Appendix A
The Bochner integral

A.1. Integrals over measurable real sets

We recall here the few elements of Bochner integral theory that are used in these notes. Extended treatments, with proofs, may be found in the books [11, 39, 40, 94]. A nice treatment of the spaces $W^{s,p}(a,b;X)$ is in the appendix of [35].

X is any real or complex Banach space. We consider the usual Lebesgue measure in $\mathbb{R}$, and we denote by $\mathcal{M}$ the σ-algebra consisting of all Lebesgue measurable subsets of $\mathbb{R}$. If $A \subset \mathbb{R}$, $1\!\!1_A$ denotes the characteristic function of the set A.

Definition A.1. A function $f : \mathbb{R} \mapsto X$ is said to be *simple* if there are $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, $A_1, .., A_n \in \mathcal{M}$, with meas $A_i < \infty$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, such that

$$f = \sum_{i=1}^{n} x_i 1\!\!1_{A_i}.$$

If $I \in \mathcal{M}$, a function $f : I \mapsto X$ is said to be *Bochner measurable* if there is a sequence of simple functions $\{f_n\}$ such that

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \text{for almost all } t \in I.$$

It is easy to see that every continuous function is measurable.

If $f = \sum_{i=1}^{n} x_i 1\!\!1_{A_i}$ is a simple function we set

$$\int_{\mathbb{R}} f(t)dt = \sum_{i=1}^{n} x_i \text{ meas} A_i. \tag{A.1}$$

Definition A.2. Let $f : \mathbb{R} \mapsto X$. f is said to be *Bochner integrable* if there is a sequence of simple functions $\{f_n\}$ converging to f almost everywhere, such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \|f_n(t) - f(t)\| dt = 0.$$

Then $n \mapsto \int_{\mathbb{R}} f_n(t)dt$ is a Cauchy sequence in X. We set

$$\int_{\mathbb{R}} f(t)dt = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)dt. \tag{A.2}$$

Arguing as in the case $X = \mathbb{R}$, one sees that $\int_{\mathbb{R}} f(t)dt$ is independent of the choice of the sequence $\{f_n\}$. If f is defined on a measurable set, the above definition can be extended as follows.

Definition A.3. If $I \in \mathcal{M}$ and $f : I \mapsto X$, f is said to be integrable over I if the extension $\tilde{f}$ defined by

$$\tilde{f}(t) \begin{cases} = f(t), & if\ t \in I \\ = 0, & if\ t \notin I \end{cases}$$

is integrable. In this case we set

$$\int_I f(t)dt = \int_{\mathbb{R}} \tilde{f}(t)dt. \tag{A.3}$$

If $I = (a, b)$, with $-\infty \le a \le b \le +\infty$, we set as usual

$$\int_{(a,b)} f(t)dt = \int_a^b f(t)dt; \quad \int_b^a f(t)dt = -\int_a^b f(t)dt\,.$$

A simple criterion for establishing whether a function is integrable is stated in the following proposition.

Proposition A.4. *Let $I \in \mathcal{M}$, and let $f : I \mapsto X$. Then f is integrable if and only if f is measurable and $t \mapsto \|f(t)\|$ is Lebesgue integrable on I. Moreover,*

$$\left\| \int_I f(t)dt \right\| \le \int_I \|f(t)\|dt. \tag{A.4}$$

From the definition it follows easily that if Y is another Banach space and $A \in \mathcal{L}(X, Y)$, then for every integrable $f : I \mapsto X$ the function $Af : I \mapsto Y$ is integrable, and

$$\int_I Af(t)dt = A \int_I f(t)dt.$$

In particular, if $\varphi \in X'$ then for every integrable $f : I \mapsto X$ the function $t \mapsto \langle f(t), \varphi\rangle$ is integrable, and

$$\left\langle \int_I f(t)dt, \varphi \right\rangle = \int_I \langle f(t), \varphi\rangle dt.$$

It follows that for every couple of integrable functions f, g it holds

$$\int_I (\lambda f(t) + \mu g(t))dt = \lambda \int_I f(t) + \mu \int_I g(t)dt, \quad \forall\, \lambda, \mu \in \mathbb{C}.$$

Another important commutativity property is the following one.

Proposition A.5. *Let X, Y be Banach spaces, and let $A : D(A) \subset X \mapsto Y$ be a closed operator. Let $I \in \mathcal{M}$, let $f : I \mapsto X$ be an integrable function such that $f(t) \in D(A)$ for almost all $t \in I$, and $Af : I \mapsto Y$ is integrable. Then the integral $\int_I f(t)dt$ belongs to $D(A)$, and*

$$A \int_I f(t)dy = \int_I Af(t)dt.$$

A.2. L^p and Sobolev spaces

On the set of all measurable functions on I we define the equivalence relation

$$f \sim g \iff f(t) = g(t) \text{ for almost all } t \in I. \tag{A.5}$$

Definition A.6. $L^1(I; X)$ is the set of all equivalence classes of integrable functions $f : I \mapsto X$, with respect to the equivalence relation $(A.5)$.

Since no confusion will arise, in the sequel we shall identify the equivalence class $[f] \in L^1(I; X)$ with the function f itself. We define a norm on $L^1(I, X)$ by setting

$$\|f\|_{L^1(I;X)} = \int_I \|f(t)\|dt. \tag{A.6}$$

We define now the spaces $L^p(I; X)$ for $p > 1$.

Definition A.7. Let $p \in (1, +\infty]$, and $I \in \mathcal{M}$. $L^p(I; X)$ is the set of all equivalence classes of measurable functions $f : I \mapsto X$,with respect to the equivalence relation $(A.5)$, such that $t \mapsto \|f(t)\|$ belongs to $L^p(I)$. $L^p(I; X)$ is endowed with the norm

$$\|f\|_{L^p(I;X)} = \left(\int_I \|f(t)\|^p dt\right)^{1/p}, \quad \text{if } p < \infty \tag{A.7}$$

$$\|f\|_{L^\infty(I;X)} = \text{ess sup}\,\{\|f(t)\| : t \in I\} \tag{A.8}$$

In the following, if there is no danger of confusion, we shall write $\|f\|_p$ instead of $\|f\|_{L^p(I;X)}$.

An extension of a well known property for $X = \mathbb{R}$ to general X is in the next proposition.

Proposition A.8. *Let $1 \le p < \infty$, and let I, J be two real intervals. Then*

$$L^p(I; L^p(J; X)) \subset L^p(I \times J, X)$$

if we identify any function $f \in L^p(I; L^p(J; X))$ with the function $(t,s) \mapsto f(t)(s)$, $t \in I$, $s \in J$.

To introduce the Sobolev space $W^{1,p}(a, b; X)$ we need a lemma.

Lemma A.9. *Let $p \in [1, \infty)$. Then the operator*

$$L_0 : D(L_0) = C^1([a, b]; X) \mapsto L^p(a, b; X), \quad L_0 f = f'$$

is preclosed in $L^p(a,b;X)$, that is the closure of its graph is the graph of a closed operator.

Definition A.10. Let $L : D(L) \subset L^p(a,b;X)$ be the closure of the operator L_0 defined in Lemma (A.9). We set

$$W^{1,p}(a, b; X) = D(L)$$

and we endow it with the graph norm. For every $f \in W^{1,p}(a, b; X)$, Lf is said to be the *strong derivative* of f, and we denote it by f'.

In other words, $f \in W^{1,p}(a, b; X)$ if and only if there is a sequence $\{f_n\} \subset C^1([a, b]; X)$ such that $f_n \to f$ in $L^p(a, b; X)$ and $f_n' \to g$ in $L^p(a, b; X)$, and in this case $g = f'$. Moreover we have

$$\|f\|_{W^{1,p}(a,b;X)} = \|f\|_{L^p(a,b;X)} + \|f'\|_{L^p(a,b;X)} \quad \forall f \in W^{1,p}(a, b; X).$$

Since L is a closed operator, then $W^{1,p}(a, b; X)$ is a Banach space.

Let $f \in W^{1,p}(a, b; X)$, and let $\{f_n\} \subset C^1([a, b]; X)$ be such that $f_n \to f$ and $f_n' \to f'$ in $L^p(a, b; X)$. From the equality

$$f_n(t) - f_n(s) = \int_s^t f_n'(\sigma) d\sigma \tag{A.9}$$

we get, integrating with respect to s in (a, b) and letting $n \to \infty$,

$$f(t) = \frac{1}{b-a}\left(\int_a^b f(s) ds + \int_a^t (\sigma - a) f'(\sigma)\, d\sigma \right), \quad \text{a.e. in } (a, b).$$

Therefore, $W^{1,p}(a, b; X)$ is continuously embedded in $C([a, b]; X)$. Letting $n \to \infty$ in (A.9) we get also

$$f(t) - f(s) = \int_s^t f'(\sigma)\, d\sigma.$$

Sometimes it is easier to deal with weak (or distributional) derivatives, defined as follows.

Definition A.11. Let $f \in L^p(a,b;X)$. A function $g \in L^1(a,b;X)$ is said to be the weak derivative of f in (a,b) if

$$\int_a^b f(t)\varphi'(t)\,dt = -\int_a^b g(t)\varphi(t)\,dt, \quad \forall \varphi \in C_0^\infty(a,b).$$

It can be shown that weak and strong derivatives do coincide. More precisely, the following proposition holds.

Proposition A.12. *Let $f \in W^{1,p}(a,b;X)$. Then f is weakly differentiable, and f' is the weak derivative of f.*

Conversely, if $f \in L^p(a,b;X)$ admits a weak derivative $g \in L^p(a,b;X)$, then $f \in W^{1,p}(a,b;X)$, and $g = f'$.

For $0 < \alpha < 1$ and $1 \le p < \infty$, the fractional Sobolev space $W^{\alpha,p}(a,b;X)$ is defined as the subspace of $L^p(a,b;X)$ consisting of the functions f such that

$$[f]_{W^{\alpha,p}} := \int_a^b \int_a^b \frac{\|f(t)-f(s)\|}{(t-s)^{1+\alpha p}}\,ds\,dt < \infty.$$

A.3. Weighted L^p spaces

Let I be an interval contained in $(0,+\infty)$. For $1 \le p \le \infty$ we denote by $L^p_*(I)$ the space of the real or complex valued L^p functions in I with respect to the measure dt/t, endowed with its natural norm

$$\|f\|_{L^p_*(I)} = \left(\int_0^{+\infty} |f(t)|^p \frac{dt}{t}\right)^{1/p}, \quad \text{if } p < \infty,$$

$$\|f\|_{L^\infty_*(I)} = \text{ess sup}_{t\in I}\, |f(t)|.$$

Dealing with L^p_* spaces, the Hardy-Young inequalities are often more useful than the Hölder inequality. They hold for every positive measurable function $\varphi : (0,a) \mapsto \mathbb{R}$, $0 < a \le \infty$, and every $\alpha > 0$, $p \ge 1$. See [54, p. 245-246].

$$\begin{cases} \text{(i)} \displaystyle\int_0^a t^{-\alpha p}\left(\int_0^t \varphi(s)\frac{ds}{s}\right)^p \frac{dt}{t} \le \frac{1}{\alpha^p}\int_0^a s^{-\alpha p}\varphi(s)^p \frac{ds}{s}, \\ \text{(ii)} \displaystyle\int_0^a t^{\alpha p}\left(\int_t^a \varphi(s)\frac{ds}{s}\right)^p \frac{dt}{t} \le \frac{1}{\alpha^p}\int_0^a s^{\alpha p}\varphi(s)^p \frac{ds}{s} \end{cases} \tag{A.10}$$

The measure $m(dt) = dt/t$ is the Haar measure of the multiplicative group $\mathbb{R}_+$. So, it is invariant under multiplication: $m(A) = m(\lambda A)$, for every measurable $A \subset \mathbb{R}_+$ and $\lambda > 0$, and $\|\varphi\|_{L^p_*(0,\infty)} = \|\varphi(\lambda\cdot)\|_{L^p_*(0,\infty)}$.

For every $\alpha \neq 0$ the space $L^p_*(0,\infty)$ is invariant under the change of variable $t \mapsto t^\alpha$, in the sense that $\varphi \in L^p_*(0,\infty)$ iff $t \mapsto \varphi(t^\alpha) \in L^p_*(0,\infty)$, and

$$\|\varphi\|_{L^p_*(0,\infty)} = |\alpha|^{1/p} \|t \mapsto \varphi(t^\alpha)\|_{L^p_*(0,\infty)}.$$

(This is obviously true also for $p = \infty$, with the usual convention $1/\infty = 0$). In particular, for $\alpha = -1$ we get an isometry:

$$\|\varphi\|_{L^p_*(0,\infty)} = \|t \mapsto \varphi(t^{-1})\|_{L^p_*(0,\infty)}.$$

Moreover, the change of variable $t \mapsto t^{-1}$ is an isometry also between $L^p_*(1,\infty)$ and $L^p_*(0,1)$.

If X is any Banach space and $1 \leq p \leq \infty$ the space $L^p_*(I; X)$ is the set of all Bochner measurable functions $f : I \mapsto X$, such that $t \mapsto \|f(t)\|_X$ is in $L^p_*(I)$. It is endowed with the norm

$$\|f\|_{L^p_*(I;X)} = \|t \mapsto \|f(t)\|_X\|_{L^p_*(I)}.$$

In Chapters 1 and 3 we have used the following consequence of inequality (A.10)(i).

Corollary A.13. *Let u be a function such that $t \mapsto u_\theta(t) = t^\theta u(t)$ belongs to $L^p_*(0, a; X)$, with $0 < a \leq \infty$, $0 < \theta < 1$ and $1 \leq p \leq \infty$. Then also the mean value*

$$v(t) = \frac{1}{t}\int_0^t u(s)ds, \quad t > 0 \tag{A.11}$$

has the same property, and setting $v_\theta(t) = t^\theta v(t)$ we have

$$\|v_\theta\|_{L^p_*(0,a;X)} \leq \frac{1}{1-\theta}\|u_\theta\|_{L^p_*(0,a;X)} \tag{A.12}$$

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