Appendix A

A.1. Continuity and differentiability of functions depending on a parameter

In this section we consider the following problem: we are given a metric space (X, d) and a measure space $(Y, \mathscr{F}, \mu)$. Given $f : X \times Y \to \mathbb{R}$, we assume that for all $x \in X$ the function $f(x, \cdot)$ is μ–integrable, so that the function $F : X \to \mathbb{R}$ given by

$$F(x) := \int_Y f(x, y)\, d\mu(y) \qquad x \in X$$

is well defined. We would like to understand under which conditions F, an integral depending on the parameter x, is continuous. When X is an open subset of $\mathbb{R}^n$ endowed with the Euclidean distance, it is also natural to investigate the differentiability properties of F.

Theorem A.1 (Continuity of F). *Assume that $f(\cdot, y)$ is continuous in X for μ-almost all $y \in Y$ and that there exists $m \in L^1(Y, \mu)$ satisfying*

$$\sup_{x \in X} |f(x, y)| \le m(y) \qquad \text{for } \mu\text{–a.e. } y \in Y. \tag{A.1}$$

Then F is bounded and continuous in X.

Proof. It is clear that $|F(x)| \le \|m\|_1$ for all $x \in X$. Continuity is a simple consequence of the dominated convergence theorem: indeed, if $x_n \in X$ converge to x, then $f(x_n, y)$ converge to $f(x, y)$ for μ-almost every y and the convergence is dominated because of (A.1). It follows that $F(x_n) \to F(x)$. $\square$

A more expressive way to state the continuity of F is to say that limit and integral commute, namely

$$\lim_{h \to \infty} \int_Y f(x_h, y)\, d\mu(y) = \int_Y \lim_{h \to \infty} f(x_h, y)\, d\mu(y).$$

The following example shows that if no uniform upper bound is imposed on f, then continuity might fail:

Example A.2. Let $X = Y = \mathbb{R}$, $\mu = \mathscr{L}^1$ and

$$f(x, y) := \begin{cases} |x|(1 - |y||x|) & \text{if } |y||x| < 1; \\ \\ 0 & \text{if } |y||x| \geq 1. \end{cases}$$

Then $F(x) = 1$ for all $x \neq 0$, while $F(0) = 0$. In this case the smallest possible function satisfying (A.1) is $|y|^{-1}$ which is not integrable.

Next, we assume that X is an open set of $\mathbb{R}^n$ endowed with the Euclidean distance and we investigate the differentiability of F. Under suitable assumption, we can commute derivative and integral, namely

$$\frac{\partial}{\partial x_i} \int_Y f(x, y)\, d\mu(y) = \int_Y \frac{\partial f}{\partial x_i}(x, y)\, d\mu(y) \qquad \forall x \in X, i = 1, \ldots, n. \tag{A.2}$$

Theorem A.3 (Differentiability of F). *Assume that for μ-almost all $y \in Y$ the function $f(\cdot, y)$ is differentiable in X with a continuous gradient $\nabla_x f(x, y)$ and that, for any ball $B_r(x_0) \subset X$, there exists $m \in L^1(Y, \mu)$ satisfying*

$$|f(x_0, y)| + \sup_{x \in B_r(x_0)} |\nabla_x f|(x, y) \leq m(y) \qquad \text{for } \mu\text{–a.e. } y \in Y. \tag{A.3}$$

Then $F \in C^1(X)$ and (A.2) *holds.*

Proof. We fix $x_0 \in X$, $i \in \{1, \ldots, n\}$ and $x_i = x + t_i e_i$ with $t_i \neq 0$ and $t_i \to 0$. The mean value theorem, applied for any y such that $f(\cdot, y) \in C^1(X)$, gives $\theta_i(y) \in (0, 1)$ satisfying

$$\frac{F(x_0 + t_i e_i) - F(x_0)}{t_i} = \int_Y \frac{\partial f}{\partial x_i}(x_0 + \theta_i(y) t_i e_i, y)\, d\mu(y)$$

For i large enough (as soon as $|t_i| < r$) the functions of y inside the integral are dominated by the function m in (A.3), hence we can pass to the limit with the dominated convergence theorem to get (notice that the measurability of $\partial f / \partial x_i(x_0, \cdot)$ follows by the same limiting process)

$$\frac{\partial F}{\partial x_i}(x_0) = \int_Y \frac{\partial f}{\partial x_i}(x_0, y)\, d\mu(y).$$

Finally, continuity of partial derivatives of F is a consequence of the previous theorem. □

Of course similar statements can be given for k-th order derivatives of F, provided $f(\cdot, y)$ is k times differentiable and, for any ball $B_r(x_0) \subset \mathbb{R}^n$ there exists $m \in L^1(Y, \mu)$ satisfying

$$|f(x_0, y)| + \sup_{x \in B_r(x_0)} \sup_{|p| \le k} |D^p f|(x, y) \le m(y) \qquad \text{for } \mu\text{–a.e. } y \in Y$$

(here $p = (p_1, \ldots, p_n)$ and $|p| = p_1 + \cdots + p_n$). Under this assumption one obtains that

$$D^p \int_Y f(x, y)\, d\mu(y) = \int_Y D_x^p f(x, y)\, d\mu(y) \qquad \text{whenever } |p| \le k.$$

A.2. The dual space of continuous functions

In this section we want to characterize the space $(C(X))^*$, dual space of $C(X)$, with (X, d) compact metric space. Recall that $C(X)$ is a Banach space, when endowed with the sup norm, regardless of any assumption on (X, d). Some knowledge of the basic terminology of Banach spaces (dual space, dual norm) is needed for this section.

We start with some notation: we shall denote by $\mathscr{M}(X)$ the space of *signed* measures μ, i.e. the real-valued and σ-additive set functions μ, defined on $\mathscr{B}(X)$, of the form $\mu = \mu^+ - \mu^-$ with $\mu^\pm$ positive and finite Borel measures satisfying $\mu^+ \perp \mu^-$.

This orthogonality condition ensures uniqueness of the decomposition of μ, as we will see in a moment; existence, instead, is just a consequence of the σ-additivity (see Section 6.5), but we shall not use this fact in the sequel.

For $\mu \in \mathscr{M}(X)$ we denote $|\mu| = \mu^+ + \mu^-$ its total variation measure, as in Section 6.5, and set

$$\|\mu\| := |\mu|(X) = \mu^+(X) + \mu^-(X). \tag{A.4}$$

In the next proposition we show that the decomposition $\mu = \mu^+ - \mu^-$ is unique, so that (A.4) is well posed, and that $\mathscr{M}(X)$ is a normed space. The completeness of $\mathscr{M}(X)$ will be a consequence of Theorem A.6, since any dual space is complete.

Proposition A.4. *For any $\mu \in \mathscr{M}(X)$ the decomposition $\mu = \mu^+ - \mu^-$ is unique. In addition $\mathscr{M}(X)$, endowed with the norm* (A.4)*, is a normed space.*

Proof. Assume that $\mu = \mu^+ - \mu^- = \tilde{\mu}^+ - \tilde{\mu}^-$, with orthogonal decompositions. Let A be a Borel set where μ^+ is concentrated, so that μ^- is concentrated on $X \setminus A$, and let $\tilde{A}$ be an analogous Borel set for $\tilde{\mu}^\pm$. Since

$\mu \geq 0$ (respectively $\mu \leq 0$) on the subsets of A (respectively of $X \setminus A$) and the same property holds for $\tilde{A}$, we obtain that μ (and therefore $\mu^{\pm}$ and $\tilde{\mu}^{\pm}$) vanishes on subsets of $A \setminus \tilde{A}$ and of $\tilde{A} \setminus A$. On the other hand, if $B \subset A \cap \tilde{A}$ we have $\mu^-(B) = \tilde{\mu}^-(B) = 0$ and

$$\mu^+(B) = \mu(B) = \tilde{\mu}^+(B).$$

Analogously, if $B \subset (X \setminus A) \cap (X \setminus \tilde{A})$ we have $\mu^+(B) = \tilde{\mu}^+(B) = 0$ and $\mu^-(B) = \tilde{\mu}^-(B)$. This proves that $\mu^{\pm} = \tilde{\mu}^{\pm}$.

Now, stability of $\mathscr{M}(X)$ under multiplication with real constants and 1-homogeneity of the norm are obvious. Let us prove stability under addition and subadditivity of the norm: if $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ we can write as before $\mu = f|\mu|$ and $\nu = g|\nu|$ with $f,\ g : X \to [-1, 1]$. Then, setting $\sigma = |\mu| + |\nu|$, the Radon–Nikodým theorem gives $|\mu| = a\sigma$ and $|\nu| = b\sigma$ for suitable $a,\ b : X \to [0, 1]$, so that

$$\mu + \nu = f|\mu| + g||\nu| = (fa + gb)\sigma$$

and we may take $(fa + gb)^{\pm}\sigma$ as positive and negative parts of $\mu + \nu$. We obtain also

$$\|\mu + \nu\| = \int_X |fa + gb|\, d\sigma \leq \int_X |a| + |b|\, d\sigma = \|\mu\| + \|\nu\|.$$

This completes the proof of the proposition. □

We shall also denote by $\mathscr{A}(X)$ the collection of open subsets of X and use the following characterization of set functions defined on $\mathscr{A}(X)$ which are restrictions of σ-additive measures defined on the Borel σ-algebra.

Proposition A.5. *Let (X, d) be a compact metric space and let $\alpha : \mathscr{A}(X) \to [0, +\infty]$ be a nondecreasing set function satisfying $\alpha(\emptyset) = 0$ and:*

(i) *(continuity) if $A_n \in \mathscr{A}(X)$, $n \in \mathbb{N}$, monotonically converge from below to A, then $\alpha(A_n) \uparrow \alpha(A)$;*
(ii) *(subadditivity) $\alpha(A_1 \cup A_2) \leq \alpha(A_1) + \alpha(A_2)$ for all $A_1,\ A_2 \in \mathscr{A}(X)$;*
(iii) *(additivity on disjoint sets) $\alpha(A_1 \cup A_2) = \alpha(A_1) + \alpha(A_2)$ whenever $A_1 \in \mathscr{A}(X)$ and $A_2 \in \mathscr{A}(X)$ are disjoint.*

Then

$$\tilde{\alpha}(B) := \inf\{\alpha(A) :\ A \in \mathscr{A}(X),\ A \supset B\} \tag{A.5}$$

is a σ-additive extension of α to $\mathscr{B}(X)$.

Proof. Notice first that α is σ–subadditive on $\mathscr{A}(X)$: indeed, if $A \subset \cup_i A_i$ and B is an open set with compact closure in A, then B is contained in the union of finitely many A_i's, so that (ii) gives

$$\alpha(B) \leq \sum_{i=1}^{\infty} \alpha(A_i).$$

Since B is arbitrary, (i) gives $\alpha(A) \leq \sum_i \alpha(A_i)$.

Now, if we take (A.5) as the definition of $\tilde{\alpha}$ for all subsets of X, Proposition 1.16 gives that $\tilde{\alpha}$ extends α and is σ–subadditive. Then, Theorem 1.17 gives that α is σ–additive on the Borel σ–algebra, provided we are able to show that any Borel set is $\tilde{\alpha}$–additive. Since the class of additive sets is a σ–algebra, suffices to show that any closed set is $\tilde{\alpha}$–additive.

To this aim, we first show that $\tilde{\alpha}$ is additive on distant sets, namely (recall that dist(U, V) is the infimum of the distances $d(x, y)$ for $x \in U$ and $y \in V$)

$$\tilde{\alpha}(B_1 \cup B_2) = \tilde{\alpha}(B_1) + \tilde{\alpha}(B_2) \qquad \text{whenever } \operatorname{dist}(B_1, B_2) > 0. \quad \text{(A.6)}$$

Indeed, if $A \supset B_1 \cup B_2$ is open we can consider the disjoint open sets

$$A_1 := \{x \in A : \ \operatorname{dist}(x, B_1) < \operatorname{dist}(x, B_2)\},$$

$$A_2 := \{x \in A : \ \operatorname{dist}(x, B_2) < \operatorname{dist}(x, B_2)\}$$

containing B_1 and B_2 respectively to get

$$\alpha(A) \geq \alpha(A_1 \cup A_2) = \alpha(A_1) + \alpha(A_2) \geq \tilde{\alpha}(B_1) + \tilde{\alpha}(B_2).$$

Since A is arbitrary the inequality $\geq$ in (A.6) follows, while the converse one is a consequence of subadditivity.

Let $F \subset X$ be closed, $B \subset X$ and let us prove that $\tilde{\alpha}(B \cap F) + \tilde{\alpha}(B \setminus F) \leq \tilde{\alpha}(B)$ (the opposite inequality follows by subadditivity). Assuming with no loss of generality $\tilde{\alpha}(B) < \infty$ and setting

$$B_h := \left\{x \in B : \ 2^h > \operatorname{dist}(x, F) \geq 2^{h-1}\right\} \qquad h \in \mathbb{Z}$$

the additivity on distant sets gives

$$\sum_{h \in \mathbb{Z}} \tilde{\alpha}(B_{2h}) \leq \tilde{\alpha}(B) < \infty, \qquad \sum_{h \in \mathbb{Z}} \tilde{\alpha}(B_{2h+1}) \leq \tilde{\alpha}(B) < \infty$$

because all finite sums are made on distant sets, all contained in B. We have then that $\sum_{h\in\mathbb{Z}}\tilde\alpha(B_h)$ is convergent and, since the sets B_h are a partition of $B\setminus F$, using once more the additivity on distant sets we get

$$\begin{aligned}\tilde\alpha(B\cap F)+\tilde\alpha(B\setminus F) &\le \tilde\alpha(B\cap F)+\tilde\alpha\Bigl(\bigcup_{h=-\infty}^{N} B_h\Bigr)+\sum_{h=N+1}^{\infty}\tilde\alpha(B_h)\\ &=\tilde\alpha\Bigl((B\cap F)\cup\bigcup_{h=-\infty}^{N} B_h\Bigr)+\sum_{h=N+1}^{\infty}\tilde\alpha(B_h)\\ &\le \tilde\alpha(B)+\sum_{h=N+1}^{\infty}\tilde\alpha(B_h)\end{aligned}$$

for any $N\ge 1$. Letting $N\to\infty$ the inequality follows. □

For $g\in C(X)$ we can define

$$\int_X g\,d\mu := \int_X g\,d\mu^+ - \int_X g\,d\mu^-.$$

In this way $\int g\,d\mu$ is linear w.r.t. g; in addition, since

$$\begin{aligned}\int_X h\,\mu &= \int_0^\infty \mu^+(\{h>t\})\,dt-\int_0^\infty \mu^-(\{h>t\})\,dt\\ &=\int_0^\infty \mu(\{h>t\})\,dt\end{aligned}$$

whenever h is nonnegative, splitting g in positive and negative part we obtain that $\int_X g\,d\mu$ is also linear w.r.t. to μ. Since

$$\left|\int_X g\,d\mu\right|\le\int_X |g|\,d\mu^+ + \int_X |g|\,d\mu^- \le \max|g|\|\mu\| = \|g\|\,\|\mu\| \qquad \forall g\in C(X)$$

the functional

$$L_\mu(g):=\int_X g\,d\mu \qquad g\in C(X) \tag{A.7}$$

belongs to $(C(X))^*$ and satisfies $\|L_\mu\|\le\|\mu\|$. The remarkable fact is that any element in the dual is representable in this form, and that equality holds. This will also prove that $\mathscr{M}(X)$ is a Banach space (with the definition of $\mathscr{M}(X)$ given above, independent of Section 6.5, it is not even totally obvious that it is a linear space!).

Theorem A.6 (Riesz). *Let (X, d) be a compact metric space. The space $(C(X))^*$ is, via* (A.7)*, isomorphic and isometric to $\mathscr{M}(K)$. That is: all functionals L_μ belong to $(C(X))^*$ and, for any $L \in (C(X))^*$, there exists a unique $\mu \in \mathscr{M}(K)$ satisfying $L = L_\mu$. Finally, $\|L_\mu\| = \|\mu\|$.*

Proof. The proof will be achieved in three steps. In the first one we build an auxiliary positive finite measure μ^* and prove in the second one that μ^* provides the desired representation of L when L is nondecreasing. In the last one we achieve the general case and provide equality of the norms.

Step 1. Let $\alpha^* : \mathscr{A}(X) \to [0, +\infty)$ be defined by

$$\alpha^*(A) := \sup \{|L(g)| : \ |g| \le 1, \ \operatorname{supp} g \subset A\} .$$

Notice that $\alpha^*(X) \le \|L\|$ and that $\alpha^*(\emptyset) = 0$. Notice also that we can equivalently replace $|L(g)|$ with $L(g)$ inside the supremum and that a simple approximation argument gives

$$\alpha^*(A) \ge |L(g)| \qquad \text{whenever } |g| \le \mathbb{1}_A. \tag{A.8}$$

Indeed, if $|g| \le \mathbb{1}_A$ we can find continuous functions $g_n : X \to [-1, 1]$ convergent to g and with support contained in A. In addition, if L is monotone we have also

$$\alpha^*(A) \le L(\chi) \qquad \text{whenever } \mathbb{1}_A \le \chi. \tag{A.9}$$

We claim that α^* satisfies all the assumption of Proposition A.5. Indeed, if $g \in C(X)$ has support contained in A, since the support is compact we have $K \subset A_i$ for i large enough; it follows that $L(g) \le \alpha^*(A_i) \le \sup_j \alpha^*(A_j)$ and since g is arbitrary the continuity follows. In order to prove the subadditivity, given a continuous $g : X \to [-1, 1]$ with support K contained in $A_1 \cap A_2$, we can consider the disjoint compact sets $K \setminus A_1$ and $K \setminus A_2$ and a continuous function $\chi : X \to [0, 1]$ identically equal to 1 in a neighbourhood of $K \setminus A_1$ and identically equal to 0 in a neighbourhood of $K \setminus A_2$. It follows that $(1 - \chi)g$ has support contained in A_1 and χg has support contained in A_2, hence

$$L(g) = L((1 - \chi)g) + L(\chi g) \le \alpha^*(A_1) + \alpha^*(A_2).$$

Since g is arbitrary, the subadditivity of α^* follows. Finally, to prove the additivity on disjoint sets it suffices to notice that, given g_i with support in A_i and $|g_i| \le 1$, the function $g = g_1 + g_2$ has support in $A_1 \cup A_2$ and satisfies $L(g) = L(g_1) + L(g_2)$ and $|g| \le 1$.

By Proposition A.5 we obtain that α^* is the restriction to $\mathscr{A}(X)$ of a positive measure μ^*. Notice also that μ^* is finite, since

$$\mu^*(X) = \alpha^*(X) = \|L\|. \tag{A.10}$$

Step 2. Now we claim that $L_{\mu^*} \geq |L|$, namely $L_{\mu^*}(g) \geq |L(g)|$ for any nonnegative $g \in C(X)$. Also, we shall prove that if L is nondecreasing, namely $L(g) \geq 0$ whenever $g \in C(X)$ is nonnegative, then L_{μ^*} coincides with L. This proves already Riesz theorem for positive functionals.

By homogeneity, in the proof of the inequality $L_{\mu^*}(g) \geq |L(g)|$, it is not restrictive to assume $0 \leq g \leq 1$. Given an integer $N \geq 1$, let us consider the open sets $A_i := \{g > i/N\}$, $i = 0, \ldots, N-1$, and notice that

$$\frac{1}{N} + \sum_{i=1}^{N-1} \frac{1}{N} \mathbb{1}_{A_i} \geq g \geq \sum_{i=1}^{N-1} \mathbb{1}_{A_i}. \tag{A.11}$$

Now, given continuous functions $\chi_i : X \to [0,1]$ satisfying $\mathbb{1}_{A_i} \leq \chi_i \leq \mathbb{1}_{A_{i-1}}$, $i = 1, \ldots, N$, we can use (A.8) to estimate

$$L_{\mu^*}(g) \geq \sum_{i=1}^{N-1} \frac{1}{N} \mu^*(A_i) \geq \sum_{i=1}^{N-1} \frac{1}{N} |L(\chi_{i+1})| \geq \left| L\left(\frac{1}{N} \sum_{i=2}^{N} \chi_i \right) \right|.$$

But, since

$$\frac{1}{N} + \sum_{i=1}^{N-1} \frac{1}{N} \chi_i \geq g \geq \sum_{i=1}^{N-1} \frac{1}{N} \chi_{i+1} \tag{A.12}$$

we can let $N \to \infty$ and use the continuity of L to get $L_{\mu^*}(g) \geq |L(g)|$.

If L is also monotone we can use the inequality (A.9) to get

$$L_{\mu^*}(g) - \frac{1}{N} \leq \sum_{i=1}^{N-1} \frac{1}{N} \mu^*(A_i) \leq \sum_{i=1}^{N-1} \frac{1}{N} L(\chi_i) = L\left(\frac{1}{N} \sum_{i=1}^{N-1} \chi_i \right).$$

Again we can let $N \to \infty$ and use (A.12) to get $L_{\mu^*}(g) = L(g)$.

Step 3. Now we define linear continuous functionals $L^\pm : C(X) \to \mathbb{R}$ by

$$L^+(g) := \frac{L_{\mu^*}(g) + L(g)}{2}, \qquad L^-(g) := \frac{L_{\mu^*}(g) - L(g)}{2}.$$

We have $L^+ + L^- = L_{\mu^*}$ and $L^+ - L^- = L$. In addition, by Step 2, $L^\pm$ are monotone.

Now we can apply the construction of Step 1 and use monotonicity in Step 2 to find positive finite measures $\mu^\pm$ such that $L^\pm = L_{\mu^\pm}$. It follows that

$$L = L^+ - L^- = L_{\mu^+} - L_{\mu^-} = L_\mu$$

and the representation of L follows. Analogously, we obtain that

$$L_{\mu^*} = L^+ + L^- = L_{\mu^+} + L_{\mu^-} = L_{\mu^+ + \mu^-}$$

so that $\mu^* = \mu^+ + \mu^-$. To conclude, we identify $\|\mu\|$ with $\|L\|$ and show that μ^+ and μ^- are orthogonal. The bound on $\|\mu\|$ follows by (A.10):

$$\|\mu\| = \mu^+(X) + \mu^-(X) = L_{\mu^+}(1) + L_{\mu^-}(1) = L_{\mu^*}(1) = \|L\|.$$

In order to show that $\mu^+ \perp \mu^-$, write $\mu^\pm = a^\pm \mu^*$ and use the identity $\mu^* = \mu^+ + \mu^-$ to get $a^+ + a^- = 1$ μ^*–a.e. in X. On the other hand the density of $C(X)$ in $L^2(X, \mu^*)$ and a truncation argument provide a sequence of continuous functions $g_n : X \to [-1, 1]$ convergent in $L^2(X, \mu^*)$ to the sign of $a^+ - a^-$, so that

$$\|L\| = \sup_{|g| \leq 1} |L_\mu(g)| = \sup_{|g| \leq 1} \left| \int (a^+ - a^-) g \, d\mu^* \right| = \int_X |a^+ - a^-| \, d\mu^*.$$

Hence

$$\int_X (1 - |a^+ - a^-|) \, d\mu^* = \mu^*(X) - \|L\| \leq 0.$$

Since $|a^+ - a^-| \leq 1$ it must be $|a^+ - a^-| = 1$ μ^*–a.e. in X. Since $a^\pm \in [0, 1]$ μ–a.e., this can only happen if $a^+ a^- = 0$ μ^*–a.e. in X, which means that μ^+ is orthogonal to μ^-. □

Remark A.7. A similar result holds, with minor changes in the proof, if (X, d) is locally compact and separable, namely there exists an non-decreasing sequence of open sets with compact closure whose union is the whole of X. In this case $C(X)$ has to be replaced by $C_0(X)$, namely the closure in $C(X)$ of the space $C_c(X)$ of compactly supported functions, while $\mathscr{M}(X)$ remains unchanged.

Solutions of some exercises

In this chapter we provide solutions to the main exercises proposed in the text, and in particular of those marked with one or two ★.

Chapter 1

Exercise 1.1. All verifications are very simple and we omit them.

Exercise 1.2. We prove the statement for the translations, the proof for the dilations being similar. Fix $h \in \mathbb{R}$ and consider the class

$$\mathscr{F} := \{A \in \mathscr{B}(\mathbb{R}) \ : \ A + h \in \mathscr{B}(\mathbb{R})\} .$$

Then $\mathscr{F}$ is a σ–algebra containing the intervals, because the class $\mathscr{I}$ of intervals is invariant under translations. Therefore $\mathscr{F} \supset \sigma(\mathscr{I}) = \mathscr{B}(\mathbb{R})$. This proves that $A + h$ is Borel whenever A is Borel.

Exercise 1.3. Set $X = \mathbb{N}$ and $\mu := \sum_n \delta_n$. Then the sets $A_n := \{n, n+1, \ldots\}$ satisfy $\mu(A_n) = +\infty$, but their intersection is empty.

Exercise 1.4. Let $A_n \uparrow A$ with $A_n,\ A \in \mathscr{A}$. Then the sets $B_n := A \setminus A_n$ satisfy $B_n \downarrow \emptyset$, so that by assumption $\mu(B_n) \downarrow \mu(\emptyset) = 0$. Since μ is finite, $\mu(B_n) = \mu(A) - \mu(A_n)$, so that $\mu(A_n) \uparrow \mu(A)$.

Exercise 1.5. For any $n \in \mathbb{N}^*$ the set A_n of all atoms x such that $\mu(\{x\}) \geq 1/n$ has at most cardinality $n\mu(X)$: indeed, if we choose k elements $x_1, \ldots, x_k$ in this sets, adding the inequalities $\mu(\{x_i\}) \geq 1/n$ we find $k/n \leq \mu(X)$, whence the upper bound on the cardinality of A_n follows.

If μ is σ–finite, we choose $X_i \uparrow X$ with $X_i \in \mathscr{E}$ and $\mu(X_i) < \infty$ and repeat the previous argument with the sets $A_{i,n} := \{x \in A_\mu \cap X_i \ : \ \mu(\{x\}) \geq 1/n\}$, whose union gives A_μ. If not finiteness assumption is made, the statement fails: take $X = \mathbb{R}$, $\mathscr{E} = \mathscr{P}(\mathbb{R})$ and $\mu(A) = 0$ if $A = \emptyset$ and $\mu(A) = +\infty$ otherwise.

Exercise 1.6. Let μ be diffuse. First we prove that for all $\tau \in (0, 1)$ and all $A \in \mathscr{E}$ there exists a subset $B \in \mathscr{E}$ with $0 < \mu(B) < \tau\mu(A)$. Indeed, if this property fails for some τ and A, for all subsets B either $\mu(B)=0$ or $\mu(B)\geq\tau\mu(A)$. Now, choose $B_1\subset A$ with $\mu(B_1)\in(0,\mu(A))$ (this is possible by assumption), then $B_2\subset A\setminus B_1$ with $\mu(B_2)\in(0,\mu(B_1))$ and so on. Since all these sets are contained in A, we have $\mu(B_i)\geq\tau\mu(A)$, and this contradicts the fact that they are disjoint.

Now, given $t\in(0,\mu(X))$ we define a sequence of pairwise disjoint sets B_i and numbers s_i as follows: first set

$$s_1:=\sup\{\mu(B):\ \mu(B)\leq t\}$$

and then choose B_1 with $t\geq\mu(B_1)>s_1/2$; then recursively set

$$s_{n+1}:=\sup\left\{\mu(B):\ B\subset B_n^c,\ \mu(B)\leq t-\mu(B_n)\right\}$$

and choose $B_{n+1}\subset B_n^c$ with $t-\mu(B_n)\geq\mu(B_{n+1})>s_{n+1}/2$. We now claim that $\mu(\cup_i B_i)=t$. If this property fails, then $\sum_i\mu(B_i)<t$ and the convergence of the series implies that $s_i\to 0$. On the other hand

$$s_i\geq\sup\left\{\mu(B):\ B\subset X\setminus\bigcup_i B_i,\ \mu(B)\leq t-\sum_i\mu(B_i)\right\}$$

The previous property with $A=X\setminus\cup_i B_i$ and $\tau=(t-\sum_i\mu(B_i))/\mu(A)$ shows that the supremum in the right hand side (independent of i) is positive, contradicting the fact that $s_i\to 0$.

Exercise 1.7. Let X be a separable metric space and let $\mathscr{E}=\mathscr{B}(X)$. If $\mu(\{x\})>0$ for some $x\in X$, obviously μ is not diffuse. Conversely, if $A\in\mathscr{B}(X)$ is given, with $\mu(A)>0$ and $\mu(B)\in\{0,\mu(A)\}$ for all $B\subset A$, we can fix a countable dense set $(x_i)\subset X$ and define

$$r_0:=\sup\left\{r\geq 0\ :\ \mu(A\cap\overline{B}_r(x_0))=0\right\}.$$

Since $r\mapsto\mu(A\cap\overline{B}_r(x_0))$ is right continuous, the maximality of r_0 easily implies that $\mu(A\cap\overline{B}_{r_0}(x_0))>0$, and therefore $\mu(A\cap\overline{B}_{r_0}(x_0))=\mu(A)$. Now we iterate this construction, setting $A_1:=A\cap\overline{B}_{r_0}(x_0)$, defining

$$r_1:=\sup\left\{r\geq 0\ :\ \mu(A_1\cap\overline{B}_r(x_1))=0\right\},$$

so that $\mu(A_1\cap\overline{B}_{r_1}(x_1))=\mu(A_1)=\mu(A)$. Continuing in this way, we have a nonincreasing family of sets (A_i) with $\mu(A_i)=\mu(A)$; it follows that $\mu(\bigcap_i A_i)=\mu(A)>0$. On the other hand, any point $x\in\bigcap_i A_i$ satisfies

$$d(x,x_i)=r_i\qquad\forall i\in\mathbb{N}.$$

By the density of the family (x_i), this intersection contains at most one point (and at least one, because the measure is positive). It follows that this point is an atom of μ.

Exercise 1.8. Cantor's middle third set can be obtained as follows: let $C_0 = [0,1]$, let C_1 the set obtained from C_0 by removing the interval $(1/3, 2/3)$, let C_2 be the set obtained from C_1 by removing the intervals $(1/9, 2/9)$ and $(7/9, 8/9)$, and so on. Each set C_n consists of 2^n disjoint closed intervals with length 3^{-n}, so that $\lambda(C_n) = (2/3)^n \to 0$. If follows that the intersection C of all sets C_n is a closed and λ–negligible set.

In order to show that C has the cardinality of continuum (at this stage it is not even obvious that $C \neq \emptyset$!) we recall that numbers $x \in [0,1]$ can be represented with a ternary, instead of a decimal, expansion: this means that we can write

$$x = \sum_{i \geq 1} a_i 3^{-i} = 0, a_1 a_2 a_3 \ldots$$

with the ternary digits $a_i \in \{0, 1, 2\}$. As for decimal expansions, this representation is not unique; for instance $1/3$ can be written either as 0.1 or as $0.0222\ldots$, and $2/3$ can be written either as 0.2 or as $0.1222\ldots$. It is easy to check that C_1 corresponds to the set of numbers that can be expressed by a ternary representation not having 1 as first digit, C_2 corresponds to the set of numbers that admit a representation not having 1 as a first or second digit, and so on. It follows that C is the set of numbers that admit a ternary representation not using the digit 1: since the map

$$(a_1, a_2, \ldots) \in \{0, 2\}^{\mathbb{N}^*} \mapsto x = \sum_{i=1}^{\infty} a_i 3^{-i}$$

provides a bijection of $\{0, 2\}^{\mathbb{N}^*}$ with C, and the cardinality of $\{0, 2\}^{\mathbb{N}^*}$ is the continuum, this proves that C has the cardinality of continuum.

Exercise 1.9. Let $\{q_n\}_{n \in \mathbb{N}}$ be an enumeration of the rational numbers in $[0,1]$, and set

$$A := \bigcup_{n=0}^{\infty} (q_n - \frac{\varepsilon}{4} 2^{-n}, q_n + \frac{\varepsilon}{4} 2^{-n}).$$

Then $A \subset \mathbb{R}$ is open and $\lambda(A) < \sum_n \varepsilon 2^{-n-1} = \varepsilon$ (why is the inequality strict ?). Therefore $[0,1] \setminus A$ has Lebesgue measure strictly less than ε and an empty interior, because $[0,1] \setminus A$ does not intersect $\mathbb{Q}$.

Exercise 1.11 Let $\{I_n\}_{n \in \mathbb{N}}$ be an enumeration of the open intervals with rational endpoints of $(0,1)$. By the construction in Exercise (1.9), for any interval I and any $\delta \in (0, \lambda(I))$ we can find a compact set $C \subset I$ with an empty interior such that $0 < \lambda(C) < \delta$. We will define

$$E := \bigcup_{i=0}^{\infty} C_i$$

where $C_n \subset I_n$ are compact sets with an empty interior, $\lambda(C_n) > 0$ and $\lambda(C_n) < \delta_n$. The choice of C_n and δ_n will be done recursively. Notice first that

$$\lambda(E \cap I_n) \geq \lambda(C_n) > 0 \qquad \forall n \in \mathbb{N},$$

so we have only to take care of the condition $\lambda(E \cap I_n) < \lambda(I_n)$. Set $\beta_n = \lambda(I_n \setminus \cup_0^n C_i)$ and notice that $\beta_n > 0$ because all C_i have an empty interior. Since

$$\lambda(I_n \cap E) \leq \lambda(I_n \cap \bigcup_0^n C_i) + \sum_{i=n+1}^{\infty} \delta_i = \lambda(I_n) - \beta_n + \sum_{i=n+1}^{\infty} \delta_i$$

it suffices to choose δ_n (and C_n) in such a way that $\sum_{n+1}^{\infty} \delta_i < \beta_n$. This is possible, choosing for instance $\delta_{n+1} > 0$ satisfying

$$\delta_{n+1} < \max\left\{\frac{1}{2}\beta_n, \frac{1}{4}\beta_{n-1}, \ldots, \frac{1}{2^{n+1}}\beta_0\right\},$$

to get $\delta_i < 2^{n-i}\beta_n$ for $i > n$.

Exercise 1.12. Let A be μ–measurable and let $B, C \in \mathscr{E}$ be satisfying $A \Delta B \subset C$ and $\mu(C) = 0$. For any set $D \subset X$ we have, by monotonicity of μ^*,

$$\mu^*(D \cap A) + \mu^*(D \setminus A) \leq \mu^*(D \cap (B \cup C)) + \mu^*((D \setminus B) \cup C).$$

Since $\mu^*(D \cap C) \leq \mu^*(C) = \mu(C) = 0$, by using twice the subadditivity of μ^* and then the additivity of B we get

$$\mu^*(D \cap A) + \mu^*(D \setminus A) \leq \mu^*(D \cap B) + \mu^*(D \setminus B) = \mu^*(D).$$

Since D is arbitrary, this proves that A is additive.

Exercise 1.13. The statement is trivial if $\mu^*(A) = \infty$. If not, for any $n \in \mathbb{N}^*$ we can find, by the definition of μ^*, a countable union A_n of sets of $\mathscr{N}$ such that $A_n \supset A$ and $\mu(A_n) \leq \mu^*(A) + 1/n$. Then, setting $B := \bigcap_n A_n$ we have $B \supset A$ and $\mu(B) \leq \inf_n \mu^*(A) + 1/n = \mu^*(A)$. The inequality $\mu(B) \geq \mu^*(B)$ follows by the monotonicity of μ^*, taking into account that $\mu^*(B) = \mu(B)$.

Exercise 1.14. $\mathscr{E}_\mu$ is a σ–algebra: stability under complement is immediate, because $A^c \Delta B^c = A \Delta B$; if $A_i \Delta B_i \subset C_i$, then $(\bigcup_i A_i)\Delta(\bigcup_i B_i) \subset \bigcup_i C_i$, and since μ–negligible sets are stable under countable unions, this proves that $\mathscr{E}_\mu$ is stable under countable unions.

The extension $\mu(A) := \mu(B)$, where $B \in \mathscr{E}$ is any set such that $A \Delta B$ is contained in a μ–negligible set of $\mathscr{E}$, is well defined and σ–additive on $\mathscr{E}_\mu$: if $A \Delta B \subset C$ and $A \Delta B' \subset C'$, then $B \Delta B' \subset C \cup C'$; consequently, if $\mu(C) = \mu(C') = 0$ it must be $\mu(B) = \mu(B')$. The σ–additivity can be proven with an argument analogous to the one used to show that $\mathscr{E}_\mu$ is a σ–algebra.

μ–negligible sets of $\mathscr{E}_\mu$ are characterized by the property of being contained in a μ–negligible set of $\mathscr{E}$: if $A \in \mathscr{E}_\mu$ is μ–negligible, there exist μ–negligible sets B, $C \in \mathscr{E}$ with $A \Delta B \subset C$; as a consequence A is contained in the μ–negligible set $B \cup C \in \mathscr{E}$. Conversely, if $A \subset X$ is contained in a μ–negligible set $C \in \mathscr{E}$ we may take $B = \emptyset$ to conclude that $A \in \mathscr{E}_\mu$ and $\mu(A) = 0$.

Exercise 1.15. Let A be additive; by Exercise 1.13 we can find a set $B \in \mathscr{E}$ containing A with $\mu(B) = \mu^*(A)$. The additivity of A and the equality $\mu^*(B) = \mu(B)$ give

$$\mu(B) = \mu^*(A) + \mu^*(B \setminus A).$$

As a consequence $\mu^*(B \setminus A) = 0$. Now we apply Exercise 1.13 again, to find a μ–negligible set $C \in \mathscr{E}$ containing $A \setminus B$. It follows that $A \Delta B$ is contained in C, and therefore A is μ–measurable.

Exercise 1.16. Let us first build a family of pairwise disjoints sets $\{A_i\}_{i \in I} \subset \mathscr{P}(\mathbb{N})$, with I and all sets A_i having an infinite cardinality and $\bigcup_i A_i = \mathbb{N}$ (the construction of the σ–algebra will be more clear if we keep I and $\mathbb{N}$ distinct). The family $\{A_i\}$ can be obtained, for instance, through a bijective correspondence S between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, setting $A_i := S(\{i\} \times \mathbb{N})$. Then, we define $\pi : \mathbb{N} \to I$ by

$$\pi(n) = i, \text{ where } i \in I \text{ is the unique index such that } n \in A_i$$

and (with the convention $\pi^{-1}(\emptyset) = \emptyset$)

$$\mathscr{F} := \left\{\pi^{-1}(J) \ : \ J \subset I\right\}.$$

It is immediate to check that $\mathscr{F}$ is a σ–algebra, that $A_i = \pi^{-1}(\{i\}) \in \mathscr{F}$ and that any nonempty set in $\mathscr{F}$ contains one of the sets A_i. Therefore $\mathscr{F}$ contains infinitely many sets, and all of them except $\emptyset$ have an infinite cardinality.

Exercise 1.17. It suffices to define $\mu(A) = 0$ if A has a finite cardinality, and $+\infty$ otherwise. A finite union of sets has an infinite cardinality if and only if at least one of the sets has an infinite cardinality, and this shows that μ is additive.

The solutions of the next exercises require a more advanced knowledge of set theory, and in particular the theory of ordinals, the transfinite induction, the behavior of cardinality under unions and products, and Zorn lemma. We shall denote by ω the smallest uncountable ordinal and by χ the cardinality of continuum.

Exercise 1.18. Notice that $\mathscr{F}^{(j)} \subset \sigma(\mathscr{K})$ implies

$$\left\{ \bigcup_{k=0}^{\infty} A_k,\ B^c\ :\ (A_k) \subset \mathscr{F}^{(j)},\ B \in \mathscr{F}^{(j)} \right\} \subset \sigma(\mathscr{K}).$$

Therefore, if i is the successor of j, we obtain $\mathscr{F}^{(i)} \subset \sigma(\mathscr{K})$; analogously, if i has no predecessor, and $\mathscr{F}^{(j)} \subset \sigma(\mathscr{K})$ for all $j \in i$, then $\bigcup_{j\in i} \mathscr{F}^{(j)}$, namely $\mathscr{F}^{(i)}$, is contained in $\sigma(\mathscr{K})$. Using these two facts, one obtains by transfinite induction that $\mathscr{F}^{(i)} \subset \sigma(\mathscr{K})$ for all $i \in \omega$. An analogous induction argument shows that $\mathscr{F}^{(i)} \subset \mathscr{F}^{(j)}$ whenever $i \in j$.

So, the union $\mathscr{U} := \bigcup_{i\in\omega} \mathscr{F}^{(i)}$ is contained in $\sigma(\mathscr{K})$ and, to prove that equality holds, it suffices to show that this union is a σ–algebra. Let $(B_k) \subset \mathscr{U}$ and let $i_k \in \omega$ be such that $B_k \in \mathscr{F}^{(i_k)}$. Since i_k are countable and ω is uncountable we have $i := \bigcup_k i_k \in \omega$ and all sets B_k belong to $\mathscr{F}^{(i)}$. It follows that their union belongs to $\mathscr{F}^{(j)}$, where j is the successor of i, and therefore to $\mathscr{U}$. An analogous (and simpler) argument proves that $\mathscr{U}$ is stable under complement.

Exercise 1.19. Obviously $\mathscr{B}(\mathbb{R})$ has at least the cardinality of continuum, so we need only to show an upper bound on the cardinality of $\mathscr{B}(\mathbb{R})$. The proof is based on the fact that a union $\bigcup_{i\in J} X_i$ and a product $\bigtimes_{i\in J} X_i$ have cardinality not greater than χ if the index set J and all sets X_i have cardinality not greater than χ. Let $\mathscr{F}^{(i)}$ be defined as in Exercise 1.19, with $\mathscr{K}$ having at most the cardinality of continuum. Using the previous property of products, with J even countable, one can prove by transfinite induction that, for all $i \in \omega$, $\mathscr{F}^{(i)}$ has at most cardinality χ. If we choose as $\mathscr{K}$ the class of intervals, whose cardinality is (at most) χ, we find

$$\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{K}) = \bigcup_{i\in\omega} \mathscr{F}^{(i)}.$$

Now we use the above mentioned property of unions, with $J = \omega$ and $X_i = \mathscr{F}^{(i)}$, to conclude that $\mathscr{B}(\mathbb{R})$ has at most the cardinality of continuum.

Exercise 1.20. Obviously $\mathscr{L}$ has a cardinality not greater than the cardinality of $\mathscr{P}(\mathbb{R})$; by Bernstein theorem[(1)] it suffices to show that the cardinality of $\mathscr{P}(\mathbb{R})$ is not greater than the cardinality of $\mathscr{L}$: if C is the Cantor set of Exercise 1.8, we know that $\mathscr{P}(\mathbb{R})$ is in one-to-one correspondence of $\mathscr{P}(C)$, because C has the cardinality of continuum; on the other hand, any subset of C obviously belongs to $\mathscr{L}$, because C has null Lebesgue measure.

Exercise 1.21. Let $\mathscr{E} \subset \mathscr{P}(X)$ be a σ–algebra. Assume by contradiction that $\mathscr{E}$ is infinite and countable. We define the equivalence relation

$$y \sim y' \qquad \text{if and only if} \quad ((y \in B \;\Leftrightarrow\; y' \in B) \quad \forall B \in \mathscr{E})$$

and let $\mathscr{F}$ be the partition of X in equivalence classes. We now prove that $\mathscr{F} \subset \mathscr{E}$. Indeed, let $F \in \mathscr{F}$, fix $f \in F$, for any $x \notin F$ we have $f \not\sim x$ so there must be $B \in \mathscr{E}$ such that $f \in B$ and $x \notin B$ (or the opposite, but then we may consider B^c); given this set B, for any $g \in F$ we have that $g \sim f$ implies $g \in B$, so that $F \subset B$. Since x is arbitrary we conclude that

$$F = \bigcap_{B \in \mathscr{E}, F \subset B} B.$$

Now, since $\mathscr{E}$ is countable, it follows that $F \in \mathscr{E}$. We eventually note that any set in $\mathscr{E}$ is union of sets in $\mathscr{F}$: but then, if $\mathscr{F}$ were finite then $\mathscr{E}$ would be finite, whereas if $\mathscr{F}$ were infinite then $\mathscr{E}$ would be uncountable.

Exercise 1.22 We define $\mathscr{F}$ as in the solution of the previous exercise, in this case it has finite cardinality, say n; consequently, there are 2^n sets in $\mathscr{E}$.

Exercise 1.23 We define $\mathscr{F}$ as in the solution of Exercise 1.21; we also adapt the above argument to show again that $\mathscr{F} \subset \mathscr{E}$. Indeed, let $F \in \mathscr{F}$, fix $f \in F$, for any $x \notin F$ we have $f \not\sim x$ so there must be $B = B_{F,x} \in \mathscr{E}$ such that $f \in B$ and $x \notin B$; and again $F \subset B_{F,x}$. Hence

$$F = \bigcap_{x \in X,\, x \notin F} B_{F,x}$$

and this proves that $\mathscr{F} \subset \mathscr{A}$, since X is countable. We then use the Axiom of Choice to define a function $\phi : \mathscr{F} \to X$ such that $\phi(F) \in F$, and eventually define $\tilde{\mu} = \sum_{F \in \mathscr{F}} \mu(F)\delta_{\phi(x)}$.

(1) If A has cardinality not greater than B, and B has cardinality not greater than A, then there exists a bijection between A and B

Exercise 1.24. We begin our construction with an algebra τ_0 in $\mathscr{P}(\mathbb{N})$ and $\mu_0 : \tau_0 \to \{0, 1\}$ which is additive but not σ–additive. For instance we may take as τ_0 the algebra generated by singletons $\{x\}$ with $x \in \mathbb{N}$ (*i.e.* the sets $A \subset \mathbb{N}$ such that either A or A^c are finite) and set

$$\mu_0(A) := \begin{cases} 0 & \text{if } A \text{ is finite;} \\ 1 & \text{if } A^c \text{ is finite.} \end{cases}$$

We will extend μ_0 to an additive function, that we still denote by μ_0, defined on the whole of $\mathscr{P}(\mathbb{N})$. If such an extension exists, it can't be σ–additive, because $\mu_0(\{n\}) = 0$ for all $n \in \mathbb{N}$, while $\mu_0(\mathbb{N}) = 1$.

In the class $\mathscr{C}$ of pairs (τ, μ) with τ algebra and $\mu : \tau \to \{0, 1\}$ additive, we define the partial order relation $(\tau, \mu) \le (\tau', \mu')$ by $\tau \subset \tau'$ and $\mu'_{|\tau} = \mu$; then we consider the class $\mathscr{C}_0$ of all (τ, μ) satisfying $(\tau, \mu) \ge (\tau_0, \mu_0)$. By Zorn lemma, we can find a maximal $(\bar\tau, \bar\mu)$ in this class: indeed, it is easy to check that any totally ordered chain $I \subset \mathscr{C}_0$ has an upper bound (τ', μ'), defined by

$$\tau' := \bigcup_{(\tau,\mu)\in I} \tau \qquad \text{and} \qquad \mu'(A) := \mu(A) \quad \text{where} \quad A \in \tau, (\tau, \mu) \in I.$$

We will show that the maximality of $(\bar\tau, \bar\mu)$ forces $\bar\tau$ to coincide with $\mathscr{P}(\mathbb{N})$, so that $\bar\mu$ will be the desired extension of μ_0.

Let us assume by contradiction that $\bar\tau \subsetneq \mathscr{P}(\mathbb{N})$ and choose $Z \subset \mathbb{N}$ with $Z \notin \bar\tau$. We notice that

$$\left\{(A_1 \cap Z) \cup (A_2 \cap Z^c) \ : \ A_1,\ A_2 \in \bar\tau\right\}$$

is the algebra generated by $\bar\tau \cup \{Z\}$. Moreover, either Z or Z^c satisfy the following property

$$\text{for all } A \in \bar\tau \text{ with } \bar\mu(A) = 1,\ Z \cap A \neq \varnothing. \tag{A.13}$$

If not, we would be able to find $A_1,\ A_2 \in \bar\tau$ with $A_1 \cap Z = A_2 \cap Z^c = \varnothing$ and $\bar\mu(A_1) = \bar\mu(A_2) = 1$, so that A_1 and A_2 would be disjoint and $\bar\mu(A_1 \cup A_2) = 2$, contradicting the fact that $\bar\mu$ maps $\bar\tau$ into $\{0, 1\}$. Possibly replacing Z by its complement we shall assume that Z fulfils (A.13).

Now we extend $\bar\mu$ to the algebra generated by $\bar\tau \cup \{Z\}$, as follows:

$$\tilde\mu(B) := \bar\mu(A_1) \text{ whenever } A_1,\ A_2 \in \bar\tau \text{ and } B = (A_1 \cap Z) \cup (A_2 \cap Z^c). \tag{A.14}$$

Let us check that $\tilde\mu$ is well defined and additive.

1. $\tilde{\mu}$ is well defined: if

$$B = (A_1 \cap Z) \cup (A_2 \cap Z^c) = (A_3 \cap Z) \cup (A_4 \cap Z^c)$$

then $(A_1 \cap Z) = (A_3 \cap Z)$, and if $\bar{\mu}(A_1) \neq \bar{\mu}(A_3)$ then one of the two numbers, say $\bar{\mu}(A_1)$, equals 1, while $\bar{\mu}(A_3) = 0$. Defining $A := A_1 \setminus A_3$ we have $\bar{\mu}(A) = 1$ and $A \cap Z = \emptyset$, contradicting (A.13).
2. Suppose $B,\ B' \in \bar{\tau}$ are disjoint. Let $B = (A_1 \cap Z) \cup (A_2 \cap Z^c)$ and $B' = (A_1' \cap Z) \cup (A_2' \cap Z^c)$. Then $A_1 \cap A_1' \cap Z = \emptyset$. Setting $A_1'' := A_1' \setminus A_1$ we still have $B' = (A_1'' \cap Z) \cup (A_2' \cap Z^c)$, and then we can use the additivity of $\bar{\mu}$ to conclude that

$$\tilde{\mu}(B \cup B') = \bar{\mu}(A_1 \cup A_1'') = \bar{\mu}(A_1) + \bar{\mu}(A_1'') = \tilde{\mu}(B) + \tilde{\mu}(B').$$

If $B \in \tau$ we can choose $A_1 = A_2 = B$ in (A.14) to obtain that $\tilde{\mu}(B) = \bar{\mu}(B)$, so that $\tilde{\mu}$ extends $\bar{\mu}$ to the algebra generated by $\bar{\tau} \cup \{Z\}$. This violates the maximality of $(\bar{\tau}, \bar{\mu})$.

Exercise 1.25 We obviously need only to show that the cardinality of C is at least equal to the continuum. By the inner regularity of λ we can assume with no loss of generality that C is closed. Now, we define $A = (0, 1) \setminus C$ and

$$g(t) := \lambda\big([0, t] \cap C\big) \qquad t \in [0, 1].$$

This continuous function maps continuously $[0, 1]$ onto $[0, \lambda(C)]$, and it is constant in any connected component of A, so that $g(A)$ is at most countable. Since $g(C)$ contains $[0, \lambda(C)] \setminus g(A)$ we obtain that C has cardinality at least equal to the continuum (one can actually see that $g(C) = g([0, 1])$).

Exercise 1.26 Since K is totally bounded, for all $\epsilon > 0$ there exist finitely many balls $B_1, \ldots, B_N$ with radius ϵ whose union covers K. The properties of μ imply the existence of an index i such that $\mu(\{n : x_n \in B_i\}) = 1$. Now we start with $\epsilon = 1$ and find a closed ball $B^{(1)}$ with radius 1 such that $\mu(\{n : x_n \in B^{(1)}\}) = 1$. Repeating this construction in $B^{(1)}$ we find a closed ball $B^{(2)}$ with radius $1/2$ contained in $B^{(1)}$ with $\mu(\{n : x_n \in B^{(2)}\}) = 1$. Continuing in this way, if z is the common point of the balls $B^{(i)}$, we find x_n μ-converges to z.

Chapter 2

Exercise 2.1 The verification is straightforward and is omitted.

Exercise 2.2 Let $\varphi,\ \psi\colon X \to \mathbb{R}$ be $\mathscr{E}$–measurable. If $\varphi(x) + \psi(x) < t$ we can find a rational number r such that $\varphi(x) < r$ and $\psi(x) < t - r$, hence

$$\{\varphi + \psi < t\} = \bigcup_{r \in \mathbb{Q}} [\{\varphi < r\} \cap \{\psi < t - r\}]\,.$$

This proves that $\varphi + \psi$ is $\mathscr{E}$–measurable. Analogously, since

$$\{\varphi^2 > a\} = \{\varphi > \sqrt{a}\} \cup \{\varphi < -\sqrt{a}\}, \quad a \geq 0$$

we obtain that φ^2 is measurable. Considering the difference $(\varphi + \psi)^2 - (\varphi - \psi)^2$ we obtain that $\varphi\psi$ is $\mathscr{E}$–measurable.

Exercise 2.3. (i) The verification of the axioms of distance is immediate. In order to prove the compactness of $\overline{\mathbb{R}}$, let us consider a sequence $(x_n) \subset \overline{\mathbb{R}}$. If $\sup_n x_n = +\infty$ we can find for any k an index $n(k)$ such that $x_{n(k)} \geq k$; it follows that $d(x_{n_k}, +\infty) = |\arctan x_{n(k)} - \pi/2|$ tends to 0, so that $x_{n(k)} \to +\infty$ in the metric space. Analogously, if $\inf_n x_n = -\infty$ we can find a subsequence converging to $-\infty$ in $(\overline{\mathbb{R}}, d)$. Finally, if both $\sup_n x_n$ and $\inf_n x_n$ are finite, the sequence (x_n) is bounded and we can extract, thanks to the Bolzano–Weierstrass theorem, a subsequence $x_{n(k)}$ converging to $x \in \mathbb{R}$. The continuity of $z \mapsto \arctan z$ implies that $x_{n(k)} \to x$ in $(\overline{\mathbb{R}}, d)$. To prove the equivalence of the two topologies, let us work with closed sets: if $C \subset \mathbb{R}$ is closed with respect to the $(\overline{\mathbb{R}}, d)$ topology, then it is closed with respect to the Euclidean topology, because $|x_n - x| \to 0$ implies $|\arctan x_n - \arctan x| \to 0$. On the other hand, if $|\arctan x_n - \arctan x| \to 0$ then for n large enough $\arctan x_n$ belongs to an interval $I := (\arctan x - \varepsilon, \arctan x + \varepsilon) \subset (-\pi/2, \pi/2)$; the continuity of $y \mapsto \tan y$ in I implies that $x_n \to x$. This proves the converse implication, and the equivalence of the two topologies.
(ii) We notice first that, according to (i), $\mathscr{B}(\mathbb{R})$ and $\{-\infty\}, \{+\infty\}$ belong to $\mathscr{B}(\overline{\mathbb{R}})$. Therefore, if f is measurable between $\mathscr{E}$ and the Borel σ–algebra of $(\overline{\mathbb{R}}, d)$, then it is $\mathscr{E}$–measurable according to (2.2). According to the measurability criterion, in order to prove the converse implication it suffices to show that $\mathscr{B}(\overline{\mathbb{R}})$ is generated by $\mathscr{B}(\mathbb{R}) \cup \{-\infty\} \cup \{+\infty\}$: this follows by the fact that if $C \subset \overline{\mathbb{R}}$ is closed, then

$$C = (C \cap \mathbb{R}) \cup (C \cap \{-\infty\}) \cup (C \cap \{+\infty\})$$

(again by (i)) belongs to the algebra generated by $\mathscr{B}(\mathbb{R}) \cup \{-\infty\} \cup \{+\infty\}$, therefore the σ–algebra generated by this family of sets contains $\mathscr{B}(\overline{\mathbb{R}})$.

Exercise 2.4. If $\{f \neq g\}$ is contained in a μ–negligible set C of $\mathscr{E}$, for some $\mathscr{E}$–measurable function g, then $\{f > t\} \Delta \{g > t\} \subset C$ for all $t \in \mathbb{R}$, and since $\{g > t\} \in \mathscr{E}$ it follows that $\{f > t\} \in \mathscr{E}_\mu$; this means that f is $\mathscr{E}_\mu$–measurable. Conversely, assume that f is $\mathscr{E}_\mu$–measurable and find for all $q \in \mathbb{Q}$ a set $B_q \in \mathscr{E}$ and a μ–negligible set $C_q \in \mathscr{E}$ with $\{f > q\}\Delta B_q \subset C_q$. We define

$$g(x) := \sup\left\{q \in \mathbb{Q} \;:\; x \in B_q\right\}, \qquad C := \bigcup_{q\in\mathbb{Q}} C_q.$$

Since $\{g \le t\} = \bigcap_{q\le t} B_q$ we have that g is $\mathscr{E}$–measurable. Let us prove that $f(x) = g(x)$ for all $x \notin C$: for any such x we have $x \in B_q$ for all $q < f(x)$, therefore $g(x) \ge f(x)$; if the inequality were strict, there would exist $q \in \mathbb{Q}$ with $x \in B_q$ and $q > f(x)$, therefore x would be in $B_q \setminus \{f > q\} \subset C_q \subset C$.

Exercise 2.5. If $\sigma \le \tau$ we can find a nondecreasing family of partitions $\sigma_1, \ldots, \sigma_n$ with $\sigma_1 = \sigma$, $\sigma_n = \tau$ and $\sigma_{i+1} \setminus \sigma_i$ containing just one point. Therefore, in the proof of the monotonicity of $\sigma \mapsto I_\sigma(f)$ we need only to show that $I_\sigma(f) \le I_{\sigma\cup\{t\}}(f)$ whenever $t \in (0,\infty) \setminus \sigma$. Let $\sigma = \{t_0, \ldots, t_N\}$ and let i be the last index such that $t_i < t$. If $i < N$ we use the inequality

$$\begin{aligned}(t_{i+1} - t_i) f(t_{i+1}) &= (t_{i+1} - t) f(t_{i+1}) + (t - t_i) f(t_{i+1})\\ &\le (t_{i+1} - t) f(t_{i+1}) + (t - t_i) f(t)\end{aligned}$$

adding to both sides $\sum_{j\ne i}(t_{j+1}-t_j) f(t_{j+1})$ we obtain $I_\sigma(f) \le I_{\sigma\cup\{t\}}(f)$. If $i = N$ the argument is even easier, because the difference $I_{\sigma\cup\{t\}}(f) - I_\sigma(f)$ is given by $(t - t_N) f(t)$.

Now, let $f,\ g : (0,+\infty) \to [0,+\infty)$ be given; since $I_\sigma(f + g) = I_\sigma(f) + I_\sigma(g)$ we get $I_\sigma(f + g) \le \int_0^\infty f(t)\,dt + \int_0^\infty g(t)\,dt$. Since $\sigma \in \Sigma$ is arbitrary, this proves that

$$\int_0^\infty f(t) + g(t)\,dt \le \int_0^\infty f(t)\,dt + \int_0^\infty g(t)\,dt.$$

In order to prove the converse inequality, fix $L < \int_0^\infty f(t)\,dt$, $M < \int_0^\infty g(t)\,dt$ and find $\sigma,\ \eta \in \Sigma$ with $I_\sigma(f) > L$ and $I_\eta(g) > M$; then

$$\begin{aligned}\int_0^\infty f(t) + g(t)\,dt &\ge I_{\sigma\cup\eta}(f + g) = I_{\sigma\cup\eta}(f) + I_{\sigma\cup\eta}(g)\\ &\ge I_\sigma(f) + I_\eta(g) > L + M.\end{aligned}$$

Letting $L \uparrow \int_0^\infty f(t)\,dt$ and $M \uparrow \int_0^\infty g(t)\,dt$ the inequality is proved.

Exercise 2.6. We will prove that f_* is lower semicontinuous, the proof of the upper semicontinuity of f^* being analogous. Let $(x_n) \subset \mathbb{R}$ be converging to x and use the definition of $f_*(x_n)$ to find $y_n \in \mathbb{R}$ such that

$$|x_n - y_n| < \frac{1}{n} \qquad \text{and} \qquad f(y_n) \le f_*(x_n) + \frac{1}{n}.$$

Then (y_n) still converges to x, so that

$$f_*(x) \le \liminf_{n\to\infty} f(y_n) \le \liminf_{n\to\infty} f_*(x_n) + \frac{1}{n} = \liminf_{n\to\infty} f_*(x_n).$$

Exercise 2.7. Let $t \in \mathbb{R}$ and let $(x_n) \subset \{f_* \le t\}$ be convergent to x. Then, the lower semicontinuity of f_* gives

$$f_*(x) \le \liminf_{n\to\infty} f_*(x_n) \le t.$$

This proves that $x \in \{f_* \le t\}$, so that $\{f_* \le t\}$ is closed. The proof for f^* is similar. Since the Borel σ–algebra is generated by halflines, it follows that f^* and f_* are Borel, and the same is true for the set $\{f_* = f^*\}$, that coincides with Σ.

Exercise 2.8. Set $\varphi_0 := \varphi$, $A_0 := \{\varphi_0 \ge a_0\}$ and $\varphi_1 := \varphi - a_0 1\!\!1_{A_0} \ge 0$. Then, set $A_1 := \{\varphi_1 \ge a_1\}$ and $\varphi_2 := \varphi_1 - a_1 1\!\!1_{A_1}$ and so on. If $\varphi(x) = +\infty$ then $\varphi_n(x) = +\infty$ for all n, so that x belongs to all sets A_i and $\sum_{i=0}^n a_i 1\!\!1_{A_i}(x) = +\infty$. We then assume that $\varphi(x) < +\infty$ in the following. By construction we have that $0 \le \varphi_{i+1} \le \varphi_i \le \cdots \le \varphi_0 = \varphi$, hence

$$\varphi = \varphi_{n+1} + \sum_{i=0}^{n} (\varphi_i - \varphi_{i+1}) = \varphi_{n+1} + \sum_{i=0}^{n} a_i 1\!\!1_{A_i}.$$

This proves that $\varphi \ge \sum_i a_i 1\!\!1_{A_i}$. If the inequality were strict for some $x \in X$ with $\varphi(x) < +\infty$, we could find $\varepsilon > 0$ such that $\varphi_i(x) \ge \varepsilon$ for all $i \in \mathbb{N}$, and since $a_i < \varepsilon$ for i large enough, we would get $x \in A_i$ for i large enough. But since the series $\sum_i a_i$ is not convergent, we would get $\sum_i a_i 1\!\!1_{A_i}(x) = \infty$, a contradiction.

Exercise 2.9. Assume by contradiction that the absolute continuity property fails. Then, for some $\varepsilon > 0$ we can find A_i with $\mu(A_i) < 2^{-i}$ and $\int_{A_i} |\varphi|\, d\mu \ge \varepsilon$. It follows that the set $B := \limsup_i A_i$ is μ–negligible, and

$$B_n := \bigcup_{i\ge n} A_i \setminus B \downarrow \varnothing.$$

Since $\int_{B_n} |\varphi|\, d\mu \ge \int_{A_n} |\varphi|\, d\mu \ge \varepsilon$ we find a contradiction with the dominated convergence theorem applied to the functions $1\!\!1_{B_n}|\varphi|$, pointwise converging to 0.

Exercise 2.10. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that $\int_A |\varphi|\, d\mu < \varepsilon/2$ whenever $A \in \mathscr{E}$ and $\mu(A) < \delta$. The triangle inequality gives, with the same choice of A, $\int_A |\varphi_n|\, d\mu < \varepsilon$ for $n > n_0$, provided $\|\varphi_n - \varphi\|_1 < \varepsilon/2$ for $n > n_0$. Since $\varphi_1, \ldots, \varphi_{n_0}$ are integrable, we can find $\delta_i > 0$ such that $\int_A |\varphi_i|\, d\mu < \varepsilon$ whenever $A \in \mathscr{E}$ and $\mu(A) < \delta_i$. If

$\delta_0 = \min\{\delta, \min_i \delta_i\}$, we have $\int_A |\varphi_n|\,d\mu < \varepsilon/2$ whenever $n \in \mathbb{N}$, $A \in \mathscr{E}$ and $\mu(A) < \delta$.
A possible example for the second question is $\Omega = [0,1]$, $\mu = \lambda$ the Lebesgue measure, and $\varphi_n = \frac{2^n}{n}\mathbb{1}_{[2^{-n},2^{1-n})}$. The uniform integrability is a direct consequence of the convergence of φ_n to 0 in L^1. If $\varphi_n \le g$, then

$$\sum_{n=1}^{\infty} \varphi_n = \bigvee_{n=1}^{\infty} \varphi_n \le g$$

but $\int \sum_{n=1}^{\infty} \varphi_n = \sum_1^{\infty} 1/n = +\infty$.
Exercise 2.11. (a) For any $y \in X$ we have

$$g_\lambda(x) \le g(y) + \lambda d(x,y) \le g(y) + \lambda d(x',y) + \lambda d(x,x').$$

Since y is arbitrary we get $g_\lambda(x) \le g_\lambda(x') + \lambda d(x,x')$. Reversing the roles of x and x' the inequality is achieved.

(b) Clearly the family (g_λ) is monotone with respect to λ, and since we can always choose $y = x$ in the minimization problem we have $g_\lambda(x) \le g(x)$. Assume that $\sup_\lambda g_\lambda(x)$ is finite (otherwise the statement is trivial) and let x_λ such that $g_\lambda(x) + \lambda^{-1} \ge g(x_\lambda) + \lambda d(x,x_\lambda)$. This inequality implies that $x_\lambda \to x$ as $\lambda \to \infty$ and, now neglecting the term $\lambda d(x,x_\lambda)$, that

$$g_\lambda(x) + \frac{1}{\lambda} \ge g(x_\lambda).$$

Passing to the limit in this inequality as $\lambda \to \infty$ and using the lower semicontinuity of g we get $\sup_\lambda g_\lambda(x) \ge g(x)$.
Exercise 2.12. Let us first assume that f is bounded. For $\varepsilon > 0$ we consider the functions

$$f_\varepsilon(x,y) := \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(x',y)\,dx'.$$

Since $x \mapsto f(x,y)$ is continuous, we can apply the mean value theorem to obtain that $f_\varepsilon(x,y) \to f(x,y)$ as $\varepsilon \downarrow 0$. So, in order to show that f is a Borel function, we need only to show that f_ε are Borel.

We will prove indeed that f_ε are continuous: let $x_n \to x$ and $y_n \to y$; since $f(x',y_n) \to f(x',y)$ for all $x' \in \mathbb{R}$, we have

$$\mathbb{1}_{[x_n-\varepsilon,x_n+\varepsilon]}(x')f(x',y_n) \to \mathbb{1}_{[x-\varepsilon,x+\varepsilon]}(x')f(x',y)$$

for all $x' \in \mathbb{R} \setminus \{x-\varepsilon, x+\varepsilon\}$. Therefore, since f is bounded, the dominated convergence theorem yields

$$\begin{aligned} f_\varepsilon(x,y) &= \frac{1}{2\varepsilon}\int_{\mathbb{R}} \mathbb{1}_{[x-\varepsilon,x+\varepsilon]}(x')f(x',y)\,dx' \\ &= \frac{1}{2\varepsilon}\lim_{n\to\infty}\int_{\mathbb{R}} \mathbb{1}_{[x_n-\varepsilon,x_n+\varepsilon]}(x')f(x',y_n)\,dx' = \lim_{n\to\infty} f_\varepsilon(x_n,y_n). \end{aligned}$$

In the general case when f is not bounded we approximate it by the bounded functions $f_h(x) := \max\{-h, \min\{f(x), h\}\}$, with $h \in \mathbb{N}$, that are still separately continuous, and therefore Borel.

Chapter 3

Exercise 3.1. On the real line, endowed with the Lebesgue measure, the function $(1 + |x|)^{-1}$ belongs to L^2, but not to L^1, and the function $|x|^{-1/2}\mathbb{1}_{(0,1)}(x)$ belongs to L^1, but not to L^2. Turning back to the general case, if $\varphi \in L^{p_1} \cap L^{p_2}$ with $p_1 \le p_2$, from the inequality

$$|\varphi|^p \le \max\{|\varphi|^{p_1}, |\varphi|^{p_2}\} \le |\varphi|^{p_1} + |\varphi|^{p_2} \qquad \forall p \in [p_1, p_2]$$

(that can be verified considering separately the cases $|\varphi| \le 1$ and $|\varphi| > 1$) we get that $\varphi \in L^p$ for all $p \in [p_1, p_2]$.

Exercise 3.2. The statement is trivial if $\|f\|_q = 0$, so we assume that $\|f\|_q > 0$. For $\epsilon > 0$ the set $X_\epsilon := \{|f| > \epsilon\}$ has finite μ–measure, by the Markov inequality, hence the inclusion between L^r spaces for finite measures gives that $|f|\mathbb{1}_{X_\epsilon} \in L^p(X, \mathscr{E}, \mu)$. Since the dominated convergence theorem gives

$$\lim_{\epsilon\downarrow 0} \int_X |f - f\mathbb{1}_{X_\epsilon}|^q \, d\mu = \lim_{\epsilon\downarrow 0} \int_{X\setminus X_\epsilon} |f|^q \, d\mu = 0$$

we can choose $\tilde{f} = f\mathbb{1}_{X_\epsilon}$ for $\epsilon > 0$ small enough.

Exercise 3.3. By homogeneity we can assume that $\|\varphi\|_p = 1$ and $\|\psi\|_q = 1$. Since

$$\int_X \left(\frac{|\varphi|^p}{p} + \frac{|\psi|^q}{q} - |\varphi||\psi|\right) d\mu = \frac{\|\varphi\|_p}{p} + \frac{\|\psi\|_q}{q} - 1 = 0$$

and the function among parentheses is nonnegative, it follows that if vanishes μ–a.e. In particular, for μ–a.e. x, $|\varphi(x)|$ is a minimizer of

$$y \mapsto \frac{y^q}{q} - |\psi(x)|y$$

in $[0, +\infty)$. But this problem has a unique minimizer, given by $|\psi(x)|^{q-1}$, and we conclude.

Exercise 3.4. It suffices to apply Hölder's inequality to the functions $|\varphi|^r$ and $|\psi|^r$, with the dual exponents p/r and q/r, to obtain

$$\|\varphi\psi\|_r^r \le \||\varphi|^r\|_{p/r}\||\psi|^r\|_{q/r} = \|\varphi\|_p^r\|\psi\|_q^r.$$

Exercise 3.5. The positive part and the negative part of $\varphi - \varphi_n$ have the same integral, hence

$$\int_X |\varphi - \varphi_n| \, d\mu = 2 \int_X (\varphi - \varphi_n)^+ \, d\mu.$$

The condition $\liminf_n \varphi_n \geq \varphi$ ensures that $(\varphi - \varphi_n)^+$ is pointwise convergent to 0; in addition, since φ_n are nonnegative, the functions are dominated by φ^+. Therefore the dominated convergence theorem gives the result.

Exercise 3.6. If $\psi_n \to \psi$ μ–a.e. we apply Fatou's lemma to the functions $\psi_n + \varphi_n$ to obtain

$$\liminf_{n\to\infty} \int_X \psi_n + \varphi_n \, d\mu \geq \int_X \psi + \varphi \, d\mu.$$

Therefore

$$\limsup_{n\to\infty} \int_X \psi_n \, d\mu + \liminf_{n\to\infty} \int_X \varphi_n \, d\mu \geq \int_X \varphi \, d\mu + \int_X \psi \, d\mu.$$

Subtracting $\int \psi \, d\mu$ from both sides the statement is achieved. In the general case, let $n(k)$ be a subsequence such that $\lim_k \int \varphi_{n(k)} \, d\mu = \liminf_n \int_X \varphi_n$, and let $n(k(s))$ be a further subsequence converging to φ μ–a.e. Then

$$\begin{aligned}\liminf_{n\to\infty} \int_X \varphi_n \, d\mu &= \lim_{s\to\infty} \int_X \varphi_{n(k(s))} \, d\mu \geq \int_X \liminf_n \varphi_{n(k(s))} \, d\mu \\ &\geq \int_X \liminf_{n\to\infty} \varphi_n \, d\mu.\end{aligned}$$

Exercise 3.7. We show only how (3.13) implies $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$ for all $x, y \in J$ and $t \in [0,1]$. We prove first, by induction on m, that

$$g\left(\sum_{i=1}^{2^m} \frac{1}{2^m} x_i\right) \leq \sum_{i=1}^{2^m} \frac{1}{2^m} g(x_i)$$

for all $x_1, \ldots, x_{2^m} \in J$. The case $m = 1$ is (3.13) and the induction step can be achieved grouping the terms as follows:

$$\sum_{i=1}^{2^m} \frac{1}{2^m} x_i = \frac{1}{2}\left(\sum_{i=1}^{2^{m-1}} \frac{1}{2^{m-1}} x_i + \sum_{i=1}^{2^{m-1}} \frac{1}{2^{m-1}} x_{2^{m-1}+i}\right).$$

Now, considering the case when $x_i = x$ for $1 \le i \le k$ and $x_i = y$ otherwise, we get

$$g(tx + (1-t)y) \le tg(x) + (1-t)g(y) \qquad \text{with } t = \frac{k}{2^m}.$$

Since g is continuous, by approximation we get $g(tx + (1-t)y) \le tg(x) + (1-t)g(y)$ for all $x,\ y \in J$ and $t \in [0,1]$.

Exercise 3.8. Let us first show the existence of z_0. Let $A = g(\mathbb{R})$ and let $u_n = g(z_n)$ with $u_n \downarrow \inf A$. Since u_n is uniformly bounded from above, our assumption on g ensures that (z_n) is bounded. By the Bolzano-Weierstrass theorem we can find a subsequence $z_{n(k)}$ convergent to $z \in \mathbb{R}$. The continuity of g gives that $u_{n(k)} = g(z_{n(k)})$ converge to $g(z)$. It follows that $\inf A$ is finite and coincides with $g(z)$. Now, by applying the convexity inequality of the previous exercise with $x = z_2$, $y = z_0$ and $t = (z_1 - z_0)/(z_2 - z_0)$, we get

$$\frac{g(z_2) - g(z_1)}{z_2 - z_1} \ge \frac{g(z_1) - g(z_0)}{z_1 - z_0} \ge 0$$

for $z_0 < z_1 < z_2$, proving the monotonicity of g in $[z_0, +\infty)$. The argument in $(-\infty, z_0]$ is analogous.

Exercise 3.9. Fatou's lemma gives $\liminf_n \int \varphi_n \, d\mu \ge \int \liminf_n \varphi_n \, d\mu \ge \int \varphi \, d\mu$. Therefore $t_n := \int \varphi_n \, d\mu \to t := \int \varphi \, d\mu$; we can apply Exercise 3.5 to the functions φ_n/t_n to obtain that $\varphi_n/t_n \to \varphi/t$ in L^1. From this, taking into account that $t_n \to t$, the convergence of φ_n to φ in L^1 follows.

Exercise 3.10. Let $\Psi(c) := \Phi(c)/c$ and notice that $|\varphi_i| \le c\Phi(|\varphi_i|)/\Phi(c) = \Phi(|\varphi_i|)/\Psi(c)$ on $\{|\varphi_i| \ge c\}$. Therefore

$$\int_A |\varphi_i| \, d\mu \le \int_{A \cap \{|\varphi_i| \ge c\}} \frac{\Phi(|\varphi_i|)}{\Psi(c)} \, d\mu + \int_{A \cap \{|\varphi_i| < c\}} |\varphi_i| \, d\mu \le \frac{M}{\Psi(c)} + c\mu(A).$$

Let us choose c sufficiently large, such that $M/\Psi(c) < \varepsilon/2$, and then $\delta > 0$ such that $c\delta < \varepsilon/2$. The inequality above yields $\int_A |\varphi_i| \, d\mu < \varepsilon$ whenever $\mu(A) < \delta$.

Exercise 3.11. Let $(f_n) \subset C_b(X)$ be converging in L^1 to f, and let $f_{n(k)}$ be a subsequence pointwise convergent μ–a.e. to f. Then, given any $\varepsilon > 0$, by Egorov theorem we can find a Borel set $B \subset X$ with $\mu(B) < \varepsilon$ and $f_{n(k)} \to f$ uniformly on B^c. By the inner regularity of the measure we can find a closed set $C \subset B^c$ such that $\mu(X \setminus C) < \varepsilon$. The function f restricted to C, being the uniform limit of bounded continuous functions, is bounded and continuous.

Chapter 4

Exercise 4.1. Notice that $\langle\cdot,\cdot\rangle$ is obviously symmetric, that $\langle x,-y\rangle = -\langle x,y\rangle = \langle -x,y\rangle$ and that $\langle x,x\rangle = \|x\|^2 \ge 0$, with equality only if $x=0$. Notice that the parallelogram identity gives

$$\begin{aligned}\|x+x'+2y\|^2+\|x-x'\|^2 &= 2\|x+y\|^2+2\|x'+y\|^2\\ &= 8\langle x,y\rangle+8\langle x',y\rangle-2\|x-y\|^2-2\|x'-y\|^2\end{aligned}$$

and

$$\begin{aligned}\|x+x'-2y\|^2+\|x-x'\|^2 &= 2\|x-y\|^2+2\|x'-y\|^2\\ &= 8\langle x,-y\rangle+8\langle x',-y\rangle-2\|x+y\|^2-2\|x'+y\|^2.\end{aligned}$$

Subtracting and dividing by 4 we get

$$\langle x+x',2y\rangle = 4\langle x,y\rangle+4\langle x',y\rangle-2\langle x,y\rangle-2\langle x',y\rangle.$$

So, we proved that $\langle x+x',2y\rangle = 2\langle x,y\rangle+2\langle x',y\rangle$. Using the relation $\langle u,2v\rangle = 4\langle u/2,v\rangle$ (due to the definition of $\langle\cdot,\cdot\rangle$ and the homogeneity of $\|\cdot\|$), we get

$$\left\langle \frac{x+x'}{2},y\right\rangle = \frac{1}{2}\langle x,y\rangle+\frac{1}{2}\langle x',y\rangle.$$

Setting $x=t_1v$, $x'=t_2v$, and defining the continuous function $\phi(t) = \langle tv,y\rangle$, we get

$$\phi\left(\frac{t_1+t_2}{2}\right) = \frac{1}{2}\phi(t_1)+\frac{1}{2}\phi(t_2).$$

This means that ϕ and $-\phi$ are convex in $\mathbb{R}$, so that ϕ is an affine function, and since $\phi(0)=0$ we get $\phi(t)=t\phi(0)$, *i.e.* $\langle tu,y\rangle = t\langle u,y\rangle$. Coming back to the identity above, we get $\langle x+x',y\rangle = \langle x,y\rangle+\langle x',y\rangle$.

Exercise 4.2. Assume that $y=\pi_K(x)$. For all $z\in K$ and $t\in[0,1]$ we have $y+t(z-y)$ belongs to K, so that

$$\|y+t(z-y)-x\|^2 \ge \|y-z\|^2.$$

Expanding the squares we get

$$t^2\|z-y\|^2+2t\langle z-y,y-x\rangle \ge 0 \qquad \forall t\in[0,1].$$

This implies (either dividing by $t>0$ and passing to the limit as $t\downarrow 0$, or computing the right derivative at $t=0$) that $\langle z-y,x-y\rangle \le 0$. Conversely, if for some $y\in K$ this condition holds for all $z\in K$, the argument can be reversed to get $\|y + t(z - y) - x\| \geq \|y - x\|$ for all $t \geq 0$. Choosing $t = 1$ we get $\|z - x\| \geq \|y - x\|$, proving that $y = \pi_K(x)$.

Exercise 4.3. Let Y_k be the vector space spanned by $\{f_1, \ldots, f_k\}$ and let us prove by induction on $k \geq 1$ that f_i is orthogonal to f_j whenever $1 \leq i < j \leq k$. First we observe that if this property holds for some k, then Y_k is k-dimensional and coincides with the vector space spanned by $\{v_1, \ldots, v_k\}$ (being contained in it, and with the same dimension).

The orthogonality of the vectors f_i can be obtained just noticing that

$$f_k = v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i.$$

So, $f_k = v_k - \pi_{Y_{k-1}}(v_k)$ is orthogonal to all vectors in Y_{k-1}. It follows that $\langle e_k, e_i \rangle = 0$ for all $i < k$.

Exercise 4.4. Let $y = x - \sum_k \langle x, e_k \rangle e_k$; we know that the series converges in H by Bessel's inequality. In order to show that $\sum_k \langle x, e_k \rangle e_k = \pi_X(x)$ it suffices to prove that y is orthogonal to all vectors in X. But since any vector $v \in X$ can be represented as a series, it suffices to show that $\langle v, e_i \rangle = 0$ for all i. The continuity and linearity of the scalar product give

$$\langle y, e_i \rangle = \langle x, e_i \rangle - \sum_{k=0}^{\infty} \langle x, e_k \rangle \langle x, e_i \rangle = \langle x, e_i \rangle - \langle x, e_i \rangle = 0.$$

Exercise 4.5 Since X and its scalar product coincide with $L^2([0, 1], \mathscr{P}([0, 1]), \mu)$, where μ is the counting measure in $[0, 1]$, we obtain that X is an Hilbert space. Let us prove by contradiction that X is not separable. If $S = \{f_n\}_{n \geq 1}$ were a dense subset, it could be possible to find a countable set $D \subset [0, 1]$ such that $f_n(x) = 0$ for all n and all $x \in [0, 1] \setminus D$. Since $[0, 1]$ is not countable we can find $x_0 \in [0, 1] \setminus D$ and define $g_0(x)$ equal to 1 if $x = x_0$ and equal to 0 if $x \neq x_0$. We claim that g_0 does not belong to the closure of S. If this property fails, we can find a sequence $(f_{n(k)}) \subset S$ convergent to g_0 μ–a.e. in $[0, 1]$; but, convergence μ–a.e. corresponds to pointwise convergence and since $g_0(x_0) \neq 0$, while $f_{n(k)}(x_0) = 0$ for all k, we obtain a contradiction.

Exercise 4.6. By Parseval identity we know that $x \mapsto (\langle x, e_i \rangle)$ is a linear isometry from H to ℓ_2. As a consequence, taking the parallelogram identity into account, the scalar product is preserved.

Exercise 4.7. We consider the class of orthonormal systems $\{e_i\}_{i \in I}$ of H, ordered by inclusion. Zorn's lemma ensures the existence of a maximal system $\{e_i\}_{i\in I}$. Let V be the subspace spanned by e_i, let Y be its closure (still a subspace) and let us prove that $Y = H$. Indeed, if Y were a proper subspace of H, we would be able to find, thanks to Corollary 4.5, a unit vector e orthogonal to all vectors in Y, and in particular to all vectors e_i. Adding e to the family $\{e_i\}_{i\in I}$ the maximality of the family would be violated. Now, by the just proved density of V in H, given any $x \in H$ we can find a sequence of vectors (v_n), finite combinations of vectors e_i, such that $\|x - v_n\| \to 0$. If we denote by $J_n \subset I$ the set of indexes used to build the vectors $\{v_1, \dots, v_n\}$, and by H_n the vector space spanned by $\{e_i\}_{i\in J_n}$, we know by Proposition 4.6 that

$$\|x - \sum_{i\in J_n} \langle x, e_i\rangle e_i\| \le \|x - v_n\| \to 0.$$

As a consequence, setting $J = \cup_n J_n$, we have $x = \sum_{i\in J} \langle x, e_i\rangle e_i$.

Chapter 5

Exercise 5.1. The functions $\sin mx \cos lx$ are odd, therefore their integral on $(-\pi, \pi)$ vanishes. To show that $\sin mx$ is orthogonal to $\sin lx$ when $l \neq m$, we integrate twice by parts to get

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin lx \, dx &= \frac{m}{l} \int_{-\pi}^{\pi} \cos mx \cos lx \, dx \\ &= \frac{m^2}{l^2} \int_{-\pi}^{\pi} \sin mx \sin lx \, dx.\end{aligned}$$

The integrals of products $\cos mx \cos lx$ can be handled analogously.

Exercise 5.2. Since for $N < M$ we have

$$\|\sum_{n=0}^{N} x_n - \sum_{n=0}^{M} x_n\| \le \sum_{i=N+1}^{M} \|x_i\| \le \sum_{i=N+1}^{\infty} \|x_i\|$$

we obtain that $(\sum_0^N x_i)$ is a Cauchy sequence in E. Therefore the completeness of E provides the convergence of the series. Passing to the limit as $N \to \infty$ in the inequality $\|\sum_0^N x_i\| \le \sum_0^N \|x_i\|$ and using the continuity of the norm we obtain (5.15).

Exercise 5.3. We consider only the first system $g_k = \sqrt{2/\pi} \sin kx$, the proof for the second one being analogous. The fact that (g_k) is orthonormal can be easily checked noticing that g_k are restrictions to $(0, \pi)$ of odd functions, and using the orthogonality of $\sin kx$ in $L^2(-\pi, \pi)$. Analogously, if $f \in L^2(0, \pi)$ let us consider its extension $\tilde{f}$ to $(-\pi, \pi)$ as an odd function and its Fourier series, which obviously contains no cosinus. In $(0, \pi)$ we have

$$\sum_{k=1}^{N} b_k \sin kx = \sum_{k=1}^{N} \langle f, g_k \rangle g_k,$$

where the scalar products are understood in $L^2(0, \pi)$. Therefore, from the convergence of the Fourier series in $L^2(-\pi, \pi)$ to $\tilde{f}$, which implies convergence in $L^2(0, \pi)$ to f, the completeness follows.

Exercise 5.4. Clearly $\langle e_k, e_k \rangle = 1$, while

$$\int_{-\pi}^{\pi} e^{ikx} e^{-ilx}\, dx = \frac{1}{i(k-l)} \left[e^{i(k-l)x}\, dx \right]_{-\pi}^{\pi} = 0 \qquad \text{whenever } k \neq l.$$

As a consequence (e_k) is an orthonormal system.
Since the Fourier series $S_N f = \sum_{-N}^{N} \langle f, e_k \rangle e_k$ of f depends linearly on f, in order to show completeness we need only to show $S_N f \to f$ when f is real-valued and when f is imaginary-valued (i.e. if is real-valued). We consider only the first case, the second one being analogous. Setting $c_k = \langle f, e_k \rangle$, we have

$$c_k = \frac{1}{\sqrt{2\pi}} \int_{\pi}^{\pi} f(x) \cos kx - i f(x) \sin kx\, dx.$$

As a consequence, for $k \geq 1$ we have $\sqrt{2/\pi}\, c_k = a_k - i b_k$, where a_k and b_k are the coefficients of the real Fourier series of f, and for $k \leq -1$ we have $\sqrt{2/\pi}\, c_k = a_{-k} + i b_{-k}$. For $k = 0$, instead, we have $\sqrt{2/\pi}\, c_0 = a_0$. Taking into account these relations and setting $b_0 = 0$, we have

$$\begin{aligned}
\sum_{k=-N}^{N} c_k \frac{e^{ikx}}{\sqrt{2\pi}} &= \frac{1}{2} \Bigg\{ \sum_{k=1}^{N} (\cos kx + i \sin kx)(a_k - i b_k) \\
&\qquad + \sum_{k=-N}^{-1} (\cos kx + i \sin kx)(a_{-k} - i b_{-k}) \Bigg\} \\
&= \frac{a_0}{2} + \mathrm{Re} \left(\sum_{k=1}^{N} (\cos kx + i \sin kx)(a_k - i b_k) \right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx,
\end{aligned}$$

and the convergence of $S_N f$ to f follows by the convergence in the real-valued case.

Exercise 5.5. It suffices to note that

$$\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx\right)^2 = (\langle f, e_k\rangle)^2,$$

where (e_k) is the orthonormal system of Exercise 5.4 and to use its completeness.

Exercise 5.6. From the identity $\sum_{i=0}^{2N} e^{ikz} = (e^{i(2N+1)z} - 1)/(e^{iz} - 1)$, we get

$$\begin{aligned}\sum_{k=-N}^{N} e^{ikz} &= e^{-iNz}\sum_{k=0}^{2N} e^{ikz} = e^{-iNz}\frac{e^{i(2N+1)z} - 1}{e^{iz} - 1} = \\ &= \frac{e^{i(N+1/2)z} - e^{-i(N+1/2)z}}{e^{iz/2} - e^{-iz/2}} = \frac{\sin((N+1/2)z)}{\sin(z/2)} \qquad \text{(A.15)}\end{aligned}$$

and we call this term $G_N(z)$. Hence

$$\begin{aligned}S_N f(x) &= \sum_{k=-N}^{N} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy\right)e^{ikx} \\ &= \sum_{k=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{ik(x-y)}\,dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)G_N(x-y)\,dy.\end{aligned}$$

Using the fact that $\sin((N+1/2)z)/\sin(z/2)$ has, still because of (A.15), mean value 1 on $(-\pi, \pi)$, we get

$$f(x) - S_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - f(y))G_N(x-y)\,dy.$$

Exercise 5.7. We apply the Parseval identity to the function $f(x) = x^2$, whose Fouries series contains no sinus. It is simple to check, by integration by parts, that $a_0 = 2\pi^2/3$ and that $a_k = 4k^{-2}\cos kx$ for $k \geq 1$. We have then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2}{5}\pi^4 = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 = \frac{4}{18}\pi^4 + \sum_{k=1}^{\infty}\frac{16}{k^4}.$$

Rearranging terms, we get $\sum_1^{\infty} k^{-4} = \pi^4/90$.

Exercise 5.8. The polynomials P_n are given by $Q_n/\|Q_n\|_2$, where Q_n are recursively defined by $Q_0 = 1$ and

$$Q_n(x) := x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, Q_k\rangle}{\langle Q_k, Q_k\rangle} Q_k(x) = x^n - \sum_{k=0}^{n-1} \langle x^n, P_k\rangle P_k(x) \quad \forall n \geq 1.$$

(a) Since $Q_0 = 1$, $P_0 = 1/\sqrt{2}$ and $Q_1 = x - \langle x, P_0\rangle P_0 = x$, because $\langle x, P_0\rangle = 0$. As a consequence $P_1(x) = \sqrt{3/2}x$. Since $\langle x^2, P_1\rangle = 0$, we have also

$$Q_2(x) = x^2 - \langle x^2, P_0\rangle P_0 - \langle x^2, P_1\rangle P_1 = x^2 - \frac{1}{3}$$

and this leads, with simple calculations, to $P_2(x) = \sqrt{45/8}(x^2 - 1/3)$.
(b) Let H be the closure of the vector space spanned by C_n. This space contains all monomials x^n, and therefore all polynomials. Since the polynomials are dense in $C([a, b])$, for the sup norm, they are also dense in $L^2(a, b)$. It follows that $H = L^2(a, b)$. By Proposition 4.13 we conclude that (C_n) is complete.
(c) Set

$$z_n := \sqrt{\frac{2n+1}{2}\frac{1}{2^n n!}}, \qquad \tilde{P}_n(x) := z_n \frac{d^n}{d^n x}(x^2 - 1)^n$$

Clearly the polynomial $\tilde{P}_n$ has degree n. So, in order to show that $\tilde{P}_n = P_n$, we have to show that $\tilde{P}_n$ is orthogonal to all monomials x^k, $k = 0, \ldots, n-1$, and that $\|\tilde{P}_n\|_2 = 1$. Since $\tilde{P}_n$ has zeros at ± 1 with multiplicity n, all its derivatives at ± 1 with order less than n are zero. Therefore, for $k < n$ we have

$$\begin{aligned}
\langle \tilde{P}_n, x^k\rangle &= z_n \left\{ \left[x^k \frac{d^{n-1}}{d^{n-1}x}(x^2-1)^n \right]_{-1}^{1} - k \int_{-1}^{1} x^{k-1} \frac{d^{n-1}}{d^{n-1}x}(x^2-1)^n \, dx \right\} \\
&= \cdots \\
&= (-1)^k k! z_n \left[\frac{d^{n-k}}{d^{n-k}x}(x^2-1)^n \right]_{-1}^{1} = 0.
\end{aligned}$$

In order to prove that $\|\tilde{P}_n\|_2 = 1$, still integrating by parts we have

$$\begin{aligned}
\langle \tilde{P}_n, \tilde{P}_n\rangle &= -z_n^2 \int_{-1}^{1} \frac{d^{n-1}}{d^{n-1}x}(x^2-1)^n \frac{d^{n+1}}{d^{n+1}x}(x^2-1)^n \, dx = \cdots \\
&= z_n^2 \int_{-1}^{1} (1-x^2)^n \frac{d^{2n}}{d^{2n}x}(x^2-1)^n \, dx.
\end{aligned} \tag{A.16}$$

On the other hand

$$\int_{-1}^{1}(1-x^2)^n\,dx = 2n\int_{-1}^{1}(1-x^2)^{n-1}x^2\,dx$$
$$= -2n\int_{-1}^{1}(1-x^2)^n\,dx + 2n\int_{-1}^{1}(1-x^2)^{n-1}\,dx,$$

so that

$$\int_{-1}^{1}(1-x^2)^n\,dx = \frac{2n}{2n+1}\int_{-1}^{1}(1-x^2)^{n-1}\,dx = \cdots$$
$$= \frac{(2n)!!}{(2n+1)!!}\int_{-1}^{1}(1-x^2)^0\,dx = \frac{2(2n)!!}{(2n+1)!!}.$$

Taking into account that

$$\frac{d^{2n}}{d^{2n}x}(x^2-1)^n = (2n)! = (2n)!!(2n-1)!! = 2^n n!(2n-1)!!$$

from (A.16) we get

$$\langle \tilde{P}_n, \tilde{P}_n\rangle = \frac{2n+1}{2}\frac{1}{2^{2n}(n!)^2}\frac{2(2n)!!}{(2n+1)!!}2^n n!(2n-1)!! = 1.$$

Exercise 5.9. Recall that

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

Integrating by parts once and using that $f(-\pi) = f(\pi)$ we get

$$c_k = \frac{1}{ik}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-ikx}\,dx.$$

Continuing in this way, in m steps we get

$$c_k = \frac{1}{(ik)^m}\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(m)}(x)e^{-ikx}\,dx.$$

Chapter 6

Exercise 6.1. Let us prove the inclusion

$$(\mathscr{F}_1 \times \mathscr{F}_2) \times \mathscr{F}_3 \subset \mathscr{F}_1 \times (\mathscr{F}_2 \times \mathscr{F}_3),$$

the proof of the converse one being analogous. We have to show that all products $A \times B$, with $A \in \mathscr{F}_1 \times \mathscr{F}_2$ and $B \in \mathscr{F}_3$ belong to $\mathscr{F}_1 \times (\mathscr{F}_2 \times \mathscr{F}_3)$. Keeping B fixed, the class of sets A for which this property holds is a σ–algebra that contains the π–system of measurable rectangles $A = A_1 \times A_2$ (because $A \times B = A_1 \times (A_2 \times B)$ and $A_2 \times B \in \mathscr{F}_2 \times \mathscr{F}_3$), and therefore the whole product σ–algebra $\mathscr{F}_1 \times \mathscr{F}_2$.
For all A in the product σ–algebra we have

$$
\begin{aligned}
(\mu_1 \times \mu_2) \times \mu_3(A) &= \int_{X_1 \times X_2} \mu_3(A_{x_1 x_2})\, d\mu_1 \times \mu_2(x_1, x_2) \\
&= \int_{X_1} \int_{X_2} \mu_3(A_{x_1 x_2})\, d\mu_2(x_2)\, d\mu_1(x_1) \\
&= \int_{X_1} \mu_2 \times \mu_3(A_{x_1})\, d\mu_1(x_1) = \mu_1 \times (\mu_2 \times \mu_3)(A).
\end{aligned}
$$

Exercise 6.2. Obviously the cubes belong to $\bigtimes_1^n \mathscr{B}(\mathbb{R})$, and thanks to Lemma 6.9 the same is true for the open sets. It follows that $\mathscr{B}(\mathbb{R}^n)$ is contained in $\bigtimes_1^n \mathscr{B}(\mathbb{R})$. Let us consider the class

$$
\mathscr{M} := \left\{ B \subset \mathbb{R} \,:\, B \times \mathbb{R} \times \cdots \times \mathbb{R} \in \mathscr{B}(\mathbb{R}^n) \right\}.
$$

This class contains the open sets (because the product of open sets is open) and it is a σ–algebra, so it contains $\mathscr{B}(\mathbb{R})$. We have thus proved that all rectangles $B_1 \times \mathbb{R} \times \cdots \times \mathbb{R}$, with B_1 Borel belong to $\mathscr{B}(\mathbb{R}^n)$. By a similar argument we can show that all rectangles

$$
\mathbb{R} \times \cdots \times \mathbb{R} \times B_i \times \mathbb{R} \times \cdots \times \mathbb{R}
$$

are Borel. Intersecting rectangles in these families we obtain that all rectangles with Borel sides belong to $\mathscr{B}(\mathbb{R}^n)$ and we conclude.

Exercise 6.3. Assume that $A, B \in \mathscr{L}_1$; then there exist Borel sets A', B' and Borel Lebesgue negligible sets N_A, N_B with $A \Delta A' \subset N_A$ and $B \Delta B' \subset N_B$. Since $A' \times B' \in \mathscr{B}(\mathbb{R}^2)$, by the previous exercise,

$$
(A \times B)\Delta(A' \times B') \subset (N_A \times \mathbb{R}) \cup (\mathbb{R} \times N_B)
$$

and $N_A \times \mathbb{R}$ and $\mathbb{R} \times N_B$ are $\mathscr{L}^2$ negligible, we obtain that $A \times B \in \mathscr{L}_2$. This proves that $\mathscr{L}_2$ contains the generators of $\mathscr{L}_1 \times \mathscr{L}_1$, and therefore the whole σ–algebra. In order to show the strict inclusion, we consider the set $E = F \times \{0\}$, where $F \subset \mathbb{R}$ is not Lebesgue measurable. Since E is $\mathscr{L}^2$–negligible we have $E \in \mathscr{L}_2$. On the other hand, since the 0 section E^0 coincides with F, and therefore it does not belong to $\mathscr{L}_1$, the set E can't belong to the product of the two σ–algebras.

Exercise 6.4. Let $\mathscr{A}$ be the σ–algebra generated by these sets; since these sets are obviously cylindrical, $\mathscr{A}$ is contained in the product σ–algebra. The class of sets $B \subset \times_1^n X_i$ such that $B \times X_{n+1} \times X_{n+2} \times \cdots \in \mathscr{A}$ is a σ–algebra containing the measurable rectangles $A_1 \times \cdots \times A_n$, and therefore contains the product σ–algebra $\times_1^n \mathscr{F}_i$. Therefore $\mathscr{A}$ contains the cylindrical sets and, by definition, the whole product σ–algebra.

Exercise 6.5. The sections $T_y := \{(x,z) \ : \ (x,y,z) \in T\}$ are squares with length side $2\sqrt{r^2 - |y|^2}$ for $0 \le |y| \le r$, hence

$$\mathscr{L}^3(T) = \int_{-r}^{r} \mathscr{L}^2(T_y)\, dy = 8\int_0^r (r^2 - y^2)\, dy = 8(r^3 - \frac{1}{3}r^3) = \frac{16}{3}r^3.$$

Exercise 6.6. For $x \in \mathbb{R}^n$ (with $n \ge 3$) let

$$r := (x_1^2 + x_2^2)^{1/2}, \quad A_r := \left\{(x_3, \ldots, x_n) : \ (x_3^2 + \cdots + x_n^2) < 1 - r^2\right\}.$$

Then, using polar coordinates we get

$$\begin{aligned}\omega_n &= \int_{\{r<1\}} \mathscr{L}^{n-2}(A_r)\, dx_1 dx_2 = 2\pi\omega_{n-2}\int_0^1 r(1-r^2)^{(n-2)/2}\, dr \\ &= \frac{2\pi}{n}\omega_{n-2}.\end{aligned}$$

Therefore

$$\omega_{2k} = \frac{2^{k-1}\pi^{k-1}}{2k(2k-2)\cdots 4}\,\omega_2 = \frac{\pi^k}{k!}$$

and an analogous argument gives $\omega_{2k+1} = 2^{k+1}\pi^k/(2k+1)!!$.

Exercise 6.7. In order to show that $\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ we show that the right hand side satisfies the same recursion formula of the previous exercise. Since (thanks to the identities $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$) the formula holds when $n = 1, 2$, this will prove that the identity holds for all n. For $n \ge 2$ we have

$$\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = \frac{\pi \cdot \pi^{(n-2)/2}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{2\pi}{n}\frac{\pi^{(n-2)/2}}{\Gamma(\frac{(n-2)}{2}+1)}.$$

Exercise 6.8. We know, by Exercise 2.4, that there exist a λ–negligible set $N \in \mathscr{F} \times \mathscr{G}$ and a $\mathscr{F} \times \mathscr{G}$–measurable function $\tilde{F} : X \times Y \to [0, +\infty]$ such that $\{F \ne \tilde{F}\}$ is contained in N. By applying the Fubini–Tonelli theorem to $\mathbb{1}_N$ we obtain that N_x is ν–negligible in Y for μ–a.e. $x \in X$. Since $\{F(x,\cdot) \neq \tilde F(x,\cdot)\} \subset N_x$, still Exercise 2.4 gives that $F(x,\cdot)$ is ν–measurable for μ–a.e. $x \in X$. This proves statement (i). Since, still for μ–a.e. $x \in X$, the integral on Y (with respect to ν) of $F(x,\cdot)$ coincides with the integral of $\tilde F(x,\cdot)$, statements (ii) and (iii) follow by applying the Fubini–Tonelli theorem to $\tilde F$.

Exercise 6.9. Indeed, $\mu(D_y) = \mu(\{y\}) = 0$ for all $y \in Y$, so that $\int_Y \mu(D_y)\,d\nu(y) = 0$. On the other hand, $\nu(D_x) = \nu(\{x\}) = 1$ for all $x \in X$, so that $\int_X \nu(D_x)\,d\mu(x) = 1$.

Exercise 6.10. Let $(h(k))$ be a subsequence such that $\sum_k \|f_{h(k)} - f\|_1$ is convergent. Then the Fubini–Tonelli theorem gives

$$\int_X \left(\sum_{k=0}^{\infty} \int_Y |f_{h(k)}(x,y) - f(x,y)|\,d\nu(y)\right) d\mu(x)$$
$$= \sum_{k=0}^{\infty} \int_{X\times Y} |f_{h(k)}(x,y) - f(x,y)|\,d\mu\times\nu < \infty.$$

It follows that $\sum_k \|f_{h(k)}(x,\cdot) - f(x,\cdot)\|_{L^1(\nu)}$ is finite for μ–a.e. $x \in X$, and for any such x the functions $f_{h(k)}(x,\cdot)$ converge to f in $L^1(\nu)$. Choosing $Y = \{\bar y\}$ and $\nu = \delta_{\bar y}$, to provide a counterexample it is sufficient to consider any example (see Remark 3.7) of a sequence converging in L^1 but not μ–almost everywhere.

Exercise 6.11. It suffices to apply (6.15) to $|h|$ to show that $\int |h|\,df\mu$ is finite if and only if $\int |h| f\,d\mu$ is finite.

Exercise 6.12. We prove the property for the sup, the property for the inf being analogous. If $A = B_1 \cup B_2$ with $B_1 \in \mathscr{F}$ and $B_2 \in \mathscr{F}$ disjoint, we have

$$\begin{aligned} f\mu(B_1) + g\mu(B_2) &= \int_{B_1} f\,d\mu + \int_{B_2} g\,d\mu \\ &\le \int_{B_1} f\vee g\,d\mu + \int_{B_2} f\vee g\,d\mu \\ &= \int_A f\vee g\,d\mu. \end{aligned}$$

The arbitrariness of this decomposition, proves that $[(f\mu)\vee(g\mu)](A) \le (f\vee g)\mu(A)$. The converse inequality can be obtained noticing that, in the chain of equalities-inequality above, the inequality becomes an equality if we choose $B_1 = A\cap\{f \ge g\}$ and $B_2 = A\cap\{f > g\}$.

Exercise 6.13. It is easy to check that $\underline{\mu} \le \mu_i$ (respectively, $\overline{\mu} \ge \mu_i$) for all $i \in I$, and that any measure ν with this property is less than $\underline{\mu}$ (resp.

greater than $\overline{\mu}$): just write $\nu(B) = \sum_k \nu(B_k) \le \sum_k \mu_{i(k)}(B_k)$ (resp. $\ge \sum_k \mu_{i(k)}(B_k)$. So, it remains to show that $\underline{\mu}$ and $\overline{\mu}$ are σ-additive. For any map $i : \mathbb{N} \to I$, $A_1,\ A_2 \in \mathscr{F}$ disjoint and any countable $\mathscr{F}$–measurable partition of $A_1 \cup A_2$ we have

$$\sum_{k=0}^{\infty} \mu_{i(k)}(B_k) = \sum_{k=0}^{\infty} \mu_{i(k)}(B_k \cap A_1) + \sum_{k=0}^{\infty} \mu_{i(k)}(B_k \cap A_2).$$

Estimating the right hand side from below with $\underline{\mu}(A_1) + \underline{\mu}(A_2)$ we get (because (B_k) is arbitrary) that $\underline{\mu}$ is superadditive, i.e. $\underline{\mu}(A_1 \cup A_2) \ge \underline{\mu}(A_1) + \underline{\mu}(A_2)$. With a similar argument one can prove not only that $\overline{\mu}$ is subadditive, but also that $\overline{\mu}$ is σ–subadditive (it suffices to consider a countable $\mathscr{F}$–measurable family, instead of 2 sets).

Now, let us prove that $\underline{\mu}$ is subadditive and $\overline{\mu}$ is superadditive. Let $A_1,\ A_2 \in \mathscr{F}$ be disjoint and let $B_k^1,\ B_k^2$ be countable $\mathscr{F}$–measurable partitions of A_1 and A_2 respectively. If $i_1,\ i_2 : \mathbb{N} \to I$ we define $i(2k) = i_1(k)$, $B_{2k} = B_k^1$ and $i(2n+1) = i_2(n)$, $B_{2k+1} = B_k^2$, so that

$$\underline{\mu}(A_1 \cup A_2) \le \sum_{k=0}^{\infty} \mu_{i(k)}(B_k) = \sum_{k=0}^{\infty} \mu_{i_1(k)}(B_k^1) + \sum_{k=0}^{\infty} \mu_{i_2(k)}(B_k^2).$$

By the arbitrariness of B_k^1, B_k^2, i_1 and i_2 we conclude that $\underline{\mu}(A_1 \cup A_2) \le \underline{\mu}(A_1) + \underline{\mu}(A_2)$. With a similar argument one can prove that $\underline{\mu}$ is even σ–subadditive (one has to use a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$) and that $\overline{\mu}$ is superadditive.

Exercise 6.14. If for all $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$A \in \mathscr{F},\ \ \mu(A) < \delta \qquad \Longrightarrow \qquad \nu(A) < \varepsilon$$

then $\nu \ll \mu$: indeed, if $\mu(A) = 0$ the implication above holds for all $\varepsilon > 0$, hence $\nu(A) = 0$. If ν is finite, to prove the converse we argue by contradiction. Assume that, for some ε_0, we can find sets $A_n \in \mathscr{F}$ with $\mu(A_n) < 2^{-n}$ and $\nu(A_n) \ge \varepsilon_0$. Then, by the Borel–Cantelli lemma the set $A := \limsup_n A_n$ is μ–negligible. On the other hand, we have

$$\nu\Big(\bigcup_{m=n}^{\infty} A_m\Big) \ge \nu(A_n) \ge \varepsilon_0$$

and therefore (here we use the assumption that ν is finite) $\nu(A) \ge \varepsilon_0$, contradicting the absolute continuity of ν with respect to μ.

Exercise 6.15. Let $B \in \mathscr{F}$ be a μ–negligible set where ν is concentrated. Then $\nu(E) = \nu(E \cap B)$ for all $E \in \mathscr{F}$. But, by the absolute continuity of ν with respect to μ, we have $\nu(E \cap B) = 0$ because $E \cap B \subset B$ is μ–negligible.

Exercise 6.16. Let $B \in \mathscr{F}$ be a ν–negligible set where σ is concentrated. Then

$$\sigma(E) = \sigma(E \cap B) \le \mu(E \cap B) + \nu(E \cap B) \le \mu(E) \qquad \forall E \in \mathscr{F},$$

where we used the fact that $\nu(E \cap B) = 0$ because $E \cap B \subset B$ is ν–negligible.

Exercise 6.17. It is easy to check that the class of functions f satisfying $f\mu \le \nu$ is a lattice. Hence, given a maximizing sequence (f_h) in (6.20), possibly replacing f_h by $\max_{i \le n} f_i$, we can assume that $f_h \uparrow f$. The monotone convergence theorem gives that f is a maximizer.

In order to show that $\nu = f\mu$ we set $\sigma = \nu - f\mu \ge 0$ and notice that σ satisfies the following property:

$$t > 0, \quad B \in \mathscr{F}, \quad t\mathbb{1}_B\mu \le \sigma \qquad \Longrightarrow \qquad \mu(B) = 0. \tag{A.17}$$

Indeed, the integrals $\int_X (f + t\mathbb{1}_B)\, d\mu$ and $\int_X f\, d\mu$ have to coincide, because $(f + t\chi_B)\mu \le \nu$.

Exercise 6.18. We have to prove that any measure σ satisfying (A.17) is concentrated on a μ-negligible set. To this aim, let us consider the problem

$$\inf\left\{\mu(A) : \ A \in \mathscr{F}, \ \ \sigma \text{ is concentrated on } A\right\}.$$

By taking the intersection of a minimizing sequence it is easy to check that also this problem has a solution A; we have to show that $\mu(A) = 0$. By the minimality of A, the implication

$$\mathscr{F} \ni B \subset A, \quad \mu(B) > 0 \qquad \Longrightarrow \qquad \sigma(B) > 0 \tag{A.18}$$

holds. Let us consider the numbers

$$\xi_h := \sup\left\{\mu(B) : \ \mathscr{F} \ni B \subset A, \ \ \chi_B\mu \ge 2^h \mathbb{1}_B\sigma\right\}$$

and let us prove that $\xi_h \to 0$ as $h \to \infty$. Given maximizers $B_h \subset A$, whose existence is easy to check, we have $\mu(B_h) \ge 2^h\sigma(B_h)$ and in particular $\sum_h \sigma(B_h) < \infty$. Hence

$$\sigma\left(\limsup_{h\to\infty} B_h\right) = 0$$

and (A.18) tells us that necessarily

$$0 = \mu\left(\limsup_{h\to\infty} B_h\right) \geq \limsup_{h\to\infty} \mu(B_h).$$

Let us show now that the maximality of B_h implies that $\mu(C) \leq 2^h\sigma(C)$ for any set $C \subset A \setminus B_h$, i.e. $t\mathbb{1}_{A\setminus B_h}\mu \leq \sigma$. Indeed, if there is $C_0 \subset A \setminus B_h$ with $\mu(C_0) > 2^h\sigma(C_0)$, the maximality of B_h provides a minimal integer $h_1 \geq 1$ and $C_1 \subset C_0$ satisfying $\mu(C_1) \leq 2^h\sigma(C_1) - 1/h_1$. Let us consider $C_0 \setminus C_1$; we still have $\mu(C_0 \setminus C_1) > 2^h\sigma(C_0 \setminus C_1)$ and the maximality of B_h provide a minimal integer $h_2 \geq h_1$ and $C_2 \subset C_0 \setminus C_1$ satisfying $\mu(C_2) \leq 2^h\sigma(C_2) - 1/h_2$. Continuing in this way we have a nondecreasing sequence (h_i) of integers and $(C_i) \subset \mathscr{F}$ such that $\mu(C_i) \leq 2^h\sigma(C_i) - 1/h_i$ and $C_i \subset C_0 \setminus \cup_{j=1}^{i-1} C_j$ for all $i \geq 2$; moreover h_i is the least integer for which there is such C_i. Now $\lim_i h_i = \infty$, since the C_i are pairwise disjoint. Setting $C = C_0 \setminus \cup_1^\infty C_i$, for all $F \in \mathscr{F}$ contained in C, since $F \subset C_0 \setminus \cup_1^{i-1} C_j$ for all $i \geq 2$, we have $\mu(F) \geq 2^h\sigma(F) - 1/(h_i - 1)$ (if $h_i \geq 2$) and then $\mu(F) \geq 2^h\sigma(F)$. Hence $B_h \cup C$ is an admissible set for the maximum problem defining ξ_h, against the maximality of B_h.

We choose h in such a way that $\xi_h < \mu(A)$ and set $t = 2^{-h}$, $B = A\setminus B_h$ in (A.17). From (A.17) we conclude that $\mu(B) = 0$, contradicting the fact that $\mu(B) = \mu(A) - \xi_h > 0$.

Exercise 6.19. Let $\nu = \nu^+ - \nu^-$ and let $\nu^+ = \nu_a^+ + \nu_s^+$, $\nu^- = \nu_a^- + \nu_s^-$ be the Lebesgue decompositions with respect to μ of ν^+ and ν^- respectively. Then, $\nu_a := \nu_a^+ - \nu_a^-$ and $\nu_s := \nu_s^+ - \nu_s^-$ provide a decomposition $\nu = \nu_a + \nu_s$ with ν_a, ν_s signed, $|\nu_a| \ll \mu$ and $|\nu_s| \perp \mu$.

If μ is signed and A provides a Hahn decomposition of μ (i.e. $\mu^+(E) = \mu(E\cap A)$ and $\mu^-(E) = -\mu(E\cap A^c)$), we repeat the decomposition above in A, relative to ν and μ^+, and in $B = A^c$, relative to ν and μ^-. Denoting by $\nu_a^A + \nu_s^A$ and $\nu_a^B + \nu_s^B$ the two decompositions obtained,

$$\nu_a(E) := \nu_a^A(E\cap A) + \nu_a^B(E\cap B), \qquad \nu_s(E) := \nu_s^A(E\cap A) + \nu_s^B(E\cap B)$$

provides the desired decomposition $\nu = \nu_a + \nu_s$ with $|\nu_a| \ll |\mu|$ and $|\nu_s| \perp |\mu|$.

The uniqueness of these decompositions can be proved with the same argument used in the case of nonnegative measures.

Exercise 6.20. Let $B \in \mathscr{F}$ and let (B_i) be a $\mathscr{F}$–measurable partition of B; since

$$\sum_{i=0}^\infty |f\mu(B_i)| = \sum_{i=0}^\infty \left|\int_{B_i} f\,d\mu\right| \leq \sum_{i=0}^\infty \int_{B_i} |f|\,d\mu = \int_B |f|\,d\mu,$$

we obtain that $|f\mu|(B) \le |f|\mu(B)$. To prove the converse inequality fix $\varepsilon > 0$ and define $B_i = B \cap f^{-1}(I_i)$, where $I_i = \varepsilon[i, i+1)$, $i \in \mathbb{Z}$. Since the oscillation of $|f - \varepsilon i|$ and $||f| - \varepsilon|i||$ in $f^{-i}(I_i)$ are less than ϵ, we get

$$\left|\int_{B_i} f\,d\mu - \varepsilon i\mu(B_i)\right| \le \varepsilon\mu(B_i), \qquad \left|\int_{B_i} |f|\,d\mu - \varepsilon|i|\mu(B_i)\right| \le \varepsilon\mu(B_i),$$

hence

$$\left|\int_{B_i} |f|\,d\mu - \left|\int_{B_i} f\,d\mu\right|\right| \le 2\varepsilon\mu(B_i).$$

It follows that

$$\begin{aligned}\sum_{i\in\mathbb{Z}} |f\mu(B_i)| &= \sum_{i\in\mathbb{Z}}\left|\int_{B_i} f\,d\mu\right| \ge \sum_{i\in\mathbb{Z}}\int_{B_i} |f|\,d\mu - 2\varepsilon\mu(B_i)\\ &= \int_B |f|\,d\mu - 2\varepsilon\mu(B).\end{aligned}$$

Since ε is arbitrary the converse inequality follows.

Exercise 6.21. If $x < 0$ or $x \ge 1$ all repartition functions are respectively equal to 0 or 1, so we need to consider only the case $x \in [0,1)$. The repartition function of $\mathbb{1}_{[0,1]}\mathscr{L}^1$ obviously is equal to x, while

$$\mu_h((-\infty, x]) = \frac{\#\{i \in [1,h] \,:\, i \le hx\}}{h} = \frac{[hx]}{h},$$

where $[s]$ denotes the integer part of s. Using the inequalities $s - 1 < [s] \le s$ with $s = hx$ we obtain that $\mu_h((-\infty, x]) \to x$.

Exercise 6.22. The argument is similar to the one used in the proof of Theorem 6.27: if $y < x < y'$ and $y,\ y' \in D$ we have

$$\begin{aligned}F(y) = \lim_{h\to\infty} F_h(y) &\le \liminf_{h\to\infty} F_h(x) \le \limsup_{h\to\infty} F_h(x)\\ &\le \lim_{h\to\infty} F_h(y') = F(y').\end{aligned}$$

Letting $y \uparrow x$ and $y' \downarrow x$, we conclude.

Exercise 6.23. We define $a_{-h^2} = \mu((-\infty, -h])$ and, for $-h^2 < i \le h^2$, $a_i = \mu((i-1)/h, i/h])$. Let us denote by μ_h the measure obtained in this way. If $x \in (-h, h]$ and i is the smallest integer in $(-h^2, h^2]$ such that $x \le i/h$, we have

$$\mu\left(\left(-\infty, x - \frac{1}{h}\right]\right) \le \mu\left(\left(-\infty, \frac{i-1}{h}\right]\right) = \sum_{j=-h^2}^{i-1} a_i \le \mu_h((-\infty, x]).$$

If x is not an atom of μ, this proves that

$$\liminf_h \mu_h((-\infty, x]) \geq \mu((-\infty, x]).$$

Analogously

$$\mu\left(\left(-\infty, x + \frac{1}{h}\right]\right) \geq \mu\left(\left(-\infty, \frac{i}{h}\right]\right) = \sum_{j=-h^2}^{i} a_i \geq \mu_h((-\infty, x]).$$

If x is not an atom of μ, this proves that

$$\limsup_h \mu_h((-\infty, x]) \leq \mu((-\infty, x]).$$

Exercise 6.24. Let us assume that (6.31) holds. If $F_i(x) \to 1$ as $x \to +\infty$ uniformly in $i \in I$, for any $\varepsilon > 0$ we can find x such that $1 - F_i(x) < \varepsilon/2$ for all $i \in I$. Analogously, we can find $y < x$ such that $F_i(y) < \varepsilon/2$ for all $i \in I$. Then, the interval $I = (y, x]$ satisfies $\mu_i(I) > 1 - \varepsilon$ for all $i \in I$, because $I^c = (-\infty, y] \cup (x, +\infty)$.

Exercise 6.25. If μ is the weak limit and $\varepsilon > 0$ is given, let us choose an integer $n \geq 1$ such that $\mu([1-n, n-1]) > 1 - \varepsilon$ and points $x \in (-n, 1-n)$ and $y \in (n-1, n)$ where the repartition functions of μ_h are converging to the repartition function of μ. Then, since $\mu((\infty, x]) + 1 - \mu((-\infty, y]) = \mu(\mathbb{R} \setminus (x, y)) < \varepsilon$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\sup_{n \geq n_\varepsilon} \mu_n((\infty, x]) + 1 - \mu_n((-\infty, y]) < \varepsilon$. Let now x' and y' be satisfying

$$\mu_n((\infty, x']) + 1 - \mu_n((-\infty, y']) < \varepsilon \qquad \forall n = 0, \ldots, n_\varepsilon - 1.$$

Then, the interval $I = [\min\{x, x'\}, \max\{y, y'\}]$ satisfies $\inf_n \mu_n(I) > 1 - \varepsilon$.

Exercise 6.26.

(a) $\lim_h \int_{\mathbb{R}} g\, d\mu_h = \int_{\mathbb{R}} g\, d\mu \qquad \forall g \in C_b(\mathbb{R})$ (that is, (6.32));
(b) $\lim_h \int_{\mathbb{R}} g\, d\mu_h = \int_{\mathbb{R}} g\, d\mu \qquad \forall g \in C_c(\mathbb{R})$;
(c) F_h converge to F on all points where F is continuous;
(d) F_h converge to F on a dense subset of $\mathbb{R}$;
(e) $\lim_h \mu_h(\mathbb{R}) = \mu(\mathbb{R})$;
(f) (μ_h) is tight.

We consider the functions $\rho_h(x) := \rho(x+h)$, where $\rho(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}$ is the Gaussian, and $\mu_h = \rho_h \lambda$ (λ being the Lebesgue measure), $\mu = 0$.

In this case (c), (d), do not hold, because $F_h(x) \to 1 \neq 0 = F(x)$ for all $x \in \mathbb{R}$, (e) does not hold and (b) holds.
$a \Rightarrow b,\ e$. This is easy, because $C_c(\mathbb{R}) \subset C_b(\mathbb{R})$ and $1_{\mathbb{R}} \in C_b(\mathbb{R})$.
$a \Rightarrow c$. This follows by second part of the proof of Theorem 6.28.
$d \Leftrightarrow c$. This is Exercise 6.22.
$b \wedge e \Rightarrow c$. This follows by the same argument used in the proof of second part of Theorem 6.28: the sequence (g_k) monotonically convergent to 1_A can be chosen in $C_c(\mathbb{R})$, and this shows that $\liminf_h \mu_h(A) \geq \mu(A)$ for all $A \subset \mathbb{R}$ open. Using (e) and passing to the complementary sets, we obtain $\limsup_h \mu_h(C) \leq \mu(C)$ for all $C \subset \mathbb{R}$ closed.
$d \Rightarrow f$. This follows by the same argument used in the solution of Exercise 6.25.
$d \wedge f \Rightarrow e$. For all $x \in D$, with D dense, we have $\lim_h \mu_h((-\infty, x]) = \mu((-\infty, x])$. Since $\mu_h((-\infty, x]) \to \mu_h(\mathbb{R})$ as $x \to +\infty$ uniformly in h, we can pass to the limit as $x \in D \to +\infty$ to obtain $\lim_h \mu_h(\mathbb{R}) = \lim_{x\to+\infty} \mu((-\infty, x]) = \mu(\mathbb{R})$.
$d \wedge f \Rightarrow a$. This follows by the same argument used in the first part of the proof of Theorem 6.28, choosing the points t_i in the partitions to be in the dense set where convergence occurs.

Exercise 6.27. Set

$$g(\xi) := \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{i\xi x} e^{-x^2/(2\sigma^2)}\, dx.$$

Notice that $g(0) = 1$, and that differentiation theorems under the integral sign[(2)] and an integration by parts give

$$\begin{aligned} g'(\xi) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} i e^{i\xi x} (x e^{-x^2/(2\sigma^2)})\, dx \\ &= \frac{\sigma^2}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} i \frac{d}{dx} e^{i\xi x} e^{-x^2/(2\sigma^2)}\, dx \\ &= -\frac{\xi\sigma^2}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{i\xi x} e^{-x^2/(2\sigma^2)}\, dx. \end{aligned}$$

Therefore g satisfies the linear differential equation $g'(\xi) = -\sigma^2 \xi g(\xi)$, whose general solution is $g(\xi) = c e^{-\sigma^2\xi^2/2}$. Taking into account that $g(0) = 1$, $c = 1$.

(2) In this case, the application of the theorem is justified by the fact that $\sup_{\xi\in I} |\frac{d}{d\xi} e^{i\xi x} e^{-x^2/(2\sigma^2)}|$ is Lebesgue integrable for all bounded intervals I

Exercise 6.28. Let us approximate μ by $\mu_n = \mathbb{1}_{(-n,n)}\mu$; using the inequality

$$|e^{i\xi x} - e^{i\eta x}| \le |x||\xi - \eta| \qquad x,\ \xi,\ \eta \in \mathbb{R}$$

we obtain that

$$|\hat{\mu}_n(\xi) - \hat{\mu}_n(\eta)| \le |\xi - \eta| \int_{\mathbb{R}} |x|\, d\mu_n(x) \le n|\xi - \eta|,$$

therefore $\hat{\mu}_n$ is uniformly continuous. Since $|\hat{\mu}_n(\xi) - \hat{\mu}(\xi)| \le \mu(\mathbb{R} \setminus [-n, n])$, we have that $\hat{\mu}_n \to \hat{\mu}$ uniformly as $n \to \infty$, therefore $\hat{\mu}$ is uniformly continuous (indeed, given $\varepsilon > 0$, find n such that $\sup|\hat{\mu}_n - \hat{\mu}| < \varepsilon/2$ and $\delta = \varepsilon/(2n)$ to obtain $|\hat{\mu}_n(\xi) - \hat{\mu}_n(\eta)| \le \varepsilon/2$ whenever $|\xi - \eta| < \delta$, and then $|\hat{\mu}(\xi) - \hat{\mu}(\eta)| < \varepsilon$).

Exercise 6.29. Obviously $|\hat{\mu}(\xi_0)| = 1$, and we set $c = \hat{\mu}(\xi_0) = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Since

$$\int_{\mathbb{R}} |1 - \bar{c}e^{ix\xi_0}|^2\, d\mu(x) = 2 - \bar{c}c - c\bar{c} = 0,$$

we obtain that $e^{ix\xi_0} = c$ for μ–a.e. $x \in \mathbb{R}$. This implies that $x\xi_0 - \theta \in 2\pi\mathbb{Z}$ for μ–a.e. $x \in \mathbb{R}$, so that μ is concentrated on the set of points $\{(2n\pi + \theta)/\xi_0\}_{n\in\mathbb{N}}$, and it suffices to set $x_0 = \theta/\xi_0$ to obtain the stated representation of μ as a sum of Dirac masses.
Obviously $|\hat{\mu}| \equiv 1$ if μ is a Dirac mass. Conversely, if $|\hat{\mu}| \equiv 1$, we find x_0 with $\mu(\{x_0\}) > 0$ and $\xi_0,\ \xi_0' \in \mathbb{R} \setminus \{0\}$ with $\xi_0/\xi_0' \notin \mathbb{Q}$ to obtain that μ is concentrated on the set $\{2n\pi/\xi_0 + x_0\}_{n\in\mathbb{N}}$ and on the set $\{2n\pi/\xi_0' + x_0\}_{n\in\mathbb{N}}$. By our choice of ξ_0 and ξ_0', the intersection of the two sets is the singleton $\{x_0\}$, and this proves that $\mu = \delta_{x_0}$.

Chapter 7

Exercise 7.1. Let $C > 0$ be such that $|H(x) - H(y)| \le C|x - y|$ for all $x,\ y \in \mathbb{R}$. Let $\varepsilon > 0$ and let $\delta > 0$ be such that $\sum_i |f(b_i) - f(a_i)| < \varepsilon/C$ whenever $\sum_i (b_i - a_i) < \delta$. We have $\sum_i |H(f(b_i)) - H(f(a_i))| \le C\sum_i |f(b_i) - f(a_i)|$ whenever $\sum_i (b_i - a_i) < \delta$. In particular, choosing $f(t) = t$, we see that Lipschitz functions are absolutely continuous.

Exercise 7.2. We assume that both $\mathscr{L}^1(E) > 0$ and $\mathscr{L}^1(\mathbb{R} \setminus E) > 0$. Let $a \in \mathbb{R}$ be such that $\mathscr{L}^1((a, \infty) \cap E) > 0$ and $\mathscr{L}^1((a, \infty) \setminus E) > 0$, and define $F(t) = \mathscr{L}^1(E \cap (a, t))$. By our choice of a, $F(t)$ and $(t - a) - F(t)$ are not identically 0 in $(a, +\infty)$.

If $t > a$ is a rarefaction point of E, we have

$$F_+'(t) = \lim_{h\downarrow 0} \frac{F(t + h) - F(t)}{h} = \lim_{h\downarrow 0} \frac{\mathscr{L}^1((t, t + h) \cap E)}{h} = 0.$$

Analogously, $F'_-(t) = 0$ and we find that F' is equal to 0 at all rarefaction points. A similar argument proves that $F' = 1$ at all density points. Let now $t_0 \in (a, \infty)$ where $0 < F(t_0) < (t_0 - a)$ and apply the mean value theorem to obtain $t_0' \in (a, t_0)$ such that

$$F(t_0) = (t_0 - a)F'(t_0').$$

By our choice of t_0 it follows that $F'(t_0') \in (0, 1)$, a contradiction (because either t_0' is a density point or a rarefaction point).

Exercise 7.3. Assume first that φ is continuous and bounded. Let $H(z) := \int_{f(a)}^{z} \varphi(y)\, dy$. By the (classical) fundamental theorem of the integral calculus, H is differentiable and $H'(z) = \varphi(z)$ for all $z \in f(I)$. By the chain rule and Exercise 7.1, the function

$$F(t) := \int_{f(a)}^{f(t)} \varphi(y)\, dy = H(f(t))$$

is absolutely continuous and it has derivative equal to $H'(f(t))f'(t) = \varphi(f(t))f'(t)$ at all points t where f is differentiable. On the other hand, still by the fundamental theorem of the integral calculus, the function

$$G(t) := \int_a^t \varphi(f(x))f'(x)\, dx$$

has derivative equal to $(\varphi \circ f)f'$ $\mathscr{L}^1$–a.e. in $[a, b]$. Since both F and G vanish at $t = a$, they coincide.

By the dominated convergence theorem, the identity of the two functions persists if $\varphi = \mathbb{1}_A$, with A open (because $\mathbb{1}_A$ is the pointwise limit of continuous functions). By applying Dynkin's theorem to the class $\mathscr{M}$ of the sets $E \in \mathscr{B}(f(I))$ such that $\int_{f(a)}^{f(t)} \mathbb{1}_E(y)\, dy = \int_a^t \mathbb{1}_E(f(x))f'(x)\, dx$ we obtain that the formula holds for all $\varphi = \mathbb{1}_E$ with E Borel. Eventually we obtain it for simple functions and, by uniform approximation, for bounded Borel functions.

Exercise 7.4. Choosing $g = \mathbb{1}_N$, by Exercise 7.3 we get $\int_a^b \mathbb{1}_{f^{-1}(N)} f'\, dx = 0$, because $\mathbb{1}_N \circ f = \mathbb{1}_{f^{-1}(N)}$. Let h^+ and h^- be respectively the positive and negative part of $f'\mathbb{1}_{f^{-1}(N)}$. Since

$$\int_a^b h^+\, dx - \int_a^b h^-\, dx = \int_a^b f'\mathbb{1}_{f^{-1}(N)}\, dx = 0$$

for all intervals (a, b), it follows that $h^+ = h^-$ $\mathscr{L}^1$–a.e. in $\mathbb{R}$. As a consequence, $f' = 0$ $\mathscr{L}^1$–a.e. in $f^{-1}(N)$.

Chapter 8

Exercise 8.1. Both are measures in $(Z, \mathscr{H})$. If $B \in \mathscr{H}$ then $g \circ f_\# \mu(B) = \mu(f^{-1}(g^{-1}(B)))$, because $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. On the other hand,

$$g_\#(f_\# \mu)(B) = f_\# \mu(g^{-1}(B)) = \mu(f^{-1}(g^{-1}(B))).$$

Exercise 8.2. Let $n \geq 1$ integer, $0 \leq k < 2^n$ and let us consider the interval $I = [k/2^n, (k+1)/2^n)$. Then, $f^{-1}(I)$ is the cylindrical set of all binary sequences $a_0 a_1 \cdots$ such that $a_0 \cdots a_{n-1}$ is the binary expression of k. It follows that

$$\mathop{\times}_{i=0}^{\infty} \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \left(f^{-1}(I)\right) = \mathscr{L}^1(I).$$

because their common value is 2^{-n}. On the other hand, $f^{-1}(\{1\})$ consists of a single point and therefore the identity above holds for $I = \{1\}$, the common value being 0. By additivity the identity holds for finite unions of sets of this type, a family stable under finite intersections. By the coincidence criterion the two measures coincide.

Exercise 8.3. Let $A \subset \mathbb{R}$ be a dense open set whose complement C has strictly positive Lebesgue measure (Exercise 1.9), and let

$$\varphi(t) := \min\{1, \mathrm{dist}(t, C)\} \qquad t \in \mathbb{R}.$$

By construction the function φ is continuous, nonnegative, bounded by 1, and vanishes precisely on C. Then, set

$$F(t) := \begin{cases} \displaystyle\int_0^t \varphi(s)\, ds & \text{if } t \geq 0; \\ \displaystyle -\int_t^0 \varphi(s)\, ds & \text{if } t < 0. \end{cases}$$

We have $F' = \varphi$, so that $F \in C^1$ and its critical set $C_F = C$ has positive Lebesgue measure. It follows that $F_\# \mathscr{L}^1$ is not absolutely continuous with respect to $\mathscr{L}^1$. Finally, since $\int_a^b \varphi\, dt > 0$ whenever $a < b$ (because $A \cap (a, b) \neq \emptyset$) we obtain that F is strictly increasing.

Exercise 8.4. Recall that $F(C_F)$ is always Lebesgue negligible, regardless of any injectivity assumption on U. Hence, possibly replacing U by $U \setminus C_F$ we can assume with no loss of generality that $C_F = \emptyset$, *i.e.* $DF(x)$ is nonsingular at any $x \in U$. Recall that, according to the local invertibility theorem, for any $x \in U$ there exists a ball $B_r(x)$ contained in U such that the restriction to F is injective. Now, following the strategy of

Lemma 6.9 we can cover U by a sequence of right open cubes $\{Q_i\}_{i\in I}$, pairwise disjoint, such that the restriction of F to a neighbourhood of Q_i is injective (we keep dividing a cube until this property is achieved). Let $Q_i = \times_{i=1}^n [a_i, a_i+\delta)$; for $b_i < a_i$ sufficiently close to a_i and $\tilde{Q}_i = \times_{i=1}^n (b_i, b_i+\delta)$ we have (by injectivity of F on $\tilde{Q}_i$)

$$F_\#(\mathbb{1}_{\tilde{Q}_i}\mathscr{L}^n) = \frac{1}{|J_F|\circ F^{-1}}\mathbb{1}_{F(\tilde{Q}_i)}\mathscr{L}^n$$

and therefore we can pass to the limit to get

$$F_\#(\mathbb{1}_{Q_i}\mathscr{L}^n) = \frac{1}{|J_F|\circ F^{-1}(y)}\mathbb{1}_{F(Q_i)}\mathscr{L}^n.$$

If we add both sides with respect to $i \in I$ we get

$$F_\#(\mathbb{1}_U\mathscr{L}^n) = \sum_{i\in I}\frac{1}{|J_F|\circ F^{-1}(y)}\mathbb{1}_{F(Q_i)}\mathscr{L}^n = \sum_{x\in F^{-1}(y)}\frac{1}{|J_F|(x)}\mathbb{1}_{F(U)}\mathscr{L}^n.$$

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