].

several important classes of spaces [countably compact spaces (Fréchet 1906), compact spaces (Vietoris 1921), Lindelöf spaces (Alexandroff and Urysohn 1929), separable spaces (Fréchet 1906), CCC (Suslin 1920), locally compact spaces (Alexandroff 1921), metrizable spaces (Fréchet 1906), first countable spaces (Hausdorff 1914), second countable spaces (Hausdorff 1914), Moore spaces (Moore 1916), developable spaces (Alexandroff and Urysohn 1923), semi-metrizable spaces (Wilson 1931), connected spaces (Hausdorff 1914), various axioms of separation], mappings of different forms [continuous mappings (Fréchet 1910), homeomorphic mappings (Fréchet 1910), closed mappings (Hurewicz 1926), open mappings (Aronszajn 1931), quotient mappings (Baer and Levi 1932), monotone mappings (Whyburn 1934)], the topology of pointwise convergence and the compact-open topology in the sets of continuous functions (Fox 1945) etc., have become important tools for the modern general topology research.

B.2 The Foundation Laying Period

The spaces in this appendix are at least T_2 spaces, τ and τ^c represent the topology and the set of closed subsets for a space respectively, and all mappings are assumed to be continuous and onto. The hypothesis of separation properties for individual space is slightly different of those in the original literature.

B.2.1 The Introduction of Paracompactness

In the research of the theory of topological spaces, the key breakthrough on the restriction of finiteness or countability for families of sets in spaces is the wild usages of locally finite (locally countable) families and point-finite (point-countable) families.

Definition B.2.1 Suppose $\mathscr{P}$ is a family of sets in a space X. $\mathscr{P}$ is said to be *locally finite* [5] (resp. *locally countable*) in X if every point of X has a neighborhood which only meets at most finitely (resp. countably) many elements of $\mathscr{P}$; $\mathscr{P}$ is said to be *point-finite* (resp. *point-countable*) in X if every point of X only belongs to at most finitely (resp. countably) many elements of $\mathscr{P}$.

Definition B.2.2 A space X is called a *paracompact space* [112] if every open cover of X has a locally finite open refinement.

It was established by Alexandroff, in modern terminology, that every separable metrizable space is paracompact [5]. "In the late 1940s and early 1950s, general topology had a strong upsurge brought on by the definition of paracompactness" [411]. Compactness $\Rightarrow$ paracompactness $\Rightarrow$ normality. In 1940, Tukey [469] introduced fully normal spaces which are similar to paracompact spaces by the concept of normal coverings and proved that every metrizable space is a fully normal space. In 1948, Stone [440] proved that the class of fully normal spaces is equivalent to the class of paracompact spaces, and hence every metrizable space is a paracompact space. The class of paracompact spaces is a class between the class of metrizable spaces or compact spaces and the class of normal spaces. The properties of paracompact spaces were concerned about by many topologists.

Every paracompact countably compact space is a compact space and this proposition implies the following general question: What kinds of countably compact spaces are compact spaces? Arens and Dugundji [20] introduced metacompact (or weakly paracompact) spaces by point-finite families, and proved that every metacompact countably compact space is a compact space.

Definition B.2.3 A space X is called a *metacompact space* [20] if every open cover of X has a point-finite open refinement.

The topological properties defined by open covers and their refinements, such as paracompactness, metacompactness etc., are collectively referred to as *covering properties*.

B.2.2 The Metrization Problems

In view of the role played by the metric spaces in many fields of mathematics, seeking metrization theorems for general topological spaces is of great significance.

Definition B.2.4 Suppose $\mathscr{P}$ is a family of sets in a space X. $\mathscr{P}$ is said to be *discrete* in X [62] if each point in X has a neighborhood which only meets at most one element of $\mathscr{P}$. Let Φ be a property of families. Then the family $\mathscr{P}$ in X is called a σ-Φ *family* if $\mathscr{P}$ the union of countably many families in X with the Φ property.

Both σ-discrete families and σ-locally finite families are generalizations of countable families. By these concepts, Bing [62], Nagata [370] and Smirnov [428] obtained the following excellent metrization theorem in general topology.

Theorem B.2.5 (The Bing-Nagata-Smirnov metrization theorem) *A regular space X is metrizable if and only if X satisfies one of the following conditions:*

(1) X has a σ-discrete base.
(2) X has a σ-locally finite base.

Just like this, in 1964, S. Wang [474] for the first time gave a characterization of ω_μ*-metrizable spaces* as follows: a regular space X is ω_μ-metrizable if and only if X has a ω_μ-base. Morita and Hanai [363] and Stone [441] proved independently that metrizability is invariant under perfect mappings. Several years later, various generalizations of metrizability were obtained. By excitation of the Hanai-Morita-Stone theorem, topologists widely investigated the metrization problem for images of metrizable spaces and the topological properties preserved by perfect mappings. For example, in 1964 Y. Liu and L. Liu [315] proved that an adjunction space of two metrizable spaces is metrizable if and only if it is first countable.

The obtaining of several metrization theorems for general spaces does not declare that the metrization problems are completely solved. In some sense, the metrization theorems of general spaces is only a foundation to get metrization theorems of spaces with specific topological properties, because the metrization problems for spaces with specific topological properties are far from resolved and more important in real life applications. Many classic metrization problems, such as the Alexandroff problem that whether every perfectly normal topological manifold is metrizable posed in 1923 [6], and the question whether every normal Moore space is a metrizable space raised by Jones [214], have been bothering and motivating the general topologists contemporary. Just the *normal Moore space conjecture* dominated the study of metrization problems and promoted the progress of related fields in general topology for half a century. In his famous paper on metrization of topological spaces [62], Bing defined collectionwise normality and give a partial answer to the normal Moore space conjecture. Around this conjecture, Alexandroff [2] introduced the concept of uniform bases.

Definition B.2.6 A base $\mathscr{B}$ for a space X is said to be a *uniform base* [2] if, for every $x \in U \in \tau$, $\{B \in \mathscr{B} : x \in B \not\subset U\}$ is finite.

Alexandroff made a conjecture that every normal space with a uniform base is a metrizable space. Heath [177] gave another form of the Alexandroff conjecture that every metacompact normal Moore is a metrizable space. As a generalization of spaces with a uniform base, Arhangel'skiĭ [27] introduced the concept of BCO (i.e. base of countable order) spaces.

Definition B.2.7 A base $\mathscr{B}$ for a space X is called a *BCO* [27] if for each $x \in X$, $\{B_i\}$ is a neighborhood base of x in X whenever $\{B_i\}$ is a decreasing sequence of sets in $\mathscr{B}$ containing x. A space with a BCO is called a BCO space.

Every developable space is a BCO space and every paracompact BCO space is a metrizable space. However, a collectionwise normal BCO space may not be a metrizable space. The another generalization of spaces with a uniform base is spaces with a σ-point-finite base [27]. Although a paracompact space with a σ-point-finite base may not be a metrizable space, every collectionwise normal perfect space with a σ-point-finite base is a metrizable space. The following classic problem posed by Heath [191, 384] is relevant to this result: Is every perfectly normal paracompact space with a point-countable base metrizable? Todorčevič [468] gave a negative answer to this problem in 1991.

The great enthusiasm of topologists for paracompactness could not be separated form a series of basic characterizations of paracompact spaces obtained in the 1950s. The beautiful characterizations of paracompactness by means of discrete families, locally finite families, closure preserving families and cushioned families given by Michael [325–327] is a breakthrough in the study of paracompactness.

Definition B.2.8 Suppose $\mathscr{P}$ is a family of sets in a space X. $\mathscr{P}$ is called a *closure-preserving family* [326] in X if, for any $\mathscr{P}' \subset \mathscr{P}$, $\overline{\cup \mathscr{P}'} = \cup \overline{\mathscr{P}'}$. A cover $\mathscr{B}$ of X is called a *cushioned refinement* of $\mathscr{P}$ [327] if, for each $B \in \mathscr{B}$, there is $P_B \in \mathscr{P}$ such that $\overline{\cup \mathscr{B}'} \subset \cup\{P_B : B \in \mathscr{B}'\}$ for any $\mathscr{B}' \subset \mathscr{B}$.

Theorem B.2.9 *For every space X, the following are equivalent:*

(1) X is a paracompact space.
(2) Every open cover of X has a locally finite closed refinement.
(3) Every open cover of X has a closure-preserving closed refinement.
(4) Every open cover of X has a cushioned refinement.
If further assume X is a regular space, then they are equivalent to the following:
(5) Every open cover of X has a σ-discrete open refinement.
(6) Every open cover of X has a σ-locally finite open refinement.
(7) Every open cover of X has a σ-closure-preserving open refinement.
(8) Every open cover of X has a σ-cushioned open refinement.

A direct corollary of Theorem B.2.9 is that paracompactness is invariant under closed mappings [326]. This is the beginning of investigation of the problem if some particular covering properties are invariant under closed mappings. By the enlightenment of Michael's characterizations of paracompactness and the Bing-Nagata-Smirnov metrization theorem, Ceder [92] introduced M_i-spaces ($i = 1, 2, 3$) in a classic paper titled "Some generalizations of metric spaces".

Definition B.2.10 Suppose $\mathscr{B}$ is a family of ordered pairs $B = (B_1, B_2)$ of sets in a space X. $\mathscr{B}$ is said to be a *pair-base* for X if, for each $B \in \mathscr{B}$, $B_1 \in \tau$, and for every $x \in U \in \tau$, there is $B \in \mathscr{B}$ such that $x \in B_1 \subset B_2 \subset U$. $\mathscr{B}$ is called a *cushioned family* in X if for any $\mathscr{B}' \subset \mathscr{B}$, $\overline{\cup\{B_1 : B \in \mathscr{B}'\}} \subset \cup\{B_2 : B \in \mathscr{B}'\}$. A family $\mathscr{P}$ of sets in X is called a *quasi-base* for X if $\{(P^\circ, P) : P \in \mathscr{P}\}$ is a pair-base for X.

Definition B.2.11 ([92]) Let X be a regular space. X is called an M_1-*space* if X has a σ-closure-preserving base; X is called an M_2-*space* if X has a σ-closure-preserving quasi-base; X is called an M_3-*space* if X has a σ-cushioned base.

Every M_1-space is an M_2-space, and every M_2-space is an M_3-space. However, is every M_3-space an M_2-space? Is every M_2-space an M_1-space? These famous questions were raised by Ceder [92]. Ceder's paper has opened the prelude of the study on generalized metric spaces. The study of M_i-spaces is one of the main clues for the development of the theory of generalized metric spaces.

B.2.3 Paracompactness of Product Spaces

After proving that every product of a compact space and a paracompact space is a paracompact space, Dieudonné [112] asked whether every product of two paracompact spaces is a paracompact space. Sorgenfrey [433] constructed a paracompact space (which was called the Sorgenfrey line later) such that the product of the space multiplied by itself is not even a normal space. Compared with metric spaces or compact spaces, the fatal defect of paracompact spaces is that not every product of two paracompact spaces is a paracompact space. To investigate the normality of product spaces, Dowker [116] and Katětov [224] introduced the class of countable paracompact spaces independently.

Definition B.2.12 A space X is called a *countably paracompact space* [116, 224] if every countable open cover of X has a locally finite open refinement.

Dowker proved that a normal space X is a countably paracompact space if and only if the product of the space X and the unit closed interval $\mathbb{I}$ is a normal space. After a discussion on the precise relationship between normality and countably paracompactness, Dowker raised the question whether there is a normal space which is not countably paracompact. Such spaces are called *Dowker spaces* and a Dowker space was constructed by Rudin [409] until 1971.

Michael [325] asked whether every product of a paracompact space and a metrizable space is a paracompact space. Ten years later, Michael [329] constructed a paracompact space (it was called the Michael line later), such that, the product of this space and a separable metrizable space is not a normal space. Tamano [446] raised the question of seeking a necessary and sufficient condition for a space X, such that, $X \times Y$ is a normal space for any metrizable space Y. A paper of Morita [360] gave an appropriate solution of the Tamano problem.

Definition B.2.13 A space X is called a *Morita space* (or *Morita's P-space*[2] [360]) if, for every family of open sets $\{G(\alpha_1, \ldots, \alpha_i) : \alpha_1, \ldots, \alpha_i \in \Omega, i \in \mathbb{N}\}$ such that $G(\alpha_1, \ldots, \alpha_i) \subset G(\alpha_1, \ldots, \alpha_{i+1})$, there is a family $\{F(\alpha_1, \ldots, \alpha_i) : \alpha_1, \ldots, \alpha_i \in \Omega, i \in \mathbb{N}\}$ of closed sets in X with the following properties:

(1) $F(\alpha_1, \ldots, \alpha_i) \subset G(\alpha_1, \ldots, \alpha_i)$;
(2) if $\{\alpha_i\}$ is a sequence in Ω such that $X = \bigcup_{i\in\mathbb{N}} G(\alpha_1, \ldots, \alpha_i)$, then $X = \bigcup_{i\in\mathbb{N}} F(\alpha_1, \ldots, \alpha_i)$.

Morita proved that a space X is a normal Morita space if and only if any product space of X and a metrizable space is a normal space. Frolík [135] proved that any product space of countably many Čech-complete paracompact spaces is a paracompact space. Frolík's theorem is undoubtedly the first satisfactory result about the paracompactness of product spaces. Its shortcoming is that a metrizable space may not be a Čech-complete space. For this reason, Morita [360] defined the following M-spaces.

Definition B.2.14 A space X is called an *M-space* [360] if, there is a normal sequence $\{\mathscr{U}_i\}$ of open covers of X, such that, for each $x \in X$ and any sequence $\{x_i\}$ in X, $\{x_i\}$ has an accumulation point in X whenever $x_i \in \mathrm{st}(x, \mathscr{U}_i)$.

[2] A space X is called a *P-space* if every G_δ-set of X is open in X, i.e., every point of X is a P-point.

The concepts of M-spaces and Čech-complete spaces are independent. However, they are coincide in the class of paracompact spaces. Countably compact spaces, metrizable spaces and Čech-complete paracompact spaces are all M-spaces, and every M-space is a Morita space. Any product space of countably many paracompact M-spaces is a paracompact M-space.

The above discussion is based on the fact that metrizability is countably productive. For the normality of product spaces of the uncountably many metrizable spaces, Stone [440] obtained the following equivalent conditions.

Theorem B.2.15 *For every family $\{X_\alpha\}_{\alpha \in \Lambda}$ of nonempty metrizable spaces, the following are equivalent:*

(1) $\prod_{\alpha \in \Lambda} X_\alpha$ *is a normal space.*
(2) $\prod_{\alpha \in \Lambda} X_\alpha$ *is a paracompact space.*
(3) There are at most countably many α in Λ such that the X_α is not a compact space.

Thus, $\mathbb{N}^{\aleph_1}$ is not a normal space. The researchers transferred interest of normality or paracompactness of product spaces of uncountably many spaces into normality or paracompactness of suitable subspaces of product spaces. By this idea, Corson [108] introduced the concepts of σ-products and Σ-products, as generalizations of finite products and countable products respectively.

Definition B.2.16 Suppose $X = \prod_{\alpha \in \Lambda} X_\alpha$ is a product of infinitely many spaces. Take $b = (b_\alpha) \in X$. For each $x \in X$, denote $\text{supp}(x) = \{\alpha \in \Lambda : x_\alpha \neq b_\alpha\}$. Let

$$\sigma(b) = \{x \in X : |\text{supp}(x)| < \aleph_0\},$$
$$\Sigma(b) = \{x \in X : |\text{supp}(x)| \leqslant \aleph_0\}.$$

Then the subspaces $\sigma(b)$ and $\Sigma(b)$ of X are called a *σ-product* and a *Σ-product* of spaces $\{X_\alpha : \alpha \in \Lambda\}$ with the based point b respectively. Generally, they are denoted as $\sigma\{X_\alpha : \alpha \in \Lambda\}$ and $\Sigma\{X_\alpha : \alpha \in \Lambda\}$ respectively.

Corson proved that every σ-product of separable metrizable spaces is a Lindelöf space, and every Σ-product of complete metrizable spaces is a normal space. The problem if every Σ-product of metrizable spaces is a normal space was a most attractive famous open problem. In 1977, Gul'ko [168] solved this problem positively. In 1983, Rudin [413] proved that every Σ-product of metrizable spaces is a shrinking space.

B.2.4 Spaces and Classifications

In fact, early work on continuous functions, quotient mappings, open mappings and closed mappings done by researchers in general topology has become a driving force for the development of this field.

Since 1944, some of more powerful mappings are constantly emerging, such as compact mappings (Vaĭnšteĭn 1947), perfect mappings (Vaĭnšteĭn 1947), π-mappings (Ponomarev 1960), pseudo-open mappings (Arhangel'skiĭ 1963), compact-covering mappings (Michael 1966) an so on. Mappings and spaces are mutually dependent, and hence in order to promote the vigorous development of mappings, it is necessary to discuss the bridge bond between mappings and spaces. In 1961, one important event which had great influence on the development of general topology was that the international topological symposium named "General Topology and its Relations to Modern Analysis and Algebra" held in Prague for the first time (the conference holds every 5 years once since then) [111]. In this conference, Alexandroff [3] put forward the idea of investigating spaces by mappings, namely, to connect various classes of spaces by using mappings as a link, then according to the differences between classes of spaces and mappings to arranged the study. The Alexandroff idea was based on several known results. For example, Gale [136] characterized k-spaces as quotient images of locally compact spaces; Ponomarev [401] characterized first countable spaces as open images of metrizable spaces; Frolík [135] characterized Čech-complete paracompact spaces as perfect preimages of complete metrizable spaces. Alexandroff made a conjecture that paracompact spaces can be characterized as perfect preimages of metrizable spaces. Although this conjecture is incorrect, the conjecture led Arhangel'skiĭ [25] to introduce the concept of p-spaces.

Definition B.2.17 A completely regular space X is called a *p-space* [25] if there is a sequence $\{\mathscr{U}_n\}$ of families of open sets in βX covering X, such that, for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) \subset X$. The sequence $\{\mathscr{U}_n\}$ is called a pluming of X.

Both Čech-complete spaces and completely regular Moore spaces are p-spaces. Arhangel'skiĭ characterized perfect preimages of metrizable spaces as paracompact p-spaces. Another important role played by p-spaces is that each product space of countably many (paracompact) p-spaces is a (paracompact) p-space. Although the class of p-spaces and the class of M-spaces are independent, a theorem was proved that these two classes of spaces coincide in the class of paracompact spaces.

In a word, Alexandroff's mutual classification idea of mappings and spaces has become an important source for further study of general topology, and has been reached in many concrete spaces. For example,

(1) Ponomarev [401] characterized spaces with a point-countable base as open s-images of metrizable spaces;
(2) Arhangel'skiĭ [23] characterized spaces with a uniform base as open compact images of metrizable spaces;
(3) Arhangel'skiĭ [26] characterized Fréchet–Urysohn spaces as pseudo-open images of metrizable spaces;
(4) Morita [360] characterized M-spaces as quasi-perfect preimages of metrizable spaces;
(5) Franklin [133] characterized sequential spaces as quotient images of metrizable spaces;

(6) Heath [179] characterized developable spaces as open π-images of metrizable spaces.

As mentioned above, the development of general topology in the 20 years from 1944 to 1964 laid a solid foundation for the investigation of generalized metric properties and established a research framework of generalized metric properties. The applications of discrete families, locally finite families and closure-preserving families etc. in treating uncountable cases of families were the main features of this period of research. The introduction of paracompactness generalized a lot of important theorems on compact spaces or metrizable spaces to paracompact spaces (These generalizations are often essential), numbers of nice properties of paracompact spaces have been applied in many areas outside general topology, which accelerated the infiltration of general topology, as a basis branch, to other areas. The successful derivation of metrization theorems for general spaces made people understand the essence of metric spaces more profoundly, and brought a bright prospect on a more in-depth and meticulous study of metric properties. More importantly, it allows researchers to avoid the difficulties and complex structures of distance functions in the process of seeking metrization theorems for specific topological spaces, and at the same time is helpful for all kinds of generalizations and applications of metric properties. The central topic in this period was mining properties of paracompact spaces and seeking more metrization theorems. Topologists gradually shifted their early interest and attention for compact spaces and metric spaces to paracompact spaces and generalized metric spaces. Specifically, the scholars focused on the following four problems simultaneously:

(1) Among which class of spaces with weaker topological properties, countable compactness and compactness are equivalent to each other?
(2) Among which class of more general spaces, paracompactness is closed under countable products?
(3) Seeking more metrization theorems for spaces with some specific topology properties.
(4) Classifying spaces by mappings.

B.3 The Formation Period

In 1965 and 1966, Arhangel'skiĭ, Borges, Michael, Wicke, Worrell and others published a number of important papers, which steered the study of general topology into a stage of rapid development and formed the theory of generalized metric spaces.

B.3.1 Isocompactness and Normality

Bacon [49] called a space X isocompact if every countably compact closed subspace of X is compact.

The role in metrization problems played by developable spaces and BCO spaces were indicated by the work in the foundation laying period. What kind of covering properties and base properties do developable spaces have? The article titled "Characterizations of developable topological spaces" written by Worrell and Wicke [481] is one of the most outstanding papers on developable spaces. The concepts of θ-refinable property and θ-base introduced by them gave a satisfactory answer to the above question.

Definition B.3.1 A space X is said to be a *θ-refinable space* [481] if every open cover of X has a sequence $\{\mathscr{U}_n\}$ of open refinements, such that, for each $x \in X$, there is $i \in \mathbb{N}$ with $\mathrm{ord}(x, \mathscr{U}_i) < \aleph_0$, where $\mathrm{ord}(x, \mathscr{U}_i) = |\{U \in \mathscr{U}_i : x \in U\}|$.

Definition B.3.2 A family $\bigcup_{i \in \mathbb{N}} \mathscr{B}_i$ of open sets in a space X is said to be a *θ-base* [481] if for every $x \in U \in \tau$, there exist $i \in \mathbb{N}$ and $B \in \mathscr{B}_i$ such that $x \in B \subset U$ and $\mathrm{ord}(x, \mathscr{B}_i) < \aleph_0$.

Every developable space is a θ-refinable space with a θ-base, and the class of developable spaces is equivalent to the class of θ-refinable BCO spaces. In 1977, Y. Liu [314] introduced the concept of *strictly quasi-paracompact spaces*. Every θ-refinable space is a strictly quasi-paracompact space and every strictly quasi-paracompact space is an isocompact space. In 1976, Reed and Zenor [407] proved that every normal developable manifold is metrizable, which substantively opened a path on metrization of topological manifolds. With a slight modification of the definition of developable spaces, Bennett [55] defined quasi-developable spaces in 1968.

Definition B.3.3 A space X is called a *quasi-developable space* [55] if, there is a sequence $\{\mathscr{B}_i\}$ of families of open sets in X, such that, for each $x \in X$, $\{\mathrm{st}(x, \mathscr{B}_i) : i \in \mathbb{N}, \mathrm{st}(x, \mathscr{B}_i) \neq \varnothing\}$ is a neighborhood base of x in X.

Different from developable spaces, a quasi-developable paracompact space may not be a metrizable space. However, a space is a developable space if and only if it is a perfect quasi-developable space. Bennett and Lutzer [59] proved that the quasi-developable property is equivalent to the θ-base property. Smith [430] positively answered the following question raised by Bennett [57]: Is every quasi-developable space an isocompact space?

The concepts of point-countable bases and θ-bases are independent. If the assumption $\mathrm{ord}(x, \mathscr{U}_i) < \aleph_0$ in the definition of θ-bases is replaced with $\mathrm{ord}(x, \mathscr{U}_i) \leqslant \aleph_0$, then the topological property produced is that of *$\delta\theta$-bases* defined by Aull [44]. Obviously, every space with a point-countable base or a θ-base has a $\delta\theta$-base. Many conclusions on spaces with a point-countable base or a θ-base are also true for spaces with a $\delta\theta$-base. For example, every space with a $\delta\theta$-base is an isocompact space.

In the process of discussion on metrization of compact spaces, Arhangel'skiĭ and Projzvolov [314] introduced the concept of pseudo-bases, and proved that every compact space with a point-countable pseudo-base is metrizable.

Definition B.3.4 A family $\mathscr{B}$ of open sets in a space X is said to be a *pseudo-base* [41] for X if, for every $x, y \in X$ with $x \neq y$, there is $B \in \mathscr{B}$ such that $x \in B \subset X - \{y\}$.

Shiraki [422] pointed out that every space with a point-countable pseudo-base is an isocompact space.

Definition B.3.5 A space X is said to have a *G_δ-diagonal* [92] (resp. *quasi-G_δ-diagonal* [192]) if, there is a sequence $\{\mathscr{U}_n\}$ of open covers of X (resp. families of open sets in X) such that $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) = \{x\}$ (resp. $\bigcap_{n\in\mathbb{N}}\{\mathrm{st}(x, \mathscr{U}_n) : x \in \cup\mathscr{U}_n\} = \{x\}$) for each $x \in X$.

Šneĭder [432] proved every compact space with a G_δ-diagonal is a metrizable space. Anderson [16], Bennett [57] and Heath [183] asked whether every countably compact space with a G_δ-diagonal or a quasi-G_δ-diagonal is a compact space. Chaber [93] answered this question positively. Bennett, Byerly and Lutzer [58] proved the following result.

Theorem B.3.6 *Every countably compact subset of a space X with a quasi-G_δ-diagonal is a compact metrizable G_δ-subset of X.*

So far, the research on the isocompactness of generalized metric spaces can be temporarily come to an end. One good result on covering properties is that Uspenskii [470] proved every σ-metacompact pseudo-compact space is a compact space. Seeking characterizations of isocompact spaces is still an open problem.

For product spaces, normality and paracompactness is a pair of twin brothers, but the former one seems more essential and difficult. In the 20th century, the excellent work done by Dowker in the 50s and by Morita in the 60s played a valuable role for further research. In 1969, Nagata [371] proved that a space X is a paracompact M-space if and only if X is homeomorphic to a closed subspace of a product of a metrizable space and a compact space. In the 1970s, a lot of excellent results emerged. Based on the work of Rudin and Starbird [415], Przymusińsky [404] made a careful study on problems of the normality of product spaces with a compact or metric factor, and gave a necessary and sufficient condition for the product spaces to be normal spaces. A series of results could be obtained as corollaries of his result. In 1977, Burke and van Douwen [86], Kato [226] constructed an M-space X such that X is not homeomorphic to any closed subspace of the product of a metrizable space and a countably compact space. The another clue on normality of product spaces is the following Morita's conjecture [362] .

Conjecture B.3.7 The following conclusions are true:

(1) A space X is a discrete space if and only if for every normal space Y, $X \times Y$ is a normal space.

(2) A space X is a metrizable space if and only if for every normal Morita space Y, $X \times Y$ is a normal space.
(3) A space X is a σ-locally compact metrizable space if and only if for every normal, countably paracompact space Y, $X \times Y$ is a normal space.

Rudin [412] confirmed Conjecture B.3.7(1). Morita [362] proved that if for every non-discrete space Y, there is a normal space X such that $X \times Y$ is not a normal space, then Conjecture B.3.7(3) is true (and hence by Rudin [412], Conjecture B.3.7(3) is actually true). In 2001, Balogh [52] proved Conjecture B.3.7(2) holds.

In the following, we introduce the normality of Σ-products. A Σ-product of compact spaces may not be a normal space (see Theorem B.3.8). So, while discussing the normality of Σ-products, we generally assume that factor spaces are some class of generalized metric spaces. Kombarov [234] obtained a necessary and sufficient condition for Σ-product spaces of paracompact p-spaces to be normal spaces.

Theorem B.3.8 *Suppose $\{X_\alpha\}_{\alpha \in \Lambda}$ is a family of paracompact p-spaces and $X = \Sigma\{X_\alpha : \alpha \in \Lambda\}$. Then the following are equivalent:*

(1) X is a normal space.
(2) X is a collectionwise normal space.
(3) X is of countable tightness.
(4) Every X_α is of countable tightness.

Comparing to paracompactness of product spaces of countably many spaces, Theorem B.3.8 enlighten people to discuss normality of Σ-product spaces of paracompact Σ-spaces. Yajima [483] constructed an example to show that a Σ-product of M_1-spaces may not be a normal space. In 1995, Eda, Gruenhage, Koszmider, Tamano and Todorčevič [118] proved that under the assumption CH, the Σ-product of Lašnev spaces S_{ω_2} is not a normal space. H. Teng [463] proved that every Σ-product of semi-stratifiable spaces is a collectionwise subnormal space.

B.3.2 Generalizations of Bases—Networks

The Bing-Nagata-Smirnov metrization theorem evoked enthusiasm on spaces with various base properties. However, one of the shortcomings of spaces defined by bases is that when considering operations relevant to the "closeness", it is inconvenient, sometimes even very difficult. Therefore, it is necessary to make some appropriate generalizations for the concept of bases. In order to prove the Alexandroff–Urysohn addition theorem of arbitrary cardinal numbers, Arhangel'skiĭ [21] made a successful attempt.

Definition B.3.9 A family $\mathscr{P}$ of sets in a space X is called a *network* [21] for X if every open subset of X is the union of some elements of $\mathscr{P}$.

Arhangel'skiĭ [21, 22] studied spaces with a countable network, proved every space with a countable network is preserved by continuous mappings, and applied the notion of networks to give a short and natural proof of the following theorem: if a compact space X is the union of two subspaces of weight not greater than an infinite cardinal number κ, then the weight of X also does not exceed κ. Every p-space with a countable network has a countable base [25]. It is well-known that the covering dimension $\dim X$, the small inductive dimension $\operatorname{ind} X$, and the large inductive dimension $\operatorname{Ind} X$ of a separable metric space X coincide. Arhangel'skiĭ [31] asked whether they agree in the class of regular continuous images of separable metric spaces, or *cosmic spaces* as they are often called [331]. Cosmic spaces are also characterized as regular spaces which have a countable network [331]. For a cosmic space X it is known that $\operatorname{ind} X = \operatorname{Ind} X$, so the question is whether $\dim X = \operatorname{ind} X$. In 2006, Charalambous [100] gave a cosmic space X such that $\dim X = 1$ and $\operatorname{ind} X = 2$.

In order to investigate quotient images of separable metrizable spaces, Michael [331] introduced the concept of strict k-networks.

Definition B.3.10 A family $\mathscr{P}$ of sets in a space X is said to be a *strict k-network* [331] for X if, for every compact set K with $K \subset U \in \tau$ in X, there is $P \in \mathscr{P}$ such that $K \subset P \subset U$. A regular space with a countable strict k-network is called an $\aleph_0$-*space*.

Every $\aleph_0$-space can be characterized as a regular compact-covering image of a separable metrizable space. However, a metrizable space may not be an $\aleph_0$-space. O'Meara [387] introduced the concept of k-networks as a common generalization of bases and strict k-networks.

Definition B.3.11 A family $\mathscr{P}$ of sets in a space X is said to be a *k-network* [387] for X if, for every compact set K with $K \subset U \in \tau$ in X, there is a finite subfamily $\mathscr{P}'$ of $\mathscr{P}$ such that $K \subset \cup\mathscr{P}' \subset U$. A regular space with a σ-locally finite k-network is called an $\aleph$-*space*.

Strict k-networks were called *pseudo-bases* in [331]. Because the term "pseudo-base" already has been used with a different meaning by Arhangel'skiĭ and Projzvolov [41] (see Definition B.3.4), and the concept of pseudo-bases is stronger than that of k-networks, we use the term "strict k-network" in this book.

The concept of $\aleph$-spaces generalizes that of metrizable spaces and $\aleph_0$-spaces. With the help of the concept of networks, Okuyama [385] defined σ-spaces, which generalizes $\aleph$-spaces and cosmic spaces.

Definition B.3.12 A regular space with a σ-locally finite network is called a σ-*space*.

After investigating several basic properties of σ-spaces, Okuyama raised the following two questions:

(1) Is every regular space with a σ-closure-preserving network a σ-space?
(2) Is every regular closed image of a σ-space a σ-space?

Arhangel'skiĭ [31] discussed spaces with a σ-discrete network. Siwiec and Nagata [426] proved a beautiful characterization theorem of σ-spaces, and solved the above two questions.

Theorem B.3.13 (The Nagata-Siwiec theorem) *For every regular space X, the following are equivalent:*

(1) X is a σ-space.
(2) X has a σ-discrete network.
(3) X has a σ-closure-preserving network.

Because of the characterizations of paracompact spaces given by Michael, the questions on M_i-spaces raised by Ceder and the Nagata-Siwiec theorem, it is natural for researchers to concern about whether the class of σ-space is equivalent to the class of the regular spaces with a σ-cushioned network. In order to describe the problem more clearly, we introduce the innovation work on M_3-spaces done by Borges [65].

Definition B.3.14 A space X is called a *stratifiable space* [65] if, there is a function $G : \mathbb{N} \times \tau^c \to \tau$ on X such that

(1) $H \subset G(n, H)$ and $H = \bigcap_{n\in\mathbb{N}} \overline{G(n, H)}$;
(2) $H \subset F \Rightarrow G(n, H) \subset G(n, F)$.

The class of stratifiable spaces is equivalent to the class of M_3-spaces. Thus, Borges solved some problems on M_i-spaces raised by Ceder [92], for example whether M_3-spaces are preserved by closed mappings. In 1967, Michael defined the concept of semi-stratifiable spaces [110].

Definition B.3.15 A space X is said to be a *semi-stratifiable space* [110] if, there is a function $G : \mathbb{N} \times \tau^c \to \tau$ on X such that

(1) $H = \bigcap_{n\in\mathbb{N}} G(n, H)$;
(2) $H \subset F \Rightarrow G(n, H) \subset G(n, F)$.

Creede [109, 110] studied semi-stratifiable spaces systematically. Stratifiable spaces, σ-spaces and semi-metrizable spaces are all semi-stratifiable spaces. Spaces having a σ-cushioned network were named as *pseudo-stratifiable spaces* by Kofner [231]. He proved that the class of pseudo-stratifiable spaces is equivalent to the class of semi-stratifiable spaces and at the same time constructed a regular Lindelöf pseudo-stratifiable space which is not a σ-space.

Okuyama [385] proved that paracompact σ-spaces are countably productive in his first paper about σ-spaces. Since the class of (paracompact) M-spaces and the class of (paracompact) σ-spaces are independent, another question one met after the appearing of σ-spaces is to look for the class of spaces containing both the class of M-spaces and the class of σ-spaces, and paracompactness is countably productive in this class. Nagami [368] successfully introduced the class of Σ-spaces satisfying the requirements.

Definition B.3.16 Suppose $\mathscr{P}$ is a cover of a space X. For each $x \in X$, let $C(\mathscr{P}, x) = \cap(\mathscr{P})_x$. A *$\Sigma$-network* for X is a sequence $\{\mathscr{P}_i\}$ of locally finite closed covers of X, such that, if x is a point and $\{x_i\}$ is a sequence in X with $x_i \in C(\mathscr{P}_i, x)$, then $\{x_i\}$ has an accumulation point in X. A space with a Σ-network is called a *Σ-space* [368]. For a Σ-network for X, let

$$C(x) = \cap\{C(\mathscr{P}_i, x) : i \in \mathbb{N}\},\ x \in X.$$

Then $C(x)$ is a countably compact closed set in X. When every $C(x)$ is a compact set in X, $\{\mathscr{P}_i\}$ is said to be a strong Σ-network in X and X is called a *strong Σ-space* [368].

Every M-space or every σ-space is a Σ-space, and every Σ-space is a Morita space. Paracompact Σ-spaces are countably productive. Nagami proved Σ-spaces are preserved by perfect mappings. However, Michael [333] constructed a locally compact paracompact space (hence a strong Σ-space) such that one closed image of it is not a Σ-space. In view of this, Michael [333] introduced the concept of strong $\Sigma^\sharp$-spaces.

Definition B.3.17 A family $\mathscr{P}$ of sets in a space X is called a *(mod k)-network* [333] if, there is a cover $\mathscr{K}$ consisting of compact closed subsets of X, such that, for each $K \in \mathscr{K}$, $\{P \in \mathscr{P} : K \subset P\}$ is a network of K in X.

The class of strong Σ-spaces is equivalent to the class of spaces with a σ-locally finite closed (mod k)-network. Michael [333] called a space with a σ-closure-preserving closed (mod k)-network a *strong $\Sigma^\sharp$-space*. Obviously, strong $\Sigma^\sharp$-spaces are preserved by closed mappings. Michael further pointed out that whether or not the class of strong $\Sigma^\sharp$-spaces is valuable to study deeply, depends on whether the paracompact strong $\Sigma^\sharp$-spaces are countably productive. In 1984, Patsei [390] proved that paracompact strong $\Sigma^\sharp$-spaces are countably productive. Okuyama [386] investigated spaces relevant to strong Σ-spaces systematically. In order to unify our description, if we replace $\mathscr{K}$ in the definition of (mod k)-networks with a cover of X consisting of some countably compact closed subsets of X, then $\mathscr{P}$ is said to be a *quasi-(mod k)-network* for X. So every Σ-network is just a σ-locally finite closed quasi-(mod k)-network.

Definition B.3.18 [240] A family $\mathscr{P}$ of sets in a space X is said to be a *hereditarily closure-preserving family* in X if $\{H_P : P \in \mathscr{P}\}$ is a closure-preserving family whenever $H_P \subset P \in \mathscr{P}$.

Definition B.3.19 [386] A space with a σ-hereditarily closure-preserving closed quasi-(mod k)-network is called a *Σ^*-space*, and a space with a σ-closure-preserving closed quasi-(mod k)-network is called a *$\Sigma^\sharp$-space*.

Every Σ-space is a Σ^*-space and every Σ^*-space is a $\Sigma^\sharp$-space. But neither of the opposite implications is true. Both Σ^*-spaces and $\Sigma^\sharp$-spaces are preserved by closed mappings. Okuyama proved that a paracompact space X is a Σ-space if and only if $X \times \mathbb{I}$ is a Σ^*-space. It is worth mentioning here that paracompactness is not countably productive in the class of semi-stratifiable spaces. Under the assumption

CH, Michael [334] constructed a hereditarily Lindelöf semi-metrizable space such that the product multiplied by itself is not a normal space.

Open (mod k)-networks are called *(mod k)-bases* [333]. In 1970, Lutzer and Michael [333] proved independently that a space X is a paracompact M-space if and only if X is a regular space with a σ-locally finite (mod k)-base.

By means of various generalizations of bases, such as networks, k-networks and (mod k)-networks, a series of generalized metric spaces were obtained, such as cosmic spaces, σ-spaces, semi-stratifiable spaces, $\aleph$-spaces, Σ-spaces and so on. Naturally, this causes the following two problems:

(1) What are the metrization theorems for these classes of spaces?
(2) What are the "factor decomposition" theorems for these classes of spaces?

Look at the metrization problems first. It is reflected in the metrization problems of semi-stratifiable spaces and Σ-spaces. Every semi-stratifiable space has a G_δ-diagonal and Chaber [93] proved that every M-space with a G_δ-diagonal is a metrizable space. For Σ-spaces, main discussion is the problem which class of Σ-spaces is that of σ-spaces. This also can be regarded as a form of factor decomposition theorems of σ-spaces. Siwiec and Nagata [426] defined the concepts of pseudo-networks (it was called *ct-networks* at that time) and $\sigma^\sharp$-spaces.

Definition B.3.20 A family $\mathscr{P}$ of sets in a space X is called a *pseudo-network* for X if for every $x, y \in X$ with $x \neq y$, there is $P \in \mathscr{P}$ such that $x \in P \subset X - \{y\}$. A space with a σ-closure-preserving closed pseudo-network is called a *$\sigma^\sharp$-space*.

Shiraki [422] proved that every σ-space can be decomposed as a space which is both a regular $\Sigma^\sharp$-space and a $\sigma^\sharp$-space, and every regular Σ-space with a point-countable pseudo-base is a σ-space. Burke and Lutzer [88] constructed a Moore space which has no point-countable pseudo-base, and answered an open question of Reed [406] negatively.

The research on seeking this kind of theorems of generalized metric spaces is very fruitful, such as

(1) a space is a σ-space if and only if it is a regular c-semi-stratifiable $\Sigma^\sharp$-space [319];
(2) a space is a semi-metrizable space if and only if it is a first countable semi-stratifiable space [110];
(3) a space is a Nagata space if and only if it is a first countable stratifiable space [65].

Ceder [92] asked that whether every Nagata space can be decomposed as a paracompact semi-metrizable space. After constructing a counterexample for Ceder's question, Heath [180] asked: What is the necessary and sufficient condition for a paracompact semi-metrizable space being a Nagata space? Lutzer [316] gave the definition of a k-semi-stratifiable space.

Definition B.3.21 A regular space X is said to be a *k-semi-stratifiable space* if there is a function $G : \mathbb{N} \times \tau^c \to \tau$ on X satisfying the requirement for being a semi-stratifiable space, and if $K \cap H = \varnothing$ for a compact set K and a closed set H in X, then there is $n \in \mathbb{N}$ such that $K \cap G(n, H) = \varnothing$.

Lutzer proved that a space is a Nagata space if and only if it is a semi-metrizable k-semi-stratifiable space. Its essence is that a space is a Nagata space if and only if it is a first countable k-semi-stratifiable space. None of the above results gives a decomposition of stratifiable spaces. Arhangel'skiĭ [31] asked: Which class of cosmic spaces is the class of stratifiable spaces? Heath, Lutzer and Zenor [186] gave the definition of monotonically normal spaces. They proved that a space is a stratifiable space if and only if it is a monotonically normal semi-stratifiable space, and a space is metrizable if and only if it is a monotonically normal p-space with a G_δ-diagonal.

Definition B.3.22 ([186]) A space X is called a *monotonically normal space* if for each pair H, F of disjoint closed sets in X, there is an open set $D(H, F)$ in X such that

(1) $H \subset D(H, F) \subset \overline{D(H, F)} \subset X - F$;
(2) $H \subset H',\ F \supset F' \Rightarrow D(H, F) \subset D(H', F')$.

For M_i-space problems, Gruenhage [160] and Junnila [216] proved that every M_3-space is an M_2-space independently. Since then, a lot of progresses have been made on the problem whether every M_3-space is an M_1-space. In 2008, Tamano [451] proved that if X is a *Polish space* (i.e. separable complete metric space), then $C_k(X)$ is an M_1-space, and hence $C_k(\mathbb{P})$ is an M_1-space. Further more, Reznichenko [408] proved that for every separable metric space X, $C_k(X)$ is a stratifiable space if and only X is a Polish space. It remains unsolved whether $C_k(X)$ is hereditarily M_1 for any Polish space X [451].

B.3.3 The Alexandroff Idea

After Alexandroff [3] proposed his idea, results on classifying many specific spaces by mappings emerged. However, only after Arhangel'skiĭ [31] was published, classifying spaces by mappings formed a research direction in general topology.

In 1966, Arhangel'skiĭ [31] published a historic literature entitled "Mappings and spaces", which created a new era of classifying spaces by means of mappings. This article systematically summarized the important results achieved in the mapping theory in half a century since the birth of general topology. More importantly, it gave the concrete problems on investigating various spaces by mappings, which formed the famous Alexandroff-Arhangel'skiĭ's questions. Its core content is to establish the relationships between the class of metric spaces and classes of spaces with certain topological properties by mappings. These problems are outstanding contributions to general topology, and make the idea of classifying spaces by means of mapping become an important integral part of the theory of generalized metric space [4].

In the view of describing metric spaces by distance functions, symmetrizability, Δ-metrizability [382] and semi-metrizability [477] are the direct generalizations of metrizability, and they are the earliest generalized metric properties. Niemytzki [382] did pioneering work on generalized metric spaces, for example, he defined *Δ-metrics* as the "metrics" that do not have to satisfy the symmetric condition and also introduced symmetrics as the "metrics" that need not satisfy the triangle inequality.

Definition B.3.23 ([31]) A family $\mathscr{P}$ of sets in a space X is called a *weak base* for X if, there is $\mathscr{P}_x \subset \mathscr{P}$ for each $x \in X$ such that

(1) $x \in \cap \mathscr{P}_x$ and $\mathscr{P} = \bigcup_{x \in X} \mathscr{P}_x$;
(2) $U, V \in \mathscr{P}_x \Rightarrow$ there is $W \in \mathscr{P}_x$ such that $W \subset U \cap V$;
(3) F is a closed subset of X if and only if for each $x \in X - F$, there is $P \in \mathscr{P}_x$ such that $F \cap P = \varnothing$.

If X has a weak base $\mathscr{P}$ such that every $\mathscr{P}_x$ is countable, then X is called a *g-first countable space* (or *weakly first countable space*, *gf-countable space*).

Every symmetrizable space is a g-first countable space, and every first countable space is equivalent to a g-first countable Fréchet–Urysohn space. By means of symmetrizability, one can obtain more general metrization theorems than by means of semi-metrizability and also can effectively characterize certain quotient images of metrizable spaces. For example,

(1) a space X is a quotient π-image of a metrizable space if and only if X is a symmetrizable space satisfying the weak Cauchy condition [232];
(2) a space X is a pseudo-open π-image of a metrizable space if and only if X is a semi-metrizable space [1, 78].

As a generalization of bases, weak bases provide a way for people to investigate more general generalized metric spaces. For example, Siwiec [425] defined g-metrizable spaces as regular spaces with a σ-locally finite weak base; in 1982, Foged [127] proved that every g-metrizable space has a σ-discrete weak base, and hence answered a question of Siwiec; in 2005, C. Liu [305] proved that the class of g-metrizable spaces is equivalent to the class of regular spaces with a σ-hereditarily closure-preserving weak base, and hence answered a question of Tanaka. It is still an open problem that whether every regular space with a σ-compact-finite weak base is g-metrizable [299].

One important result obtained in exploring metrizable spaces is that metrizability implies paracompactness. Developable spaces, as a generalization of metrizable spaces, may not be metacompact spaces. Except the θ-refinable property, does every developable space have some covering properties parallel to those possessed by every metrizable space?

Arhangel'skiĭ [31] gave the definition of σ-paracompact spaces.

Definition B.3.24 ([31]) A space X is called a *σ-paracompact space* if, every open cover $\mathscr{U}$ of X has a sequence $\{\mathscr{U}_n\}$ of open refinements, such that, for each $x \in X$, there exist $n \in \mathbb{N}$ and $U \in \mathscr{U}$ with $\mathrm{st}(x, \mathscr{U}_n) \subset U$.

When investigating semi-metrizable spaces, McAuley [234] found that semi-metrizability implies so called *F_σ-screenable property* of a space X, i.e., every open cover of X has a σ-discrete closed refinement.

Burke [75] established the precise relationships among these properties.

Theorem B.3.25 *For every space X, the following are equivalent:*

(1) X is a σ-paracompact space.
(2) X is an F_σ-screenable space.
(3) Every open cover of X has a σ-locally finite closed refinement.
(4) Every open cover of X has a σ-closure-preserving closed refinement.

Burke called a space with one of the above properties a *subparacompact space*. Metacompactness and subparacompactness are independent and every subparacompact space is a θ-refinable space. Burke [76] and Katuta [227] raised the following question: if every open cover of a space has a σ-cushioned refinement, then is it a subparacompact space? In 1978, Junnila [217] gave this question a positive answer.

Although every paracompact p-space can be characterized as a perfect preimage of a metrizable space, Arhangel'skiĭ [31] pointed out that a p-space may not be any perfect preimage of a developable space. Associated with this, Arhangel'skiĭ [31] introduced the concept of strict p-spaces.

Definition B.3.26 ([31]) A completely regular space X is called a *strict p-space* if, there is a sequence $\{\mathscr{U}_n\}$ of families of open sets in βX covering X satisfying $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{U}_n)} \subset X$ for every $x \in X$.

Strict p-spaces are stronger than p-spaces strictly. Burke [77] proved every θ-refinable p-space is a strict p-space. Burke [80] raised the question whether every strict p-space is a θ-refinable space. In 1986, S. Jiang [213] answered this question positively. On the other hand, every completely regular perfect preimage of a developable space is a subparacompact p-space. Isiwata [206] gave an example to illustrate the converse is not true and also gave a characterization of perfect preimages of developable spaces. A characterization of perfect preimages of σ-spaces was given by Suzuki [445].

Every closed image of a paracompact space is also a paracompact space, but a closed image of a metrizable space may not be a metrizable space. Arhangel'skiĭ asked in [31]: What is the intrinsic characterization of closed images of metrizable spaces? Lašnev [240] studied this question first, and hence closed images of metrizable spaces are called *Lašnev spaces* by the researchers later. Slaughter [427] proved that every Lašnev space is an M_1-space. The concept of hereditarily closure-preserving families of sets (see Definition B.3.18) was introduced by Lašnev when he investigated closed images of metrizable spaces. He characterized every closed image of a metrizable space as a Fréchet–Urysohn space with an almost refining sequence of hereditarily closure-preserving coverings comprising a network for the space. The importance of this characterization is the introduction of concept of hereditarily closure-preserving families. For example, Morita and Rishel [365] gave a characterization of closed images of (paracompact) M-spaces by using Lašnev's method; the concept of Σ^*-spaces introduced by Okuyama [386] is based on hereditarily closure-preserving families of sets. More interestingly, Burke, Engelking and Lutzer [87] established a new metrization theorem.

Theorem B.3.27 (The Burke-Engelking-Lutzer metrization theorem) *A space X is a metrizable space if and only if X is a regular space with a σ-hereditarily closure-preserving base.*

Metrizable spaces and paracompact M-spaces are preserved by perfect mappings, but an M-space may not be preserved by a perfect mapping [361].

Definition B.3.28 [201] A space X is called an *M^*-space* if, there is a sequence $\{\mathscr{F}_i\}$ of locally finite closed covers of X, such that, for every $x \in X$ and any sequence $\{x_i\}$, $\{x_i\}$ has an accumulation point in X if $x_i \in \mathrm{st}(x, \mathscr{F}_i)$.

Morita and Rishel [365] and Nagata [372] proved that the class of perfect images of M-spaces is just the class of M^*-spaces. If "locally finite" in the definition of M^*-spaces is replaced with "closure-preserving", then we obtain the class of *$M^\sharp$-spaces* defined by Siwiec and Nagata [426]. The question whether every $M^\sharp$-space is an M^*-space raised by Morita and Rishel [365] is still open.

In the above, we mainly discuss closed images of spaces. Investigation on open images of spaces also has triggered a lot of profound work on generalized metric spaces.

The best known for this is the class MOBI of Arhangel'skiĭ [31] and the five quotient mappings of Michael [336].

Definition B.3.29 The *class MOBI* is the smallest class of spaces such that

(1) every metric space is in this class;
(2) this class is closed under open compact mappings.

The following question has not been solved: What kind of intrinsic characterization does the class MOBI have? The following proposition of Bennett [56] is a direct corollary of Definition B.3.29.

Proposition B.3.30 *A space X belongs to the class MOBI if and only if there is a metrizable space M and a finite set $\{f_1, f_2, \ldots, f_n\}$ of open compact mappings such that $X = f_n \circ f_{n-1} \ldots \circ f_1(M)$.*

This proposition established a fine relationship among the class MOBI, the class of metrizable spaces and the class of open compact mappings. In a period of twenty years from the 1970s to the 1980s, all the research work on the class MOBI took this proposition as a basis, and the result that every space in the class MOBI has a point-countable base was obtained [31]. After a series of work, most questions raised by Arhangel'skiĭ at that time were answered negatively. For example, Chaber [99] showed that a space in the class MOBI can be neither a weakly θ-refinable space nor a $\sigma^{\sharp}$-space. However, several difficult questions, such as whether the class MOBI is preserved under perfect mappings, whether every completely regular metacompact space is an open compact image of a paracompact space and so on, are still open [35].

The class of quotient mappings is a class of relatively weak mappings, and hence specific quotient images of metrizable spaces have been widely noticed. For example, Arhangel'skiĭ [31] raised the question for seeking the intrinsic characterizations of quotient s-images of metrizable spaces; Michael and Nagami [340] asked whether every quotient s-image of a metrizable space is a compact-covering quotient s-image of a metrizable space. Hoshina [194], Gruenhage, Michael and Tanaka [167] gave different answers for the former question. The latter one was answered negatively by H. Chen [101, 102]. For generalizations of open mappings, except quotient mappings and pseudo-open mappings mentioned above, we also have bi-quotient mappings (or limit lifting mappings) defined by Hájek [171], Filippov [123] and Michael [332], and countably bi-quotient mappings defined by Siwiec [424]. In a survey paper [336], Michael summarized the research work of this direction done from the 1950s to the early 1970s, and gave intrinsic characterizations for open images, bi-quotient images, countably bi-quotient images, pseudo-open images and quotient images of locally compact metrizable spaces, locally compact paracompact spaces, separable metrizable spaces, metrizable spaces, paracompact M-spaces and M-spaces systematically. He also characterized generalized sequentiality properties in terms of such classes of spaces and mappings.

In the following, we take sum theorems as an example to illustrate applications of the mapping theory. The content that discussed in the *sum theorems* is that under what conditions certain topological properties of spaces can be transferred to their sum space. That is, suppose $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ and each subspace X_α of the space X has the topological property Φ. Then under what conditions does X have the property Φ as well? The most simple and primitive statement in this respect should be the following question given by Alexandroff and Urysohn [8]: if a compact space X is the union of two subspaces and each of them has a countable base, then does X have a countable base? Smirnov [429] gave this question a positive answer for the countable sum case which is more general. Stone [442] studied the sum theorem for metrizable spaces. In the following, we focus on "open sum theorems" and "closed sum theorems".

The early research on sum theorems is about individual topological properties. From Hodel's paper start [188], researchers realized the intrinsic relations among mappings, spaces and topological properties. Investigating sum theorems by means of mappings has become a trend. It also shows the importance of mapping methods in general topology.

Definition B.3.31 A topological property Φ is said to satisfy the point-finite *open sum theorem* if, X has property Φ whenever $\{X_\alpha\}_{\alpha\in\Lambda}$ is a point-finite open cover of a space X and each X_α has property Φ.

Tanaka [452] first discussed the point-finite open sum theorem of σ-spaces. Gittings [157] investigated the point-finite open sum theorem and proved a topological property Φ satisfies the point-finite open sum theorem if Φ is invariant under finite-to-one open mappings and preserved by topological sums.

Definition B.3.32 Suppose $\mathscr{P}$ is a property of families of sets. A topological property Φ is said to satisfy the $\mathscr{P}$ *closed sum theorem* if, a space X has property Φ whenever $\{X_\alpha\}_{\alpha\in\Lambda}$ is a closed cover of X with property $\mathscr{P}$ and every X_α has property Φ.

When property $\mathscr{P}$ is countable, locally finite, hereditarily closure-preserving and closure-preserving, the corresponding theorems are called the countable closed sum theorem, the locally finite closed sum theorem, the hereditarily closure-preserving closed sum theorem and the closure-preserving closed sum theorem respectively. One of the driving forces of investigating closed sum theorems is the following question of Tamano [447]: Does paracompactness satisfy the closure-preserving closed sum theorem? Potoczny [403] constructed a space with a closure-preserving cover consisting of finite sets which is not a paracompact space.

The relationship between closed sum theorems and mappings is revealed by the following theorem.

Definition B.3.33 Suppose Φ is a topological property preserved by topological sums.

(1) If Φ is invariant under finite-to-one closed mappings, then Φ satisfies the locally finite closed sum theorem.
(2) If Φ is invariant under closed mappings, then Φ satisfies the hereditarily closure-preserving closed sum theorem.

One concept more general than that of hereditarily closure-preserving families is the concept of dominating families introduced by Morita [359].

Definition B.3.34 Suppose $\mathscr{P}$ is a closed cover of a space X. We say that X is *dominated* by $\mathscr{P}$ if, a subset Z of X is closed in X if and only if there is a subfamily $\mathscr{P}'$ of $\mathscr{P}$ such that $\mathscr{P}'$ covers Z and $P \cap Z$ is closed in X for every $P \in \mathscr{P}'$.

Morita raised the question what kind of topological properties satisfies the dominating sum theorem and proved that several topological properties, such as normality etc., satisfy the dominating sum theorem. Singal and Arya [423] investigated general dominating sum theorems and sum theorems for adjunction spaces.

B.3.4 g-Functions

At Wisconsin conference of set theoretic topology in 1955, McAuley [322] raised the question of seeking pure topological characterizations of semi-metrizable space. Brown [72] also asked the following question: in order to make a semi-metrizable space be a developable space, what kind of topological properties should be added? In 1962, Heath [176] introduced a class of set valued functions, which is called g-functions later, and gave satisfactory answers to these two questions.

Definition B.3.35 Let X be a space. A function $g : \mathbb{N} \times X \to \tau$ is called a *g-function* on X if $x \in g(n+1, x) \subset g(n, x)$ for every $x \in X$ and $n \in \mathbb{N}$.

The importance of g-functions is reflected in the pioneering method of using this function developed by Heath. After solving the questions of McAuley and Brown, Heath [182] characterized stratifiable spaces and σ-spaces in terms of g-functions. He proved every stratifiable space is a σ-space, and hence solved a question raised by Arhangel'skiĭ [31], which displayed the effect of g-functions.

In order to discuss the metrization problems of spaces with a G_δ-diagonal, Borges [66] introduced the class of $w\Delta$-spaces as a generalization of the class of developable spaces. For investigating the precise relationships among developable spaces, $w\Delta$-spaces and semi-stratifiable spaces, Hodel [189] gave the definition of α-spaces and proved that a space is a developable space if and only if it is both a $w\Delta$-space and an α-space. Hodel [189] also defined β-spaces in order to generalize $w\Delta$-spaces and semi-stratifiable spaces, and proved that a space is a semi-stratifiable space if and only if it is both a β-space and an α-space. In order to investigate metrization problems of M-spaces, Ishii [202] introduced wM-spaces as a generalization of $M^\sharp$-spaces. The following question raised by Ishii [202] is still open: is every wM-space with a G_δ-diagonal metrizable?

Hodel [190] defined wN-spaces as generalizations of wM-spaces and Nagata spaces, and proved a space is a metrizable space if and only if it is a developable wN-space. Hodel [190] also defined $w\gamma$-spaces, θ-spaces etc., and proved that a space is a wM-space if and only if it is both a $w\gamma$-space and a wN-space. He also proved that a space is a developable space if and only if it is a semi-stratifiable θ-space. The following definition gives a list of spaces described by g-functions.

Definition B.3.36 Suppose g is a g-function on a space X. Consider the following additional conditions:

(1) $p \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \varnothing$ and $y_n \in g(n, x_n) \Rightarrow \{x_n\}$ has an accumulation point.
(2) $g(n, p) \cap g(n, x_n) \neq \varnothing \Rightarrow \{x_n\}$ has an accumulation point.
(3) $p \in g(n, x_n) \Rightarrow \{x_n\}$ has an accumulation point.
(4) $y_n \in g(n, p)$ and $x_n \in g(n, y_n) \Rightarrow \{x_n\}$ has an accumulation point.
(5) $\{p, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, p) \Rightarrow \{x_n\}$ has an accumulation point.
(6) $x_n \in g(n, p) \Rightarrow \{x_n\}$ has an accumulation point.
(7) $\{p, x_n\} \subset g(n, y_n) \Rightarrow \{x_n\}$ has an accumulation point.

Spaces satisfying the above additional conditions are called *wM-spaces*, *wN-spaces*, *β-spaces*, *wγ-space*s, *wθ-spaces*, *q-spaces* and *wΔ-spaces* in turn. If "$\{x_n\}$ has an accumulation point" in the above additional conditions is strengthen to "$\{x_n\}$ converges to p", then spaces obtained are metrizable spaces, Nagata spaces, semi-stratifiable spaces, γ-space, θ-spaces, first countable spaces and developable spaces in turn.

Each of the above additional conditions says that every sequence in a space satisfying certain conditions has an accumulation point. Such spaces are called *generalized* *countably compact spaces* by House [195]. By this idea, several classes of generalized metric spaces can be generated in terms of sequences of covers of spaces.

Definition B.3.37 Suppose $\{\mathscr{U}_n\}$ is a sequence of open covers of a space X. Consider the following additional conditions:

(1) $x_n \in \mathrm{st}(p, \mathscr{U}_n) \Rightarrow \{x_n\}$ has an accumulation point.
(2) $\mathscr{U}_{n+1}$ is a star-refinement of $\mathscr{U}_n$ and $x_n \in \mathrm{st}(p, \mathscr{U}_n) \Rightarrow \{x_n\}$ has an accumulation point.
(3) $x_n \in \mathrm{st}^2(p, \mathscr{U}_n) \Rightarrow \{x_n\}$ has an accumulation point.

Spaces satisfying the above additional conditions are *$w\Delta$-spaces*, *M-spaces* and *wM-spaces* in turn. If "$\{x_n\}$ has an accumulation point" in the above additional conditions is strengthen to "$\{x_n\}$ converges to p", then spaces defined are developable spaces, metrizable spaces and metrizable spaces in turn.

CWC-functions are functions relevant to g-functions. It plays a special role in the metrization problem.

Definition B.3.38 [241] Let X be a space. A function $g : \mathbb{N} \times X \to \mathscr{P}(X)$ (the power set of X) is called a *CWC-function* if, $x \in g(n+1, x) \subset g(n, x)$ for every $x \in X$ and $n \in \mathbb{N}$, and $U \in \tau$ if and only if for each $x \in U$, there is $n \in \mathbb{N}$ such that $g(n, x) \subset U$. A CWC-function is also called a weak base g-function [149].

The period from 1965 to 1975 is 10 years for rapid development of the theory of generalized metric spaces. Especially, the survey paper entitled "Recent advances in the theory of generalized metric spaces" written by Burke and Lutzer [88] in 1976 declared the formation of the theory of generalized metric spaces, and established the status of this theory in general topology. Journals "General Topology and its Applications" founded in Holland in 1971 and "Topology Proceedings" founded in USA in 1976 both fully reflected this trend. A significant symbol is that the research work for solving four problems formed in the foundation period greatly enriched the theory of generalized metric spaces. At the same time, many new problems should be solved. By the stimulating of the Bing-Nagata-Smirnov metrization theorem and the organic combining of various concepts produced by generalizations of bases, the research on generalized metric spaces merged into a torrent and was pushed to a climax. Theorems of Siwiec and Nagata on characterizations of σ-spaces [426], theorems of Nagami on countable products of paracompact Σ-spaces [368] and theorems of Foged on characterizations of $\aleph$-spaces [128] later, formed a wonderful chapter in these important progresses. Through the systematic investigations on the Alexandroff idea in Prague, mappings played an important role in the big stage of general topology. Combined with many classic methods, mappings have become essential means in studying the theory of spaces [4]. The deep problems and incisive arguments of Arhangel'skiĭ in "Mappings and spaces" [31], a comprehensive description of Michael in "A quintuple quotient quest" [336], became one of the important sources for general topology to move forward.

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Symbols

S. Lin and Z. Yun, *Generalized Metric Spaces and Mappings*,
Atlantis Studies in Mathematics 6, DOI 10.2991/978-94-6239-216-8

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